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Stability of the Convex Pompeiu Sets (*).

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1. – Introduction.

We will say that a bounded set Ω in the plane is a Pompeiu set if the function $f \equiv 0$ is the only continuous function on the plane for which

$$\int_{\sigma(\Omega)} f(x)\, dx = 0 ,$$

$\forall$ rigid motion σ of the plane.

In 1929, D. POMPEIU [9] posed the problem (afterwards called Pompeiu problem (P.P.)) to find all the bounded sets of the plane which are Pompeiu sets. From now on, open bounded simply connected subsets of the plane will be called domains. In 1944 CHAKALOV [3] discovered that the disks are not Pompeiu sets, and in [4] it was conjectured that (modulo sets of measure zero) the disks are the only domains which are not Pompeiu sets. The depth of P.P. was finally clarified in 1976 by WILLIAMS [10]. A connected bounded set Ω of the plane is called a Schiffer set if

$$\text{(1.1)} \qquad \begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \ \ \lambda > 0 , \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0 , & u|_{\partial\Omega} = \text{constant} , \end{cases}$$

implies $u \equiv 0$.

WILLIAMS proved the equivalence between Pompeiu sets and Schiffer sets. In 1877, LORD RAYLEIGH [8] conjectured that if there exists a non-trivial solution to (1.1), then Ω is a disk. Many progress about P.P. were done since 1972. In that year, ZALCMAN [12] published a seminal work in which, first, he made use of the Fourier transform to attack P.P. Using ideas closely related to those presented in [12], BROWN, SCHREIBER and TAYLOR [2] gave the deepest contribution to the problem in 1973. They proved that Ω is a Pompeiu set if and only if $\hat{\chi}_\Omega$ does not vanish identically on every circle of C^2,

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where χ_Ω is the characteristic function of Ω and $\widehat{\chi}_\Omega$ denotes its Fourier transform.

The use of the technique of Riemann method of steepest descent, to analyze the asymptotic behaviour of $\widehat{\chi}_\Omega$, and the theorem of Brown, Schreiber and Taylor, enabled the author and GAROFALO to give the contributions [4]-[7] to P.P. (for complete references about P.P. see the survey of ZALCMAN [13]).

In this paper, we will prove that Pompeiu convex domains are stable, i.e. if Ω is a convex Pompeiu domain, then every convex domain sufficiently close of Ω (in the sense of small deformation, see Section 2) is a Pompeiu set. In this way we give a positive answer to a question posed by BERENSTEIN in 1980 [1], in the case of convex domains. We hope that our contribution will be a useful step toward the solution of the Pompeiu problem.

2. – Main theorem.

In 1981, WILLIAMS [11] proved that if Ω is not a disk and its boundary is Lipschitz but not analytic, then Ω is a Pompeiu set. Therefore we can limit the study of P.P. to the class of analytic domains.

In this paper we prove the following theorem about stability of Pompeiu sets.

From now on, «convex» will mean «strictly convex».

Let Ω_0 be a convex set with $0 \in \Omega$. For small real λ, we put $\Omega_\lambda = (1+\lambda)\Omega_0$.

We fix a family F of convex domains Ω such that

$$\inf_F (\min k_\Omega) > 0\,, \qquad \sup_F \|x_\Omega\|_{C^5} < +\infty\,,$$

where k_Ω is the curvature of $\partial\Omega$, $s \mapsto x_\Omega(s)$ describes $\partial\Omega$ and $\|\cdot\|_{C^5}$ denotes the C^5-norm.

THEOREM 2.1. – *Let $\Omega_0 \in F$ be a convex Pompeiu domain. There exists $\varepsilon > 0$ such that every convex domain $\Omega \in F$, with*

$$\Omega_{-\varepsilon} \subset \Omega \subset \Omega_\varepsilon\,,$$

is a Pompeiu set.

Obviously, ε is invariant under rigid motions of the set Ω_0.

3. – A description of the zeros of $\widehat{\chi}_{\partial\Omega}(\zeta)$ for large $|\zeta|$.

