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STABILITY OF THE DIVISOR CLASS GROUP UPON COMPLETION

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In honor of Phil Griffith's contributions to mathematics

Abstract. If A is a local, normal approximation domain with finite divisor class group, then $Cl(A) \cong Cl(\widehat{A})$.

1. Introduction

Let $(A, \mathfrak{M})$ be a local, analytically normal domain and $\widehat{A}$ its completion with respect to the maximal ideal $\mathfrak{M}$. We shall use $\mathrm{cl}(\mathfrak{a})$ to denote the equivalence class of a divisorial A-ideal $\mathfrak{a}$ in the divisor class group, $\mathrm{Cl}(A)$, of A. It is well known that the canonical group homomorphism $i : \mathrm{Cl}(A) \to \mathrm{Cl}(\widehat{A})$, defined by $i(\mathrm{cl}(\mathfrak{a})) := \mathrm{cl}(\widehat{\mathfrak{a}})$, is one-to-one, but, in general, far from being onto, whence the question arises for which rings might one expect the above map to be an isomorphism.

Danilov investigated this question as part of a cycle of related problems in a series of papers [4], [5], [6], [7]. Bingener [1], and Bingener and Storch [2], also gave some thought to the vanishing of $\mathrm{Coker}(i)$. In both cases, the authors used geometric techniques (and, in particular, required resolution of singularities and an algebraically closed residue field) to guarantee the surjectivity of i for approximation rings with finitely generated divisor class group.

On the other hand, Griffith and Weston [9, 4.1] leaned on algebraic methods to establish the surjectivity of the map $i| : \mathrm{Cl}(A)_{\mathrm{tor}} \to \mathrm{Cl}(\widehat{A})_{\mathrm{tor}}$ for a local $\mathbb{Q}$-algebra A, at once a normal approximation domain. (Here, $i|$ stands for the restriction of i to the torsion subgroups of the relevant divisor class groups. We mention that the condition in 4.1 on primitive roots of unity can be eliminated.) The gist of the proof involved the manipulation and descent of a Galois extension of $\widehat{A}$, module-isomorphic to $\widehat{A} \oplus \mathfrak{b} \oplus \mathfrak{b}^{(2)} \oplus \cdots \oplus \mathfrak{b}^{(e-1)}$ for a divisorial torsion ideal $\mathfrak{b}$ of $\widehat{A}$ with e the order of $\mathrm{cl}(\mathfrak{b})$ in $\mathrm{Cl}(\widehat{A})$.

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Rotthaus' paper [15] represented yet another algebraic approach to the same problem. The author introduced the technique of complete induction which she applied to, among others, the issue of the bijectivity of $i : \mathrm{Cl}(A) \to \mathrm{Cl}(\widehat{A})$ and $i| : \mathrm{Cl}(A)_{\mathrm{tor}} \to \mathrm{Cl}(\widehat{A})_{\mathrm{tor}}$. The setting was that of a local, normal domain A with the complete approximation property (e.g., if A is both a $\mathbb{Q}$-algebra and an approximation ring) satisfying the R_2-regularity condition and with $\mathrm{Cl}(A)$ and $\mathrm{Cl}(A)_{\mathrm{tor}}$ a finite group, respectively.

In the following, we eliminate the R_2-regularity condition, and ask merely for the approximation (rather than the complete approximation) property to conclude that i is an isomorphism when $\mathrm{Cl}(A)$ is finite. The present line of argumentation was motivated by a desire to generalize Hochster's proof (oral communication) of the fact that a local approximation unique factorial domain remains a factorial domain upon completion. It is tempting to generalize Danilov's result [7, Proof of Theorem 1] with the same techniques and claim that i is an isomorphism when A is a local, normal approximation domain with finitely generated divisor class group. However, the methods below, while natural for torsion elements, do not adapt with ease to the torsion free case.

2. Terminology and facts

We present some notation that will be used without repeated mention:

The set of natural numbers $\{0, 1, 2, \dots\}$ will be denoted by $\mathbb{N}$, and the set of positive integers $\{1, 2, \dots\}$ by $\mathbb{P}$. For a ring A, $\mathcal{U}(A)$ will denote the set of invertible elements and $X^1(A)$ the set of all height one prime ideals of A. If A is a domain, K will stand for the field of quotients $A_{(0)}$. When A is a normal domain and $\mathfrak{a}$ a fractional ideal of A, then $\mathfrak{a}^{**}$ is a divisorial ideal (here, $\mathfrak{a}^* := A : \mathfrak{a}$) and $\mathfrak{a}^{**} = \bigcap_{\mathfrak{p} \in X^1(A)} \mathfrak{a}_{\mathfrak{p}}$.

What follows is a pedestrian overview of facts used often in the proof, and provided here for the convenience of the reader. These can be gleaned from standard texts on the topic, such as [3], [8], [11], [12].

Theorem 2.1 (Approximation Theorem for Krull Domains [8, 5.8]). *Let A be a Krull domain with $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ distinct height one prime ideals of A and $n_1, \dots, n_r$ integers. Then there is a non-zero element x in K such that $v_{\mathfrak{p}_i}(x) = n_i$ and $v_{\mathfrak{p}}(x) \geq 0$ for $\mathfrak{p} \in X^1(A) \backslash \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$.*

(Here, $v_{\mathfrak{p}}$ is the valuation associated to the principal valuation ring $A_{\mathfrak{p}}$ and $v_{\mathfrak{p}}(\mathfrak{a}) := \inf\{v_{\mathfrak{p}}(x) | x \in \mathfrak{a}\}$ for a non-zero fractional ideal $\mathfrak{a}$ of A.)

