Annals of Global Analysis and Geometry **15:** 299–312, 1997.

Stability of the p-Curvature Positivity under Surgeries and Manifolds with Positive Einstein Tensor

MOHAMMED-LARBI LABBI
International Center for Theoretical Physics, Mathematics Section, P.O. Box 586, 34100 Trieste, Italy
(E-mail: labbi@ictp.trieste.it)

Abstract. We establish the stability of the class of manifolds with positive p-curvature under surgeries in codimension $\geq p+3$. As a consequence of this result, we first obtain the classification of compact 2-connected manifolds of dimension ≥ 7 with positive Einstein tensor; and secondly the existence of metrics with positive Einstein tensor on any compact, simply connected, non-spin manifold of dimension ≥ 7 whose second homotopy group is isomorphic to Z_2.

Key words: curvature, Einstein tensor, surgery

1991 Mathematics Subject Classification: 53B, 53C

1. Introduction and Statement of Results

The principal ingredient in the proof of the well known classification of compact simply connected manifolds of positive scalar curvature (due to Gromov and Lawson [3] and completed later by Stolz [13] is the following surgery theorem due to Gromov and Lawson [3] and Schoen and Yau [12]:

If a manifold M is obtained from a manifold N by surgery in codimension ≥ 3, and N admits a metric of positive scalar curvature, then so does M.

This theorem implies that if M and N are compact, simply connected non-spin manifolds of dimension ≥ 5, which represent the same class in the oriented bordism ring, and N admits a metric of positive scalar curvature, then so does M.

Gromov and Lawson showed that a collection of oriented manifolds known to generate the oriented bordism rings admits metrics of positive scalar curvature and thus proved the following result:

Every compact simply connected n-manifold, $n \geq 5$, which is non-spin, carries a metric of positive scalar curvature.

On the other hand, in the case of spin manifolds, Stolz [13] proved the following result:

Let M be a simply connected, closed, spin manifold of dimension $n \geq 5$. Then M carries a metric with positive scalar curvature if and only if $\alpha(M) = 0$,

where $\alpha(M)$ is the KO-characteristic number, see [5, 13].

In this paper, we first prove the following generalization of the previous surgery theorem in the case of p-curvature, which is a generalization of the scalar curvature proposed by Gromov (see next section):

MAIN THEOREM. *If a manifold M is obtained from a manifold N by surgery in codimension $\geq p+3$, and N admits a metric of positive p-curvature, then so does M.*

In particular, if X is a compact manifold which carries a Riemannian metric of positive Einstein tensor, then any manifold which can be obtained from X by surgeries in codimension ≥ 4 also carries a metric with positive Einstein tensor.

Using the same source of ideas, we are able to prove the following interesting consequences of the main theorem:

THEOREM I. *A compact 2-connected manifold of dimension ≥ 7 admits a Riemannian metric with positive Einstein tensor if and only if $\alpha(M) = 0$.*

In particular, any compact 2-connected manifold of dimension 7 admits a metric with positive Einstein tensor.

THEOREM II. *Any compact non-spin simply connected manifold V of dimension ≥ 7, such that $\pi_2(V) \cong Z_2$ admits a Riemannian metric of positive Einstein tensor.*

It is still an open question to find the classification of simply connected manifolds with positive Einstein tensor. This will be possible if one is able to prove a generalization of the Lichnerowicz vanishing theorem to the case of Einstein tensor.

The following generalization seems plausible:

Any compact spin^c manifold with positive Einstein tensor does not admit any complex harmonic spinor.

One can prove that this is true for Kahlerian manifolds of dimension 4 with canonical spin^c structures, see [5, 8].

2. The *p*-Curvature of a Riemannian Manifold

The p-curvature is an extension of the scalar curvature proposed by Gromov. Since it is not well known, we will explain its geometric meaning and its relation with well known curvatures in the following section.

2.1. DEFINITION

The p-*curvature* of an n-dimensional Riemannian manifold (M, g), denoted by s_p, $0 \leq p \leq n-2$, is a function defined on the p-Grassmannian bundle of M as follows:

Let P be a p-plane in the tangent space $T_m M$ of M at m; $s_p(P)$ is the average of the sectional curvature of (M, g) at m in the direction of $P^{\perp}$ that is:

$$s_p(P) = \sum_{j,k=p+1}^{n} K_m(e_j, e_k),$$

where $(e_{p+1}, \ldots, e_n)$ is an orthonormal basis of $P^{\perp}$ (the orthogonal subspace to P in $T_m M$) and K_m is the sectional curvature of (M, g).

