Differential and Integral Equations, Volume 1, Number 3, July 1988, pp. 253 - 279

STABILITY OF THE PERIODIC SOLUTIONS TO FULLY NONLINEAR PARABOLIC EQUATIONS IN BANACH SPACES

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(Submitted by: V. Lakshmikantham)

Abstract. We study the stability properties of the periodic solutions to a class of nonlinear abstract parabolic equations:

$$u'(t) = f(t, u(t))$$

where $f : \mathbf{R} \times D \to X$ is either periodic with respect to time or independent of time and D and X are Banach spaces with $D \hookrightarrow X$. We give applications to fully non linear parabolic p.d.e. and systems.

0. Introduction In this paper we give stability and instability results for the periodic solutions to a class of nonlinear equations in general Banach space X :

$$u'(t) = f(t, u(t)), \quad t \in \mathbf{R} \tag{0.1}$$

Here $f : \mathbf{R} \times D \to X, \quad (t,x) \to f(t,x)$, is a regular function, and D is a continuously embedded (not necessarily dense) subspace of X. We assume a parabolicity condition: for any $t \in \mathbf{R},\ x \in D$, the linear operator $f_x(t,x) : D \to X$ generates an analytic semigroup in X, and the graph norm of $f_x(t,x)$ is equivalent to the D-norm. Equation (0.1) is an abstract model for a large class of quasilinear and fully nonlinear parabolic p.d.e. (see §3).

The initial value problem for equation (0.1) has been studied in [DPG], [L5], [L3]. In particular, in [L3] it is shown that for any $u_0 \in D$ and $t_0 \in \mathbf{R}$ such that the necessary compatibility condition $f(t_0, u_0) \in \overline{D}$ holds, equation (0.1) has a local solution $u \in C\big([t_0, t_0 + \tau[;\, D\big) \cap C^1\big([t_0, t_0 + \tau[;, X\big)$ (with $\tau = \tau(t_0, u_0) > 0$) such that $u(t_0) = u_0$.

The main results of this paper may be summarized as follows: assume that f is either T-periodic with respect to time, or independent of time, and that (0.1) has a T-periodic solution $\overline{u}$, belonging to $C^\alpha(\mathbf{R}; D)$. Set $A(t) \doteq f_x(t, \overline{u}(t)),\ t \in \mathbf{R}$. Then $\{A(t); t \in \mathbf{R}\}$ is a family of linear operators in $L(D, X)$, each of them generates an analytic semigroup,

Received August 21, 1987.
†The author is a member of G.N.A.F.A. of C.N.R.
Key Words: Nonlinear parabolic equations, asymptotic stability, orbital stability, Hopf bifurcation.
AMS(MOS) Subject Classifications: 34G20, 35K99, 58E07.

and the function $t \to A(t)$ is α-Hölder continuous and T-periodic; we denote by $G(t,s)$ the corresponding evolution operator (a construction of the evolution operator when the domain of $A(t)$ is not dense in X may be found in [AT2], [L1]). The spectra of the linear operators $V(t) \doteq G(t+T,t)$, $t \in \mathbf{R}$, determine the stability properties of $\overline{u}$. If for each t the spectrum $\sigma(V(t))$ of $V(t)$ is contained in a circle centered at 0 with radius $\rho < 1$ (this may happen in the case that f is non autonomous), then $\overline{u}$ is exponentially asymptotically stable; if $\sigma(V(t))$ has some element with modulus greater than 1 and the rest of the spectrum is far from the unit circle, then $\overline{u}$ is unstable. In the autonomous case $f = f(u)$, it can be shown, as expected, that 1 is an eigenvalue of $V(t)$ for every t, so that the previous arguments cannot be used. We show that, if 1 is a simple eigenvalue and the rest of the spectrum of $V(t)$ lies in a circle centered at 0 with radius $\rho < 1$, then $\overline{u}$ is exponentially asymptotically orbitally stable with asymptotic phase: this means that, denoting by Γ the orbit $\{\overline{u}(t);\, t \in \mathbf{R}\}$, there is a neighborhood U of Γ in D and there are ω, $M > 0$ such that for any $u_0 \in U$ with $f(u_0) \in \overline{D}$, the solution $u(\cdot, u_0)$ of $u'(t) = f(u(t))$, $u(0) = u_0$, is defined for each $t \geq 0$, and there is $\theta = \theta(u_0) \in [0, T[$ such that $\|u(t,u_0) - \overline{u}(t+\theta)\|_D \leq Me^{-\omega t}\operatorname{dist}\{u_0, \Gamma\}$ (the distance is in the D-norm). The corresponding instability result is the following: if $\sigma(V(t))$ has some element with modulus greater than 1, and the rest of the spectrum (except the point $z = 1$) is far from the unit circle, then $\overline{u}$ is orbitally unstable.