Theorem 2.2 (Nagata's Theorem [8, 7.2]). *Let S be a multiplicatively closed subset of a Krull domain A. Then $1 \to \langle \mathrm{cl}(\mathfrak{p}) | \mathfrak{p} \in X^1(A)$ and $S \bigcap \mathfrak{p} \neq \emptyset \rangle \to \mathrm{Cl}(A) \to \mathrm{Cl}(S^{-1}A) \to 1$ is a short exact sequence of abelian groups.*

COROLLARY 1. *Let A be a normal domain with $\mathrm{Cl}(A)$ finitely generated. Then there is an element $a \in A\backslash\{0\}$ such that A_a is a unique factorization domain.*

Proof. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ be divisorial ideals of A with $\mathrm{Cl}(A) = \langle \mathrm{cl}(\mathfrak{a}_1), \ldots, \mathrm{cl}(\mathfrak{a}_k)\rangle$. Then let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} = \{\mathfrak{p} \in X^1(A) \mid v_{\mathfrak{p}}(\mathfrak{a}_i) \neq 0$ for some $i = 1, \ldots, k\}$.

By Theorem 1, there are $\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_s \in X^1(A)\backslash\{\mathfrak{p}, \ldots, \mathfrak{p}_r\}$ and $n_{r+1}, \ldots, n_s \in \mathbb{P}$ such that $(\bigcap_{i=1}^r \mathfrak{p}_i) \bigcap (\bigcap_{j=r+1}^s \mathfrak{p}_j^{(n_j)}) = aA$ for some $a \in A$.

By Theorem 2, $\mathrm{Cl}(A_a) = 0$ since $\mathrm{Cl}(A) = \langle \mathrm{cl}(\mathfrak{p}_1), \ldots, \mathrm{cl}(\mathfrak{p}_r)\rangle$. □

We refer the reader to [2] or [8] for the following result.

FACT 1. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of A.*

(i) *Then,* $\mathrm{div}(\mathfrak{a} \cdot \mathfrak{b}) = \mathrm{div}(\mathfrak{a}) + \mathrm{div}(\mathfrak{b})$.

(ii) *If $v_{\mathfrak{p}}(\mathfrak{a}) \cdot v_{\mathfrak{p}}(\mathfrak{b}) = 0$ for all $\mathfrak{p} \in X^1(A)$, then $(\mathfrak{a} \cdot \mathfrak{b})^{**} = \mathfrak{a}^{**} \bigcap \mathfrak{b}^{**}$.*

(iii) *For $c \in K\backslash\{0\}$, we have $(c\mathfrak{a})^{**} = c(\mathfrak{a}^{**})$.*

Consequently, we have:

FACT 2. *Let $P_1, \ldots, P_m \in X^1(A)$ be distinct prime ideals such that $\mathrm{cl}(P_1) = \cdots = \mathrm{cl}(P_m)$. (Say $P_i = \alpha_i \mathfrak{p}$ for $i = 1, \ldots, m$ with $\alpha_i \in K\backslash\{0\}$.) Let $k_1, \ldots, k_m \in \mathbb{P}$.*

(i) *Then $\bigcap_{i=1}^m P_i^{(k_i)} = (\Pi_{i=1}^m P_i^{k_i})^{**} = (\Pi_{i=1}^m \alpha_i^{k_i} \mathfrak{p}^{k_i})^{**} = \alpha \mathfrak{p}^{(k)}$, where $\alpha := \Pi_{i=1}^m \alpha_i^{k_i}$ and $k := \sum_{i=1}^m k_i$.*

(ii) *Suppose* $\mathrm{Cl}(A)$ *is a finite group of order $t \leq m$. Let $b \in A$ be such that $\mathfrak{p}^{(t)} = bA$. Then $\bigcap_{i=1}^t P_i = \gamma A$ with $\gamma = b \cdot \alpha_1 \cdot \cdots \cdot \alpha_t$. Since $\bigcap_{i=1}^t P_i \subset A$, then $\gamma \in A\backslash\mathcal{U}(A)$.*

(iii) *Suppose that $t \lneqq m$. Define $\mathfrak{a} := (\bigcap_{i=1}^t P_i^{(k_i-1)}) \bigcap (\bigcap_{i=t+1}^m P_i^{(k_i)})$ and note that $\mathfrak{a}$ is a divisorial ideal, properly contained in A. Then $\bigcap_{i=1}^m P_i^{(k_i)} = \gamma \mathfrak{a}$ for γ defined as in* (ii).

Let

(i) $\underline{X}$ stand for a sequence of variables $X_1, \ldots, X_s$;

(ii) $\underline{f}$ stand for a sequence of polynomial functions $f_1(\underline{X}), \ldots, f_t(\underline{X}) \in A[\underline{X}] = A[X_1, \ldots, X_s]$;

(iii) $\underline{x}$ stand for a sequence of elements $x_1, \ldots, x_s$ in A;

(iv) $\underline{\tilde{x}}$ stand for a sequence of elements $\underline{\tilde{x}}_1, \ldots, \underline{\tilde{x}}_s$ in $\widehat{A}$;

(v) $\underline{x} \equiv \underline{\tilde{x}} \bmod \widehat{\mathfrak{M}}^n$ stand for a sequence of congruences $x_1 \equiv \tilde{x}_1 \bmod \widehat{\mathfrak{M}}^n$, $\ldots$, $x_s \equiv \tilde{x}_s \bmod \widehat{\mathfrak{M}}^n$.

This notation will facilitate the statement of the following definition.

DEFINITION. Let $(A, \mathfrak{M})$ be a local, Noetherian ring.

(i) Suppose that given any $\underline{f} \in A[\underline{X}]$ having a solution $\underline{\tilde{x}} \in \widehat{A}^s$, and given any $n \in \mathbb{P}$, there is a solution $\underline{x} \in A^s$ for $\underline{f}$ such that $\underline{x} \equiv \underline{\tilde{x}} \bmod \widehat{\mathfrak{M}}^n$. Then A is called an *approximation ring*.

