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STABILITY OF RANDOM VARIABLES AND ITERATED LOGARITHM LAWS FOR MARTINGALES AND QUADRATIC FORMS

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Strong laws of large numbers, obtained for positive, independent random variables, are utilized to prove iterated logarithm laws (with a nonrandom normalizing sequence) for a class of martingales. A law of the iterated logarithm is also established for certain random quadratic forms.

1. Introduction. Prokhorov [6] used an exponential inequality (see [6], [8] or Lemma 2.1 below) to give necessary and sufficient conditions for the classical strong law of large numbers for suitably bounded independent random variables. In this paper Prokhorov's inequality is exploited to give conditions under which $S_n/\varphi(b_n) \to 0$ a.s., where $S_n = \Sigma_1^n X_j$, $\{X_j, j \geq 1\}$ are independent random variables with $EX_j = 0$, $0 < b_n \uparrow \infty$ is a sequence of real numbers, and φ is a function of polynomial order. As a consequence (Corollary 2.3) $\Sigma_1^n X_i^2/s_n^2 \to 1$ a.s., where $s_n^2 = \Sigma_1^n EX_i^2$ under the hypotheses of Kolmogorov's law of the iterated logarithm (LIL) [4] and this is extended to certain unbounded random variables. Under similar conditions $\Sigma_1^n X_i^2/s_n^2 \log_2 s_n^2 \to 0$ a.s., where $\log_2 x = \log\log x$.

Using different techniques, stability results are shown to hold for weighted independent identically distributed (i.i.d.) random variables in Section 3. In Section 4, the results of the previous sections are utilized to obtain a LIL for a class of martingales (U-statistics). Finally, in Section 5 a LIL is proved for a class of random quadratic forms.

2. Stability results for sums of independent random variables. Throughout this section $\{X_n, n \geq 1\}$ will be a sequence of independent random variables with $EX_n = 0$, $EX_n^2 = \sigma_n^2 < \infty$ and $s_n^2 = \Sigma_1^n \sigma_i^2 \to \infty$. Let $S_n = \Sigma_1^n X_i$. The following lemma is due to Prokhorov [6].

LEMMA 2.1. *Let* $c > 0$ *and suppose that for all* n, $|X_i| \leq cs_n$ a.s. *for all* i, $1 \leq i \leq n$. *Then if* $\varepsilon > 0$

$$P\{S_n/s_n \geq \varepsilon\} \leq \exp[(-\varepsilon/2c)\operatorname{arcsinh}(\varepsilon c/2)].$$

PROOF. See Prokhorov [6] or Stout [8, page 262]. ☐

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THEOREM 2.2. *Let φ be a function defined for positive x such that $\varphi(x)/x$ is nondecreasing and $\varphi(x)/x^{\beta}$ is decreasing for some $\beta > 1$; let $0 < b_n \uparrow \infty$ be a sequence of numbers satisfying $b_{n+1}/b_n \to 1$. Suppose that $|X_n| \leqslant \varphi(b_n)/\log_2 b_n$ a.s. for all $n \geqslant 1$ and that $s_n^2 = o(\varphi^2(b_n)/\log_2 b_n)$. Then*

$$S_n/\varphi(b_n) \to 0 \text{ a.s.}$$

PROOF. Let $p > 1$ and for each $k \geqslant 1$, set $n_k = \inf\{n: b_n \geqslant p^k\}$. Since $b_{n+1}/b_n \to 1$ it follows that $b_{n_k} \sim p^k$ for all large k whence $(p+1)/2 \leqslant \varphi(b_{n_{k+1}})/\varphi(b_{n_k}) \leqslant 2p^{\beta}$ for all large k. Hence $\varphi(b_{n_k})$ satisfies the hypotheses of Corollary 1, Loeve [5, page 253] and so it suffices to prove that

$$T_k/\varphi(b_{n_k}) \to 0 \quad \text{a.s.} \quad \text{as } k \to \infty \tag{2.1}$$

where

$$T_k = \Sigma_{n_{k-1}+1}^{n_k} X_j.$$

Let $V_k^2 = ET_k^2$. Then for $n_{k-1} < i \leqslant n_k$,

$$|X_i|/V_k \leqslant \varphi(b_{n_k})/V_k \log_2 b_{n_k} \quad \text{a.s.},$$

so if $\varepsilon > 0$, Lemma 2.1 ensures

$$\begin{aligned} p_k &= P\{|T_k| > \varepsilon\varphi(b_{n_k})\} \\ &\leqslant 2 \exp\left[\left(-\frac{\varepsilon}{2}\log_2 b_{n_k}\right) \operatorname{arcsinh}\left[\frac{\varepsilon\varphi^2(b_{n_k})}{2V_k^2 \log_2 b_{n_k}}\right]\right]. \end{aligned}$$

