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Stability of Picard operators under operator perturbations

Adrian Petruşel and Ioan A. Rus

Dedicated to Professor Mihail Megan on the occasion of his 70th birthday

Abstract. In this paper we study the following problems:
I. Let (M, d) be a complete metric space and $f, g : M \to M$ be two operators. We suppose that:
(a) f is a Picard operator with its unique fixed point x_f^*;
(b) there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for every $x \in M$.
The problem consists in estimating $d(g^n(x), x_f^*)$, for $x \in M$ and $n \in \mathbb{N}^*$.
II. Let B be a Banach space and $f, g : B \to B$ be two operators. We suppose that f is a Picard operator. The problem is to find sufficient conditions which guarantee that $f + g$ is a Picard operator.

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1 Introduction

There exist various aspects of the stability problem for the solutions of differential and integral equations (see [1], [5], [18], [21], [26], [31], [37], ...), of operatorial equations ([7], [2], [4], [6], [8], [12], [17], [18], [23], [25], [27], [28],

[29], [30], [33], [34], [36], ...) and of functional-operatorial equations ([3], [10], [11], [14], [15], [16], [20], [22], [24], [32], [33], ...).

In this paper, we will study the following two abstract problems:

Problem I. (see Problem 1.4.2 in [30]) Let (M,d) be a complete metric space and $f,g : M \to M$ be two operators. We suppose that:

(a) f is a Picard operator with its unique fixed point x_f^*;

(b) there exists $\eta > 0$ such that $d(f(x),g(x)) \leq \eta$, for every $x \in M$.

The problem consists in estimating $d(g^n(x),x_f^*)$, for $x \in M$ and $n \in \mathbb{N}^*$.

Problem II. (see Problem 55 in [34]) Let B be a Banach space and $f,g : B \to B$ be two operators. We suppose that f is a Picard operator. The problem is to find sufficient conditions which guarantee that $f+g$ is a Picard operator.

We will present first some basic notions and results which are essential for the main part of this paper.

Let X be a nonempty set and $f : X \to X$ be an operator. We consider the fixed point equation

$$x = f(x),\ x \in X. \tag{1.1}$$

We denote by F_f the fixed point set of f, i.e., $F_f := \{x \in X \mid f(x) = x\}$. If (M,d) is a metric space, then, by definition, f is a Picard operator if $F_f = \{x^*\}$ and

$$f^n(x) \to x^* \text{ as } n \to \infty,\ \text{for all } x \in M.$$

A Picard operator f for which there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and satisfying $\psi(0) = 0$, such that

$$d(x,x^*) \leq \psi(d(x,f(x)),\ \text{for all } x \in X,$$

is called a ψ-Picard operator. In particular, if ψ has a linear form (i.e., $\psi(t) = ct, t \in \mathbb{R}_+$, for some $c > 0$), then f is called a c-Picard operator.

Definition 1.1.

Let (M,d) be a metric space and $f : M \to M$ be an operator. Then, f is called:

(i) an α-contraction if $\alpha \in]0,1[$ and

$$d(f(x),f(y)) \leq \alpha d(x,y),\ \textit{for every } x,y \in M.$$

(ii) a Kannan operator if there exists $\beta \in]0,\frac{1}{2}[$ such that

$$d(f(x),f(y)) \leq \beta\left(d(x,f(x)) + d(y,f(y))\right),\ \textit{for every } x,y \in M.$$

(iii) a φ-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function (i.e., φ is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, for every $t \in \mathbb{R}_+$) and

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for every } x, y \in M.$$

(iv) a Hardy-Rogers operator if there are $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha+2\beta+2\gamma < 1$, such that, for every $x, y \in M$, we have

$$\begin{aligned} d(f(x), f(y)) \leq & \alpha d(x, y) + \beta\left(d(x, f(x)) + d(y, f(y))\right) \\ & + \gamma\left(d(x, f(y)) + d(y, f(x))\right). \end{aligned}$$

Example 1.1.
1) If (M, d) is a complete metric space and $f : M \to M$ is an α-contraction, then f is a $\frac{1}{1-\alpha}$-Picard operator.
2) If (M, d) is a complete metric space and $f : M \to M$ is Kannan operator with constant β, then f is a $\frac{1-\beta}{1-2\beta}$-Picard operator.
3) If (M, d) is a complete metric space and $f : M \to M$ is Hardy-Rogers operator with constants α, β, γ, then f is a $\frac{1-\beta-\gamma}{1-\alpha-2\beta-2\gamma}$-Picard operator.
4) If (M, d) is a complete metric space and $f : M \to M$ is φ-contraction, such that φ is a strict comparison function (i.e., φ is a comparison function and $t - \varphi(t) \to \infty$ as $t \to \infty$, then the operator f is a ψ_φ-Picard operator, with $\psi_\varphi(t) := \sup\{s \in \mathbb{R}_+ : s - \varphi(s) \leq t\}$.

