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Stability of solitary waves for nonlinear Schrödinger equations with inhomogeneous nonlinearities

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Abstract

We show that the ground-state solitary waves of the *critical* nonlinear Schrödinger equation $i\psi_t(t,r)+\Delta\psi+V(\epsilon r)|\psi|^{4/d}\psi = 0$ in dimension $d \geq 2$ are orbitally stable as $\epsilon \to 0$ if $V(0)V^{(4)}(0) < G_d[V''(0)]^2$, where G_d is a constant that depends only on d.

Keywords: Solitary waves; Nonlinear Schrödinger equation; Inhomogeneous nonlinearities; Stability

1. Introduction

The critical nonlinear Schrödinger equation

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + |\psi|^{4/d}\psi = 0, \qquad \psi(0,\mathbf{x}) = \psi_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{1.1}$$

models the propagation of intense laser beams in a homogeneous bulk medium with a Kerr nonlinearity. It is well known that solutions of (1.1) can become singular in finite time if $|\psi_0|_2^2 \geq N_c$, where $|\psi_0|_2^2 = \int |\psi_0|_2^2\,d\mathbf{x}$ is the input beam power, and N_c, the *critical power* for singularity formation, is a constant which depends only on d. The critical power N_c, thus, sets an upper limit on the amount of *power* ($|\psi|_2^2$) that can be propagated with a single beam. The critical NLS (1.1) admits solitary waves $\psi = e^{i\omega t}R_\omega(\mathbf{x})$ whose power is exactly equal to the critical power, i.e., $|R_\omega|_2^2 \equiv N_c$ [17]. These solitary waves are, however, strongly unstable. As a result, it is not possible to realize stable high-power propagation in a homogeneous bulk media.

A few years ago, it was suggested that stable high-power propagation can be achieved in plasma by sending a preliminary laser beam that creates a channel with a reduced electron density, and thus reduces the nonlinearity inside the channel [4,8]. Under these conditions, beam propagation can be modeled, in the simplest case, by the inhomogeneous nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi + V(\epsilon\mathbf{x})|\psi|^{4/d}\psi = 0, \tag{1.2}$$

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where $V(\epsilon\mathbf{x})$ is proportional to the electron density and ϵ is a small parameter. It is possible to set the experimental system so that both the potential V and the initial condition ψ_0 are radially symmetric, i.e., $V = V(r)$ and $\psi_0 = \psi_0(r)$, where $r = |\mathbf{x}|$. In this case, the equation for ψ is

$$\mathrm{i}\psi_t(t,r) + \Delta\psi + V(\epsilon r)|\psi|^{4/d}\psi = 0, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}. \tag{1.3}$$

Existence and nonexistence of blowup solutions of (1.2) were studied by Merle for certain types of inhomogeneities [10]. These results imply that a necessary condition for blowup in the radially symmetric case (1.3) is that $|\psi_0|_2^2 \geq N_{\mathrm{c}}/V^{d/2}(0)$. For comparison, in the absence of the preliminary beam $V \equiv V(\infty)$ and the critical power is $N_{\mathrm{c}}/V^{d/2}(\infty)$. We thus see that it is possible to raise the critical power for blowup by lowering the magnitude of the nonlinearity near the origin. In particular, when $V(0) = 0$ all solutions of (1.3) exist globally.

The solitary waves of (1.3) are given by $\psi = \mathrm{e}^{\mathrm{i}\omega t}\phi_\omega(r)$, where ϕ_ω is the solution of

$$\Delta\phi_\omega(r) - \omega\phi_\omega + V(\epsilon r)\phi_\omega^{4/d+1} = 0, \quad \phi'_\omega(0) = 0, \ \ \phi_\omega(\infty) = 0. \tag{1.4}$$

The following theorem gives the existence of positive (ground-state) solitary waves.

Theorem 1. *Let* $0 < V(r) < C$ *and let* $\omega > 0$. *Then,*

(1) *there exists a positive solution to* (1.4).
(2) *the positive solution is unique when* ϵ *is small enough*.

See Section 3 for the proof of part (1) and Section 4 for the proof of part (2).

We note that existence of solutions of (1.4) was proved in [15] in the framework of a more general equation. Our proof is, however, considerably simpler because of radial symmetry. The method used in the existence proof was originally due to Strauss [14] and to Berestycki and Lions [1].

When the inhomogeneity is induced by the preliminary laser beam, $V(r)$ increases monotonically from $V(0)$ to $V(\infty)$. In that case,

$$\lim_{\epsilon\to 0}|\phi_\omega|_2^2 = \frac{N_{\mathrm{c}}}{V^{d/2}(0)} > \frac{N_{\mathrm{c}}}{V^{d/2}(\infty)} = \lim_{\epsilon\to\infty}|\phi_\omega|_2^2. \tag{1.5}$$

Therefore, it is reasonable to assume that when $0 < \epsilon \ll 1$, the power of the solitary waves will be below the critical power for blowup $N_{\mathrm{c}}/V^{d/2}(0)$. In that case, one can expect the solitary waves to be stable, because solitary waves in NLS equations are typically unstable if and only if a small perturbation can lead to singularity formation. Surprisingly, however, our results show that monotonicity of V is not the correct condition for stability.

Throughout this paper, we make the following assumptions on V:

$$V > 0, \qquad V \in C^4 \cap L_\infty, \qquad |V^{(i)}(r)| \leq C\,\mathrm{e}^r \quad \text{for } i = 1, 2, 3, 4, \tag{1.6}$$

where $V^{(i)}$ is the ith derivative of V. We note that these assumptions are consistent with the electron density induced by the preliminary beam.

The natural definition of stability of solitary waves is the one of *orbital stability*.