Now, since $V_k^2 \leqslant s_{n_k}^2 = o(\varphi^2(b_{n_k})/\log_2 b_{n_k})$, it follows that for all large k,

$$\operatorname{arcsinh}\left[\frac{\varepsilon\varphi^2(b_{n_k})}{2V_k^2 \log_2 b_{n_k}}\right] \geqslant 4/\varepsilon$$

so

$$\begin{aligned} p_k &\leqslant 2 \exp(-2 \log_2 b_{n_k}) \\ &\leqslant 2(k \log p)^{-2} \end{aligned}$$

and the Borel-Cantelli lemma implies (2.1). □

Theorem 2.2 can be applied to find norming constants for the stabilization of $\Sigma_1^n X_i^2$.

COROLLARY 2.3. *If*

(i) $X_n^2 \leqslant K s_n^2/\log_2 s_n^2$ a.s. *for all $n \geqslant 1$ and some $K > 0$ and*

$$\Sigma_{i=1}^n \operatorname{Var} X_i^2 = o(s_n^4/\log_2 s_n^2)$$

or

(ii) $X_n^2 < k_n s_n^2/\log_2 s_n^2$ a.s. *for positive constants k_n such that $k_n \to 0$ as $n \to \infty$ then*

$$\Sigma_{i=1}^n X_i^2/s_n^2 \to 1 \quad \text{a.s.}$$

PROOF. Under the hypothesis of (i), apply Theorem 2.2 to the sequence $\{X_n^2 - \sigma_n^2\}$ with $b_n = s_n^2$ and $\varphi(x) = x$. Then $\sigma_n^2 = o(s_n^2)$ implying $s_{n+1}^2/s_n^2 \to 1$ and the conclusion follows.

Under (ii), clearly $\sigma_n^2 = o(s_n^2)$ and

$$\begin{aligned}\Sigma_{i=1}^n \operatorname{Var} X_i^2 &\leqslant (s_n^2/\log_2 s_n^2)\Sigma_{i=1}^n k_i \sigma_i^2 \\ &= o(s_n^4/\log_2 s_n^2)\end{aligned}$$

so the portion already proved yields the result. □

COROLLARY 2.4. *Let* $X_n^2 \leqslant K s_n^2$ a.s. *for all* $n \geqslant 1$ *and some* $K > 0$ *and suppose that* $s_{n+1}^2/s_n^2 \to 1$. *Then*

$$\Sigma_{i=1}^n X_i^2 / (s_n^2 \log_2 s_n^2) \to 0 \quad \text{a.s.}$$

PROOF. We can apply Theorem 2.2 to $\{X_n^2 - \sigma_n^2\}$ with $b_n = s_n^2$ and $\varphi(x) = x \log_2 x$ in view of $\Sigma_1^n \operatorname{Var} X_i^2 = o(s_n^4 \log_2 s_n^2)$. □

Similar results can also be obtained for certain sequences of unbounded independent random variables.

THEOREM 2.5. *Let* $\{b_n\}$ *and* φ *be as in Theorem 2.2. Suppose* $s_n^2 = O(\varphi(b_n))$ *and*

$$\Sigma_{n=1}^\infty P\{X_n^2 > \varphi(b_n)\} < \infty, \tag{2.2}$$

$$\Sigma_{i=1}^n \int_{[x^2 > \varphi(b_i)]} x^2 \, dF_i(x) = o(\varphi(b_n)), \tag{2.3}$$

$$\Sigma_{n=1}^\infty (1/\varphi(b_n)) \int_{[\varepsilon\varphi(b_n)/\log_2 b_n < x^2 \leqslant \varphi(b_n)]} x^2 \, dF_n(x) < \infty \text{ for all } \varepsilon > 0 \tag{2.4}$$

where F_n *is the distribution function of* X_n. *Then*

$$\Sigma_{i=1}^n (X_i^2 - \sigma_i^2)/\varphi(b_n) \to 0 \quad \text{a.s.} \tag{2.5}$$