In other words, $s_p(P)$ is the scalar curvature at m of the Riemannian submanifold $\exp_m(V)$, where V is a neighborhood of 0 in $P^{\perp} \subset T_m M$.

For $p = 0$ (respectively $p = n-2$) it is the scalar curvature of (M, g) (respectively the sectional curvature). Furthermore, for $p = 1$, we have:

$$s_1(\langle e_i \rangle) = \text{scal} - 2\text{Ric}(e_i) = 2\left(\frac{\text{scal}}{2} - \text{Ric}(e_i)\right),$$

where scal (respectively Ric) is the scalar curvature (respectively the Ricci curvature) of (M, g) and e_i denote a unit vector. Then one can consider the 1-curvature (up to a factor 2) as the quadratic form associated to the Einstein Tensor $(\text{scal}/2)g - \text{Ric}$.

However, for $p = n-4$, the $(n-4)$-curvature coincides with the isotropic curvature introduced by Micallef and Moore [9] modulo the Weil-curvature, in fact one can prove without difficulties that in the case of conformally flat manifolds (i.e. the Weil curvature is zero), the isotropic curvature is exactly $(1/6)s_{n-4}$.

EXAMPLES. If (M, g) is a Riemannian manifold with constant sectional curvature, then s_p is constant for all p, and equal to:

$$s_p = (n-p)(n-p-1)k.$$

2. If (M, g) is Einstein with constant r, then the 1-curvature is also constant and equal to:

$$s_1 = (n-2)r.$$

Remark. Note that the converse in these two examples is also true, that is, if the 1-curvature is constant, then so is the Ricci curvature, and if there exists p; $p \geq 2$, such that s_p is constant, then so is the sectional curvature.

2.2. GEOMETRIC INTERPRETATION OF THE p-CURVATURE

The following proposition gives a geometric meaning to the p-curvature:

PROPOSITION. *Let P be a p-plane in T_mM, then*

$$s_p(P) = \lim_{r\to 0} \frac{6(n-p)}{r^2}\left(1 - \frac{\mathrm{vol}(S^{n-p-1}(r))}{\mathrm{vol}_e(S^{n-p-1}(r))}\right),$$

where $\mathrm{vol}(S^{n-p-1}(r))$ *(respectively* $\mathrm{vol}_e(S^{n-p-1}(r))$ *is the volume of the sphere* $S^{n-p-1}(r) = \{\exp_m(x)/x \in P^{\perp}, \|x\| = r\}$ *(respectively the Euclidean sphere with ray r).*

Proof. This follows immediately from the fact that the p-curvature is the scalar curvature of the Riemannian submanifold $\exp_m(V)$ of (M, g). Then it suffices to apply the well known result concerning the determination of the scalar curvature by the volume of small spheres. □

As immediate corollaries of this fact, we first obtain the following geometric characterization of Einstein manifolds by the volume of small spheres:

COROLLARY 1. *A Riemannian manifold (M, g) of even dimension $2p$ is Einstein if and only if for every tangent p-plane P to M we have*

$$\mathrm{vol}(S^{p-1}(r)) = \mathrm{vol}(S_{\perp}^{p-1}(r)) + o(r^{p+1}),$$

where $S^{p-1}(r)$ (respectively $S_{\perp}^{p-1}(r)$) is the sphere of ray r and of dimension $p-1$ in P (respectively, in $P^{\perp}$).

Proof. First, one can easily prove that (M, g) is Einstein if and only if for all tangent p-planes P we have $s_p(P) = s_p(P^{\perp})$, and then the result follows immediately from the previous proposition. □

COROLLARY 2. *The p-curvature of an n-dimensional Riemannian manifold (M, g) is positive (respectively negative) if and only if*

$$\mathrm{vol}(S^{n-p-1}(r)) < \mathrm{vol}_e(S^{n-p-1}(r)) \quad \textit{(respectively} > \mathrm{vol}_e(S^{n-p-1}(r))\textit{)}$$

for all small spheres of dimension $n-p-1$, as in the proposition.