Definition. Let ϕ_ω be a solution of (1.4). We say that $\psi(r,t) = \mathrm{e}^{\mathrm{i}\omega t}\phi_\omega(r)$ is an orbitally stable solution of (1.3) if $\forall\epsilon > 0, \exists\,\delta > 0$ such that for any $\tilde\psi(r,0) \in H^1(R^n)$ which satisfies $\inf_\theta|\tilde\psi(r,0) - \mathrm{e}^{\mathrm{i}\theta}\phi_\omega|_{H^1} < \delta$, the corresponding solution $\tilde\psi(r,t)$ of (1.3) satisfies

$$\sup_t \inf_\theta |\tilde\psi(r,t) - \mathrm{e}^{\mathrm{i}\theta}\phi_\omega|_{H^1} < \epsilon.$$

Our stability proof follows [5,7,9,12]. We define

$$d(\omega) = E(\phi_\omega) + \omega Q(\phi_\omega), \tag{1.7}$$

where

$$E(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{4/d+2}\int V(\epsilon r)|u|^{4/d+2}, \quad Q(u) = \frac{1}{2}\int |u|^2. \tag{1.8}$$

We recall that the generic condition for stability of solitary waves is $d''(\omega) > 0$ [13,18]. We have the following lemma.

Lemma 1. *Let (1.6) hold. Then $d''(\omega) > 0$ for ϵ sufficiently small if and only if*

$$V(0)V^{(4)}(0) < G_d[V''(0)]^2, \tag{1.9}$$

where

$$G_d = 6\frac{d+2}{d}\frac{\int\{r^2R^{4/d+1}\mathcal{L}_0^{-1}(r^2R^{4/d+1})\}}{\int r^4R^{4/d+2}} \tag{1.10}$$

is a constant which depends only on d, $R(r)$ is the ground-state[1] *solution of*

$$\Delta R - R + R^{4/d+1} = 0, \quad R'(0) = R(\infty) = 0, \tag{1.11}$$

and $\mathcal{L}_0 = \Delta - 1 + (4/d+1)R^{4/d}$.

We now state our main theorem which shows that the condition $d''(\omega) > 0$ indeed implies stability.

Theorem 2. *Let (1.6) and (1.9) hold, let $\omega > 0$ and let ϕ_ω be the ground-state solution of (1.4). Then $\psi = e^{i\omega t}\phi_\omega(r)$ is an orbitally stable solution of (1.3) for ϵ sufficiently small.*

This result suggests that it may be possible to produce stable high-power beam propagation in plasma by sending a preliminary beam.

In order to motivate the condition (1.9), we use perturbation analysis in Section 2 to calculate the power of ϕ_ω. Let $\phi_\omega(r;\epsilon)$ be the solution of (1.4) and let $\hat{\epsilon} = \epsilon/\sqrt{\omega}$. Then, we have, as $\hat{\epsilon} \to 0$,

$$|\phi_\omega|_2^2 = \frac{1}{V^{d/2}(0)}\left[|R|_2^2 - \hat{\epsilon}^4\frac{(2+d)\int r^4R^{4/d+2}}{24d[V(0)]^2}(G_d[V''(0)]^2 - V(0)V^{(4)}(0)) + O(\hat{\epsilon}^6)\right], \tag{1.12}$$

where R is the ground-state solution of (1.11) and G_d is defined in (1.10). Thus, the stability condition (1.9) is also a necessary and sufficient condition for the power of the solitary waves to be below the critical power $|R|_2^2/V^{d/2}(0)$. Indeed, from (1.8), (4.6) and (5.4) we have that $d'(\omega) = (1/2)|\phi_\omega|_2^2$. Therefore, the stability condition $d''(\omega) > 0$ is satisfied if and only if $|\phi_\omega|_2^2$ is monotonically increasing in ω hence monotonically decreasing in ϵ.

The failure of the reasoning leading to the 'conclusion' that monotonicity of V implies stability of solitary waves thus lies in the assumption that monotonicity of V implies that $|\phi_\omega|_2^2$ is monotonically decreasing in ϵ. Indeed, when V is monotonic then $V''(0) > 0$. In principle, this term would have given $O(\hat{\epsilon}^2)$ contributions to $|\phi_\omega|_2^2$, whereas $V^{(4)}(0)$ would only give $O(\hat{\epsilon}^4)$ contributions. However, because the $O(\hat{\epsilon}^2)$ terms due to $V''(0)$ completely balance each other, stability is determined by both $V''(0)$ and $V^{(4)}(0)$.

[1] That is, the nontrivial solution with the smallest L^2 norm.

We calculated numerically that in the physically relevant case $d = 2$,

$$\int r^4 R^4 r \, \mathrm{d}r \approx 1.4359, \qquad \int r^2 R^3 \mathcal{L}^{-1}(r^2 R^3) r \, \mathrm{d}r \approx -0.2001.$$

Therefore, $G_2 \approx -1.6723$. Since $G_2 < 0$, a necessary condition for stability is that $V^{(4)}(0)$ be negative!

We recall that the NLS

$$\mathrm{i}\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{2\sigma}\psi = 0, \quad \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}^d, \tag{1.13}$$

is called subcritical, critical, and supercritical when $\sigma d < 2$, $\sigma d = 2$, and $\sigma d > 2$, respectively. It is well known that solutions of the NLS can become singular only in the critical and in the supercritical cases, and that the solitary wave solutions are stable only in the subcritical case. Indeed, there is a general 'rule' that solitary waves of NLS equations are stable if and only if the equation does not admit blowup solutions. We see, thus, that when condition (1.9) is satisfied, Eq. (1.3) is an exception to this 'rule' as it admits blowup solutions yet its waveguides are stable.[2]

Finally, we note that inhomogeneity of the nonlinearity is unlikely to affect the orbital stability of subcritical solitary waves of (1.13) or the strong instability of supercritical ones.[3] Indeed, $d''(\omega) > 0$, $d''(\omega) = 0$ and $d''(\omega) < 0$, when the NLS (1.13) is subcritical, critical and supercritical, respectively. Our calculation (see proof of Lemma 7) shows that the effect of inhomogeneity on $d''(\omega)$ is $\mathrm{O}(\epsilon^4)$. Therefore, stability can be affected by the inhomogeneity only in the critical case.