These results are quite similar to the well known ones about semilinear equations (see [H]). Also the proofs follow the same ideas, but there are additional technical difficulties due to the fully nonlinear character of equation (0.1). Let us consider, for instance, the proof of the theorem about orbital stability, in the autonomous case. Let $u_0 \in D$ be close to the orbit Γ, and such that $f(u_0) \in \overline{D}$. Since our equation is autonomous, we may assume, without loss of generality, that u_0 is close to $\overline{u}(0)$. Then, for any $\theta \in \mathbf{R}$, the difference $z(t) = u(t,u_0) - \overline{u}(t+\theta)$ satisfies

$$\begin{aligned} z'(t) &= f'(\overline{u}(t))z(t) + \left[f(\overline{u}(t+\theta) + z(t)) - f(\overline{u}(t+\theta)) - f'(\overline{u}(t))z(t)\right] \\ &\doteq A(t)z(t) + g(t, z(t), \theta), \quad 0 \leq t < \tau \end{aligned} \tag{0.2}$$

with $\tau = \tau(u_0) > 0$. To prove our theorem, it is sufficient to show that:

(i) if $|\theta|$ is sufficiently small, then equation (0.2) has solutions defined in $[0, \infty[$ and decaying exponentially as $t \to +\infty$ in the D-norm;
(ii) if u_0 is sufficiently close to $\overline{u}(0)$, then $u(\cdot, u_0) - \overline{u}(\cdot + \theta)$ coincides with one of these solutions in the interval $[0, \tau[$.

Concerning point (i), we try to solve (0.2) by the usual linearization procedure. We recall that (under our assumptions on $A(\cdot)$) if $\phi : [0, +\infty[\to X$ is any measurable and exponentially decaying function, then all the exponentially decaying weak solutions of

$$v'(t) = A(t)v(t) + \phi(t), \quad t \geq 0 \tag{0.3}$$

are given by

$$\begin{gathered} v(t) = G(t,0)x_1 + \int_0^t G(t,s)P_1(s)\phi(s)\,ds - \int_t^{+\infty} G(t,s)P_0(s)\phi(s)\,ds, \\ t \geq 0, \;\; x_1 \in P_1(0)(X) \cap \overline{D}, \end{gathered} \tag{0.4}$$

where $P_i(s)$ $(i = 0, 1)$ are the projections associated with the decomposition $\big(\{1\};\, \sigma(V(s))\setminus \{1\}\big)$ of the spectrum of $V(s)$. Therefore, we have to solve the equation

$$\begin{aligned} z(t) = G(t,0)x_1 &+ \int_0^t G(t,s)P_1(s)g(s,z(s),\theta)\,ds \\ &- \int_t^{+\infty} G(t,s)P_0(s)g(s,z(s),\theta)\,ds \doteq (\Gamma\, z)(t) \end{aligned} \tag{0.5}$$

with x_1 and θ fixed. The first difficulty arises when we try to solve (0.5) by a fixed point argument. Actually, since $g(t,x,\theta)$ is defined for $x \in D$ (and not for x belonging to X, or to some intermediate space between D and X), we must work in a space having the so called maximal regularity property. If Y is a Banach space of functions defined in $[0,+\infty[$ with values in X, maximal regularity in Y means that, for any $\phi \in Y$, the solution v of equation (0.3) with assigned initial value $v(0) = x$ is such that both v' and $A(\cdot)v(\cdot)$ belong to Y (obviously, under the necessary compatibility conditions between x and ϕ). In our case, Y must be a space of functions decaying exponentially as $t \to +\infty$, since we are concerned with exponential stability of $\overline{u}$. The maximal regularity property is not satisfied if we choose $Y = C_\omega\big([0,+\infty[;\, X\big) = \{w \in C\big([0,+\infty[;\, X\big);\, t \to e^{\omega t}\, w(t)$ is bounded in $[0,+\infty[\}$, for any $\omega > 0$. It is satisfied if we choose