2.3. REMARKS ON THE POSITIVITY OF THE p-CURVATURE

1. The p-curvature appears as the trace of the $(p+1)$-curvature, in fact one can easily prove that if $(e_{p+1}, \ldots, e_n)$ is any orthonormal basis of $P^{\perp}$, then

$$\sum_{k=p+1}^{n} s_{p+1}(\langle P, e_k\rangle) = (n-p-2)s_p(P).$$

It follows that the positivity of the $(p+1)$-curvature implies the positivity of the p-curvature. In particular, it implies the positivity of the scalar curvature for all $0 \leq p \leq n-2$.

2. The Riemannian product of any Riemannian manifold with a small round sphere of dimension $p+2$ has positive p-curvature.

3. Let (λ_i), $i = 1, \ldots, n$, denote the eigenvalues of the Ricci form at a point $m \in M$, and let λ_j denote a maximal one. Then the positivity of the 1-curvature at m is equivalent to the following pinching condition

$$\lambda_j < \sum_{i \neq j} \lambda_i.$$

In particular, a Kahlerian manifold with positive Ricci curvature has positive 1-curvature. Furthermore, the converse is true in dimension 4. In fact, for a Kahlerian manifold each eigenvalue of the Ricci form has multiplicity at least 2.

4. In [6] we proved the following result:

Let (M, g) be a compact conformally flat n-manifold with positive p-curvature, then $H^m(M, \mathbf{R}) = 0$ for $(n-p)/2 \leq m \leq (n+p)/2$.

Since in the case of conformally flat manifolds the isotropic curvature coincides with the $(n-4)$-curvature (up to a constant 6), then the following result is an immediate consequence of the previous one:

Let (M, g) be a compact conformally flat n-manifold with positive isotropic curvature, then $H^m(M, \mathbf{R}) = 0$ for $2 \leq m \leq n-2$.

This result was first proved by Nayatani [11].

3. Proof of the Main Theorem

We proceed as in Gromov and Lawson's proof for the scalar curvature [3]. Let (X, g) be a compact n-dimensional Riemannian manifold of positive p-curvature, and let $S^m \subset X$ be an embedded sphere of codimension q with trivial normal bundle $N \cong S^m \times \mathbf{R}^q$. There exists $r_0 > 0$ such that the exponential map

$$\exp : S^m \times D^q(r_0) \subset N \longrightarrow X$$

is an embedding, where for any $x \in S^m$, $x \times D^q(r_0)$ is the closed Euclidean ball in $\mathbf{R}^q \equiv x \times \mathbf{R}^q$. Then let $\exp^* g$ be the pull back to $S^m \times D^q(r_0)$ of the metric g on X via the exponential map.

Let g^∇ denote the natural metric on the normal bundle $N \cong S^m \times \mathbf{R}^q$ defined using the normal connection. We also denote by g^∇ its restriction to the sub-bundles $S^m \times D^q(r)$ and $\partial(S^m \times D^q(r)) = S^m \times S^{q-1}(r)$.

LEMMA 3.1. *Let g_ε, $\varepsilon < r_0$, denote the induced metric on $S^m \times S^{q-1}(\varepsilon) \subset (S^m \times D^q(r_0), \exp^* g)$. Then near $\varepsilon = 0$, g_ε is close to the natural metric g^∇ on $S^m \times S^{q-1}(\varepsilon)$ in the C^2 topology. Furthermore, the second fundamental form of the hypersurface $(S^m \times S^{q-1}(\varepsilon), g_\varepsilon)$ in $(S^m \times D^q(r_0), \exp^* g)$ has the following form with respect to the decomposition into horizontal and vertical distributions for the natural Riemannian submersion $(S^m \times S^{q-1}(\varepsilon), g^\nabla) \to S^m$:*

$$\begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{Id}{\varepsilon} + O(\varepsilon) \end{pmatrix}.$$

Proof. Note first that the two metrics g_ε and g^∇ on $S^m \times S^{q-1}(\varepsilon)$ differ only by their restrictions to $S^{q-1}(\varepsilon)$ [4], but the induced metric $(S^{q-1}(\varepsilon), \exp^* g)$ converges C^2 to the standard metric when ε converges to 0 [3], which proves the first statement of the lemma.