(ii) Suppose that given any $\underline{f} \in A[\underline{X}]$, there is an increasing function $\theta : \mathbb{N} \to \mathbb{N}$ with $\theta(n) \geq n$ such that whenever $\underline{x} \in A^s$ is a solution of $\underline{f} \bmod \mathfrak{M}^{\theta(n)}$, then there is a solution $\underline{\tilde{x}} \in A^s$ of $\underline{f}$ with $\underline{x} \equiv \underline{\tilde{x}} \bmod \mathfrak{M}^n$. Then A is called a *strong approximation ring*, and θ is called a *strong approximation function with respect to* $\underline{f}$. (Note that $\underline{\tilde{x}}$ depends on n, in general.)

Pfister and Popescu [13] and van der Put [16], in a more limited setting, have shown that the above two properties yield the same class of rings. Clearly, then, all complete, local rings are strong approximation rings.

Often it will be convenient to make "index shifts" as in the following fact.

FACT 3. *Let A be a strong approximation ring, $\underline{f} = (f_1, \ldots, f_t) \in A[\underline{X}] = A[X_1, \ldots, X_s], \theta : \mathbb{N} \to \mathbb{N}$ be a strong approximation function with respect to $\underline{f}$, and suppose that $\underline{x}(n) := (x_{1n}, \ldots, x_{sn}) \in A^s$ is a solution of $\underline{f} \bmod \mathfrak{M}^n$ for $n \in \mathbb{N}$.*

(i) *Then we may assume that, for $n \gg 0$, there is a solution $\underline{x} \in A^s$ of $\underline{f}$ such that $\underline{x}(n) \equiv \underline{x} \bmod \mathfrak{M}^{\theta^{-1}(n)}$ for $n \in \mathrm{Im}(\theta)$.*

(ii) *Furthermore, we may assume that, for $n \gg 0$, there is a solution $\underline{x} \in A^s$ of $\underline{f}$ such that $\underline{x}(n) \equiv \underline{x} \bmod \mathfrak{M}^n$.*

Proof. (i) Observe that $\theta(1) \geq \theta(2) \geq \cdots$ and $\lim_{n\to\infty} \theta(n) = +\infty$. Let $l_0 := \theta(1)$. Then the statement holds for $n \geq l_0$ since we can replace $\underline{x}(n)$ with $\underline{x}(l)$, where $l = \min\{k \in \mathbb{N} \mid n \leq k \text{ and } k \in \mathrm{Im}(\theta)\}$.

(ii) Let $\underline{x}(n) \equiv \underline{x} \bmod \mathfrak{M}^{\theta^{-1}(n)}$ for $n \in \mathrm{Im}(\theta)$. We can replace $\underline{x}(n)$ with $\underline{x}(\theta(n))$. Then there is a solution $\underline{x} \in A^s$ of $\underline{f}$ such that $\underline{x}(n) \equiv \underline{x} \bmod \mathfrak{M}^n$. □

We may make use of the above artifice without explicit mention.

Most of our manoeuvres will involve the following fact:

FACT 4. *Let $(A, \mathfrak{M})$ be a local ring with $b, b_n \in A$ for $n \in \mathbb{N}$ such that $b_n \equiv b \bmod \mathfrak{M}^n$ and such that b_n enjoys a property $\mathcal{P}$ for infinitely many n. Then we may assume that b_n enjoys the property $\mathcal{P}$ for all n and that $b_n \equiv b \bmod \mathfrak{M}^n$.*

Proof. If b_n does not satisfy $\mathcal{P}$, then replace b_n with b_l, where $l = \min\{k \in \mathbb{N} \mid n \leq k \text{ and } b_k \text{ satisfies } \mathcal{P}\}$. □

To avoid cumbersome extra notation, we shall frequently avail ourselves of the following fact:

FACT 5. *Let* $\mathfrak{p} \in X^1(A)$, $w \in A$, *and* $s \in \mathbb{P}$. *Then the statement* $w \in \mathfrak{p}^{(s)}$ *is synonymous with an equation. (Namely,* $w = \sum_{i=1}^{k} c_i r_i$ *for some* $r_i \in A$, *if* $\mathfrak{p}^{(s)} = (c_1, \ldots, c_k)A$.*)*

Our main proof will hinge on two additional facts.

FACT 6. *Let* $(A, \mathfrak{M})$ *be a local Noetherian ring. Let* $b, b_n \in A$ *be ruled by* $b_n \equiv b \bmod \mathfrak{M}^n$ *for* $n \in \mathbb{N}$.

(i) *Then for each* $\mathfrak{p} \in X^1(A)$ *there are positive integers* k *and* m *for which* $b \in \mathfrak{p}^{(m)} \backslash \mathfrak{p}^{(m+1)}$ *and* $b_n \notin \mathfrak{p}^{(m+1)}$ *for all* $n \geq k$. *(Obviously,* k *and* m *depend on* $\mathfrak{p} \in X^1(A)$.*)*
(ii) *If* $b_n \in \mathfrak{p}^{(t)}$ *for some* $t \in \mathbb{N}$, *for* $n \gg 0$, *then* $b \in \mathfrak{p}^{(t)}$.

Proof. Let $\mathfrak{I} \subseteq A$ be an ideal of A with $b \notin \mathfrak{I}$. Then there is a $k \in \mathbb{N}$ such that $b \notin \mathfrak{I} + \mathfrak{M}^k$. Thus $b_n \notin \mathfrak{I}$ for $n \geq k$.

(i) Since $\bigcap_{n=0}^{\infty} \mathfrak{p}^{(n)} = (0)$, there is an $m \in \mathbb{N}$ with $b \in \mathfrak{p}^{(m)} \backslash \mathfrak{p}^{(m+1)}$. Let $\mathfrak{I} = \mathfrak{p}^{(m+1)}$.