PROOF. As in Theorem 1 of [10], there exists a sequence $\varepsilon_n \to 0$ such that (2.4) holds with ε replaced by ε_n. Let

$$a_n = \varepsilon_n \varphi(b_n)/\log_2 b_n$$

and set

$$\begin{aligned}X_n^2 &= X_n^2 I_{[X_n^2 \leqslant a_n]} + X_n^2 I_{[a_n < X_n^2 \leqslant \varphi(b_n)]} + X_n^2 I_{[X_n^2 > \varphi(b_n)]} \\ &= Z_n + Y_n + W_n \text{ (say).}\end{aligned}$$

Now, for all large n

$$|Z_n - EZ_n| \leqslant 2a_n \leqslant \varphi(b_n)/\log_2 b_n$$

and

$$\begin{aligned}\Sigma_{i=1}^n \operatorname{Var} Z_i &\leqslant 4\varepsilon_n(\varphi(b_n)/\log_2 b_n)\Sigma_{i=1}^n \sigma_i^2 \\ &= o(\varphi^2(b_n)/\log_2 b_n).\end{aligned}$$

Hence by Theorem 2.2,

$$\Sigma_{i=1}^n (Z_i - EZ_i)/\varphi(b_n) \to 0 \quad \text{a.s.}$$

The ε_n version of (2.4) implies that

$$\Sigma_{n=1}^{\infty}(\text{Var } Y_n)/\varphi^2(b_n) < \infty$$

so by Kolmogorov's strong law of large numbers

$$\Sigma_{i=1}^{n}(Y_i - EY_i)/\varphi(b_n) \to 0 \quad \text{a.s.}$$

Finally, $\Sigma_1^n W_i = 0(1)$ a.s. via (2.2) and, moreover, (2.3) implies that

$$\Sigma_{i=1}^{n} EW_i = o(\varphi(b_n))$$

so

$$\Sigma_{i=1}^{n}(W_i - EW_i)/\varphi(b_n) \to 0 \quad \text{a.s.}$$

and (2.5) follows. □

COROLLARY 2.6. *If for some* $\delta > 0$

(2.6) $$\Sigma_{n=1}^{\infty} P\{X_n^2 > \delta s_n^2\} < \infty,$$

(2.7) $$\Sigma_{i=1}^{n} EX_i^2 I_{[X_i^2 > \delta s_i^2]} = o(s_n^2),$$

(2.8) $$\Sigma_{n=1}^{\infty} \frac{1}{s_n^2} EX_n^2 I_{[\varepsilon s_n^2/\log_2 s_n^2 < X_n^2 \leqslant \delta s_n^2]} < \infty, \qquad \textit{for all } \varepsilon > 0$$

then

(2.9) $$\frac{1}{s_n^2}\Sigma_{i=1}^{n} X_1^2 \to 1 \quad \text{a.s.}$$

PROOF. Set $a_n = \varepsilon s_n^2/\log_2 s_n^2 > 0$ and

$$A_1 = \{X_n^2 \leqslant a_n\}, \quad A_2 = \{a_n < X_n^2 \leqslant \delta s_n^2\}, \quad A_3 = \{X_n^2 > \delta s_n^2\}.$$

Since, via (2.7) and (2.8),

$$\sigma_n^2 = \Sigma_{i=1}^{3} EX_n^2 I_{A_i} \leqslant \frac{\varepsilon s_n^2}{\log_2 s_n^2} + o(s_n^2) = o(s_n^2),$$

necessarily $s_{n+1}^2/s_n^2 \to 1$ and Theorem 2.5 applies with $b_n = \delta s_n^2$, $\varphi(x) = x$. □

COROLLARY 2.7. *If* $s_{n+1}^2/s_n^2 \to 1$ *and for some* $\gamma > 0$

(2.10) $$\Sigma_{n=1}^{\infty} P\{X_n^2 > \gamma s_n^2 \log_2 s_n^2\} < \infty,$$