To prove the second one, let us first recall an elementary fact on the covariant derivative

$$D_X Y = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} X^i \frac{\partial Y^j}{\partial x_i} + \sum_{i,k=1}^{n} Y^j X^i \Gamma_{ij}^k \right) \frac{\partial}{\partial x_j},$$

where

$$X = \Sigma X^i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \Sigma Y^j \frac{\partial}{\partial x_j}.$$

Hence the second fundamental form of $S^m \times S^{q-1}(\varepsilon) \subset (S^m \times D^q(r_0))$ for the metric $\exp^* g$ decomposes into two parts. The first part is exactly the second fundamental form of $S^m \times S^{q-1}(\varepsilon) \subset (S^m \times D^q(r_0))$ for the natural metric g^∇, that is

$$\begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{Id}{\varepsilon} \end{pmatrix}.$$

And since the Γ_{ij}^k are the derivatives of the metric, it follows from the first statement of the lemma, that the second part is

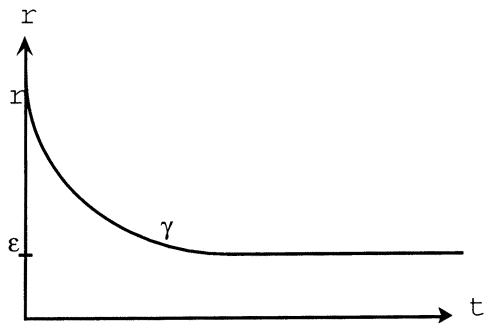
$$\begin{pmatrix} 0 & 0 \\ 0 & O(\varepsilon) \end{pmatrix}$$

which completes the proof the lemma. □

We now define a hypersurface M in the Riemannian product $(S^m \times D^q(r_0), \exp^* g) \times \mathbf{R}$ by the relation

$$M = \{((x, v), t) \in (S^m \times D^q(r_0)) \times \mathbf{R} / (\|v\|, t) \in \gamma\},$$

where γ is a curve in the (r, t)-plane as pictured below:



The important points about γ is that it is tangent to the r-axis at $t = 0$ and is constant for $t = \varepsilon > 0$. Thus the induced metric on M extends the metric $\exp^* g$ on $S^m \times D^q(r_0)$ near its boundary and finishes with the product metric

$$\partial(S^m \times D^q(\varepsilon), \exp^* g) \times \mathbf{R} = (S^m \times S^{q-1}(\varepsilon), \exp^* g) \times \mathbf{R}.$$

It follows from Lemma 3.1 that for large time (say for $t \geq t_1$), one can deform, through metrics with positive p-curvature, the metric on the tubular piece to the product of the natural metric g^{∇} on $S^m \times S^{q-1}(\varepsilon)$ and $\mathbf{R}$. This metric has positive p-curvature since $q - 1 \geq p + 2$ (see Lemma 3.2 below). For $t \leq t_1$ the metric on M remains unchanged.

As in [3] we can homotope, using only Riemannian submersions, the metric $g^{\nabla} \times \mathbf{R}$ on the tubular piece for large t (say $t \geq t_2 > t_1$) to the standard product metric on $S^m(1) \times S^{q-1}(\varepsilon) \times \mathbf{R}$ through metrics with positive p-curvature for ε small enough. In fact, each metric is the one of a total space of a Riemannian submersion having the standard sphere $S^{q-1}(\varepsilon)$ as fiber. Then it follows by Lemma 3.2 below that they have positive p-curvature when ε is small enough.

LEMMA 3.2. *Let $\pi : (M, g) \to (B, \check{g})$ be a Riemannian submersion, and let g_t be the canonical variation of the metric g, i.e., the metric on M obtained by multiplying the metric g by t^2 in the vertical directions, see [1, p. 252] and [7]. Then:*

(1) If the fibers (endowed with the induced metric) of π have positive sectional curvature and dimension $\geq p + 2$, then for all $m \in M$ there exists $t_0 > 0$ such that for all $0 \leq t \leq t_0$, the metric g_t has positive p-curvature on m.

(2) We have the same conclusion if the fibers have positive p-curvature and codimension 1.

Proof. The proof of this lemma is a direct but long calculation using the O'Neill formulas for the curvature. For a detailed proof, see [7].

The idea of the proof is that when t is small, the sectional curvatures in the vertical directions tend to high values and dominate the sectional curvatures in all other directions. Then if $p + 2$ is less than the dimension of the fibers, one can find, in the orthogonal subspace of every p-plane, at least one 2-plane which projects into a 2-plane in the vertical direction. The curvature in this direction will dominate the negative curvature of all other directions in the subspace orthogonal to the p-plane. $\square$

It remains for us to choose the curve γ such that the metric induced on M from the product metric exp$^* \, g \times \mathbf{R}$ has positive p-curvature for all points $m \in M$ with $t \leq t_1$. Let us first calculate the p-curvature of M.