Let Ω be a domain. Then by Brown, Schreiber and Taylor theorem [2], Ω is a Pompeiu set if and only if the Fourier-Laplace transform

$$\widehat{\chi}_{\partial\Omega}(\zeta) = \oint_{\partial\Omega} e^{-i\langle x,\,\zeta\rangle}(dx_1 + i dx_2)$$

does not vanish identically on every circle $\zeta_1^2 + \zeta_2^2 = \alpha^2$ in $\boldsymbol{C}^2$, with $\langle x,\,\zeta\rangle = x_1\zeta_1 + x_2\zeta_2$, $\zeta_1, \zeta_2 \in \boldsymbol{C}$. We can assume $\alpha > 0$, by Berenstein result [1].

Let Ω be a convex domain (with real analytic boundary). We put

$$E(\vartheta,\ \alpha) = \widehat{\chi}_{\partial\Omega}(\alpha\cos\vartheta,\ \alpha\sin\vartheta),$$

where $\alpha > 0$ and $0 \leqslant \vartheta \leqslant 2\pi$. The function $E(\vartheta,\ \alpha)$ is analytic in the variables ϑ, α. If we put

$$M_\alpha = \{(\alpha\cos\vartheta.\ \alpha\sin\vartheta)\,|\,\vartheta \in [0,\ 2\pi]\}$$

Ω is a Pompeiu set if and and only if $E(\vartheta,\ \alpha)$ does not vanish identiclly on every M_α.

Let s be the arc length. Then $\partial\Omega$ is described by $(x_1(s),\ x_2(s))$. We have

$$E(\vartheta,\ \alpha) = \int e^{-i\alpha(x_1(s)\cos\vartheta + x_2(s)\sin\vartheta)}(x_1'(s) + ix_2'(s))\,ds\,.$$

We apply the stationary phase method. Consider the critical points of the phase $x_1(s)\cos\vartheta + x_2(s)\sin\vartheta$. They are given by

$$\langle\tau(s),\ \xi(\vartheta)\rangle = 0 \tag{3.1}$$

where $\tau(s)$ is the tangent vector and $\xi(\vartheta) = (\cos\vartheta,\ \sin\vartheta)$.

Since Ω is convex, there are two points $X_1(\vartheta)$ and $X_2(\vartheta)$ on $\partial\Omega$ for which the tangent to $\partial\Omega$ is normal to $\xi(\vartheta)$ (see figure 1).

By an application of stationary phase method, we obtain

$$E(\vartheta,\ \alpha) = \sqrt{\frac{2\pi}{\alpha}}\,\tau^*(\vartheta)\left\{\frac{1}{\sqrt{k_1(\vartheta)}}\exp\left[i\left(\frac{\pi}{4} - \alpha X_1(\vartheta)\right)\right] + \right.$$
$$\left. - \frac{1}{\sqrt{k_2(\vartheta)}}\exp\left[-i\left(\frac{\pi}{4} + \alpha X_2(\vartheta)\right)\right] + \frac{1}{\sqrt{\alpha}}R(\vartheta,\ \alpha)\right\} \tag{3.2}$$

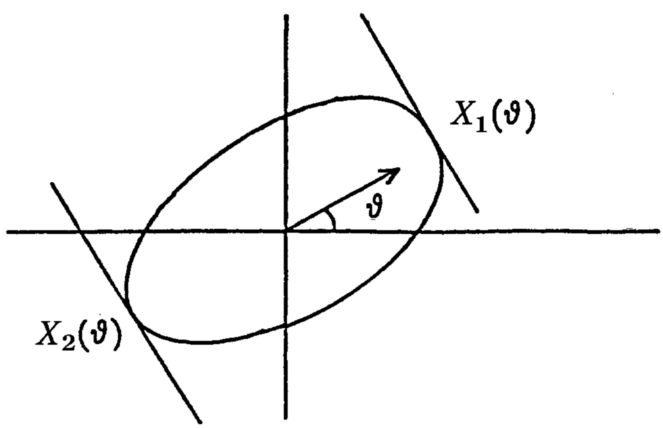

Fig. 1.

where $\tau^*(\vartheta)$ is the tangent vector to $\partial\Omega$ in $X_1(\vartheta)$ (represented by a complex number), $k_1(\vartheta)$ and $k_2(\vartheta)$ are the curvatures of $\partial\Omega$ at the points $X_1(\vartheta)$, $X_2(\vartheta)$ and $R(\vartheta, \alpha) = O(1)$ for $\alpha \to +\infty$, uniformly in ϑ.