(2.11) $$\Sigma_{n=1}^{\infty}(s_n^2 \log_2 s_n^2)^{-1} \int_{[\varepsilon s_n^2 < x^2 \leqslant \gamma s_n^2 \log_2 s_n^2]} x^2 \, dF_n(x) < \infty \textit{ for all } \varepsilon > 0$$

then

$$\Sigma_{j=1}^{n} X_j^2 / (s_n^2 \log_2 s_n^2) \to 0 \quad \text{a.s.}$$

PROOF. Apply Theorem 2.5 with $b_n = s_n^2$ and $\varphi(x) = \gamma x \log_2 x$, noting that (2.3) is automatically satisfied. □

3. The weighted i.i.d. case. Conditions under which normed weighted averages $(1/A_n)\sum_{j=1}^n a_j Y_j$ (where $a_n \geqslant 0$, $A_n = \sum_{j=1}^n a_j \to \infty$) of i.i.d. random variables converge almost surely to zero have been given by various authors [3], [1] and [2] (Theorem 5.2.3(ii)). A simple criterion is provided by the next theorem when $\beta = 0$.

THEOREM 3.1. *Let* $\{Y, Y_n, n \geqslant 1\}$ *be* i.i.d. *random variables with* $EY = 0$ *and* $\{a_n, n \geqslant 1\}$ *a sequence of positive constants. If for some* $\beta \geqslant 0$

$$na_n/A_n = O\big((\log_2 A_n)^\beta\big), \quad A_n \to \infty \tag{3.1}$$

then

$$\frac{1}{A_n(\log_2 A_n)^\beta}\sum_{j=1}^n a_j Y_j \to 0 \quad \text{a.s.} \tag{3.2}$$

PROOF. Let $B_n = \{n - 1 < |Y| \leqslant n\}$ and

$$Z_n = \frac{a_n}{A_n(\log_2 A_n)^\beta} Y_n I_{[|Y_n| < n]}.$$

Then for some positive constants C, C'

$$\sum_{n=1}^\infty \sigma_{Z_n}^2 \leqslant \sum_{n=1}^\infty \frac{a_n^2}{A_n^2(\log_2 A_n)^{2\beta}} EY^2 I_{[|Y| < n]} \leqslant C\sum_{n=1}^\infty \frac{1}{n^2}\sum_{j=1}^n \int_{B_j} Y^2$$

$$\leqslant C'\sum_{j=1}^\infty \frac{1}{j}\int_{B_j} Y^2 \leqslant C'\sum_{j=1}^\infty \int_{B_j}|Y| = C'E|Y| < \infty.$$

By the Khintchine-Kolmogorov theorem and Kronecker's lemma

$$\frac{1}{A_n(\log_2 A_n)^\beta}\sum_{j=1}^n a_j\big(Y_j I_{[|Y_j| < j]} - EYI_{[|Y| < j]}\big) \to 0 \quad \text{a.s.}$$

Since $E|Y| < \infty$, the sequences $\{Y_n\}$ and $\{Y_n I_{[|Y_n| < n]}\}$ are equivalent in the sense of Khintchine and so

$$\frac{1}{A_n(\log_2 A_n)^\beta}\sum_{j=1}^n a_j\big(Y_j - EYI_{[|Y| < j]}\big) \to 0 \quad \text{a.s.}$$

In view of $EY = 0$,

$$-\frac{1}{A_n}\sum_{j=1}^n a_j EYI_{[|Y| < j]} = \frac{1}{A_n}\sum_{j=1}^n a_j EYI_{[|Y| > j]} = o(1)$$

and (3.2) follows. □

COROLLARY 3.1. *If* $\{Y, Y_n, n \geqslant 1\}$ *are* i.i.d. *random variables and* $\{a_n, n \geqslant 1\}$ *are positive constants satisfying* (3.1) *with* $\beta = 0$, *then*

$$\frac{1}{A_n}\sum_{j=1}^n a_j Y_j^2 \to EY^2 \quad \text{a.s.} \tag{3.3}$$

PROOF. If $EY^2 < \infty$, Theorem 3.1 with $\beta = 0$ yields

$$\frac{1}{A_n}\Sigma_{j=1}^n a_j\left(Y_j^2 - EY^2\right) \to 0 \quad \text{a.s.},$$

which is tantamount to (3.3). Hence, the case $EY^2 = \infty$ follows by truncation and monotonicity. □