For every $m \in M$, we have the following orthogonal decomposition of T_mM:

$$T_mM = \mathbf{R} \cdot \tau \oplus \mathcal{H} \oplus T_mS^{q-1}(r),$$

where τ is the unit tangent vector to the curve γ in the (r, t)-plane and the second and third part are, respectively, the horizontal and vertical part for the natural Riemannian submersion $(S^m \times S^{q-1}(r), g^\nabla) \to S^p$. One can immediately prove from Lemma 3.1 that the second fundamental form of the hypersurface M is of the following form with respect to the previous orthogonal decomposition of T_mM

$$\begin{pmatrix} k & 0 \ldots\ldots\ldots\ldots 0 & \ldots\ldots\ldots 0 \\ 0 & & \\ \vdots & \bigcirc & \bigcirc \\ 0 & & \\ \vdots & \bigcirc & \left(-\dfrac{Id}{r} + O(r)\right) \sin\theta \\ 0 & & \end{pmatrix},$$

where k denotes the curvature of the curve γ in the (r, t)-plane and θ denotes the angle between the normal to M and the t-axis.

Now, let $P^\perp$ denote the orthogonal to a p-plane in T_mM, dim $P^\perp = n - p$, and let

$$V = P^\perp \cap (\mathcal{H} \oplus T_mS^{q-1}(r)).$$

Note that dim $V \geq n - p - 1$. The subspace V itself decomposes into

$$V = (V \cap \mathcal{H}) \oplus W,$$

where W is the orthogonal to $V \cap \mathcal{H}$ in $P^\perp$.

Then we have the following orthogonal decomposition of $P^\perp$.

$$P^\perp = \mathbf{R} \cdot x \oplus (V \cap \mathcal{H}) \oplus W,$$

where $\mathbf{R} \cdot x$ denotes the orthogonal subspace to V in $P^\perp$ (the vector x, if it is not zero, is a unit vector).

Since W and its orthogonal projection onto $T_mS^{q-1}(r)$ have the same dimension by construction, one can find an orthonormal basis $w_2, \ldots w_{k+1}$ of W ($k =$ dim W) such that its orthogonal projection onto $T_mS^{q-1}(r)$ is formed by orthogonal vectors. In fact, it suffices to diagonalize simultaneously two inner products on $T_mS^{q-1}(r)$.

Set $w_i = a_i x_i + b_i y_i$ where for all $i = 2, \ldots, k+1$, $a_i \neq 0$ and x_i are orthonormal vectors in $T_m S^{q-1}(r)$. Then (y_i) are orthonormal vectors in $\mathcal{H}$, but b_i can be 0.

Finally, let $y_{k+2}, \ldots, y_{l+1}$ be an orthonormal basis of $V \cap \mathcal{H}$ ($l = \dim V$). Thus $(x; (w_i)_{i=2,\ldots,k+1}; (y_i)_{i=k+2,\ldots,l+1})$ is an orthonormal basis of $P^\perp$ and it follows that

$$\begin{aligned} s_p^M(P) \;=\; & \sum_{i=2}^{k+1} K^M(x, w_i) + \sum_{i=k+2}^{l+1} K^M(x, y_i) + \sum_{i,j=2}^{k+1} K^M(w_i, w_j) \\ & + \sum_{i=2}^{k+1} \sum_{j=k+2}^{j=l+1} K^M(w_i, y_j) + \sum_{i,j=k+2}^{l+1} K^M(y_i, y_j). \end{aligned}$$

Set $x = a\tau + bx_1 + cy_1$, where x_1 (respectively y_1) is a unit vector in $T_m S^{q-1}(r)$ (respectively in $\mathcal{H}$).