Here we assume Ω is convex, in order to give (3.2) a meaning for every $\vartheta \in [0, 2\pi]$.

If we set $H(\vartheta) = \sqrt{k_1(\vartheta)/k_2(\vartheta)}$, we can write (3.2) in the form

$$E(\vartheta, \alpha) = F(\vartheta, \alpha)\left\{\exp\left[-i\left(\alpha\delta(\vartheta) - \frac{\pi}{2}\right)\right] - H(\vartheta) + \frac{1}{\sqrt{\alpha}}R(\vartheta, \alpha)\right\}. \tag{3.3}$$

Here $F(\vartheta, \alpha) \neq 0$ for $\vartheta \in [0, 2\pi]$, $\alpha > 0$, $\delta(\vartheta)$ is the diameter of Ω in the direction ϑ and the new $R(\vartheta, \alpha) = O(1)$ for $\alpha \to +\infty$, uniformly in ϑ.

The zeros of $\hat{\chi}_{\partial\Omega}$ are therefore given by

$$\exp\left[-i\left(\alpha\delta(\vartheta) - \frac{\pi}{2}\right)\right] = H(\vartheta) - \frac{1}{\sqrt{\alpha}}R(\vartheta, \alpha). \tag{3.4}$$

Equation (3.4) is solved by taking

$$\left|H(\vartheta) - \frac{1}{\sqrt{\alpha}}R(\vartheta, \alpha)\right| = 1\,, \tag{3.5}$$

$$\alpha - i\frac{\log\left(H(\vartheta) - R(\vartheta, \alpha)/\sqrt{\alpha}\right)}{\delta(\vartheta)} = \frac{2k\pi + \pi/2}{\delta(\vartheta)}\,. \tag{3.6}$$

We observe that if we have $H(\vartheta) \neq 1$ for some ϑ, then the equation (3.5) has no solutions for large α. Then we consider the case $H(\vartheta) = 1$. In this case (3.5) becomes

$$\left|1 - \frac{1}{\sqrt{\alpha}}R(\vartheta, \alpha)\right| = 1\,. \tag{3.7}$$

Therefore

$$\log\left(1 - \frac{R(\vartheta, \alpha)}{\sqrt{\alpha}}\right) = \frac{i}{\sqrt{\alpha}}G(\vartheta, \alpha)$$

for some real G. We can rewrite (3.6) in the form

$$\alpha + \frac{G(\vartheta, \alpha)}{\sqrt{\alpha}\,\delta(\vartheta)} = \frac{2k + 1/2}{\delta(\vartheta)}\pi\,. \tag{3.8}$$

From (3.8) it follows that for large α the zeros of $\hat{\chi}_{\partial\Omega}$ are described by the set of (α, ϑ)'s

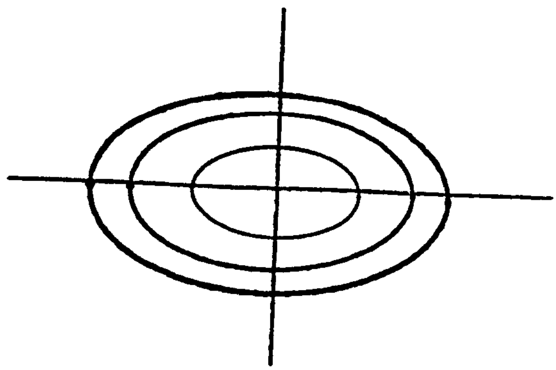

Fig. 2.