COROLLARY 3.2. *If* $\{Y, Y_n, n \geqslant 1\}$ *are* i.i.d. *random variables with* $EY^2 < \infty$ *and* $\{a_n, n \geqslant 1\}$ *are positive constants satisfying* (3.1) *for some* $\beta > 0$, *then*

$$\frac{1}{A_n(\log_2 A_n)^\beta}\Sigma_{j=1}^n a_j Y_j^2 \to 0 \quad \text{a.s.} \tag{3.4}$$

4. A LIL for a class of martingales. In contradistinction to the case of independent random variables, the formulation of an LIL for martingales [7] involves a sequence of positive random variables in the denominator as opposed to positive constants. Here, an LIL with constant denominator will be proved for a class of martingales despite the randomness of their conditional variances. The tools are an identity of [9] and the results of Sections 2 and 3.

For any sequence of random variables $\{X_n, n \geqslant 1\}$ and any positive integer k, define

$$U_{k,n} = \Sigma X_{i_1} \cdots X_{i_k}, \qquad n \geqslant k,\ U_{k,o} = 0 \tag{4.1}$$

where the sum is over $1 \leqslant i_1 < \cdots < i_k \leqslant n$. If $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$ and $\{X_n, n \geqslant 1\}$ are independent with $EX_n = 0$, $EX_n^2 = \sigma_n^2$, the identity $U_{k,n} - U_{k,n-1} = X_n U_{k-1,n-1}$ reveals that for $k \geqslant 2$, $\{U_{k,n}, \mathcal{F}_n, n \geqslant k\}$ is a martingale with

$$E\left\{(U_{k,n} - U_{k,n-1})^2 | \mathcal{F}_{n-1}\right\} = \sigma_n^2 U_{k-1,n-1}^2.$$

It will be shown under the same conditions as those of Theorem 1 of [10] for $k \geqslant 2$, $s_n^2 = \Sigma_1^n \sigma_j^2 \to \infty$ and $U_{k,n}$ as in (4.1) that

$$\limsup_{n\to\infty} \frac{U_{k,n}}{\left(2s_n^2 \log_2 s_n^2\right)^{k/2}} = \frac{1}{k!} \quad \text{a.s.} \tag{4.2}$$

Under the natural condition (*) $\sigma_n^2 = o(s_n^2)$,

$$EU_{k,n}^2 = \Sigma_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \sigma_{i_1}^2 \sigma_{i_2}^2 \cdots \sigma_{i_k}^2 \sim \frac{s_n^{2k}}{k!}$$

and so, under (*), (4.2) may be transcribed as

$$\limsup_{n\to\infty} \frac{U_{k,n}}{\left(EU_{k,n}^2(\log_2 EU_{k,n}^2)^k\right)^{\frac{1}{2}}} = \left(\frac{2^k}{k!}\right)^{\frac{1}{2}} \quad \text{a.s.} \tag{4.2$'$}$$

LEMMA 4.1. *If* $\{X_n, n \geq 1\}$ *are random variables and* $\{b_n, n \geq 1\}$ *are constants such that* $0 < b_n \uparrow \infty$ *and*

$$\limsup_{n\to\infty} b_n^{-1}\Sigma_{i=1}^n X_i = 1 = -\liminf_{n\to\infty} b_n^{-1}\Sigma_{i=1}^n X_i \quad \text{a.s.,} \tag{4.3}$$

$$b_n^{-2}\Sigma_{i=1}^n X_i^2 \to 0 \quad \text{a.s.} \tag{4.4}$$

then

$$\limsup_{n\to\infty} b_n^{-k} U_{k,n} = 1/k! \quad \text{a.s.} \tag{4.5}$$

PROOF. According to the identity (4) of [9],

$$(\Sigma_{i=1}^n X_i)^k = k! U_{k,n} + R_{k,n} \tag{4.6}$$

with $R_{k,n}$ a finite linear combination (with coefficients independent of n) of terms

$$\Pi_{i=1}^m \left(\Sigma_{j=1}^n X_j^{h_i}\right), \qquad 1 \leq h_i \leq k, \Sigma_{i=1}^m h_i = k,$$

where $1 \leq m < k$, and so, in view of (4.6) and (4.3), it suffices to show for $1 \leq m < k$ that, with probability one,

$$\Pi_{i=1}^m \left(\Sigma_{j=1}^n X_j^{h_i}\right) = o\left(b_n^k\right). \tag{4.7}$$