$$\begin{aligned} Y &= C^\alpha_\omega\big([0,+\infty[;\, X\big) \\ &\doteq \{w : [0,+\infty[\to X;\; t \to e^{\omega t}\, w(t) \in C^\alpha\big([0,+\infty[;\, X\big)\}, \end{aligned} \tag{0.6}$$

$$\|w\|_{C^\alpha_\omega([0,+\infty[;\, X)} = \|e^{\omega t}\, w(t)\|_{C^\alpha([0,+\infty[;\, X)}$$

with $\omega > 0$ sufficiently small. In this case, the necessary compatibility condition between x and ϕ is $A(0)x + \phi(0) \in D_{A(0)}(\alpha,\infty)$ (we recall that $D_{A(0)}(\alpha,\infty)$ is a real interpolation space between D and X). This condition is well understood for the linear equation (0.4). It means simply that $A(0)x_1 + P_1(0)\phi(0) \in D_{A(0)}(\alpha,\infty)$, but it is almost impossible to handle it in the nonlinear equation (0.5). Actually, to solve (0.5) by a fixed point theorem, we should work in a subset U of functions z satisfying among others, also the nonlinear condition

$$f(\overline{u}(\theta) + z(0)) \in D_{A(0)}(\alpha,\infty) \tag{0.7}$$

and we shall show that, for any $z \in U$, $\Gamma\, z$ also (defined by (0.5)) belongs to U, and, in particular, satisfies (0.7). Therefore, the choice $Y = C^\alpha_\omega\big([0,+\infty[;\, X\big)$ is not suited to our purpose. The same difficulty arises also with the choice of any other space Y, such that maximal regularity in Y requires compatibility conditions between x and ϕ. This difficulty is overcome by working in a space of functions which are not necessarily continuous up to $t = 0$; i.e., working in the space $Z^\alpha_\omega\big([0,+\infty[;\, X\big)$, defined by

$$\begin{cases} Z^\alpha_\omega\big([0,+\infty[; X\big) \doteq \Big\{w : [0,+\infty[\to X;\; w|_{[0,1]} \in Z^\alpha([0,1];\, X), \\ \qquad\qquad\qquad\qquad t \to e^{\omega t}\, w(t)|_{[1,\infty[} \in C^\alpha([1,+\infty[;\, X)\Big\} \\ \|w\|_{Z^\alpha_\omega([0,+\infty[;\, X)} \doteq \|w\|_{Z^\alpha([0,1];\, X)} + \|e^{\omega t}\, w(t)\|_{C^\alpha([1,+\infty[;\, X)} \end{cases} \tag{0.8}$$

where, for $t_0 < t_1$ we set

$$\begin{cases} Z^{\alpha}\big([t_0,t_1];\,X\big) \doteq \Big\{w \in L^{\infty}(t_0,t_1,X) \cap C^{\alpha}\big([t_0+\epsilon,\,t_1];\,X\big)\ \forall\epsilon \in]0,\,t_1-t_0[; \\ [w]_{Z^{\alpha}([t_0,t_1];\,X)} \doteq \sup_{0<\epsilon<t_1-t_0} \epsilon^{\alpha}[w]_{C^{\alpha}([t_0+\epsilon/2\,,\,t_0+\epsilon];\,X)} < +\infty\Big\} \\ \|w\|_{Z^{\alpha}([t_0,t_1];\,X)} \doteq \|w\|_{\infty} + [w]_{Z^{\alpha}([t_0,t_1];\,X)}\,. \end{cases} \tag{0.9}$$

Maximal regularity in $Z^{\alpha}_{\omega}\big([0,\,+\infty[;\,X\big)$ is proved (for each small $\omega > 0$) using previous results ([L1], [L2]) about maximal regularity in $Z^{\alpha}\big([0,1];\,X\big)$ and in $C^{\alpha}\big([t_0,\,+\infty[;\,X\big)$.