The paper is organized as follows. In Section 2, we derive (1.12) which motivates our rigorous results. In Sections 3 and 4, we prove existence and some properties (Theorem 1) of solitary wave solutions. Section 5 gives the proof of the stability results (Theorem 2).

2. Perturbation analysis

In this section, we derive (1.12) by a perturbation analysis.

Let $\phi_\omega = [\omega/V(0)]^{d/4} S(\sqrt{\omega} r)$. Then, the equation for S is

$$\Delta S(r; \epsilon) - S + \frac{V(\hat{\epsilon} r)}{V(0)} S_\epsilon^{4/d+1} = 0. \tag{2.1}$$

When $\hat{\epsilon} r \ll 1$, we can expand

$$\frac{V(\hat{\epsilon} r)}{V(0)} \sim 1 + a\hat{\epsilon}^2 r^2 + b\hat{\epsilon}^4 r^4 + \mathrm{O}(\hat{\epsilon}^6), \tag{2.2}$$

where $a = V''(0)/2V(0)$ and $b = V^{(4)}(0)/24V(0)$. We look for a solution of (2.1) of the form

$$S = R + a\hat{\epsilon}^2 g(r) + \hat{\epsilon}^4 h(r) + \mathrm{O}(\hat{\epsilon}^6). \tag{2.3}$$

Therefore,

$$S^m = R^m + \hat{\epsilon}^2 a m R^{m-1} g + \hat{\epsilon}^4 \left(m R^{m-1} h + \binom{m}{2} a^2 R^{m-2} g^2 \right) + \mathrm{O}(\hat{\epsilon}^6),$$

and the equations for R, g, and h are (1.11),

$$\Delta g - g + \left(\frac{4}{d} + 1\right) R^{4/d} g = -r^2 R^{4/d+1}, \tag{2.4}$$

[2] Another exception to this rule is the critical NLS on bounded domains [2].

[3] Strong instability of supercritical solitary waves was proved in [16].

and

$$\Delta h - h + \left(\frac{4}{d}+1\right) R^{4/d} h = -a^2 r^2 \left(\frac{4}{d}+1\right) R^{4/d} g - br^4 R^{4/d+1},$$

respectively. If we multiply (2.1) by R and integrate by parts we get that

$$-\int \nabla R \nabla S - \int RS + \int VRS^{4/d+1} = 0.$$

If we substitute (2.2) and (2.3) in this equation and collect terms, the $O(\hat{\epsilon}^2)$ and $O(\hat{\epsilon}^4)$ equations are

$$-\int \nabla R \nabla g - \int Rg + \left(\frac{4}{d}+1\right) \int R^{4/d+1} g = -\int r^2 R^{4/d+2}, \tag{2.5}$$

and

$$-\int \nabla R \nabla h - \int Rh + \left(\frac{4}{d}+1\right) \int R^{4/d+1} h$$
$$= -\binom{\frac{4}{d}+1}{2} a^2 \int R^{4/d} g^2 - a^2 \left(\frac{4}{d}+1\right) \int r^2 R^{4/d+1} g - b \int r^4 R^{4/d+2}. \tag{2.6}$$

If we multiply (1.11) by S and integrate by parts we get that

$$-\int \nabla R \nabla S - \int RS + \int SR^{4/d+1} = 0.$$

If we substitute (2.3) in this equation and collect terms, the $O(\hat{\epsilon}^2)$ and $O(\hat{\epsilon}^4)$ equations are

$$-\int \nabla R \nabla g - \int Rg + \int R^{4/d+1} g = 0 \tag{2.7}$$

and

$$-\int \nabla R \nabla h - \int Rh + \int R^{4/d+1} h = 0. \tag{2.8}$$

From (4.2) we have that

$$-\int S^2 + \frac{2}{2+d} \int VS^{4/d+2} + \frac{\hat{\epsilon}}{4/d+2} \int rV'(\hat{\epsilon} r) S^{4/d+2} = 0. \tag{2.9}$$

If we substitute (2.2) and (2.3) in this equation and collect terms, the $O(\hat{\epsilon}^2)$ and $O(\hat{\epsilon}^4)$ equations are

$$2\int Rg = \frac{2}{2+d} \int \left[r^2 R^{4/d+2} + \left(\frac{4}{d}+2\right) R^{4/d+1} g \right] + \frac{2}{4/d+2} \int r^2 R^{4/d+2}, \tag{2.10}$$

and

$$2\int Rh + a^2 \int g^2 = \frac{2+2d}{2+d} \int br^4 R^{4/d+2} + \frac{4+2d}{d} a^2 \int r^2 R^{4/d+1} g$$
$$+ \frac{4}{d} R^{4/d+1} h + \frac{2}{d}\left(\frac{4}{d}+1\right) a^2 \int R^{4/d} g^2, \tag{2.11}$$

respectively. If we subtract (2.7) from (2.5), we get that $4/d \int R^{4/d+1} g = -\int r^2 R^{4/d+2}$. Substitution into (2.10) gives that

$$\int Rg = 0. \tag{2.12}$$

If we subtract (2.8) from (2.6), we get that

$$\frac{4}{d}\int R^{4/d+1}h = -\binom{\frac{4}{d}+1}{2} a^2 \int R^{4/d} g^2 - a^2\left(\frac{4}{d}+1\right)\int r^2 R^{4/d+1} g - b\int r^4 R^{4/d+2}.$$

Substitution into (2.11) gives that

$$2\int Rh + a^2 \int g^2 = \frac{d}{2+d} b \int r^4 R^{4/d+2} + a^2 \int r^2 R^{4/d+1} g. \tag{2.13}$$