Now, if $h_i = 1$ then (4.3) guarantees that

$$\limsup_{n\to\infty} b_n^{-h_i}|\Sigma_{j=1}^n X_j^{h_i}| = 1 \quad \text{a.s.,}$$

whereas if $h_i \geq 2$,

$$0 \leq b_n^{-h_i}|\Sigma_{j=1}^n X_j^{h_i}| \leq \left(b_n^{-2}\Sigma_{j=1}^n X_j^2\right)^{h_i/2} \to 0 \quad \text{a.s.}$$

Since $m < k$ ensures that some $h_i \geq 2$, (4.7) follows. $\square$

THEOREM 4.2. *Let* $\{X_n, n \geq 1\}$ *be independent random variables with* $EX_n = 0$, $EX_n^2 = \sigma_n^2 < \infty$ *and* $s_n^2 = \Sigma_1^n \sigma_i^2 \to \infty$. *If, for some* $\delta > 0$ *and all* $\varepsilon > 0$

$$\Sigma_{n=1}^\infty P\{X_n^2 > \delta s_n^2 \log_2 s_n^2\} < \infty, \tag{4.8}$$

$$\Sigma_{i=1}^n EX_i^2 I_{[X_i^2 > \varepsilon s_i^2/\log_2 s_i^2]} = o\left(s_n^2\right), \tag{4.9}$$

$$\Sigma_{n=1}^\infty \frac{1}{s_n^2 \log_2 s_n^2} EX_n^2 I_{[\varepsilon s_n^2/\log_2 s_n^2 < X_n^2 \leq \delta s_n^2 \log_2 s_n^2]} < \infty \tag{4.10}$$

then (4.2) *holds for every positive integer* k.

PROOF. The hypotheses imply the LIL for $\{X_n\}$ and $\{-X_n\}$ (see Theorem 1 of Teicher [10]) and so (4.3) holds with $b_n = (2s_n^2 \log_2 s_n^2)^{\frac{1}{2}}$. Thus, it may and will be supposed that $k \geq 2$. Via (4.9), $\sigma_n^2 = o(s_n^2)$, that is, $s_{n+1}^2/s_n^2 \to 1$. Since (4.10) clearly guarantees (2.11), Corollary 2.7 ensures (4.4) with b_n as above whence the conclusion (4.2) follows from Lemma 4.1. $\square$

As usual, the conditions simplify greatly in the case of weighted i.i.d. random variables $X_n = \sigma_n Y_n$ where $\{Y_n, n \geq 1\}$ are i.i.d. and $\{\sigma_n, n \geq 1\}$ are constants.

THEOREM 4.3. *Let* $\{Y, Y_n, n \geqslant 1\}$ *be* i.i.d. *random variables with* $EY = 0$, $EY^2 = \sigma^2 \in (0, \infty)$ *and* $\{\sigma_n, n \geqslant 1\}$ *constants with* $s_n^2 = \Sigma_1^n \sigma_i^2 \to \infty$. *If* (i)$\gamma_n \equiv n\sigma_n^2/s_n^2 = O(1)$ *or more generally* (ii) $\sigma_n^2 = o(s_n^2/\log_2 s_n^2)$, $\gamma_n = O((\log_2 s_n^2)^\beta)$, *some* β *in* $(0, 1)$ *then* (4.2) *holds with* $X_n = \sigma_n Y_n/\sigma, n \geqslant 1$.