Now the fixed point argument works, in spite of the somewhat complicated topology of $Z^{\alpha}_{\omega}\big([0,\,+\infty[;\,X\big)$. We are able to show that for any sufficiently small $x_1 \in D \cap P_1(0)(x)$ and $\theta \in \mathbf{R}$, equation (0.5) has a unique solution $z = z(\cdot, x_1, \theta)$ belonging to a ball in $Z^{\alpha}_{\omega}\big([0,\,+\infty[;\,D\big)$. But our difficulties have not finished here. It remains to be shown that, if u_0 is sufficiently close to $\overline{u}(0)$, then there are x_1 and θ such that $u(t,u_0) - \overline{u}(t+\theta) = z(t,x_1,\theta)$ for $0 \le t < \tau$, i.e., (by uniqueness) such that $u_0 - \overline{u}(\theta) = z(0,x_1,\theta)$. To solve this equation, we need sharp estimates about the Lipschitz dependence on (x_1,θ) of $z(0,x_1,\theta) = x_1 - \int_0^{+\infty} G(0,s)P_0(s)g(s,\,z(s),\,\theta)\,ds$. These estimates are obtained using again the maximal regularity property of $Z^{\alpha}_{\omega}\big([0,\,+\infty[;\,X\big)$, so that they require a lengthy calculation.

One may ask now if it is possible to find fully nonlinear abstract parabolic equations having periodic solutions. The answer is yes. For instance, when f depends also on a real parameter λ, some of the classical bifurcation results about ordinary differential equations and systems may be extended to our situation, both in the autonomous and in the nonautonomous case. In particular, we consider the bifurcation of a branch of periodic solutions from a stationary one, under nonresonance conditions in the nonautonomous case, and under Hopf bifurcation assumptions in the autonomous case. Finally, we apply all the abstract results to some parabolic fully nonlinear equations and systems.

1. Notations and preliminaries on linear equations Let X be a real or complex Banach space with norm $\|\cdot\|$, and let $t_0 < t_1 \in \mathbf{R}$, $0 < \alpha < 1$. We shall use the following functional spaces whose definitions are well known: $C^{\alpha}\big([t_0,t_1];\,X\big)$, $C^{1,\alpha}\big([t_0,t_1];\,X\big)$, $C(I;\,X)$, $C^1(I;\,X)$, where I is either $[t_0,t_1]$, or $]t_0,t_1]$, $[t_0,+\infty[$, $]t_0,+\infty[$, $\mathbf{R}$. The space $Z^{\alpha}\big([t_0,t_1];\,X\big)$ has been defined in (0.9); we shall consider also its subspace

$$z^{\alpha}\big([t_0,t_1];\,X\big) = \Big\{\phi \in C\big([t_0,t_1];\,X\big) \cap C^{\alpha}\big([t_0+\epsilon,\,t_1];\,X\big) \forall\epsilon \in]0,\,t_1-t_0[; \\ \lim_{\epsilon\to 0^+} \epsilon^{\alpha}[\phi]_{C^{\alpha}([t_0+\epsilon/2\,,\,t_0+\epsilon];\,X)} = 0\Big\} \tag{1.1}$$

Finally, we shall use the spaces of exponentially decaying functions $C^{\alpha}_{\omega}\big([0,\,+\infty[;\,X\big)$ and $Z^{\alpha}_{\omega}\big([0,\,+\infty[;\,X\big)$ defined in the introduction, and the spaces $C^{\alpha}_{\omega}\big([t_0,\,+\infty[;\,X\big)$, $C^{\alpha}_{\omega}\big(]-\infty,\,t_0];\,X\big)$ which are defined similarly.