such that

$$\alpha = \frac{4k+1}{2\delta(\vartheta)}\pi + o(1), \tag{3.9}$$

$$H(\vartheta) = 1\,. \tag{3.10}$$

In conclusion, for large α, the zeros of $\widehat{\chi}_{\partial\Omega}$ are close to dilations of the curve

$$\vartheta \mapsto \frac{1}{\delta(\vartheta)} \qquad \text{(see figure 2)},$$

EXAMPLE 3.1. – In the case of a disk of radius R, (3.9) becomes

$$\alpha = \left(R + \frac{1}{4}\right)\pi + o(1)\,. \tag{3.11}$$

We know that the Fourier transform of the characteristic function of a circle of radius R is given by

$$\widehat{\chi}_{\partial\Omega}(\alpha\cos\vartheta,\ \alpha\sin\vartheta) = 2\pi R e^{i\vartheta} J_1(\alpha R)$$

and then $\widehat{\chi}_{\partial\Omega} = 0$ for

$$\alpha = \frac{x_k}{R} \tag{3.12}$$

where x_k is the k-th positive zero of J_1.

It is well known that (3.12) is in accordance with (3.11).

EXAMPLE 3.2. – In the case of the ellipse $\Omega = \{(x_1, x_2)\,|\,x_1^2/a^2 + x_2^2/b^2 \leqslant 1\}$, we have $\delta(\vartheta) = 2p(\vartheta)$ with $p(\vartheta) = \sqrt{a^2\cos^2\vartheta + b^2\sin^2\vartheta}$ and then (3.9) becomes

$$\alpha = \left(R + \frac{1}{4}\right)\frac{\pi}{p(\vartheta)} + o(1). \tag{3.13}$$

On the other hand, a simple calculation gives

$$\hat{\chi}_{\partial\Omega}(\alpha\cos\vartheta, \alpha\sin\vartheta) = \frac{2abe^{i\vartheta}}{p(\vartheta)} J_1(\alpha p(\vartheta))$$

and then $\hat{\chi}_{\partial\Omega} = 0$ for

$$\alpha = \frac{x_k}{p(\vartheta)} \tag{3.14}$$

where x_k is as above. Also in this case (3.14) is in accordance with (3.11).

4. – A new proof of a result due to Berenstein.

In [1] BERENSTEIN proved that if $\hat{\chi}_\Omega$ vanishes identically on infinitely many circles then Ω is a disk. Here we make use of (3.9) and (3.10) to give a different proof of this result at least for convex domains. After some modification, our proof can be extended to general domains.

For large α, the zeros of $\hat{\chi}_\Omega$ are close to the curve

$$\alpha = \frac{4k+1}{2\delta(\vartheta)}\pi \tag{4.1}$$

with the condition (3.10).

Clearly, (4.1) describes a circle if and only if

$$\delta(\vartheta) \equiv \text{constant}. \tag{4.2}$$

Now, (4.2) and (3.10) imply that Ω is a disk.

5. – Proof of Theorem 2.1.

We begin this section by proving a lemma.

Let $a, b > 0$ be fixed

LEMMA. – *Let $\varphi \in C^5([-a, b])$, $k \in C^2([-a, b])$, with $\varphi(0) = \varphi'(0) = 0$ and $\min_{[-a,b]} \varphi'' = A > 0$. Then for every $r > 0$, one has*

$$\left| \int_{-a}^{b} e^{ir\varphi(x)} k(x)\,dx - \frac{\exp(i\pi/4)}{\sqrt{4}} \frac{\sqrt{\pi}}{\sqrt{\varphi''(0)/2}} \right| \leqslant \frac{1}{A^7}\|k\|_{C^2} F(A, \|\varphi\|_{C^5})\frac{1}{\sqrt{r}}$$

where F is a continuous function of its arguments.

PROOF. – Put $J(r) = \int_{-a}^{b} e^{ir\varphi(x)} k(x) dx$ and $\varphi(s) = s^2 E(s)$.