PROOF. The proof follows in the same fashion as that of the prior theorem once it is noted that (4.3) obtains via Theorem 3 of [10] or its corollary while Corollary 3.2 with $a_j = \sigma_j^2$ yields

$$\frac{1}{s_n^2(\log_2 s_n^2)^\beta} \Sigma_{j=1}^n \sigma_j^2 Y_j^2 \to 0 \quad \text{a.s.}$$

and a fortiori (4.4) with $b_n^2 = 2s_n^2 \log_2 s_n^2$. □

It should be noted that the behavior of $U_{k,n}$ for $k \geqslant 2$ is markedly different from that of $U_{1,n} = \Sigma_1^n X_j$. In fact, via $2U_{2,n} = (\Sigma_1^n X_j)^2 - \Sigma_1^n X_j^2$ it follows that under the conditions of Corollary 2.7, with probability one,

$$\liminf_{n\to\infty} U_{2,n}/s_n^2 \log_2 s_n^2 \geqslant \tfrac{1}{2} \liminf_{n\to\infty}\left(-\Sigma_1^n X_j^2/s_n^2 \log_2 s_n^2\right) = 0.$$

In the i.i.d. case, as is well known, with probability 1 every point of $[-\sigma, \sigma]$ is a limit point of $U_{1,n}/s_n\, (2 \log_2 s_n)^{\frac{1}{2}}$ and so again via the prior identity

$$\liminf_{n\to\infty} \frac{U_{2,n}}{s_n^2 \log_2 s_n^2} = 0, \quad \text{a.s.}$$

This also extends to the non-i.i.d. case.

5. Quadratic forms in i.i.d. random variables. Let $\{a_{ij}, i \geqslant 1, j \geqslant 1\}$ be a real symmetric matrix of infinite order and $\{Y, Y_n, n \geqslant 1\}$ a sequence of i.i.d. random variables with $EY = 0$, $EY^2 = 1$. Then

$$Q_n = \Sigma_{i,j=1}^n a_{ij} Y_i Y_j, \qquad n \geqslant 1 \tag{5.1}$$

constitute random quadratic forms. Varberg [11] has studied the limiting behavior of Q_n, showing for square summable matrices $\{a_{ij}, i \geqslant 1, j \geqslant 1\}$ that $Q_n - EQ_n \to 0$ a.s. and that Q_n converges in quadratic mean if, in addition, $A_n = \Sigma_{j=1}^n a_{jj} \to A$. When $a_{ij} = \sigma_i \sigma_j$ and $A_n = s_n^2 \to \infty$, Theorem 3 of [10] furnishes conditions under which

$$\limsup_{n\to\infty} Q_n/s_n^2 \log_2 s_n^2 = \limsup_{n\to\infty} \left[\frac{\Sigma_{j=1}^n \sigma_j Y_j}{s_n(\log_2 s_n^2)^{\frac{1}{2}}}\right]^2 = 2, \quad \text{a.s.}$$

The same conclusion holds for more general matrices according to

THEOREM 5.1. *Let* Q_n *as defined by* (5.1) *be a sequence of quadratic forms in the* i.i.d. *random variables* $\{Y, Y_n, n \geqslant 1\}$ *with* $EY = 0$, $EY^2 = 1$ *where the constants*

$\{a_{ij}, i \geqslant 1, j \geqslant 1\}$ *form a real, symmetric matrix whose diagonal elements are non-negative. If* $A_n = \sum_{i=1}^n a_{ii} \to \infty$,

$$\sum_{n=2}^{\infty} \frac{1}{(A_n \log_2 A_n)^2} \sum_{i=1}^{n-1} \left(a_{in} - (a_{ii}a_{nn})^{\frac{1}{2}}\right)^2 < \infty \tag{5.2}$$

and either (5.3) $n a_{nn}/A_n = O(1)$ *or*

$$a_{nn} = o\left(\frac{A_n}{\log_2 A_n}\right), \frac{n a_{nn}}{A_n} = O\left((\log_2 A_n)^{\beta}\right), \textit{some } \beta \textit{ in } (0, 1) \tag{5.3$'$}$$

then

$$\limsup_{n\to\infty} \frac{Q_n}{2A_n \log_2 A_n} = 1, \quad \text{a.s.} \tag{5.4}$$