Therefore, combining

$$\int S^2 = \int R^2 + 2a\hat{\epsilon}^2 \int Rg + \hat{\epsilon}^4 \left[2\int Rh + a^2 \int g^2\right] + \mathrm{O}(\hat{\epsilon}^6),$$

with (2.12) and (2.13) gives that

$$\int S^2 = \int R^2 + \hat{\epsilon}^4 \left[\frac{d}{2+d} b \int r^4 R^{4/d+2} + a^2 \int r^2 R^{4/d+1} g\right] + \mathrm{O}(\hat{\epsilon}^6).$$

Therefore,

$$|R_\epsilon|_2^2 = [V(0)]^{-d/2}\left[|R|_2^2 + \hat{\epsilon}^4\left(\frac{d}{2+d}\frac{V^{(4)}(0)}{24V(0)}\int r^4 R^{4/d+2} + \frac{[V''(0)]^2}{4V^2(0)}\int r^2 R^{4/d+1} g\right) + \mathrm{O}(\hat{\epsilon}^6)\right].$$

Since $g = \mathcal{L}^{-1}(-r^2 R^{4/d+1})$, relation (1.12) follows.

3. Existence of a ground state

In order to prove Theorem 1, we introduce the minimization problem

$$M(\omega) = \inf_{u\in H^1_{\mathrm{radial}}} I_\omega(u), \tag{3.1}$$

subject to constraint $K(u) = 1$, where

$$I_\omega(u) = \int \omega|u|^2 + |\nabla u|^2, \qquad K(u) = \int V(\epsilon r)|u(r)|^{4/d+2}. \tag{3.2}$$

Lemma 2. *Let $0 < V(r) < C$ and let $\omega > 0$. Then, the minimization problem (3.1) has a positive minimizer.*

Proof. Let u_n be a minimizing sequence, i.e., $I_\omega(u_n) \to M(\omega)$ and $K(u_n) = 1$. We can assume that u_n is positive. Since $|u_n|_{H^1} \le C$ uniformly in n, we have that $u_n \hookrightarrow u_\epsilon$ weakly in H^1 and thus $I_\omega(u_\epsilon) \le \lim_{n\to\infty} I_\omega(u_n) = M(\omega)$. Because the embedding $H^1_{\mathrm{radial}}(R^d) \to L^{4/d+2}$ is compact we have that $u_n \to u_\epsilon$ strongly in $L^{4/d+2}$. Since, in addition, V is bounded,

$$|K^{1/p}(u_\epsilon) - K^{1/p}(u_n)| = ||V^{1/p}u_\epsilon|_p - |V^{1/p}u_n|_p| \le |V^{1/p}(u_\epsilon - u_n)|_p \le |V|_\infty^{1/p}|u_\epsilon - u_n|_p \to 0,$$

where $p = 4/d + 2$. Therefore, $K(u_\epsilon) = 1$ and u_ϵ is a positive minimizer of (3.1). □

Proof of (1) in Theorem 1. For clarity we write from now on ϕ instead of ϕ_ω, except where we want to emphasize the parametric dependence on ω.

The Euler–Lagrange equation for the minimizer of (3.1) is

$$\Delta u_\epsilon - \omega u_\epsilon + \lambda V(\epsilon r)|u_\epsilon|^{4/d+1} = 0, \tag{3.3}$$

where λ is a Lagrange multiplier. Let

$$\phi = \lambda^{d/4} u_\epsilon. \tag{3.4}$$

Then, ϕ is a positive solution of (1.4). □

4. Several technical lemmas

In this section, we prove several technical results that are used in Section 5 and also part (2) of Theorem 1. We first note that standard calculations show that solutions of (1.4) satisfy the following identities:

$$-\int |\phi'|^2 - \omega \int |\phi|^2 + \int V(\epsilon r)\phi^{4/d+2} = 0, \tag{4.1}$$

$$\omega \int |\phi|^2 = \frac{2}{2+d} \int V(\epsilon r)\phi^{4/d+2} + \frac{1}{4/d+2} \int r \left(\frac{\mathrm{d}}{\mathrm{d}r} V(\epsilon r) \right) \phi^{4/d+2}, \tag{4.2}$$

which are usually referred to as Pohozaev identities.

Lemma 3. *Let u_ϵ be the minimizer of (3.1) and let ϕ be given by (3.4). Then,*

$$I_\omega(\phi) = K(\phi) = [M(\omega)]^{2/d+1}. \tag{4.3}$$

In addition, when $\omega = 1$ then

$$|\phi|_{H^1} = |u_\epsilon|_{H^1}^{d/2+1}. \tag{4.4}$$

Proof. The identity $I_\omega(\phi) = K(\phi)$ is simply (4.1). Since $K(u_\epsilon) = 1$, it follows from (3.4) that

$$\lambda = [K(\phi)]^{1/(d/2+1)}. \tag{4.5}$$

Therefore,

$$M(\omega) = I_\omega(u_\epsilon) = \lambda^{-d/2} I_\omega(\phi) = [K(\phi)]^{-1/(2/d+1)} I_\omega(\phi) = [I_\omega(\phi)]^{1/(2/d+1)},$$

which leads to (4.3). Eq. (4.4) follows from (4.3) since $I_1(\cdot) = |\cdot|_{H^1}$. □

Let

$$\phi(r) = \omega^{d/4} R_\epsilon(\sqrt{\omega} r). \tag{4.6}$$

Then, by (1.4),

$$\Delta R_\epsilon(r) - R_\epsilon + V\left(\epsilon \frac{r}{\sqrt{\omega}}\right) R_\epsilon^{4/d+1} = 0. \tag{4.7}$$

Lemma 4. *Let $P_\epsilon = \partial R_\epsilon / \partial \omega$. Then,*

$$\int V\left(\epsilon \frac{r}{\sqrt{\omega}}\right) R_\epsilon^{4/d+1} P_\epsilon = \frac{d}{8} \epsilon \omega^{-3/2} \int r V'\left(\epsilon \frac{r}{\sqrt{\omega}}\right) R_\epsilon^{4/d+2}. \tag{4.8}$$