We shall deal with generators of analytic semigroups. A linear operator $A : D(A) \subset X \to$

X generates an analytic semigroup if[1]

$$\begin{cases} \text{there are } \omega \in \mathbf{R},\ \ \theta \in]\pi/2\,,\,\pi],\ M > 0 \text{ such that the resolvent} \\ \text{set of } A \text{ contains the sector } S = \{\lambda \in \mathbf{C};\ \lambda \neq \omega\ ,\ |\arg(\lambda - \omega)| < \theta\ \} \text{ and} \\ \|(\lambda - \omega)(\lambda - A)^{-1}\|_{L(X)} \le M \text{ for any } \lambda \in S \end{cases} \tag{1.2}$$

Under assumption (1.2), one can show the existence of M_0, M_1, $M_2 > 0$ such that

$$\|t^k\, A^k\, e^{tA}\|_{L(X)} \le M_k\, e^{\omega t}\,, \quad t \ge 0,\ k = 0,1,2. \tag{1.3}$$

Here and in what follows we refer to [Sin] for the properties of analytic semigroups in the non-dense domain case. Due to estimates (1.3) and to the equality $\frac{d}{dt}e^{tA}x = Ae^{tA}x$ $(t > 0,\ x \in X)$, it is not difficult to see that for any $x \in X$ and $\alpha \in]0,1[$, the function $\phi(t) = e^{tA}x$ belongs to $Z^{\alpha}\big([0,T];\ X\big)$ for any $T > 0$, and if $\omega < 0$, then ϕ also belongs to $Z^{\alpha}_{-\omega}\big([0,\ +\infty[;\ X\big)$. Analogously, one can easily show that for any $x \in \overline{D(A)}$, $T > 0$, $\alpha \in]0,1[$, $\phi(t) = e^{tA}x$ belongs to $z^{\alpha}\big([0,T];\ X\big)$.

The interpolation spaces $D_A(\alpha,\infty)$ $(0 < \alpha < 1)$ are defined by

$$\begin{cases} D_A(\alpha,\infty) = \big\{x \in X;\ [x]_\alpha \doteq \sup\limits_{0<t<1} \|t^{1-\alpha}Ae^{tA}x\| < +\infty\big\} \\ \|x\|_{D_A\,(\alpha,\infty)} \doteq \|x\| + [x]_\alpha \end{cases} \tag{1.4}$$

Let now D be a Banach space being continuously embedded in X. Let us consider a family of linear operators $A(t) : D \to X$ such that

$$\begin{aligned} &\text{for any } t \in \mathbf{R},\ A(t) \text{ satisfies (1.2), and the graph} \\ &\text{norm of } A(t) \text{ is equivalent to the } D \text{ norm } \|\cdot\|_D; \end{aligned} \tag{1.5}$$

$$\begin{aligned} &\text{there is } T > 0,\ \ \alpha \in]0,1[\text{ such that the function } t \to A(t) \\ &\text{is } T\text{-periodic and belongs to } C^{\alpha}\big([0,T];\ L(D,X)\big). \end{aligned} \tag{1.6}$$

Under assumptions (1.5) and (1.6), we shall give some existence and regularity results for bounded solutions of equations

$$z'(t) = A(t)z(t) + g(t), \quad t > t_0 \tag{1.7}$$

$$w'(t) = A(t)w(t) + h(t), \quad t < t_0 \tag{1.8}$$

where g and h belong to $C\big(]t_0,\ +\infty[;\ X\big)$ and to $C\big(]-\infty,\ t_0[;\ X\big)$ respectively. A function $z \in C^1\big(]t_0,\ +\infty[;\ X\big) \cap C\big(]t_0,\ +\infty[;\ D\big)$ satisfying (1.7) is said to be a classical solution of (1.7). If g is continuous up to $t = t_0$ and z belongs to $C^1\big([t_0,\ +\infty[;\ X\big) \cap C\big([t_0,\ +\infty[;\ D\big)$ (so that it satisfies (1.7) up to $t = t_0$), then z is said to be a strict solution of (1.7). Similar definitions may be given for classical and strict solutions of (1.8).

[1]Through the whole paper we consider only complex resolvents and spectra; even if X is a real Banach space, for any linear operator L we denote by $\sigma(L)$ (resp. $\rho(L)$) the spectrum (resp. the resolvent set) of the complexification of L.