(ii) Let $\mathfrak{I} = \mathfrak{p}^{(t)}$. □

FACT 7. *Suppose that* $(A, \mathfrak{M})$ *is a local, normal approximation domain with* $a \in \mathfrak{M}$. *Let* $x \in \widehat{A}$ *and* $x_n \in A$ *be ruled by* $x_n \equiv x \bmod \widehat{\mathfrak{M}}^n$ *for all* $n \in \mathbb{N}$.

(i) *If* x *is irreducible in* $\widehat{A}$, *then* x_n *is irreducible in* A *for all* $n \gg 0$.
(ii) *If* $x \notin \mathcal{U}(\widehat{A}_a)$, *then* $x_n \notin \mathcal{U}(A_a)$ *for* $n \gg 0$.

Proof. (i) By way of contradiction, let $x_n = c_n d_n$ with $c_n, d_n \in A \backslash \mathcal{U}(A)$ for infinitely many n. We can then assume that this state of affairs holds for all n.

Thus $x \equiv c_n d_n \bmod \widehat{\mathfrak{M}}^n$ for all n. Since $\widehat{A}$ is a strong approximation ring, then there are $c, d \in \widehat{A}$ with $x = cd$ and $(c_n, d_n) \equiv (c, d) \bmod \widehat{\mathfrak{M}}^n$ for $n \gg 0$. This duet of congruences together with the assumption that $c_n, d_n \in \mathfrak{M}$ (for all n) force the conclusion that $c, d \in \mathfrak{M}$, a contradiction to the irreducibility of x in $\widehat{A}$. Hence, x_n is irreducible in A for $n \gg 0$.

(ii) Let $aA = \bigcap_{i=1}^{s} \mathfrak{p}_i^{(l_i)}$, where $\mathfrak{p}_i \in X^1(A)$ and $l_i \in \mathbb{P}$ for $i = 1, \ldots, s$. By way of contradiction, assume that $x_n \in \mathcal{U}(A_a)$ for infinitely many (and, hence, for all) $n \in \mathbb{N}$. Then $x_n A = \bigcap_{i=1}^{s} \mathfrak{p}_i^{(m_{in})}$, where $m_{in} \in \mathbb{N}$. For each $i \in \{1, \ldots, s\}$, there is an $m_i \in \mathbb{N}$ such that $m_{in} = m_i$ for $n \gg 0$ by Fact 6. So there are $x_0, u_n, v_n \in A$ for $n \gg 0$ such that $x_0 A = \bigcap_{i=1}^{s} \mathfrak{p}_i^{(m_i)}$, $x_n = x_0 u_n$ and $1 = u_n v_n$. Then $(x, 1) \equiv (x_0 u_n, u_n v_n) \bmod \widehat{\mathfrak{M}}^n$. Since $\widehat{A}$ is a strong approximation ring, then there are $u, v \in \widehat{A}$ such that $x = x_0 u$ and $uv = 1$ with $(u_n, v_n) \equiv (u, v) \bmod \widehat{\mathfrak{M}}^n$ for $n \gg 0$. But $x_0 \in \mathcal{U}(\widehat{A}_a)$ and $u \in \mathcal{U}(\widehat{A})$, contradicting that $x \notin \mathcal{U}(\widehat{A}_a)$. So $x_n \notin \mathcal{U}(A_a)$ for $n \gg 0$. □

Finally we turn to a paper by Rotthaus [14] in which the author showed that approximation rings are both excellent and Henselian. This result carries with it some crucial consequences:

Corollary 2. *A local, normal approximation domain is analytically normal.*

Corollary 3. *Let $(A, \mathfrak{M})$ be an approximation ring and $\mathfrak{p} \in \operatorname{Spec}(A)$. Then $\widehat{\mathfrak{p}} \in \operatorname{Spec}(\widehat{A})$.*

Proof. Note that $A/\mathfrak{p}$ is local, reduced, and excellent. Let $(A/\mathfrak{p})'$ stand for the integral closure of $A/\mathfrak{p}$ in its total field of fractions. By [10, 7.8.3.1 (vii)], it is known that:

(i) $\widehat{A}/\widehat{\mathfrak{p}}$ is reduced,
(ii) $(A/\mathfrak{p})'$ has the same number of maximal ideals as $\widehat{A}/\widehat{\mathfrak{p}}$ has minimal prime ideals, and
(iii) $(A/\mathfrak{p})'$ is a finitely generated $A/\mathfrak{p}$-module.

Since $A/\mathfrak{p}$ is Henselian, then $(A/\mathfrak{p})'$ is a product of a finite number of local rings by (iii).

On the other hand, $(A/\mathfrak{p})'$ is a domain, and hence local. By (ii), $\widehat{A}/\widehat{\mathfrak{p}}$ has only one minimal prime. By (i), $\widehat{\mathfrak{p}} \in \operatorname{Spec}(\widehat{A})$. □

3. Descent of $\operatorname{Cl}(\widehat{A})$

At this point, we are in the position to plunge into the proof.

Theorem 3.1. *Let $(A, \mathfrak{M})$ be a local, normal, approximation domain with finite divisor class group. Then the canonical group monomorphism $i : \operatorname{Cl}(A) \longrightarrow \operatorname{Cl}(\widehat{A})$, ruled by $i(\operatorname{cl}(\mathfrak{a})) = \operatorname{cl}(\widehat{\mathfrak{a}})$, is an isomorphism.*

Proof. By Corollary 1, there is an element $a \in A$ for which A_a is an unique factorization domain. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \in X^1(A)$ and $l_1, \ldots, l_s$ be positive integers such that $aA = \bigcap_{i=1}^s \mathfrak{p}_i^{(l_i)}$. It follows from Theorem 2 that $\operatorname{Cl}(A) = \langle \operatorname{cl}(\mathfrak{p}_i) \mid i = 1, \ldots, s \rangle$.

Also Corollary 3 implies that $a\widehat{A} = \bigcap_{i=1}^s \widehat{\mathfrak{p}}_i^{(l_i)}$.