Proof. Differentiating (4.7) with respect to ω gives

$$\Delta P_\epsilon - P_\epsilon + \left(\frac{4}{d}+1\right) V\left(\epsilon\frac{r}{\sqrt{\omega}}\right) R_\epsilon^{4/d} P_\epsilon - \frac{1}{2}\epsilon\omega^{-3/2} r V' R_\epsilon^{4/d+1} = 0. \tag{4.9}$$

If we multiply (4.9) by R_ϵ and integrate, we have

$$-\int \nabla R_\epsilon \nabla P_\epsilon - \int R_\epsilon P_\epsilon + \left(\frac{4}{d}+1\right)\int V R_\epsilon^{4/d+1} P_\epsilon - \frac{1}{2}\epsilon\omega^{-3/2}\int r V' R_\epsilon^{4/d+2} = 0.$$

If we multiply (4.7) by P_ϵ and integrate, we have

$$-\int \nabla R_\epsilon \nabla P_\epsilon - \int R_\epsilon P_\epsilon + \int V R_\epsilon^{4/d+1} P_\epsilon = 0.$$

The difference of the last two equations gives (4.8). □

Let us define the linearized operator $\mathcal{L}_\epsilon$ on $H^2_{\text{radial}}(R^d)$ by

$$\mathcal{L}_\epsilon = \Delta - 1 + \left(\frac{4}{d}+1\right) V\left(\epsilon\frac{r}{\sqrt{\omega}}\right) R_\epsilon^{4/d}. \tag{4.10}$$

We can rewrite (4.9) as

$$\mathcal{L}_\epsilon(P_\epsilon) = \tfrac{1}{2}\epsilon\omega^{-3/2} r V' R_\epsilon^{4/d+1}. \tag{4.11}$$

It is well known that $\text{Ker}(\mathcal{L}_0)$ is empty (see e.g. [11]) and that $\mathcal{L}_0^{-1}$ is bounded. Therefore, there exists a constant $C_0 > 0$ such that

$$|\mathcal{L}_0 v|_2 \ge C_0 |v|_2. \tag{4.12}$$

Lemma 5. *Let R_ϵ be the solution of (4.7), then*

(a) $|R_\epsilon|_{H^1} \le C$ *uniformly as* $\epsilon \to 0$.

(b) $$\lim_{\epsilon\to 0}\left| V\left(\epsilon\frac{r}{\sqrt{\omega}}\right) R_\epsilon^2(r) - V(0) R_0^2(r)\right|_\infty = 0. \tag{4.13}$$

(c) *Let $\omega > 0$ and let ϵ be sufficiently small. Then $\mathcal{L}_\epsilon$ is invertible and $\mathcal{L}_\epsilon^{-1}$ is bounded.*

(d) *Let R_ϵ be a positive solution of (4.7). Then there exist positive constants ϵ_0, c_0 and L such that $R_\epsilon(r) \le c_0\,\mathrm{e}^{-r/\sqrt{2}}$ for all $0 < \epsilon \le \epsilon_0$ and for all $r \ge L$.*

Proof. Without loss of generality, we can set $\omega = 1$. From (4.4) and (4.6) it follows that $|R_\epsilon|_{H^1} = |\phi_\epsilon|_{H^1} = |u_\epsilon|_{H^1}^{d/2+1}$. Therefore, it is sufficient to show that $|u_\epsilon|_{H^1}$ is uniformly bounded. To see that, let $v_\epsilon = \alpha_\epsilon u_0$, where u_0 is the minimizer of the minimization problem (3.1) when $\epsilon = 0$ and α_ϵ is chosen so that $K(v_\epsilon) = 1$. Therefore, for the minimizer u_ϵ of (3.1) we have that $|u_\epsilon|_{H^1} = I_1(u_\epsilon) \le I_1(v_\epsilon) = \alpha_\epsilon^2 I_1(u_0)$. Since $\lim_{\epsilon\to 0}\alpha_\epsilon = 1$ it follows that $|u_\epsilon|_{H^1}$ is uniformly bounded. That completes the proof of (a).

To prove (b), we note from (1) and standard elliptic regularity theory [3], we have that $R_\epsilon \to R_0$ weakly in H^1 and strongly in C^2_{loc}, where R_0 is the unique solution of (4.7) for $\epsilon = 0$. From the radial lemma of Strauss, we have that

$$|R_\epsilon| \le \frac{C}{r^{(d-1)/2}}|R_\epsilon|_{H^1} \quad \text{for } r \ge 1. \tag{4.14}$$

Again, in light of (a), we only need to prove (4.13) on a bounded domain which is now obvious.

Remark. Our stability proof is limited to the case $d \geq 2$ because we rely on the uniform decay estimate (4.14) in the proof of (b).

We now proceed to prove (c). Since $\mathcal{L}_\epsilon = \mathcal{L}_0 + (4/d+1)(V(\epsilon(r/\sqrt{\omega}))R_\epsilon^2 - V(0)R_0^2)$, we have that

$$\begin{aligned}\langle \mathcal{L}_\epsilon v, \mathcal{L}_\epsilon v\rangle &= \langle \mathcal{L}_0 v, \mathcal{L}_0 v\rangle + \left\langle \mathcal{L}_0 v, \left(\frac{4}{d}+1\right)\left(V\left(\epsilon\frac{r}{\sqrt{\omega}}\right)R_\epsilon^2 - V(0)R_0^2\right)v\right\rangle \\ &\quad + \left\langle \left(\frac{4}{d}+1\right)\left(V\left(\epsilon\frac{r}{\sqrt{\omega}}\right)R_\epsilon^2 - V(0)R_0^2\right), \mathcal{L}_0 v\right\rangle \\ &\quad + \left(\frac{4}{d}+1\right)^2\left\langle \left(V\left(\epsilon\frac{r}{\sqrt{\omega}}\right)R_\epsilon^2 - V(0)R_0^2\right)v, \left(V\left(\epsilon\frac{r}{\sqrt{\omega}}\right)R_\epsilon^2 - V(0)R_0^2\right)v\right\rangle \\ &\geq |\mathcal{L}_0 v|_2^2 - 2\left(\frac{4}{d}+1\right)\left|V\left(\epsilon\frac{r}{\sqrt{\omega}}\right)R_\epsilon^2 - V(0)R_0^2\right|_\infty |v|_2|\mathcal{L}_0 v|_2 \\ &\quad - \left(\frac{4}{d}+1\right)^2\left|V\left(\epsilon\frac{r}{\sqrt{\omega}}\right)R_\epsilon^2 - V(0)R_0^2\right|_\infty |v|_2^2.\end{aligned}$$