We obtain $\varphi'(s)/s = 2E(s) + sE'(s)$, and by Cauchy theorem $2E(s) + sE'(s) = \varphi''(z)$ for some intermediate z. Therefore

$$2E(s) + sE'(s) \geq A \,. \tag{5.1}$$

We make the change of variable $u = s\sqrt{E(s)}$ and observe that

$$u'(s) = \frac{2E(s) + sE'(s)}{2\sqrt{E(s)}} \,.$$

From (5.1) it follows that $s \to u(s)$ is an invertible map. We call $s = s(u)$ the inverse map. We have

$$J(r) = \int_{-a}^{b} e^{iru^2} k(s(u))\, s'(u)\, du$$

for some new a, b. Put $k(u) = k(s(u))s'(u)$. We can write

$$J(r) = \frac{1}{\sqrt{r}} \int_{-a\sqrt{r}}^{b\sqrt{r}} e^{iv^2} h\left(\frac{v}{\sqrt{r}}\right) dv = \tag{5.2}$$

$$= \frac{1}{\sqrt{r}} \left\{ \int_{-a\sqrt{r}}^{b\sqrt{r}} e^{iv^2} h(0)\, dv - \frac{i}{2\sqrt{r}} \int_{-a\sqrt{r}}^{b\sqrt{r}} 2ive^{iv^2} p\left(\frac{v}{\sqrt{r}}\right) dv \right\},$$

with

$$p(\sigma) = \frac{h(\sigma) - h(0)}{\sigma} \,.$$

By integrating by parts the last integral of (5.2) and by taking into account that $h(0) = k(0)s'(0)$ and $s'(0) = 1/u'(0) = 1/\sqrt{E(0)} = 1/\sqrt{\varphi''(0)/2}$, we get

$$\left| J(r) - \frac{1}{\sqrt{r}} \exp\left(i\frac{\pi}{4}\right) \frac{\sqrt{\pi}}{\sqrt{\varphi''(0)/2}} \right| \leq \frac{4}{\sqrt{r}} \|p\|_{C^1} \,. \tag{5.3}$$

From now on, C will denote a constant which is independent on k, φ.

Since

$$p(\sigma) = \frac{h(\sigma) - h(0)}{\sigma} \,, \qquad p'(\sigma) = \frac{h'(\sigma)\,\sigma - h(\sigma) + h(0)}{\sigma^2} \,,$$

by Cauchy theorem we have

$$p(\sigma) = h'(z), \qquad p'(\sigma) = \frac{h''(w)}{2}$$

for some intermediate z, w. Then

$$\|p\|_{C^1} \leqslant C\|h\|_{C^3}. \tag{5.4}$$

We recall that $h(u) = k(s(u))s'(u)$, which implies

$$h' = k's'^2 + ks'' \tag{5.5}$$

$$h'' = k''s'^3 + 3k's's'' + ks'''. \tag{5.6}$$

Since $E(s) = \varphi(s)/s^2$ we have

$$E'(s) = \frac{s\varphi'(s) - 2\varphi(s)}{s^3}$$

and by Cauchy $E'(s) = \varphi'''(z)/6$ for an intermediate z. In general, it is a simple matter to prove

$$\|E\|_{C^n} \leqslant C\|\varphi\|_{C^{n+2}}. \tag{5.7}$$

Finally

$$s' = \frac{2\sqrt{E}}{2E + sE'}$$

$$s'' = \frac{2E'(2E + sE') - 4E(3E' - sE'')}{(2E + sE')^3}$$

$$s''' = \dots$$

and by using (5.1):

$$|s^{(n)}| \leqslant \frac{G(\|E\|^{C^3})}{A^{2^n-1}}, \qquad 1 \leqslant n \leqslant 3 \tag{5.8}$$

for some continuous function G.

Formulas (5.5), (5.6) and (5.8) imply

$$\|h\|_{C^2} \leqslant \|k\|_{C^2} \frac{G(A, \|\varphi\|_{C^5})}{A^7} \tag{5.9}$$

for a new continuous G.

Now, the Lemma follows from (5.3), (5.4), (5.6).

For a given convex domain Ω, we put

$$M(\Omega) = \max \operatorname{diam} \Omega = \max \delta(\vartheta),$$

$$m(\Omega) = \min \operatorname{diam} \Omega = \min \delta(\vartheta),$$

$$S(\Omega) = \max |H_\Omega(\vartheta) - 1|,$$

where $H_\Omega(\vartheta) = \sqrt{k_{\Omega 1}(\vartheta)/k_{\Omega 2}(\vartheta)}$ and the meaning of $k_{\Omega 1}(\vartheta)$, $k_{\Omega 2}(\vartheta)$ is clear (see the definition of $k_1(\vartheta)$ and $k_2(\vartheta)$ in the Section 3).