Other examples of Picard operator involving function spaces are given now.

Example 1.2.
Let $(B, |\cdot|)$ be a Banach space and $g \in C([a, b] \times B, B)$ be an operator. We suppose that there exists $l_g \in]0, 1[$ such that

$$|g(t, u) - g(t, v) \leq l_g|u - v|, \forall t \in [a, b], \ \forall u, v \in B.$$

Consider $T : C([a, b], B) \to C([a, b], B)$ defined by

$$Tx(t) := g(t, x(t)), t \in [a, b].$$

If we consider the Banach space $\mathbb{B} := (C([a, b], B), \|\cdot\|)_\infty)$, then the operator $T : \mathbb{B} \to \mathbb{B}$ is an l_g-contraction and, hence, a Picard operator.

Example 1.3.
Let $(B, |\cdot|)$ be a Banach space and $K \in C([a, b] \times [a, b] \times B, B)$ be an operator. We suppose that there exists $l_K \in]0, 1[$ such that

$$|K(t, s, u) - K(t, s, v) \leq l_K|u - v|, \forall t, s \in [a, b], \ \forall u, v \in B.$$

Consider $S : C([a,b],B) \to C([a,b],B)$ defined by

$$Sx(t) := \int_a^t K(t,s,x(s))ds, t \in [a,b].$$

If we consider the Banach space $\mathbb{B} := (C([a,b],B), \|\cdot\|)_\tau)$ (where $\|\cdot\|_\tau$ is a Bielecki type norm with $\tau > l_K$), then the operator $S : \mathbb{B} \to \mathbb{B}$ is $\frac{l_K}{\tau}$-contraction and, hence, a Picard operator. Moreover, S is a strong contraction in the sense of Goebel. Recall that if $(B, \|\cdot\|)$ is a Banach space, then an operator $f : B \to B$ is called a strong contraction if for every $\epsilon > 0$ there exists a norm $\|\cdot\|_\epsilon$ on B, equivalent with $\|\cdot\|$, such that

$$\|f(x) - f(y)\|_\epsilon \leq \epsilon\|x - y\|_\epsilon, \text{ for every } x, y \in B.$$

For details see [13] and [34].

2 Results concerning Problem I.

For a better understanding of the problem, we present several examples.

Example 2.1.
If (M,d) is a complete metric space and $f : M \to M$ is an α-contraction, then, for Problem I, we have the following result.

Theorem 2.1. *Let (M,d) is a complete metric space, $f : M \to M$ be an α-contraction and $g : M \to M$ be an operator for which there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for every $x \in M$. Denote by x_f^* the unique fixed point of f. Then*

$$d(g^n(x), x_f^*) \leq \frac{\eta}{1-\alpha} + \frac{\alpha^n}{1-\alpha} d(x, f(x)), \forall x \in M, \forall n \in \mathbb{N}^*.$$

Proof. For $x \in M$, we have that

$$d(g^n(x), x_f^*) \leq d(f^n(x), x_f^*) + d(f^n(x), g^n(x)) \leq$$

$$\frac{\alpha^n}{1-\alpha} d(x, f(x)) + d(f^n(x), g^n(x)).$$

On the other hand

$$d(f^n(x), g^n(x)) \leq d(f(g^{n-1}(x)), g(g^{n-1}(x))) + d(f(g^{n-1}(x)), f(f^{n-1}(x))) \leq$$

$$\eta + \alpha d(f^{n-1}(x), g^{n-1}(x)) \leq \eta + \alpha\eta + \alpha^2 d(f^{n-2}(x), g^{n-2}(x)) \leq \cdots \leq \frac{\eta}{1-\alpha}.$$

Thus, the conclusion follows by the above two relations. □

Example 2.2.
Let (M, d) be a complete metric space and $f : M \to M$ be a φ-contraction. Suppose that, in addition, φ satisfies the following assumptions:
(a) $t - \varphi(t) \to \infty$, as $t \to \infty$ (i.e., φ is a strict comparison function);
(b) $\varphi(t_1 + t_2) \leq \varphi(t_1) + \varphi(t_2), \forall t_1, t_2 \in \mathbb{R}_+$.
Then, for Problem I, we have the following result.