Using (4.12), we have that

$$|\mathcal{L}_\epsilon v|_2^2 \geq |v|_2^2\left[C_0^2 - \left(2\left(\frac{4}{d}+1\right)C_0 + \left(\frac{4}{d}+1\right)^2\right)\left|V\left(\epsilon\frac{r}{\sqrt{\omega}}\right)R_\epsilon^2 - V(0)R_0^2\right|_\infty\right].$$

Therefore, in light of (b), when ϵ is sufficiently small there exists $C_1 > 0$ such that $|\mathcal{L}_\epsilon v|_2^2 \geq C_1|v|_2^2$, from which (c) follows.

Next we prove (d). Eq. (4.7) can be rewritten as

$$\Delta R_\epsilon = \left(1 - V\left(\frac{\epsilon r}{\sqrt{\omega}}\right)R_\epsilon^{4/d}\right)R_\epsilon.$$

From Lemma 5, inequality (4.14) and (1.6) it follows that there exist $\epsilon_0, L > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ and for all $r \geq L$,

$$1 - V\left(\frac{\epsilon r}{\sqrt{\omega}}\right)R_\epsilon^{4/d} \geq \frac{1}{2}.$$

Let $v_\epsilon(r) = c_0\,\mathrm{e}^{-r/\sqrt{2}} - R_\epsilon$, where c_0 is sufficiently large such that $v_\epsilon(L) > 0$. Then,

$$\Delta v_\epsilon = c_0\Delta\,\mathrm{e}^{-r/\sqrt{2}} - \Delta R_\epsilon = c_0\Delta\,\mathrm{e}^{-r/\sqrt{2}} - R_\epsilon(1 - VR_\epsilon^{4/d}) \leq \tfrac{1}{2}c_0\,\mathrm{e}^{-r/\sqrt{2}} - \tfrac{1}{2}R_\epsilon = \tfrac{1}{2}v_\epsilon.$$

Therefore, from the maximum principle for exterior domains [3], we have that $v_\epsilon \geq 0$ for all $r \geq L$, and thus that $R_\epsilon(r) \leq c_0\,\mathrm{e}^{-r/\sqrt{2}}$. This completes the proof of the lemma.

Proof of (2) in Theorem 1. We prove uniqueness of positive solutions for (4.7), which is equivalent to (1.4) up to a simple rescaling.

Let $R_{1_\epsilon}, R_{2_\epsilon}$ be two solutions of (4.7), i.e.,

$$\Delta R_{i_\epsilon} - R_{i_\epsilon} + V\left(\epsilon\frac{r}{\sqrt{\omega}}\right)R_{i_\epsilon}^{4/d+1} = 0, \quad i = 1, 2.$$

We then have,

$$\mathcal{L}_\epsilon^*(R_{1_\epsilon} - R_{2_\epsilon}) = 0, \tag{4.15}$$

where we denote

$$\mathcal{L}_\epsilon^* = \Delta - 1 + V\left(\epsilon\frac{r}{\sqrt{\omega}}\right) W(R_{1_\epsilon}, R_{2_\epsilon}),$$

and

$$R_{1_\epsilon}^{4/d+1} - R_{2_\epsilon}^{4/d+1} = (R_{1_\epsilon} - R_{2_\epsilon})W(R_{1_\epsilon}, R_{2_\epsilon}).$$

Since the positive solution R_0 of (4.7) is unique [6], then, as in Lemma 5(b) $R_{1_\epsilon}, R_{2_\epsilon} \to R_0$ uniformly. Therefore,

$$W(R_{1_\epsilon}, R_{2_\epsilon}) \to \left(\frac{4}{d} + 1\right) R_0^{4/d} \quad \text{as } \epsilon \to 0.$$

As in Lemma 5(c), we can show that $\mathcal{L}_\epsilon^*$ is invertible and $|\mathcal{L}_\epsilon^* v|_2^2 \geq C|v|_2^2$, for ϵ small enough. By (4.15), this implies $R_{1_\epsilon} = R_{2_\epsilon}$ for ϵ small enough. □

5. Orbital stability

Lemma 6. *$d(\omega)$ is differentiable and strictly increasing for $\omega > 0$.*

Proof. Using (1.7), (1.8) and (3.2), we have that

$$d(\omega) = \frac{1}{2} I_\omega(\phi) - \frac{1}{4/d + 2} K(\phi). \tag{5.1}$$

Therefore, by (4.3)

$$d(\omega) = \frac{1}{2 + d} K(\phi) = \frac{\omega}{2 + d} \int V\left(\frac{\epsilon r}{\sqrt{\omega}}\right) R_\epsilon^{4/d+2}. \tag{5.2}$$

Differentiating $d(\omega)$ with respect to ω and using (4.8) gives that

$$d'(\omega) = \frac{1}{d + 2}\left(\int V R_\epsilon^{4/d+2} + \frac{d}{4}\omega^{-1/2}\epsilon \int rV' R_\epsilon^{4/d+2}\right). \tag{5.3}$$