Consider a family $\mathcal{F}$ of convex domain with the following property

$$M(\Omega) - m(\Omega) + S(\Omega) \geqslant C > 0 \qquad \forall \Omega \in \mathcal{F}. \tag{5.10}$$

Condition (5.10) means that the domains which are in $\mathcal{F}$, are far from a disk, since disks are characterized by $M(\Omega) = m(\Omega)$ and $S(\Omega) = 0$.

Let Ω be a domain in $\mathcal{F}$. Then we have from (5.10) that one of the following conditions is true

$$S(\Omega) \geqslant \frac{C}{2}, \tag{5.11}$$

$$M(\Omega) - m(\Omega) \geqslant \frac{C}{2}, \qquad S(\Omega) < \frac{C}{2}. \tag{5.12}$$

From (3.3) we have

$$\hat{\chi}_{\partial\Omega}(\alpha\cos\vartheta, \alpha\sin\vartheta) = F_\Omega(\vartheta, \alpha)\left\{e^{-i\alpha\delta_\Omega(\vartheta)} + iH_\Omega(\vartheta) + \frac{1}{\sqrt{\alpha}} R_\Omega(\vartheta, \alpha)\right\}. \tag{5.13}$$

Note the dependence of the right hand side from Ω.

We observe that if $\Omega = V = F \cap \{\Omega\colon \Omega_{-\varepsilon_0} \subset \Omega \subset \Omega_{\varepsilon_0}\}$ for a suitably small ε_0, then from the Lemma it follows that

$$R = \sup_{\Omega\in V,\ \vartheta\in[0,2\pi],\ \alpha\in\mathbf{R}^+} |R_\Omega(\vartheta, \alpha)| < +\infty.$$

Assume (5.11) holds. Then for $\Omega \in V \cap \mathcal{F}$

$$\sup_{[0,2\pi]} \left| e^{-i\alpha\delta_\Omega(\vartheta)} + iH_\Omega(\vartheta) + \frac{1}{\sqrt{\alpha}} R_\Omega(\vartheta, \alpha)\right| \geqslant \tag{5.14}$$

$$\geqslant \sup_{[0,2\pi]} |e^{-i\alpha\delta_\Omega(\vartheta)} + iH_\Omega(\vartheta)| - \frac{1}{\sqrt{\alpha}} R \geqslant \sup_{[0,2\pi]} |H_\Omega(\vartheta) - 1| - \frac{1}{\sqrt{\alpha}} R \geqslant \frac{C}{2} - \frac{1}{\sqrt{\alpha}} R \geqslant \frac{C}{4}$$

for $\alpha \geqslant 16R^2/C^2$.

Now assume (5.12) holds. For $\Omega \in V \cap \mathcal{F}$ we have

$$(5.15)\quad \sup_{[0,2\pi]} \left| e^{-i\alpha\delta_\Omega(\vartheta)} + iH_\Omega(\vartheta) - \frac{1}{\sqrt{\alpha}} R_\Omega(\vartheta,\alpha) \right| \geqslant \sup_{[0,2\pi]} |e^{-i\alpha\delta_\Omega(\vartheta)} + iH_\Omega(\vartheta)| - \frac{1}{\sqrt{\alpha}} R \geqslant$$

$$\leqslant \sup_{[0,2\pi]} \{ |e^{-i\alpha\delta_\Omega(\vartheta)} + i| - |H_\Omega(\vartheta) - 1| \} - \frac{1}{\sqrt{\alpha}} R \geqslant \sup_{[0,2\pi]} |e^{-i\alpha\delta_\Omega(\vartheta)} + i| - \frac{C}{2} - \frac{1}{\sqrt{\alpha}} R =$$

$$= \sup_{\delta \in [m(\Omega), M(\Omega)]} |e^{-i\alpha\delta} + i| - \frac{C}{2} - \frac{1}{\sqrt{\alpha}} R = \sup_{t \in [\alpha m(\Omega), \alpha M(\Omega)]} |e^{-it} + i| - \frac{C}{2} - \frac{1}{\sqrt{\alpha}} R .$$