Under assumptions (1.5) and (1.6), there exists an evolution operator $G(t,s) \in L(X)$ ($t > s$) such that $\frac{\partial}{\partial t}G(t,s)x \doteq A(t)G(t,s)x$ for $t > s$, $x \in X$, $G(t,r)G(r,s) = G(t,s)$ for $t \geq r \geq s$ and $G(t,t) = I$ for any $t \in \mathbf{R}$ (see [L1], [AT 2]). The asymptotic behaviour of $G(t,s)$ may be described in terms of the spectrum of the linear operators

$$V(s) \doteq G(s+T,\, s), \quad s \in \mathbf{R}. \tag{1.9}$$

More precisely, we assume that there exists $\rho > 0$ such that

$$\sigma(V(s)) \cap \{\lambda \in \mathbf{C};\ |\lambda| = \rho\} = \emptyset \tag{1.10}$$

and we set $\sigma(V(s)) = \sigma_1(V(s)) \cup \sigma_2(V(s))$, where

$$\begin{cases} \sigma_1(V(s)) = \sigma(V(s)) \cap \{\lambda \in \mathbf{C};\ |\lambda| < \rho\} \\ \sigma_2(V(s)) = \sigma(V(s)) \cap \{\lambda \in \mathbf{C};\ |\lambda| > \rho\} \end{cases} \tag{1.11}$$

$\sigma_2(V(s))$ may be possibly empty, and one can easily show that it in fact does not depend on s. Moreover we have:

$$\begin{cases} \sup\{|\lambda|;\ \lambda \in \sigma_1(V(s)),\ s \in \mathbf{R}\} \doteq \rho_1 < \rho \\ \inf\{|\lambda|;\ \lambda \in \sigma_2(V(s)),\ s \in \mathbf{R}\} \doteq \rho_2 > \rho \end{cases} \tag{1.12}$$

We can define two families of operators and the corresponding ranges:

$$\begin{cases} P_1(s) = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma(0,\rho)} (\lambda - V(s))^{-1}\, d\lambda, \quad s \in \mathbf{R} \\ X_1(s) = P_1(s)(X), \quad s \in \mathbf{R} \\ P_2(s) = 1 - P_1(s), \quad s \in \mathbf{R} \\ X_2(s) = P_2(s)(X), \quad s \in \mathbf{R} \end{cases} \tag{1.13}$$

For every $s \in \mathbf{R}$, $X_2(s)$ is included in D, and the restriction of $G(t,s)$ to $X_2(s)$ may be defined also for $t < s$, still satisfying $\frac{\partial}{\partial t}G(t,s)x = A(t)G(t,s)x$ for every $x \in X_2(s)$ (for more details see [L 2]).

Let us consider now problems (1.7) and (1.8).

Proposition 1.1. *Let (1.5), (1.6), and (1.10) hold with $\rho < 1$, and let $\omega \in]0, -T^{-1}\log \rho_1[$, where ρ_1 is defined in (1.12). Then:*

(i) *If g belongs to $Z^\alpha_\omega([t_0, +\infty[;\, X)$, then all the classical solutions z of (1.7) such that $t \to e^{\omega t}z(t)$ is bounded in $[t_0, +\infty[$ are given by*

$$z(t) = G(t,t_0)x_1 + \int_{t_0}^{t} G(t,s)P_1(s)g(s)\,ds - \int_t^{+\infty} G(t,s)P_2(s)g(s)\,ds, \qquad t \geq t_0,\ x_1 \in X_1(t_0) \cap \overline{D}, \tag{1.14}$$

where $P_1(s)$, $P_2(s)$, and $X_1(t_0)$ are defined in (1.13).

(ii) *If $x_1 \in X_1(t_0) \cap D$, then every z defined in (1.14) belongs to $C([t_0, +\infty[;\, X) \cap Z^\alpha_\omega([t_0, +\infty[;\, D)$, z' belongs to $Z^\alpha_\omega([t_0, +\infty[;\, X)$ and there is $C_1 > 0$ (not depending on t_0, g, x_1) such that*

$$\|z\|_{Z^\alpha_\omega([t_0,+\infty[;D)} + \|z'\|_{Z^\alpha_\omega([t_0,+\infty[;X)} \le C_1\big(\|x_1\|_D + \|g\|_{Z^\alpha_\omega([t_0,+\infty[;X)}\big) \tag{1.15}$$