Now by a direct computation using the Gauss equation and the previous calculation for the second fundamental form of M, we obtain that

$$\begin{aligned} K^M(x, w_i) \;=\; & (1 - a^2 \sin^2\theta) K^{S^m \times D^q}(\check{x}, w_i) + a_i^2 b^2 \left(\frac{1}{r^2} + O(1) \right) \sin^2\theta \\ & + a^2 a_i^2 \left(-\frac{1}{r} + O(r) \right) k \sin\theta. \\ K^M(x, y_i) \;=\; & (1 - a^2 \sin^2\theta) K^{S^m \times D^q}(\check{x}, y_i), \\ K^M(w_i, w_j) \;=\; & K^{S^m \times D^q}(w_i, w_j) + a_i^2 a_j^2 \left(\frac{1}{r^2} + O(1) \right) \sin^2\theta, \\ K^M(w_i, y_j) \;=\; & K^{S^m \times D^q}(w_i, y_j), \\ K^M(y_i, y_j) \;=\; & K^{S^m \times D^q}(y_i, y_j), \end{aligned}$$

where

$$\check{x} = \frac{a \cos\theta (\partial/\partial r) + bx_1 + cy_1}{(1 - a^2 \sin^2\theta)^{1/2}}.$$

We have indexed by M (respectively $S^m \times D^q$) the sectional curvature of M (respectively $S^m \times D^q$). Hence the p-curvature of M is

$$\begin{aligned} S_p^M(P) \;=\; & s_p^{S^m \times D^q}(\check{P}) + O(1) \sin^2\theta \\ & + \left[\sum_{i=2}^{k+1} a_i^2 b^2 + \sum_{i,j=2}^{k+1} a_i^2 a_j^2 \right] \left(\frac{1}{r^2} + O(1) \right) \sin^2\theta \\ & + a^2 \left(\sum_{i=2}^{k+1} a_i^2 \right) \left(-\frac{1}{r} + O(r) \right) k \sin\theta, \end{aligned}$$

where $\check{P}$ is the p-plane tangent to $S^m \times D^q$ generated by $\langle \check{x}, (w_i), (y_j) \rangle$, that is, the orthogonal projection of $P^\perp$ onto the tangent to $S^m \times D^q$.

Since $\dim(V \cap \mathcal{H}) \leq \dim \mathcal{H} = n - q$ we have

$$k = \dim W = \dim V - \dim(V \cap \mathcal{H}) \geq q - p - 1.$$

Recall that for all i, $a_i \neq 0$. Let $a_0^2 = \min(a_i^2)$ for $i = 2, \ldots, k+1$; hence

$$\begin{aligned} s_p^M(P) \ \geq\ & s_p^{S^m \times D^q}(\check{P}) + O(1) \sin^2\theta \\ & + a_0^4 (q-p-1)(q-p-2) \left(\frac{1}{r^2} + O(1) \right) \sin^2\theta \\ & + a^2 \left(\sum_{i=2}^{k+1} a_i^2 \right) \left(-\frac{1}{r} + O(r) \right) k \sin\theta. \end{aligned}$$

Since $q - p - 2 > 0$, one can use exactly the same procedure of bending the curve γ as in [3] to find a curve such that $k \leq A/r_0$, for some constant A, and then the p-curvature of M is positive.

This completes the proof of the main theorem. □

4. Proof of Theorem I

The proof uses the following lemmas:

LEMMA 4.1. *Let (W, V, V') be an n-dimensional simply connected bordism, and let p be an integer, $p \leq n - 4$.*

If $\forall \lambda \leq p$ we have $H_\lambda(W, V) = 0$, then V can be obtained from V' by surgeries in codimension $\geq p + 1$.

Proof. As a consequence of the work of Smale [10], there exists a Morse function $f : W \to [0, 1]$ such that $V = f^{-1}\{0\}$ and $V' = f^{-1}\{1\}$, with no critical points of index $\leq n - p - 1$, see [14] for a detailed proof of this fact.

Thus by Morse theory (see, for example, [2, p. 198]), the manifold V can be obtained from V' by surgeries in codimension $\geq p + 1$. □

LEMMA 4.2. *Every compact 2-connected manifold of dimension ≥ 7, spin-cobordant to a manifold with positive 1-curvature, admits a metric of positive 1-curvature.*

Proof. Let V be a compact n-manifold spin-cobordant to an n-manifold V' with positive 1-curvature.

By a surgery in codimension $n - 1$, we may assume that V' is simply connected (see the main theorem).

Let W be a spin manifold such that $\partial W = V - V'$. One can kill $\pi_1(W)$ by surgeries.

