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STABILITY OF THE MILLING PROCESS

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Abstract

A new technique for determining the stability conditions of delayed differential equations with time-periodic coefficient is presented. The method is based on a special kind of approximation of the delayed term. As a practical application, the stability of the milling process with respect to the technological parameters is analysed, and an unstable zone in the domain of high cutting speed is shown.

Keywords: regenerative effect, parametric excitation, stability.

1. Introduction

Even nowadays, one of the most popular manufacturing processes is the cutting process. Vibrations arising during the cutting process have a great effect on the accuracy of the work-piece. In order to increase the efficiency and precision of manufacturing, we should get know the properties of the arising vibrations. Machine tool chatter is one of the most complicated dynamical processes, several models appeared in the specialist literature to explain and to predict the vibrations (TOBIAS, 1965, TLUSTY et al., 1962).

The 1 degree of freedom (DOF) mechanical model of the turning process leads to a retarded differential equation. The presence of the time delay results an infinite phase space in mathematical sense, and the stability investigation needs a lot of complicated calculations, but it can be done in analytic way (see STÉPÁN, 1989). The stability chart in function of the technological parameters can be given.

The 1 DOF mechanical model of the milling process leads to a retarded differential equation with a time-periodic coefficient due to the time-varying number of working teeth of the tool. The stability criteria of this kind of system cannot be given in a closed form. A method is used, which approximates the delayed part with an integral expression with respect to the past. This results a finite dimensional approximation of the infinite dimensional problem, so the stability map of the milling process as a function of the technological parameters can approximately be determined.

2. Stability of Turning Processes

The 1 DOF mechanical model of the regenerative machine tool vibration of the turning process can be seen in *Fig. 1*.

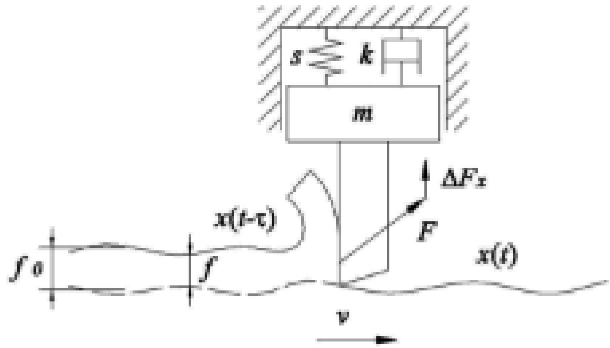

Fig. 1. Mechanical model of turning

The equation of motion is the following:

$$\ddot{x} + 2\kappa\alpha\dot{x} + \alpha^2 x = \frac{1}{m}\Delta F_x, \tag{1}$$

where $\alpha = \sqrt{s/m}$ is the natural angular frequency of the undamped free oscillating system, and $\kappa = k/(2m\alpha)$ is the a relative damping factor. The calculation of the x component of the cutting force variation ΔF_x requires an expression of the cutting force as a function of the technological parameters, primarily as a function of the chip thickness f:

$$F_x(f) = Kwf^{x_F}, \tag{2}$$

where the parameter K depends on further technological parameters, w is the chip width, and x_F is the exponent of chip thickness (a generally used value is $x_F = 0.75$).

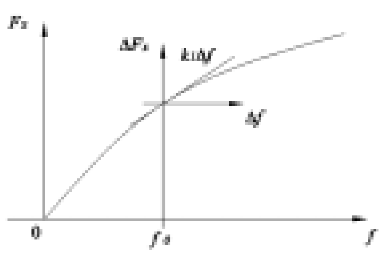

Fig. 2. Cutting force variation

The linearization of expression (2) around the prescribed chip thickness f_0 yields:

$$\Delta F_x = k_1 \Delta f, \tag{3}$$

where the so-called cutting force coefficient k_1 is linearly proportional to the chip width w:

$$k_1 = \left.\frac{\partial F_x}{\partial f}\right|_{f=f_0} = \left.\frac{\partial (Kwf^{x_F})}{\partial f}\right|_{f=f_0} = x_F K w f_0^{x_F-1}.$$