Theorem 2.2. *Let (M, d) be a complete metric space and $f : M \to M$ be a strong φ-contraction satisfying (a)-(b) from above. Denote by x_f^* the unique fixed point of f. If $g : M \to M$ is an operator for which there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$ for every $x \in M$, then we have that*

$$d(g^n(x), x_f^*) \leq \theta(\eta) + \varphi^n(\psi_\varphi(d(x, f(x)))), \forall x \in M, \forall n \in \mathbb{N}^*,$$

where $\theta(t) := \sum_{n=0}^{\infty} \varphi^n(t)$ and $\psi_\varphi(t) = \sup\{s \in \mathbb{R}_+ : s - \varphi(s) \leq t\}$, for $t \in \mathbb{R}_+$.

Proof. If φ is a strict comparison function, then the function $\psi_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\psi_\varphi(t) = \sup\{s \in \mathbb{R}_+ : s - \varphi(s) \leq t\}$ is increasing and has the property that $\psi_\varphi(t) \to 0_+$ as $t \to 0_+$. Then, for each $x \in M$, we have

$$d(g^n(x), x_f^*) \leq d(f^n(x), x_f^*) + d(f^n(x), g^n(x)).$$

On one hand

$$d(f^n(x), x_f^*) \leq \varphi^n(d(x, x_f^*))$$

with

$$d(x, x_f^*) \leq \psi_\varphi(d(x, f(x))).$$

On the other hand

$$d(f^n(x), g^n(x)) \leq d(f(g^{n-1}(x)), g(g^{n-1}(x))) + d(f(g^{n-1}(x)), f(f^{n-1}(x))) \leq$$

$$\eta + \varphi(d(f^{n-1}(x), g^{n-1}(x))) \leq \eta + \varphi(\eta + \varphi(d(f^{n-2}(x), g^{n-2}(x)))) \leq$$

$$\eta + \varphi(\eta) + \varphi^2(d(f^{n-2}(x), g^{n-2}(x))) \leq \cdots \leq \sum_{n=0}^{\infty} \varphi^n(\eta) = \theta(\eta).$$

Thus, the conclusion follows by the above two relations. □

We formulate now two open problems.

Problem A. *For which generalized contractions (Kannan operators, Hardy-Rogers operators, ...) in complete metric spaces we have similar results to the above ones ?*

Problem B. *A second open question is to study the above problem in the case of generalized metric spaces.*

Concerning the last problem, we have the following result in complete $\mathbb{R}^m_+$-metric spaces.

Example 2.3.
If (M, d) is a complete $\mathbb{R}^m_+$-metric space and $f : M \to M$ is an A-contraction (i.e., $A \in \mathcal{M}_{m,m}$ is a matrix convergent to zero and $d(f(x), f(y)) \leq Ad(x, y)$, $\forall x, y \in M$), then we have the following result.

Theorem 2.3. *Let (M, d) is a complete $\mathbb{R}^m_+$-metric space, $f : M \to M$ be an A-contraction and $g : M \to M$ be an operator for which there exists $\eta = (\eta_1, \cdots, \eta_m) \in \mathbb{R}^m_+$ with $\eta_i > 0$ for every $i \in \{1, \cdots, m\}$, such that $d(f(x), g(x)) \leq \eta$, for every $x \in M$. Denote by x^*_f the unique fixed point of f. Then, we have that*

$$d(g^n(x), x^*_f) \leq (I_m - A)^{-1}\eta + A^n(I_m - A)^{-1}d(x, f(x)), \forall x \in M, \forall n \in \mathbb{N}^*.$$

Remark 2.1.
Consider Problem I with $f : M \to M$ be a ψ-Picard operator. In this case, we have

$$d(g^n(x), x^*_f) \leq \psi(d(g^n(x), f(g^n(x))), \forall x \in M, \forall n \in \mathbb{N}^*.$$

By this relation, we obtain that, if g is continuous and the sequence $(g^n(x))_{n\in\mathbb{N}}$ is f-asymptotically regular (i.e., $d(g^n(x), f(g^n(x))) \to 0$ as $n \to \infty$ for all $x \in M$), then g is a Picard operator with $x^*_g = x^*_f$.

3 Results concerning Problem II.

We start this section by the following result of Browder-Petryshyn (see [7]), given in terms of Picard operators.

Theorem 3.1. *Let B be a Banach space and $f : B \to B$ be a bounded linear operator. Suppose that f is Picard. Let $g : B \to B$ be a constant operator, i.e., there exists $y \in B$ such that $g(x) = y$, for every $x \in B$. Then, for each $y \in (1_B - f)(B)$ the operator $f + g$ is Picard.*

The main result of this section is the following.