If $\widehat{A}_a$ is an unique factorization domain, then $\operatorname{Cl}(\widehat{A}) = \langle \operatorname{cl}(\widehat{\mathfrak{p}}_i) \mid i = 1, \ldots, s \rangle$, and hence $i : \operatorname{Cl}(A) \longrightarrow \operatorname{Cl}(\widehat{A})$ is an isomorphism. Therefore, our goal is to establish that $\widehat{A}_a$ is an unique factorization domain. Since $\widehat{A}_a$ is a Noetherian domain, it suffices to show that any irreducible element x in $\widehat{A}_a$ is a prime element in $\widehat{A}_a$. Furthermore, one can reduce the problem to proving that x divides either u or v in $\widehat{A}_a$ whenever $x \in \widehat{A}$ is an irreducible element of both $\widehat{A}$ and $\widehat{A}_a$ and $u, v \in \widehat{A}$ are such that $uv = xy$ for some $y \in \widehat{A}$.

Let $x \in \widehat{\mathfrak{p}}_i^{(m_i)} \backslash \widehat{\mathfrak{p}}_i^{(m_i+1)}$ and $u \in \widehat{\mathfrak{p}}_i^{(o_i)} \backslash \widehat{\mathfrak{p}}_i^{(o_i+1)}$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \in X^1(A)$ with $aA = \bigcap_{i=1}^s \mathfrak{p}_i^{(l_i)}$. As A is an approximation ring, there are elements u_n, v_n, x_n, y_n of A such that $u_n v_n = x_n y_n$, $x_n \in \mathfrak{p}_i^{(m_i)} \backslash \mathfrak{p}_i^{(m_i+1)}$ and $u_n \in \mathfrak{p}_i^{(o_i)} \backslash \mathfrak{p}_i^{(o_i+1)}$, and $(u_n, v_n, x_n, y_n) \equiv (u, v, x, y) \bmod \widehat{\mathfrak{M}}^n$ (see Facts 5 and 6). Also, x_n is irreducible in A and $x_n \notin \mathcal{U}(A_a)$ for $n \gg 0$ by Fact 7.

At this point we turn to the primary decomposition of $x_n A$. For each n, there are positive integers $m_n, k_{1n}, k_{2n}, \ldots, k_{m_n n}$ and distinct prime ideals $P_{1n}, \ldots, P_{m_n n} \in X^1(A)$ such that $x_n A = \bigcap_{j=1}^{m_n} P_{jn}^{(k_{jn})}$. Our strategy will consist of "stabilizing" m_n and k_{jn}, namely proving that there are positive integers $m, k_1, \ldots, k_m$ such that $x_n A = \bigcap_{j=1}^{m} P_{jn}^{(k_j)}$ for $n \gg 0$.

Suppose that $|\operatorname{Cl}(A)| = t$.

We shall first bound m_n:

Let $\mathcal{S}_n = \{P_{jn} \mid j = 1, \ldots, m_n\}$ with $|\mathcal{S}_n| = m_n$. We partition $\mathcal{S}_n$ into a disjoint union $\mathcal{S}_n = \dot{\bigcup}_{g \in \operatorname{Cl}(A)} \mathcal{S}_{gn}$, where $\mathcal{S}_{gn} := \{P_{jn} \in \mathcal{S}_n \mid \operatorname{cl}(P_{jn}) = g\}$ and $m_{gn} := |\mathcal{S}_{gn}|$. (Here we are allowing $\mathcal{S}_{gn}$ to be the empty set for some $g \in \operatorname{Cl}(A)$, as would be the case if g is the identity in $\operatorname{Cl}(A)$, for instance.)

Let $\mathcal{H}$ be the subset $\{g \in \operatorname{Cl}(A) \mid \mathcal{S}_{gn} \neq \emptyset\}$ of $\operatorname{Cl}(A)$. From now on, any mention of g will presume that $g \in \mathcal{H}$, as opposed to the larger $\operatorname{Cl}(A)$. Since $|\operatorname{Cl}(A)| = t$, then $|\mathcal{H}| < t$.

We fix $\mathfrak{p}_{gn} \in \mathcal{S}_{gn}$ so that $\operatorname{cl}(\mathfrak{p}_{gn}) = g$. For purposes of inventory, I_{gn} will denote the set $\{j \mid 1 \leq j \leq m_n \text{ and } P_{jn} \in \mathcal{S}_{gn}\}$.

If $t \lneqq m_{gn}$, then let $I_{gn}^{'}$ be a fixed subset of I_{gn} of order t, and $I_{gn}^{''} := I_{gn} \backslash I_{gn}^{'}$.

Evocative of the notation in Fact 2, we shall let

(i) $\alpha_{jn} \in K$ be such that $P_{jn} = \alpha_{jn} \mathfrak{p}_{gn}$ for $j \in I_{gn}$,

(ii) $b_{gn} \in A$ be such that $\mathfrak{p}_{gn}^{(t)} = b_{gn} A$,

(iii) $\mathfrak{a}_{gn} := (\bigcap_{j \in I_g^{'}} P_{jn}^{(k_{jn}-1)}) \bigcap (\bigcap_{j \in I_g^{''}} P_{jn}^{(k_{jn})}) \subsetneqq A$ if $t \lneqq m_{gn}$, and
$\mathfrak{a}_{gn} := \bigcap_{j \in I_g} P_{jn}^{(k_{jn})} \subsetneqq A$ if $t \geq m_{gn}$,

(iv) $\gamma_{gn} := (\prod_{j \in I_g^{'}} \alpha_{jn}) b_{gn} \in \bigcap_{j \in I_g^{'}} P_{jn} \subset A \backslash \mathcal{U}(A)$ if $t \lneqq m_{gn}$, and
$\gamma_{gn} := 1$ if $t \geq m_{gn}$.

Then by Fact 2, $x_n A = (\prod_{g \in \mathcal{H}} \gamma_{gn}) \cdot (\bigcap_{g \in \mathcal{H}} \mathfrak{a}_{gn})$.