The chip thickness variation Δf can be expressed as the difference of the delayed tool edge position $x(t-\tau)$ and the present one $x(t)$:

$$\Delta f = x(t-\tau) - x(t), \tag{4}$$

where the delay τ is the time of one revolution of work-piece. Putting (3) and (4) into (1), we get the linearized equation of motion:

$$\ddot{x}(t) + 2\kappa\alpha\dot{x}(t) + \alpha^2 x(t) = \frac{1}{m}k_1(x(t-\tau) - x(t)). \tag{5}$$

Although the analysis of this retarded differential equation leads to an infinite eigenvalue problem (HALE, 1977) and needs a lot of calculation, there exists a closed form stability criterion (see e.g. STÉPÁN, 1998) which results the stability chart shown in *Fig. 3*. The axes on the chart are related to the technological parameters, to the number of revolutions of the workpiece $\Omega = 60/\tau$ and the cutting force coefficient k_1 which depends linearly on the depth of the cut. In case of *Fig. 3*, the fixed parameters are $m = 50$ [kg], $\kappa = 0.05$ and $\alpha = 775$ [rad/s].

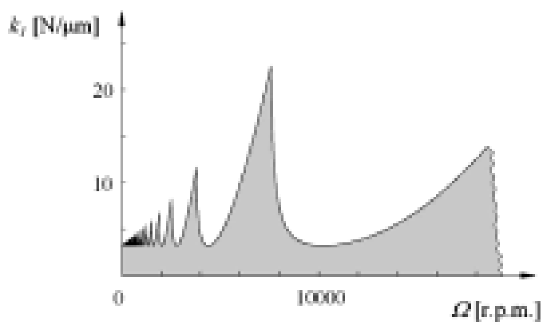

Fig. 3. Stability chart of turning

3. Mechanical Model of Milling

The 1 DOF mechanical model of the milling process can be seen in *Fig. 4*. The number of the working tool teeth varies in time. The equation of motion is the same as at the case of turning process:

$$\ddot{x} + 2\kappa\alpha\dot{x} + \alpha^2 x = \frac{1}{m}\Delta F_x, \tag{6}$$

The calculation of the x component of the cutting force variation ΔF_x is more complicated than in the case of turning.

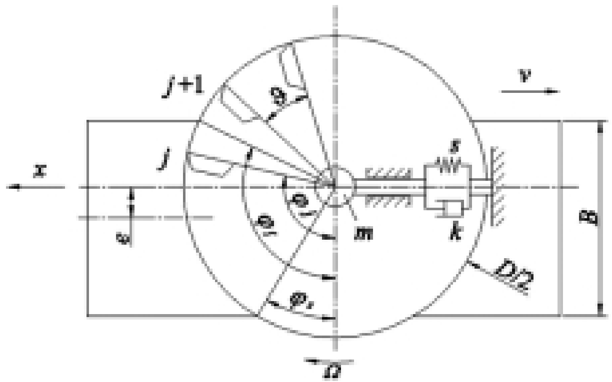

Fig. 4. Mechanical model of milling

Let the number of the tool edges be z each marked with $j = 0, 1, \ldots, z-1$. The angular position of the tooth marked j can be given in the following way:

$$\varphi_j = \Omega t + j\vartheta, \tag{7}$$

where Ω is the angular velocity of the tool, and ϑ is the angle between two edges. The edge marked j works only if its angular position fulfils the condition:

$$\varphi_s \leq \varphi_j \leq \varphi_f,$$

where angles φ_s and φ_f depend on the geometrical parameters of the manufacturing:

$$\cos\varphi_s = \frac{B+2e}{D}, \qquad \cos\varphi_f = \frac{B-2e}{D},$$

where B is the width of the work-piece, e is the distance between the centre lines of the tool and the work-piece, and D is the diameter of the tool.