Theorem 3.2. *Let B be a Banach space and $f, g : B \to B$ be two operators. We suppose:*
(1) f is a strong contraction with respect to equivalent norms $\|\cdot\|_\epsilon$, where $0 < \epsilon < 1$;
(2) g is an l_g-contraction with respect to each $\|\cdot\|_\epsilon$, for $0 < \epsilon < 1$.
Then, the operator $f + g$ is Picard.

Proof. Let $\epsilon > 0$ such that $l_g + \varepsilon < 1$. Then, the operator $f + g$ is a $(l_g + \varepsilon)$-contraction with respect to $\|\cdot\|_\epsilon$. □

Example 3.1.
Let $(B, |\cdot|)$ be a Banach space and T be given by Example 1.2. Since there exists $l_g \in]0, 1[$ such that

$$|g(t, u) - g(t, v) \leq l_g|u - v|, \forall t \in [a, b],\ \forall u, v \in B,$$

we get that the operator T is an l_g-contraction with respect to a Bielecki norm $\|\cdot\|_\tau$ on $C([a, b], B)$, for every $\tau > 0$. Let S be the operator given in Example 1.3. Then, $T + S$ is a Picard operator on $(C([a, b], B), \|\cdot\|_\infty)$. Indeed, $T + S$ is a $(l_g + \frac{l_K}{\tau})$-contraction with respect to $\|\cdot\|_\tau$, where $\tau > 0$ is such that $l_g + \frac{l_K}{\tau} < 1$. Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|_\tau$ are Lipschitz equivalent, the conclusion follows.

For other examples of this type see [32]. These examples suggest the following result.

Theorem 3.3. *Let $(B, |\cdot|)$ be a Banach space and $\mathbb{B} := (C([a, b], B), \|\cdot\|_\tau)$ be the Banach space of continuous abstract functions on $[a, b]$ with values in B endowed with a Bielecki norm $\|\cdot\|_\tau$ corresponding to $\tau > 0$. Let $T, S : \mathbb{B} \to \mathbb{B}$ be two operators. We suppose that:*
(1) there exists $l_T \in]0, 1[$ such that

$$|Tx(t) - Ty(t)| \leq l_T|x(t) - y(t)|, \forall t \in [a, b], \forall x, y \in \mathbb{B};$$

(2) there exists $c > 0$ such that

$$|Sx(t) - Sy(t)| \leq \frac{c}{\tau}\|x - y\|_\tau e^{\tau(t-a)}, \forall t \in [a, b], \forall x, y \in \mathbb{B} \text{ and } \tau > 0.$$

Then, the operators T, S and $T + S$ are Picard in $((C([a, b], B), \|\cdot\|_\infty)$

Proof. We will prove that $T+S$ is a Picard operator. Indeed, for all $t \in [a, b]$, $x, y \in \mathbb{B}$ and $\tau > 0$, we have

$$|(T + S)x(t) - (T + S)y(t)| \leq l_T|x(t) - y(t)| + \frac{c}{\tau}\|x - y\|_\tau e^{\tau(t-a)}.$$

Then
$$\|(T+S)x-(T+S)y\|_\tau \le (l_T+\frac{c}{\tau})\|x-y\|_\tau.$$
Since $l_T<1$, there exists $\tau>0$ such that $l_T+\frac{c}{\tau}<1$. This shows that $T+S$ is a contraction and hence a Picard operator. □

Example 3.2.
Let $f,g:\mathbb{R}\to\mathbb{R}$ be two mappings defined by $f(x)=\frac{1}{3}x$ and $g(x)=-x$. In this case, f is a Picard operator, g is not a Picard operator, but $f+g:\mathbb{R}\to\mathbb{R}$ is a Picard operator.

By the above considerations, the following open question also arises.
Problem C. Let B be a Banach space and $f:B\to B$ be an operator. In which conditions the operator 1_B-f is Picard ? Some basic references for this problem are [8], [9], [25], ...

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Adrian Petruşel

Department of Mathematics, Babeş-Bolyai University
M. Kogălniceanu Street, no. 1
Cluj-Napoca, Romania

Academy of Romanian Scientists,
Bucharest, Romania

E-mail: `petrusel@math.ubbcluj.ro`

Ioan A. Rus

Department of Mathematics, Babeş-Bolyai University
M. Kogălniceanu Street, nr. 1
Cluj-Napoca, Romania

E-mail: `iarus@math.ubbcluj.ro`