As a consequence of Whitney's theorem, the homotopy groups $\pi_2(W)$ and $\pi_3(W)$ are generated by embedded spheres, since $\dim W \geq 8$. However, W is spin, thus every embedded 2-sphere has trivial normal bundles [8], and so we can eliminate $\pi_2(W)$ by surgeries.

Furthermore, since every vector bundle over S^3 is trivial and $\dim W \geq 8$, we can eliminate $\pi_3(W)$ by surgeries. Hence one can assume that

$$\pi_1(W) = \pi_2(W) = \pi_3(W) = 0.$$

Then by Hurewicz's theorem we have $H_i(W) = 0$, $\forall i \leq 3$, and since V is 2-connected we also have that $H_i(V) = 0$, $\forall i \leq 2$. Then using the long exact sequence we obtain that

$$H_1(W, V) = H_2(W, V) = H_3(W, V) = 0.$$

The lemma follows immediately from Lemma 4.1 and the main theorem. □

Now, let M be a 2-connected manifold of dimension ≥ 7 (consequently it is a spin manifold) such that $\alpha(M) = 0$, then by Stolz's theorem, stated in the introduction, it admits a metric with positive scalar curvature. Due to another theorem of Stolz (Theorem B in [13]), the manifold M is then spin-cobordant to the total space N of a fiber bundle with fiber HP^2 and structural group $PSp(3)$. But the total space N admits a metric with $s_1 > 0$ (see Lemma 3.2). It follows from Lemma 4.2 that M also admits a metric with $s_1 > 0$.

Conversely, if M is with $s_1 > 0$ then it is with positive scalar curvature and hence $\alpha(M) = 0$, which proves Theorem I. □

5. Proof of Theorem II

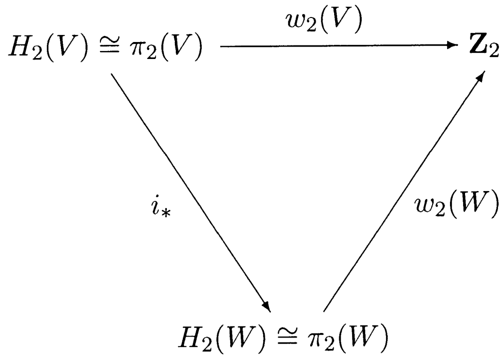
Let us start with the proof of the following lemma:

LEMMA 5.1. *Let V be a non-spin simply connected manifold of dimension ≥ 7, such that $\pi_2(V) \cong Z_2$.*

If V is oriented-cobordant to a manifold with $s_1 > 0$, then it admits a metric with $s_1 > 0$.

Proof. Let W be an orientable manifold such that $\partial W = V - V'$, where V' is a manifold with $s_1 > 0$. As in the proof of Theorem I, we can assume that $\pi_1(V') = 0$ and eliminate $\pi_1(W)$ and $\pi_3(W)$ by surgeries.

Recall that we have the following commutative diagram



where $w_2(V)$ (respectively $w_2(W)$) is the second Stiefel–Whitney class of V (respectively W) seen as an homomorphism on the homology.

On the one hand, since V is non-spin, $w_2(V)$ is surjective, then so is $w_2(W)$, but the kernel of $w_2(W)$ is generated by embedded 2-sphere with trivial normal bundle [8], thus we can kill it. Hence $w_2(W)$ will be an isomorphism.

On the other hand, since $\pi_2(V) \cong Z_2$, the homomorphism $w_2(V)$ is also an isomorphism. It follows then that $i_* : H_2(V) \to H_2(W)$ is an isomorphism.

Furthermore, since $\pi_1(W) = \pi_3(W) = 0$, the Hurewicz theorem implies that $H_3(W) = 0$.

Consequently, the long exact sequence implies that $H_\lambda(W, V) = 0$, $\forall \lambda \leq 3$. The lemma then follows immediately from Lemma 4.1 and the main theorem. □

Now to prove Theorem II, it suffices by Lemma 5.1 to note that the set of generators for Ω_*^{so}, given in [3], each of which carries a metric of positive 1-curvature. One can do this easily using Lemma 3.2. □

Acknowledgements

I would like to thank Professor J. Lafontaine for many useful discussions and the International Center for Theoretical Physics where part of this research was carried out.

Appendix

One can easily adapt the proof of the main theorem to prove similar surgery results for other curvature functions which are defined on p-planes.