The x component of the force acting on tooth j assumes the form (see BALI, 1988):

$$F_{x_j} = F_{v_j}\cos\varphi_j + F_{f_j}\sin\varphi_j, \tag{8}$$

where F_{v_j} and F_{f_j} are the tangential and axial component of the cutting force acting on tooth j, respectively (see *Fig. 5*).

The tangential component of the cutting force acting on tooth j reads:

$$F_{v_j} = \begin{cases} Kwf_j^{x_F} & \text{if} \quad \varphi_s \leq \varphi_j \leq \varphi_f \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

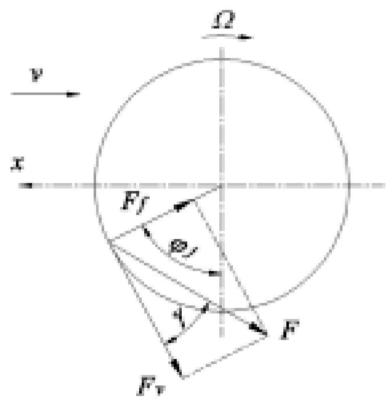

Fig. 5. Cutting force components

To compose (9) in a mathematical form, we should introduce the screen function of Laczik (LACZIK, 1986):

$$g_j(\varphi) = \frac{1}{2}(1 + \operatorname{sgn}(\sin(\varphi_j - \psi) - p)) = \begin{cases} 1 & \text{if} \quad \varphi_s \leq \varphi_j \leq \varphi_f \\ 0 & \text{otherwise} \end{cases}, \tag{10}$$

where

$$\tan\psi = \frac{\sin\varphi_s - \sin\varphi_f}{\cos\varphi_s - \cos\varphi_f}, \qquad p = \sin(\varphi_s - \psi).$$

So the cutting force acting at the tooth j is the following:

$$F_{v_j} = Kwf_j^{x_F} g_j(\varphi). \tag{11}$$

The axial component of cutting force acting on tooth j can be expressed in the following way (see *Fig. 5*):

$$F_{f_j} = F_{v_j} \tan\gamma, \tag{12}$$

where $\gamma \approx 15°$ in general. From (11) and (12) we get the x component of the force acting on edge j:

$$F_{x_j} = F_{v_j}(\cos\varphi_j + \sin\varphi_j \tan\gamma) = Kwf_j^{x_F} g_j(\varphi)(\cos\varphi_j + \sin\varphi_j \tan\gamma). \tag{13}$$

The x component of the force acting on the tool is the sum of (13) via j:

$$F_x = \sum_{j=0}^{z-1} Kwf_j^{x_F} g_j(\varphi)(\cos\varphi_j + \sin\varphi_j \tan\gamma). \tag{14}$$

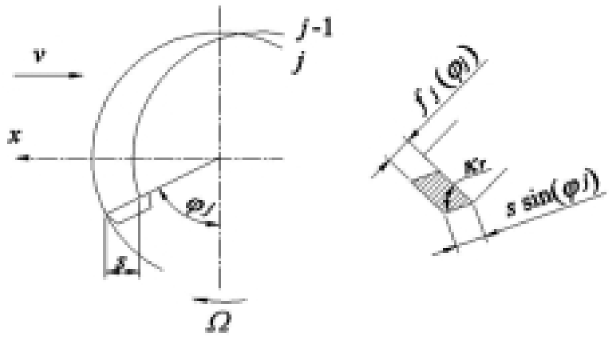

Fig. 6. Geometry of milling tool edge

Because of the vibrations of the tool, the feed s per tooth has a deviation from the prescribed value s_0 in the following way:

$$s = s_0 + x(t) - x(t - \tau),$$

so the chip thickness cut by the tooth j can be written:

$$f_j = (s_0 + x(t) - x(t - \tau)) \sin\varphi_j \sin\kappa_r, \tag{15}$$

where κ_r is the tool cutting edge angle. The ideal chip thickness reads:

$$f_{j_0} = s_0 \sin\varphi_j \sin\kappa_r.$$

The difference between the ideal and real chip thickness assumes the form:

$$\Delta f_j = (x(t) - x(t - \tau)) \sin\varphi_j \sin\kappa_r.$$

Substituting (15) into (14) we get the value of F_x in the function of the two positions of the tool $x(t)$ and $x(t - \tau)$:

$$\begin{aligned} F_x = \sum_{j=0}^{z-1} K w (s_0 + x(t) - x(t - \tau))^{x_F} \\ \times (\sin\varphi_j \sin\kappa_r)^{x_F} g_j(\varphi)(\cos\varphi_j + \sin\varphi_j \tan\gamma). \end{aligned} \tag{16}$$

The linearization of expression (16) around the prescribed feed s_0 per tooth yields:

$$\begin{aligned} \Delta F_x = \sum_{j=0}^{z-1} K w x_F s_0^{x_F - 1} (\sin\varphi_j \sin\kappa_r)^{x_F} \\ \times g_j(\varphi)(\cos\varphi_j + \sin\varphi_j \tan\gamma)(x(t) - x(t - \tau)). \end{aligned} \tag{17}$$

The substitution of (7) and (10) into (17) gives

$$\begin{aligned}\Delta F_x &= \sum_{j=0}^{z-1} \frac{1}{2} K w x_F s_0^{x_F-1} (\sin(\Omega t + j\vartheta) \sin \kappa_r)^{x_F} \\ &\quad \times (1 + \mathrm{sgn}\,(\sin(\Omega t + j\vartheta - \psi) - p)) \\ &\quad \times \cos(\Omega t + j\vartheta) + \sin(\Omega t + j\vartheta) \tan\gamma)(x(t) - x(t-\tau)).\end{aligned} \tag{18}$$

(18) can be written in a simpler form:

$$\Delta F_x = k_1(t)(x(t) - x(t-\tau)), \tag{19}$$

where $k_1(t)$ is now a time dependent cutting force coefficient:

$$\begin{aligned}k_1(t) &= \sum_{j=0}^{z-1} \frac{1}{2} K w x_F s_0^{x_F-1} (\sin(\Omega t + j\vartheta) \sin \kappa_r)^{x_F} \\ &\quad \times (1 + \mathrm{sgn}\,(\sin(\Omega t + j\vartheta - \psi) - p)) \\ &\quad \times \cos(\Omega t + j\vartheta) + \sin(\Omega t + j\vartheta) \tan\gamma).\end{aligned}$$

Using the expression (19), the equation of motion (6) gets the following form:

$$\ddot{x}(t) + 2\kappa\alpha\dot{x}(t) + \alpha^2 x(t) = \frac{1}{m} k_1(t)(x(t) - x(t-\tau)). \tag{20}$$

With the fixed parameters $D = 100$ [mm], $B = 30$ [mm], $e = 30$ [mm], $z = 12$, $\Omega = 150$ [1/s], $\gamma = 15°$, $\kappa_r = 75°$, $x_F = 0.78$, $K = 2000$ [N/mm$^{1+x_F}$], $s_0 = 0.4$ [mm], $w = 1.035$ [mm], we get the function $k_1(t)$ shown in *Fig. 7*.

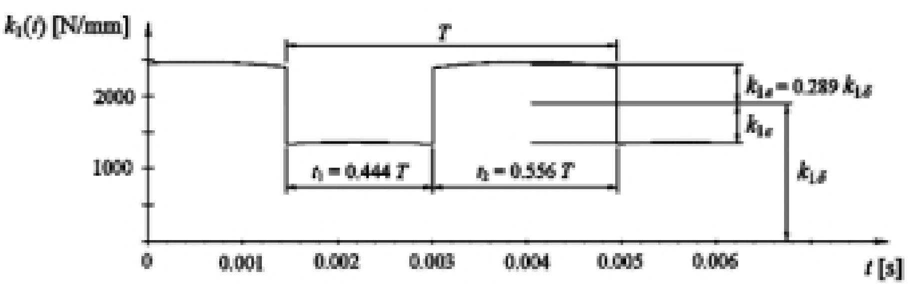

Fig. 7. Cutting coefficient variation

The time period of the function $k_1(t)$ is $T = (60)/(z\Omega) = 0.00349$ [s] which is equal to the time delay τ in case of milling. This function can well be approximated with a piecewise constant function in the following way:

$$k_1(t) = \begin{cases} k_{1\delta} - k_{1\varepsilon} & \text{if} \quad 0 \le t \le t_1 \\ k_{1\delta} + k_{1\varepsilon} & \text{if} \quad t_1 < t \le t_1 + t_2 = T \end{cases}, \tag{21}$$

where $k_{1\delta}$ is the approximated mean value, $k_{1\varepsilon}$ is the approximated amplitude of function $k_1(t)$, and $k_{1\varepsilon} = 0.289k_{1\delta}$. The lengths of the time intervals are $t_1 = 0.444T$ and $t_2 = 0.556T$.

4. Stability of Time Periodic Cutting

The stability of equation (20) cannot be determined in a closed form, approximations should be applied. The delayed term $x(t-\tau)$ can be approximated in the following way (see FARGUE, 1973):

$$x(t-\tau) \approx \int_{-\infty}^{0} x(t+\vartheta)\cdot w_n(\vartheta)\,\mathrm{d}\vartheta, \tag{22}$$

since:

$$x(t-\tau) = \lim_{n\to\infty}\int_{-\infty}^{0} x(t+\vartheta)\cdot w_n(\vartheta)\,\mathrm{d}\vartheta,$$

where $w_n(\vartheta)$ is a special weight function series coming from the product of a polynomial and an exponential expression:

$$w_n(\vartheta) = (-1)^n \frac{n^n}{\tau^n n!}\vartheta^n e^{\frac{n\cdot\vartheta}{e^{\tau}}}.$$

The function $w_n(\vartheta)$ satisfies the following properties:

$$\int_{-\infty}^{0} w_n(\vartheta)\,\mathrm{d}\vartheta = 1, \qquad \lim_{n\to\infty} w_n(\vartheta) = \delta_{-\tau}(\vartheta),$$

where $\delta_{-\tau}(\vartheta)$ is the Dirac distribution:

$$\delta_{-\tau}(\vartheta) = \begin{cases} \infty & \text{if} \quad \vartheta = -\tau \\ 0 & \text{if} \quad \vartheta \neq -\tau \end{cases}, \qquad \int_{-\infty}^{\infty} \delta_{-\tau}(\vartheta)\,\mathrm{d}\vartheta = 1.$$

Fig. 8 shows the weight function with parameters $n = 2, 10, 50, 120$ and $\tau = 1$. It can be seen that the greater n, the more correct the approximation is.

The approximation (22) can be applied in (20). A long calculation (derivations and partial integration) yields a finite dimensional system of differential equations with a time periodic coefficient matrix (see further details in INSPERGER, 1999):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) = \mathbf{A}(t)\mathbf{y}(t), \tag{23}$$