Since $\mathfrak{a}_{gn} \subseteq A$ for all $g \in \mathcal{H}$, then $\bigcap_{g \in \mathcal{H}} \mathfrak{a}_{gn} = \delta_n A$ for some $\delta_n \in A$. So $x_n = (\prod_{g \in \mathcal{H}} \gamma_{gn}) \cdot \delta_n \cdot u_n$ for some $u_n \in \mathcal{U}(A)$.

If $t \lneqq m_{gn}$ for at least one $g \in \mathcal{H}$, then $\delta_n \in A \backslash \mathcal{U}(A)$ and $\gamma_{gn} \in A \backslash \mathcal{U}(A)$, contradicting that x_n is irreducible in A for $n \gg 0$. Thus, $m_{gn} \leq t$ for all $g \in \mathcal{H}$, for $n \gg 0$.

But $m_n = \sum_{g \in \mathcal{H}} m_{gn} \leq t \cdot (t-1)$ since $|\mathcal{H}| < t$ for $n \gg 0$.

As $t^2 - t < \infty$, then there is an $m \in \{1, 2, \ldots, t^2 - t\}$ such that $m_n = m$ for $n \gg 0$ as per Fact 4.

So $x_nA = \bigcap_{j=1}^m P_{jn}^{(k_{jn})}$ for $n \gg 0$.

We now bound k_{jn}:

As before, we can find $b_{jn} \in A$ with $P_{jn}^{(t)} = b_{jn}A$. Let $q_{jn}, r_{jn} \in \mathbb{N}$ be such that $k_{jn} = t \cdot q_{jn} + r_{jn}$ with $0 \leq r_{jn} \lneqq t$. Then $x_n = b_n \cdot c_n$, where $b_n := \Pi_{j=1}^m b_{jn}^{q_{jn}}$ and $c_nA = \bigcap_{j=1}^m P_{jn}^{(r_{jn})}$. Given that x_n is irreducible for $n \gg 0$, it would seem that either $q_{jn} = 0$ for $j = 1, 2, \ldots, m$; or that $m = 1$ and k_{1n} equals the order of $\mathrm{cl}(P_{1n})$ in $\mathrm{Cl}(A)$. In either case, $k_{jn} < t$ for $j = 1, \ldots, m$. Since $t < \infty$, then there is a $k_j \in \{1, 2 \ldots, t\}$ such that $k_{jn} = k_j$ for $n \gg 0$ as per Fact 4.

So $x_nA = \bigcap_{j=1}^m P_{jn}^{(k_j)}$ for $n \gg 0$.

Without loss of generality, we may assume that $P_{jn} \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ for $j = 1, \ldots, r_n$ and that $P_{jn} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ for $j = r_n + 1, \ldots, m$. (We observe that $1 \leq r_n \leq m$ as $x_n \notin \mathcal{U}(A_a)$.) Further, since $m < \infty$, there is an $r \in \{1, \ldots, m\}$ such that $r_n = r$ for $n \gg 0$ as per Fact 4.

Then there exist $\kappa_i \in \{0, 1, \ldots, t-1\}$ such that

$$(\dagger) \qquad x_nA = \bigcap_{j=1}^{r} P_{jn}^{(k_j)} \bigcap \left(\bigcap_{i=1}^{s} \mathfrak{p}_i^{(\kappa_i)} \right) \quad \text{for } n \gg 0.$$

Since A_a is an unique factorization domain, there are $z_{jn} \in A$ with $z_{jn}A_a = P_{jn}A_a$ for $j = 1, \ldots, r$. It follows that $z_{jn}A = P_{jn} \bigcap (\bigcap_{i=1}^s \mathfrak{p}_i^{(m_{ijn})})$ with $m_{ijn} \in \mathbb{N}$ for $j = 1, \ldots, r$. We may assume that z_{jn} is irreducible in A for $j = 1, \ldots, r$, thereby forcing (by the same line of reasoning as immediately above) that $m_{ijn} < t$ for all $i = 1, \ldots, s$ and all $j = 1, \ldots, m$. Since $t < \infty$, then there are $m_{ij} \in \{1, \ldots, t-1\}$ such that $m_{ijn} = m_{ij}$ for $n \gg 0$ as per Fact 4.

So $z_{jn}A = P_{jn} \bigcap (\bigcap_{i=1}^s \mathfrak{p}_i^{(m_{ij})})$ for $n \gg 0$ with $m_{ij} \in \{0, 1, \ldots, t-1\}$ for all i and j.

Let $e \in A \bigcap \mathcal{U}(A_a)$ be such that $eA := \bigcap_{i=1}^s \mathfrak{p}_i^{(\kappa_i{}')}$ with $\kappa_i{}' \geq \kappa_i$. Then

$$\begin{aligned}
e \cdot \left(\prod_{j=1}^{r} z_{jn}^{k_j} \right) A &= \left(\bigcap_{i=1}^{s} \mathfrak{p}_i^{(\kappa_i{}')} \right) \cdot \prod_{j=1}^{r} \left(P_{jn}^{(k_j)} \bigcap \left(\bigcap_{i=1}^{s} \mathfrak{p}_i^{(k_j \cdot m_{ij})} \right) \right) \\
&= \left(\left(\left(\prod_{j=1}^{r} P_{jn}^{(k_j)} \right) \cdot \left(\prod_{i=1}^{s} \mathfrak{p}_i^{(\kappa_i)} \right) \right)^{**} \cdot \left(\prod_{i=1}^{s} \mathfrak{p}_i^{\left(\kappa_i{}' - \kappa_i + \sum_{j=1}^r k_j \cdot m_{ij}\right)} \right)^{**} \right)^{**} \\
&= \left(\left(\left(\bigcap_{j=1}^{r} P_{jn}^{(k_j)} \right) \bigcap \left(\bigcap_{i=1}^{s} \mathfrak{p}_i^{(\kappa_i)} \right) \right) \cdot \left(\bigcap_{i=1}^{s} \mathfrak{p}_i^{(\kappa_i{}' - \kappa_i + \sum_{j=1}^r k_j \cdot m_{ij})} \right) \right)^{**} \\
&= x_n \cdot \left(\bigcap_{i=1}^{s} \mathfrak{p}_i^{\left(\kappa_i{}' - \kappa_i + \sum_{j=1}^r k_j \cdot m_{ij}\right)} \right)
\end{aligned}$$

by Fact 1.