In this Appendix, we state without proof other results for two natural curvature functions:

1. Let r_p, $1 \leq p \leq n-1$, be the function defined on the p-Grassmannian bundle of a Riemannian manifold (M, g) of dimension n by

$$r_p(P) = \sum_{i \in I, j \in J} K(e_i, f_j),$$

where $(e_i)_{i\in I}$ (respectively $(f_j)_{j\in J}$) is any orthonormal basis of P (respectively of $P^\perp$), and K is the sectional curvature of (M,g).

For $p=1$ it is the Ricci curvature. It is related to the p-curvature of (M,g) by the following formula

$$r_p(P) = s_0 - s_p(P) - S_{n-p}(P^\perp).$$

In the same manner as in the proof of the main theorem one can prove the following

PROPOSITION. *Let p be an integer such that $2 \le p \le n-2$. If an n-manifold M is obtained from an n-manifold N by surgery in codimension* $\ge \max\{p+2, n-p+1\}$*, and N admits a metric with r_p positive, then so does M.*

2. Let k_p be the following function

$$k_p : G^pM \longrightarrow \mathbf{R},$$

$$k_p(P) = \sum_{i\in I} \mathrm{Ric}(e_i),$$

where $(e_i)_{i\in I}$ is an orthonormal basis of P and Ric is the Ricci curvature. For $p=1$ (respectively $p=n$) it is the Ricci curvature (respectively the scalar curvature). Similarly, we can prove the following

PROPOSITION. *Let p be an integer such that $2 \le p \le n$. If an n-manifold M is obtained from an n-manifold N by surgery in codimension* $\ge \min[n-p+3, \max\{p+2, n-p+1\}]$*, and N admits a metric with k_p positive, then so does M.*

As a consequence, one can prove the following results (see the proof of Theorems I and II)

PROPOSITION. *Let M be a compact 2-connected manifold of dimension ≥ 7, then the following conditions are equivalent*

1. *M admits a Riemannian metric with positive scalar curvature.*
2. *M admits a Riemannian metric with positive 1-curvature.*
3. *M admits a Riemannian metric with k_{n-1} positive.*

PROPOSITION. *Any compact non-spin simply connected manifold V of dimension ≥ 7, such that $\pi_2(V) \cong Z_2$ admits a Riemannian metric with k_{n-1} positive.*

References

1. Besse, A. L.: *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.
2. Doubrovine, B., Novikov, S. and Fomenko, A.: *Méthodes de la Théorie de l'Homologie*, 3ème partie, traduction française, Editions Mir, Moscow, 1987.

3. Gromov, M. and Lawson, H. B.: The classification of simply connected manifolds of positive scalar curvature, *Ann. of Math.* **111** (1980), 423–434.
4. Heintze, E. and Karcher, H.: A general comparison theorem with applications to volume estimates for submanifolds, *Ann. Scient. Ec. Norm. Sup., série 4* **11** (1978), 451–470.
5. Hitchin, N.: Harmonic spinors, *Adv. Math* **14** (1974), 1–55.
6. Labbi, M. L.: Sur les nombres de Betti des variétés conformément plates, *C.R. Acad. Sci. Paris, série I* **319** (1994), 77–88.
7. Labbi, M. L.: Actions des groupes de Lie presques simples et positivité de la p-courbure, *Annales de Toulouse*, to appear.
8. Lawson, H. B. and Michelsohn, M. L.: *Spin Geometry*, Princeton University Press, Princeton, NJ, 1989.
9. Micallef, M. J. and Moore, J. D.: Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes, *Annals of Math.* **127** (1988), 199–227.
10. Milnor, J.: *Lectures on the h-Cobordism Theorem*, Princeton University Press, Princeton, NJ, 1965.
11. Nayatani, S.: Kleinian groups and conformally flat metrics, in *Proceedings of the First MSJ International Research Institute on Geometry and Global Analysis*, July 12–23, Tohuku University, Sendai, Japan, 1993, pp. 1–9.
12. Schoen, R. and Yau, S. T.: On the structure of manifolds with positive scalar curvature, *Manuscripta Math.* **28** (1979), 159–183.
13. Stolz, S.: Simply connected manifolds of positive scalar curvature, *Annals of Math.* **136** (1992), 511–540.
14. Wall, C. T. C.: Geometrical connectivity I., *J. London Math. Soc.* **2**(3) (1971), 597–604.

(Received July 4, 1995; revised version July 8, 1996)