Therefore, from (4.2) we have that

$$d'(\omega) = Q(R_\epsilon) > 0. \quad \square \tag{5.4}$$

Lemma 7. *Let (1.6) hold. Then, $d''(\omega) > 0$ for ϵ sufficiently small if and only if*

$$V^{(4)}(0) < 6\frac{d + 2}{d} \frac{\int \{r^2 R_0^{4/d+1} \mathcal{L}_0^{-1}(r^2 R_0^{4/d+1})\}}{\int r^4 R_0^{4/d+2}} [V''(0)]^2.$$

Proof. If we differentiate (5.3) with respect to ω, use (4.8) and (4.11) and expand $V'(\epsilon r/\sqrt{\omega})$ and $V''(\epsilon r/\sqrt{\omega})$ in a Taylor series in ϵ, we get that

$$\begin{aligned}(2+d)d''(\omega) &= \frac{d}{8}\epsilon\omega^{-3/2}\int rV'R_\epsilon^{4/d+2} - \frac{d}{8}\epsilon^2\omega^{-2}\int r^2V''R_\epsilon^{4/d+2} + \frac{d}{4}\left(\frac{4}{d}+2\right)\epsilon\omega^{-1/2}\int rV'R_\epsilon^{4/d+1}P_\epsilon \\ &= \frac{d}{8}\epsilon\omega^{-3/2}\int rV'R_\epsilon^{4/d+2} - \frac{d}{8}\epsilon^2\omega^{-2}\int r^2V''R_\epsilon^{4/d+2} \\ &\quad + \frac{d}{4}\left(\frac{2}{d}+1\right)\epsilon^2\omega^{-2}\int rV'R_\epsilon^{4/d+1}\mathcal{L}_\epsilon^{-1}(rV'R_\epsilon^{4/d+1}) \\ &= -\epsilon^4\omega^{-3}\left[\frac{d}{24}\int r^4V^{(4)}(0)R_0^{4/d+2} - \frac{2+d}{4}\right. \\ &\qquad \left.\times\int (r^2V''(0)R_0^{4/d+1})\mathcal{L}_0^{-1}(r^2V''(0)R_0^{4/d+1}) + F(V_\epsilon,R_\epsilon)\right].\end{aligned}$$

Here $F(V_\epsilon, R_\epsilon)$ is the remainder from the Taylor expansion. Using Lemma 5(c) and (d), it is easy to show that $F(V_\epsilon, R_\epsilon) \to 0$ as $\epsilon \to 0$. Therefore, condition (1.9) implies that $d''(\omega) > 0$ for ϵ sufficiently small. □

Proof of Lemma 1. This follows from Lemma 7 and the rescaling $R = V^{d/4}(0)R_0$. □

Lemma 8. *Let $d''(\omega) > 0$. Then there exists $\delta = \delta(\omega) > 0$ such that for all $\tilde{\omega}$ with $|\tilde{\omega} - \omega| < \delta$,*

$$d(\tilde{\omega}) \geq d(\omega) + d'(\omega)(\tilde{\omega} - \omega) + \tfrac{1}{4}d''(\omega)|\tilde{\omega} - \omega|^2.$$

Proof. Taylor expansion. □

Given a solution ϕ of (1.4), we can define the set

$$U_{\omega,\delta} = \{u \in H^1_{\text{radial}}(R^d),\ \ |u - \phi|_{H^1} < \delta\}.$$

Since $d(\omega)$ is monotonic (Lemma 6), we can define the C^1 map

$$\omega(\cdot) : U_{\omega,\delta} \to R^+$$

by

$$\omega(u) = d^{-1}\left(\frac{1}{2+d}K(u)\right). \tag{5.5}$$

Lemma 9. *Let $d''(\omega) > 0$ for some $\omega > 0$. Then there exists $\delta = \delta(\omega) > 0$ such that for all $u \in U_{\omega,\delta}$,*

$$E(u) - E(\phi) + \omega(u)[Q(u) - Q(\phi)] \geq \tfrac{1}{4}d''(\omega)|\omega(u) - \omega|^2,$$

where $\omega(u)$ is defined in (5.5).

Proof. From (1.8) and (3.2), we have that

$$E(u) + \omega(u)Q(u) = \frac{1}{2}I_{\omega(u)}(u) - \frac{1}{4/d+2}K(u). \tag{5.6}$$

In addition, from (5.2) and (5.5), we have that $K(u) = (d+2)d(\omega(u)) = K(\phi_{\omega(u)})$. Since $\phi_{\omega(u)}$ is a minimizer of $I_{\omega(u)}(u)$ subject to the constraint $K(u) = K(\phi_{\omega(u)})$, we have that $I_{\omega(u)}(u) \geq I_{\omega(u)}(\phi_{\omega(u)})$. Therefore, using Lemma 8 and (5.1),

$$\begin{aligned}E(u)+\omega(u)Q(u) &\geq \frac{1}{2}I_{\omega(u)}(\phi_{\omega(u)}) - \frac{1}{4/d+2}K(\phi_{\omega(u)})\\ &= d(\omega(u)) \geq d(\omega)+d'(\omega)(\omega(u)-\omega)+\frac{1}{4}d''(\omega)|\omega(u)-\omega|^2.\end{aligned}$$

From (5.4), we have that $d'(\omega) = Q(\phi)$. Therefore, using (1.7),

$$\begin{aligned}E(u)+\omega(u)Q(u) &\geq E(\phi)+\omega Q(\phi)+Q(\phi)[\omega(u)-\omega]+\tfrac{1}{4}d''(\omega)|\omega(u)-\omega|^2\\ &= E(\phi)+\omega(u)Q(\phi)+\tfrac{1}{4}d''(\omega)|\omega(u)-\omega|^2.\end{aligned}$$