Let $t_i := {\kappa_i}' - \kappa_i + \sum_{j=1}^{r} k_j \cdot m_{ij}$. Then $e \cdot (\prod_{j=1}^{r} z_{jn}^{k_j})A = x_n \cdot (\bigcap_{i=1}^{s} \mathfrak{p}_i^{(t_i)})$ implies that there are $d \in A \bigcap \mathcal{U}(A_a)$ and μ_n , $\eta_n \in A$ such that for $n \gg 0$:

$$dA = \bigcap_{i=1}^{s} \mathfrak{p}_i^{(t_i)},$$

$$(*) \qquad e \cdot \left(\prod_{j=1}^{r} z_{jn}^{k_j} \right) = x_n \cdot \mu_n \cdot d,$$

$$\mu_n \cdot \eta_n = 1.$$

Then $e \cdot (\prod_{j=1}^{r} z_{jn}^{k_j}) \equiv x \cdot \mu_n \cdot d \bmod \widehat{\mathfrak{M}}^n$. Since $z_{nj} \in \mathfrak{p}_i^{(m_{ij})}$, by Fact 5, we also have $z_{nj} \in \widehat{\mathfrak{p}_i}^{(m_{ij})} \bmod \widehat{\mathfrak{M}}^n$. Using the strong approximation property of $\widehat{A}$, it follows that there are elements $\tilde{z}_{1n}, \ldots, \tilde{z}_{rn}, \tilde{\mu}_n, \tilde{\eta}_n \in \widehat{A}$ such that:

$$(**) \qquad e \cdot \left(\prod_{j=1}^{r} \tilde{z}_{jn}^{k_j} \right) = x \cdot \tilde{\mu}_n \cdot d,$$

$$\tilde{\mu}_n \cdot \tilde{\eta}_n = 1,$$

$$(***) \qquad \tilde{z}_{jn} \in \widehat{\mathfrak{p}}_i^{(m_{ij})} \quad \text{for } j = 1, \ldots, r,$$

$$(\tilde{z}_{1n}, \ldots, \tilde{z}_{rn}, \tilde{\mu}_n, \tilde{\eta}_n) \equiv (z_{1n}, \ldots, z_{rn}, \mu_n, \eta_n) \bmod \widehat{\mathfrak{M}}^{\theta^{-1}(n)}$$

for $n \gg 0$. (Here, $\theta : \mathbb{N} \to \mathbb{N}$ is the appropriate strong approximation function.)

Since $\tilde{\mu}_n \cdot d \in \mathcal{U}(\widehat{A}_a)$ and x is irreducible in $\widehat{A}_a$, then all but one of $\tilde{z}_{1n}, \ldots, \tilde{z}_{rn}$ are in $\mathcal{U}(\widehat{A}_a)$. Since $r < \infty$, then there exists a $j \in \{1, \ldots, r\}$ such that $\tilde{z}_{jn} \notin \mathcal{U}(\widehat{A}_a)$, for $n \gg 0$ as per Fact 4. Without loss of generality, let $j = 1$. By the irreducibility of x in $\widehat{A}_a$, $k_1 = 1$.

Since x and d are fixed and $\tilde{\mu}_n \in \mathcal{U}(\widehat{A})$, then there exist $\alpha_i \in \mathbb{N}$ such that $x \cdot \tilde{\mu}_n \cdot d \in \widehat{\mathfrak{p}}_i^{(\alpha_i)} \backslash \widehat{\mathfrak{p}}_i^{(\alpha_i+1)}$ for $i = 1, \ldots, s$. By equation $(**)$, it follows that $\tilde{z}_{jn} \notin \widehat{\mathfrak{p}}_i^{(\alpha_i+1)}$ for $i = 1, \ldots, s$, and $j = 1, \ldots, r$, for $n \gg 0$. Since $\alpha_i < \infty$ for all i, then there are $\beta_{ij} \in \{0, \ldots, \alpha_i\}$ for all $j = 1, \ldots, r$ such that $\tilde{z}_{jn} \in \widehat{\mathfrak{p}}_i^{(\beta_{ij})} \backslash \widehat{\mathfrak{p}}_i^{(\beta_{ij}+1)}$ for $n \gg 0$ as per Fact 4.

By $(***)$, $\beta_{ij} \geq m_{ij}$ for all i and j. We observe that $v_{\mathfrak{p}_i}(x_n \cdot \mu_n \cdot d) = v_{\widehat{\mathfrak{p}}_i}(x \cdot \tilde{\mu}_n \cdot d)$, since $\mu_n \in \mathcal{U}(A)$, $\tilde{\mu}_n \in \mathcal{U}(\widehat{A})$, and $x_n \in \mathfrak{p}_i^{(m_i)} \backslash \mathfrak{p}_i^{(m_i+1)}$, $x \in \widehat{\mathfrak{p}}_i^{(m_i)} \backslash \widehat{\mathfrak{p}}_i^{(m_i+1)}$. So

$$\begin{aligned} \sum_{j=1}^{r} k_j m_{ij} &= v_{\mathfrak{p}_i} \left(\prod_{j=1}^{r} z_{jn}^{k_j} \right) = v_{\mathfrak{p}_i}(x_n \cdot \mu_n \cdot d) - v_{\mathfrak{p}_i}(e) \\ &= v_{\widehat{\mathfrak{p}}_i}(x \cdot \tilde{\mu}_n \cdot d) - v_{\widehat{\mathfrak{p}}_i}(e) = v_{\widehat{\mathfrak{p}}_i} \left(\prod_{j=1}^{r} \tilde{z}_{jn}^{k_j} \right) = \sum_{j=1}^{r} k_j \beta_{ij}, \end{aligned}$$

and thus $m_{ij} = \beta_{ij}$ for all i and j.