where

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_{n+3}(t) \end{bmatrix},$$

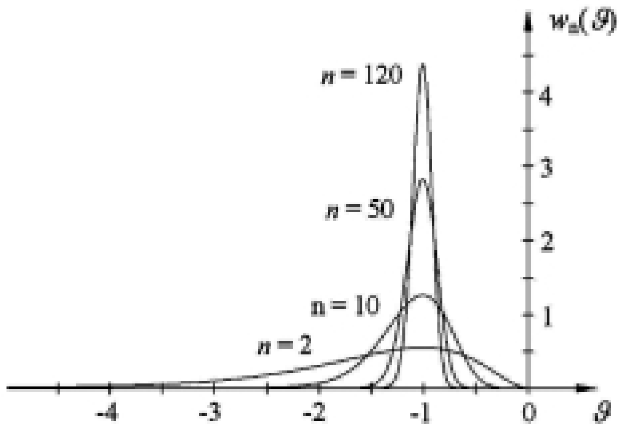

Fig. 8. Weight function

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -\left(\alpha^2 + \dfrac{k_1(t)}{m}\right) & -2\kappa\alpha & \dfrac{k_1(t)}{m} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -\dfrac{n}{\tau} & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{n}{\tau} & -1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{n}{\tau} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -\dfrac{n}{\tau} & -1 \\ (-1)^n \dfrac{n^{n+1}}{\tau^{n+1}} & 0 & 0 & 0 & 0 & \dots & 0 & -\dfrac{n}{\tau} \end{bmatrix}.$$

In the case of the turning process the precision of the method can be checked by applying the approximation for the autonomous equation (5). This yields also the system (23) with the constant value of the parameter $k_1(t) \equiv k_1$, so the coefficient matrix is constant. The necessary and sufficient condition of the asymptotic stability is that all eigenvalues of the coefficient matrix have negative real parts. The stability chart with approximation $n = 120$ can be seen in *Fig. 9*. The dashed curves are the correct stability limits (see *Fig. 3*). It can be seen that for high cutting speed ($z\Omega > 10000$ [r.p.m.]) the approximation is already good.

In the case of milling, $k_1(t)$ is given by formula (21) which yields a time dependent coefficient matrix in (23). The bases of the stability analysis for linear periodic systems are given by the Floquet theory (see FARKAS, 1994), and there are existing methods (INSPERGER and HORVÁTH, 1999) to calculate the stability chart, as it is shown in *Fig. 10*. This is an approximation of the stability chart of the time periodic retarded equation (20). Similarly to the constant k_1, the approximate

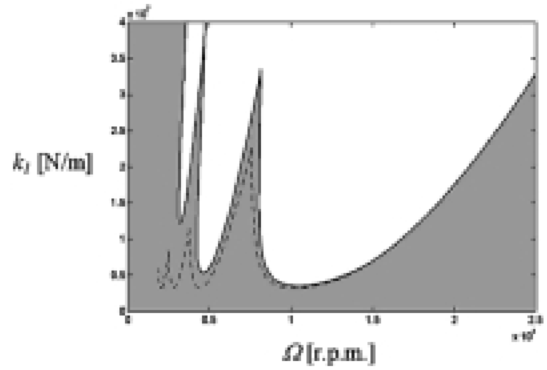

Fig. 9. Approximate stability chart of turning

mean value $k_{1\delta}$ is linearly proportional to the chip width w (or to the depth of cut). This mean value makes it possible to compare the cases of turning and milling. The stability chart in *Fig. 10* has a qualitative difference from the chart in *Fig. 9*. In milling processes, a new unstable domain arises in case of high speed cutting which can also be the reason of a new kind of machine tool chatter in milling processes related to the time periodicity of the milling.

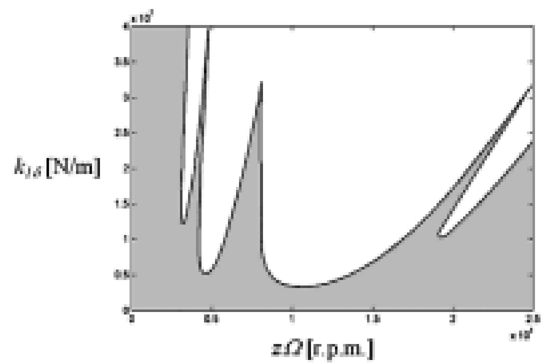

Fig. 10. Approximate stability chart of milling

5. Conclusions

Machine tool chatter is one of the most complicated dynamical phenomena since the corresponding mathematical model, a retarded differential equation has an infinite dimensional phase space. In the case of milling, parametric excitation arises due to the time-varying number of working teeth. Via an approximation of the delayed term, the stability chart in the plane of the technological parameters can be determined. The analysis resulted a new unstable domain in case of high speed cutting.

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