PROOF. If $U_n = \sum_{1 \leqslant i < j \leqslant n} c_{ij} Y_i Y_j$ and $\mathcal{F}_n = \sigma(Y_1, \cdots, Y_n)$ then $\{U_n, \mathcal{F}_n, n \geqslant 2\}$ and $\{S_n = \sum_{j=2}^n b_j^{-1}(U_j - U_{j-1}), \mathcal{F}_n, n \geqslant 2\}$ are martingales where $U_1 = 0$ and $\{b_n, n \geqslant 1\}$ are constants. In view of independence, S_n is an $\mathcal{L}_2$ bounded martingale and hence a.s. convergent provided

$$\sum_{n=2}^{\infty} b_n^{-2} \sum_{i=1}^{n-1} c_{in}^2 < \infty.$$

Hence, setting $b_n = A_n \log_2 A_n$ and $c_{ij} = a_{ij} - (a_{ii}a_{jj})^{\frac{1}{2}}$, (5.2) and Kronecker's lemma ensure that

$$\frac{U_n}{A_n \log_2 A_n} = \frac{1}{A_n \log_2 A_n} \sum_{1 \leqslant i < j \leqslant n} \left(a_{ij} - (a_{ii}a_{jj})^{\frac{1}{2}}\right) Y_i Y_j \to 0 \quad \text{a.s.}$$

Now, with the prior choice of c_{ij},

$$\frac{Q_n}{2A_n \log_2 A_n} = \left(\frac{1}{(2A_n \log_2 A_n)^{\frac{1}{2}}} \sum_{i=1}^n a_{ii}^{\frac{1}{2}} Y_i\right)^2 + \frac{U_n}{A_n \log_2 A_n} \tag{5.5}$$

and so, in view of (5.3)′ or (5.3), the upper limit of the first term on the right of (5.5) is almost surely one according to Theorem 3 and Corollary 1 of [10]. □

Clearly, symmetry of $\{a_{ij}\}$ can be dispensed with if, in addition to (5.2),

$$\sum_{n=2}^{\infty} (A_n \log_2 A_n)^{-2} \sum_{j=1}^{n-1} \left(a_{nj} - (a_{nn}a_{jj})^{\frac{1}{2}}\right)^2 < \infty.$$

It should be noted that the hypotheses of Theorem 5.1 preclude $\{a_{ij}\}$ being a diagonal matrix. In fact, (5.3) or (5.3)′ entails $a_{nn}/A_n = o(1)$ and hence $A_n \sim A_{n-1}$ whence diagonality ensures that for some C in $(0, \infty)$

$$\sum_{n=2}^{\infty} \frac{1}{(A_n \log_2 A_n)^2} \sum_{i=1}^{n-1} \left(a_{in} - (a_{ii}a_{nn})^{\frac{1}{2}}\right)^2$$

$$= \sum_{n=2}^{\infty} \frac{a_{nn} A_{n-1}}{(A_n \log_2 A_n)^2} \geqslant C \sum_{n=2}^{\infty} \frac{a_{nn}}{A_n (\log_2 A_n)^2} = \infty$$

since the partial sum of N terms of the last series exceeds $\log A_N / (\log_2 A_N)^2$.

REFERENCES

[1] Chow, Y. S. and Teicher, H. (1971). Almost certain summability of independent, identically distributed random variables. *Ann. Math. Statist.* **42** 401–404.

[2] Chow, Y. S. and Teicher, H. (1978). *Probability Theory: Independence, Interchangeability, Martingales.* Springer-Verlag, New York.

[3] Jamison, Orey and Pruitt (1965). Convergence of weighted averages of independent random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **4** 40–44.

[4] Kolmogorov, A. (1929). Uber das Gesetz des iterierten Logarithmus. *Math. Ann.* **101** 126–135.

[5] Loeve, M. (1960). *Probability Theory, 2nd ed.* Van Nostrand, Princeton, New Jersey.

[6] Prokhorov, Yu. V. (1959). Some remarks on the strong law of large numbers. *Theor. Probability Appl.* **4** 204–208.

[7] Stout, W. F. (1970). A martingale analogue of Kolmogorov's law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3** 279–290.

[8] Stout, W. F. (1974). *Almost Sure Convergence.* Academic Press, New York.

[9] Teicher, H. (1968). Some new conditions for the strong law. *Proc. Nat. Acad. Sci. U.S.A.* **59** 705–707.

[10] Teicher, H. (1974). On the law of the iterated logarithm. *Ann. Probability* **2** 714–728.

[11] Varberg, D. E. (1966). Convergence of quadratic forms in independent random variables. *Ann. Math. Statist.* **37** 567–576.

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