Since $\tilde{z}_{jn} \in \mathcal{U}(\widehat{A}_a)$ for $j \in \{2, \ldots, r\}$, we can find $z_j \in A$,and $\gamma_{jn}, \delta_{jn} \in \widehat{A}$ for $j \in \{2, \ldots, r\}$ such that $z_j A = \bigcap_{i=1}^{s} \mathfrak{p}_i^{(m_{ij})}$, $\tilde{z}_{jn} = z_j \gamma_{jn}$ and $\gamma_{jn} \cdot \delta_{jn} = 1$. Then $z_{jn} \equiv z_j \gamma_{jn} \bmod \widehat{\mathfrak{M}}^{\theta^{-1}(n)}$ for $n \gg 0$ and $j \in \{2, \ldots, r\}$.

Hence, upon a suitable shift of the index n up, as per Fact 3, $z_{jn} \equiv z_j \gamma_{jn} \bmod \widehat{\mathfrak{M}}^n$ and $\gamma_{jn}\delta_{jn} = 1$ for $n \gg 0$.

So for $j \in \{2, \ldots, r\}$, we have $z_{jn} A = z_j \cdot P_{jn}$. This equality implies that P_{jn} is principal, say $P_{jn} = p_{jn} A$ for some $p_{jn} \in P_{jn}$.

By way of contradiction we assume that $r \geq 2$. Equation (†) and the fact that $k_1 = 1$ produce the new equation

$$\begin{aligned} x_n A &= P_{1n} \bigcap \left(\bigcap_{j=2}^{r} P_{jn}^{(k_j)} \right) \bigcap \left(\bigcap_{i=1}^{s} \mathfrak{p}_i^{(\kappa_i)} \right) \\ &= \left(P_{1n} \bigcap \left(\bigcap_{i=1}^{s} \mathfrak{p}_i^{(\kappa_i)} \right) \right) \cdot \left(\prod_{j=2}^{r} p_{jn}^{k_j} \right). \end{aligned}$$

Thus $P_{1n} \bigcap (\bigcap_{i=1}^{s} \mathfrak{p}_i^{(\kappa_i)})$ is a principal ideal generated over A by some $d_n \in A$ such that $x_n = d_n \cdot (\prod_{j=2}^{r} p_{jn}^{k_j})$. Note that $d_n \in A \backslash \mathcal{U}(A)$ since $d_n \in P_{1n}$ and that $p_{jn} \in A \backslash \mathcal{U}(A)$ for $j = 2, \ldots, r$ since $p_{jn} \in P_{jn}$, contradicting the irreducibility of x_n in A.

Therefore, $r = 1$. As $x_n \notin \mathcal{U}(A_a)$, then $P_{1n} \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Also since $x_n \in \mathfrak{p}_i^{(m_i)} \backslash \mathfrak{p}_i^{(m_i+1)}$, then $\kappa_i = m_i$ for all i, and so $x_n A = P_{1n} \bigcap (\bigcap_{i=1}^{s} \mathfrak{p}_i^{(m_i)})$. Thus x_n is a prime element in A_a for $n \gg 0$. Recalling the equations $x_n y_n = u_n v_n$, we conclude then that x_n divides u_n or v_n in A_a. We may assume that x_n divides u_n in A_a for infinitely many (and, hence, for all) $n \gg 0$. So $x_n c_n = u_n a^{s_n}$ for some $c_n \in A$ not divisible by a, for some $s_n \in \mathbb{N}$ and $n \gg 0$.

Recall that $u_n \in \mathfrak{p}_i^{(o_i)} \backslash \mathfrak{p}_i^{(o_i+1)}$ for $n \gg 0$, and $a \in \mathfrak{p}_i^{(l_i)} \backslash \mathfrak{p}_i^{(l_i+1)}$ with $l_i \geq 1$. Then $v_{\mathfrak{p}_i}(c_n) = s_n l_i + o_i - m_i \geq 0$ for all i since $c_n \in A$. On the other hand, since a does not divide c_n, there must be an $i_0 \in \{1, \ldots, s\}$ such that $s_n l_{i_0} + o_{i_0} - m_{i_0} \lneqq l_{i_0}$.

Let

$$\begin{aligned} \mathcal{S} &:= \{\sigma \in \mathbb{N} \mid \sigma l_i + o_i - m_i \geq 0 \quad \text{for all } i = 1, \ldots, s\}, \\ \mathcal{T} &:= \{\sigma \in \mathbb{N} \mid \sigma l_i + o_i - m_i \lneqq l_i \quad \text{for some } i = 1, \ldots, s\}, \end{aligned}$$

Then $s_n \in \mathcal{S} \bigcap \mathcal{T}$. If we let $s_0 := \min \mathcal{S}$, it can be shown that either $s_n = 0$ for $n \gg 0$, or that $\mathcal{S} \bigcap \mathcal{T} = \{s_o\}$. Let $s = 0$ in the former case, and let $s = s_0$ in the latter. We now turn to the new equations: $x_n c_n = u_n a^s$ to conclude that $x c_n \equiv u a^s \bmod \widehat{\mathfrak{M}}^n$. By the strong approximation property of $\widehat{A}$ we can find $c \in \widehat{A}$ such that $xc = ua^s$ and $c_n \equiv c \bmod \widehat{\mathfrak{M}}^{\theta^{-1}(n)}$ for $n \gg 0$. Thus x divides u in $\widehat{A}_a$, which is what we set out to prove. $\square$

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