□

Proof of Theorem 2. Assume that ψ is unstable. From the definition of stability $\exists\delta > 0$ and initial data $\psi_k(0) \in U_{\omega,(1/k)}$ such that $\sup_{t>0}\inf_\theta|\psi_k(t) - e^{i\theta}\phi|_{H^1} \geq \delta$, where $\psi_k(t)$ is the solution of refeq13 with initial data $\psi_k(0)$. Let t_k be the first time at which

$$\inf_\theta|\psi_k(t_k) - e^{i\theta}\phi|_{H^1} = \delta. \tag{5.7}$$

Let us denote $\Phi_k(r) = \psi_k(t_k)$. Since $E(\psi(t))$ and $Q(\psi(t))$ are conserved in t and continuous in ψ, then

$$\begin{aligned}&|E(\Phi_k)-E(\phi)| = |E(\psi_k(0))-E(\phi)| \to 0,\\ &|Q(\Phi_k)-Q(\phi)| = |Q(\psi_k(0))-Q(\phi)| \to 0, \quad \text{as } k\to\infty.\end{aligned} \tag{5.8}$$

Let δ be small enough so that Lemma 9 applies. We then have that

$$E(\Phi_k)-E(\phi)+\omega_k[Q(\Phi_k)-Q(\phi)] \geq \tfrac{1}{4}d''(\omega)|\omega_k-\omega|^2, \quad \omega_k = \omega(\Phi_k). \tag{5.9}$$

From (5.7), we have that $|\Phi_k|_{H^1} \leq C$ uniformly. Since $\omega(u)$ is a continuous map, ω_k is uniformly bounded in k. Therefore, by (5.8), as $k \to \infty$ the left-hand side of (5.9) goes to zero. Since $d''(\omega) > 0$ (Lemma 1), this implies that $\lim_{k\to\infty}\omega_k = \omega$. Hence, using (5.2) and (5.5),

$$\lim_{k\to\infty} K(\Phi_k) = \lim_{k\to\infty}(d+2)d(\omega_k) = (d+2)d(\omega) = K(\phi). \tag{5.10}$$

Using this and (5.6) and (5.8), we have that

$$\begin{aligned}I_\omega(\Phi_k) &= 2[E(\Phi_k)+\omega Q(\Phi_k)] + \frac{d}{d+2}K(\Phi_k)\\ &= 2[E(\phi)+\omega Q(\phi)] + \frac{d}{d+2}K(\Phi_k) + 2[E(\Phi_k)-E(\phi)] + 2\omega[Q(\Phi_k)-Q(\phi)] \to (d+2)d(\omega).\end{aligned}$$

Since $(d+2)d(\omega) = I_\omega(\phi)$ (see (4.3) and (5.2)), we see that $I_\omega(\Phi_k) \to I_\omega(\phi)$.

Let $v_k = [K(\Phi_k)]^{-1/(4/d+2)}\Phi_k$. Then $K(v_k) = 1$ and by (4.3)

$$I_\omega(v_k) = [K(\Phi_k)]^{-1/(2/d+1)}I_\omega(\Phi_k) \to [K(\phi)]^{-1/(2/d+1)}I_\omega(\phi) = M(\omega).$$

Hence, $\{v_k\}$ is a minimizing sequence of (3.1). By uniqueness of the minimizer u_ϵ (Theorem 1), there exists a sequence $\{\theta_k\}$ such that $\lim_{k\to\infty}|v_k - e^{i\theta_k}u_\epsilon|_{H_1} = 0$. Using this and (3.4), (4.5) and (5.10), we get that

$$|\Phi_k - e^{i\theta_k}\phi|_{H^1} = |[K(\Phi_k)]^{1/(4/d+2)}v_k - e^{i\theta_k}[K(\Phi)]^{1/(4/d+2)}u_\epsilon|_{H^1} \to 0,$$

which is in contradiction with (5.7).

□

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References

[1] H. Berestycki, P.L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Ration. Mech. Anal. 82 (1983) 313–345.
[2] G. Fibich, F. Merle, Self-focusing on bounded domains, Physica D 155 (2001) 132–158.
[3] D. Gilbert, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1983.
[4] T.S. Gill, Optical guiding of laser beam in nonuniform plasma, Pramana J. Phys. 55 (2000) 842–845.
[5] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry, J. Funct. Anal. 74 (1987) 160–197.
[6] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^n, Arch. Ration. Mech. Anal. 105 (1989) 243–266.
[7] S. Levandosky, Stability and instability of fourth-order solitary waves, J. Dynam. Differential Equations 10 (1998) 151–188.
[8] C.S. Liu, V.K. Tripathi, Laser guiding in an axially nonuniform plasma channel, Phys. Plasmas 1 (1994) 3100–3103.
[9] Y. Liu, X.P. Wang, Nonlinear stability of solitary waves of a generalized Kadomtsev–Petviashvili equation, Commun. Math. Phys. 183 (1997) 253–266.
[10] F. Merle, Nonexistence of minimal blow-up solutions of equations $iu_t = -\Delta u - k(x)|u|^{4/N}u$ in R^N, Ann. Inst. Henry Poincare Phys. Theorique. 64 (1996) 33–85.
[11] W.M. Ni, I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem, Duke Math. J. 70 (1993) 247–281.
[12] J. Shatah, Stable Klein–Gordon equations, Commun. Math. Phys. 91 (1983) 313–327.
[13] J. Shatah, W.A. Strauss, Instability of nonlinear bound states, Commun. Math Phys. 100 (1985) 173–190.
[14] W.A. Strauss, Existence of solitary waves in higher dimensions, Commun. Math. Phys. 55 (1977) 149–162.
[15] X.F. Wang, B. Zeng, On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions, SIAM J. Math. Anal. 28 (1997) 633–655.
[16] K. Wang, On the strong instability of standing wave solutions of the inhomogeneous nonlinear Schrödinger equations, M.Phil. Thesis, HKUST, 2001.
[17] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Commun. Math. Phys. 87 (1983) 567–576.
[18] M.I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, Commun. Pure Appl. Math. 39 (1986) 51–68.