We observe that by (5.12), $\alpha(M(\Omega) - m(\Omega)) \geqslant \alpha(C/2) \geqslant 2\pi$ if $\alpha \geqslant 4\pi/C$. Then the interval $[\alpha m(\Omega), \alpha M(\Omega)]$, contains at least a point of the form $-\pi/2 + 2k\pi$, and therefore

$$\sup_{t \in [\alpha m(\Omega), \alpha M(\Omega)]} |e^{-it} + i| = 2$$

when $\alpha \geqslant 4\pi/C$.

By (5.15) we deduce

$$(5.16)\qquad \sup_{[0,2\pi]} \left| e^{-i\alpha\delta_\Omega(\vartheta)} + iH_\Omega(\vartheta) - \frac{1}{\sqrt{\alpha}} R(\vartheta,\alpha) \right| \geqslant 2 - \frac{C}{2} - \frac{1}{\sqrt{\alpha}} R \geqslant 1$$

for $\alpha \geqslant 4R^2/(2-C)^2$ when $\Omega \in V \cap \mathcal{F}$.

In conclusion, by (5.13), (5.14) and (5.16) $\hat{\chi}_{\partial\Omega}$ does not vanish identically for every $\Omega \in V \cap \mathcal{F}$ and $\alpha \geqslant \alpha_0$ with α_0 independent from Ω.

Now, assume that Ω_0 is a Pompeiu convex domain. Then, by (5.16) there exist $\varepsilon > 0$ and $\alpha_0 > 0$ such that

(5.17) $\hat{\chi}_{\partial\Omega}$ does not vanish identically on every circle M_α if $\Omega_{-\varepsilon} \subset \Omega \subset \Omega_\varepsilon$,

$\alpha \geqslant \alpha_0$.

Since Ω_0 is a Pompeiu set,

$$(5.18)\qquad P(\alpha) = \sup_{M_\alpha} |\hat{\chi}_{\Omega_0}| > 0 \qquad \forall \alpha > 0$$

and hence

$$(5.19)\qquad P = \inf_{[0,\alpha_0]} P(\alpha) > 0 .$$

Let Ω be a small deformation of Ω_0. We obtain for $\alpha \leqslant \alpha_0$:

$$(5.20)\qquad \sup_{M_\alpha} |\hat{\chi}_\Omega| \geqslant P - \sup_{|\zeta| \leqslant \alpha_0} |\hat{\chi}_\Omega - \hat{\chi}_{\Omega_0}| \geqslant \frac{P}{2} \text{ if } \sup_{|\zeta| \leqslant \alpha_0} |\hat{\chi}_\Omega - \hat{\chi}_{\Omega_0}| \leqslant \frac{P}{2} .$$

A simple calculation shows

$$(5.21)\qquad |\widehat{\chi}_{\Omega}(\zeta)-\widehat{\chi}_{\Omega_0}(\zeta)| =$$

$$= \left| \int_{\Omega-\Omega_0} e^{-i\langle x,\,\zeta\rangle}dx - \int_{\Omega_0-\Omega} e^{-i\langle x,\,\zeta\rangle}dx \right| \leq \mu(\Omega-\Omega_0)+\mu(\Omega_0-\Omega).$$

Therefore

$$(5.22)\qquad \sup |\widehat{\chi}_{\Omega}-\widehat{\chi}_{\Omega_0}| \leq \frac{P}{2}$$

if

$$(5.23)\qquad \mu(\Omega-\Omega_0)+\mu(\Omega_0-\Omega) \leq \frac{P}{2}\,,$$

and (5.22) is surely satisfied if

$$(5.24)\qquad \Omega_{-\delta} \subset \Omega \subset \Omega_{\delta}$$

for a sufficiently small δ.

In conclusion, for $\alpha \leq \alpha_0$

$$(5.25)\qquad \sup_{M_\alpha} |\widehat{\chi}_{\Omega}| \geq \frac{P}{2}$$

when Ω satisfies (5.24).

Finally, by (5.25) and (5.17) Theorem follows.

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