(iii) *If $g \in C^\alpha_\omega([t_0, +\infty[; X)$ and $A(t_0)x_1 + g(t_0)$ belongs to $D_{A(0)}(\alpha,\infty)$, then z is a strict solution of (1.7) and belongs to $C^\alpha_\omega([t_0, +\infty[; D)$, z' belongs to $C^\alpha_\omega([t_0, +\infty[; X)$. There is $C_2 > 0$ (not depending on t_0, g, x_1) such that*

$$\begin{aligned}\|z\|_{C^\alpha_\omega([t_0,+\infty[;D)} + \|z'\|_{C^\alpha_\omega([t_0,+\infty[;X)} \le\\ C_2\big(\|x_1\|_D + \|P_1(t_0)(A(t_0)x_1 + g(t_0))\|_{D_{A(0)}(\alpha,\infty)} + \|g\|_{C^\alpha_\omega([t_0,+\infty[;X)}\big)\end{aligned} \tag{1.16}$$

Let now (1.5), (1.6), and (1.10) hold with $\rho > 1$, and let $\omega \in]0, T^{-1}\log\rho_2[$. If h belongs to $C^\alpha_\omega(]-\infty, t_0]; X)$, then all the classical solutions w of (1.8) such that $t \to e^{-\omega t}w(t)$ is bounded in $]-\infty; t_0]$ are given by

$$\begin{aligned} w(t) = G(t,t_0)x_2 + \int_{t_0}^t G(t,s)P_2(s)h(s)\,ds + \int_{-\infty}^t G(t,s)P_1(s)h(s)\,ds\\ t \le t_0,\quad x_2 \in X_2(t_0),\end{aligned} \tag{1.17}$$

where $P_1(s)$, $P_2(s)$ and $X_2(t_0)$ are defined in (1.13). Any w given by (1.17) is also a strict solution of (1.18); it belongs to $C^\alpha_\omega(]-\infty, t_0]; D)$ and w' belongs to $C^\alpha_\omega(]-\infty, t_0]; X)$. There is $C_3 > 0$ such that

$$\|w\|_{C^\alpha_\omega(]-\infty,t_0];D)} + \|w'\|_{C^\alpha_\omega(]-\infty,t_0];X)} \le C_3\big(\|x_2\| + \|h\|_{C^\alpha_\omega(]-\infty,t_0];X)}\big). \tag{1.18}$$

Proof: We have only to show (ii), since the other statements follow from [L2, §3]. Since x_1 belongs to D, then $z(t_0)$ belongs to D; therefore z belongs to $Z^\alpha([t_0, t_0+1]; D)$, z' belongs to $Z^\alpha([t_0, t_0+1]; X)$ and

$$\|z\|_{Z^\alpha([t_0,t_0+1];D)} + \|z'\|_{Z^\alpha([t_0,t_0+1];X)} \le \text{const}\,\big(\|z(t_0)\|_D + \|g\|_{Z^\alpha([t_0,t_0+1];X)}\big) \tag{1.19}$$

thanks to [L1, prop. 2.2]. Since z belongs to $C^\alpha([t_0+1/2, t_0+1]; D) \cap C^{1,\alpha}([t_0+1/2, t_0+1]; X)$ then, by lemma 1.1 of [L1], $z'(t_0+1)$ belongs to $D_{A(0)}(\alpha,\infty)$, and

$$\|z'(t_0+1)\|_{D_{A(0)}(\alpha,\infty)} \le \text{const}\,\big(\|z\|_{C^\alpha([t_0+1/2,t_0+1];D)} + \|z'\|_{C^\alpha([t_0+1/2,t_0+1];X)}\big) \tag{1.20}$$

Therefore, by [L2, prop. 3.10 (iii)], z belongs to $C^\alpha_\omega([t_0+1, +\infty[; D)$, z' belongs to $C^\alpha_\omega([t_0, +\infty[; X)$ and

$$\begin{aligned}\|z\|_{C^\alpha_\omega([t_0+1,+\infty[;D)} + \|z'\|_{C^\alpha_\omega([t_0+1,+\infty[;X)} \le\\ \text{const}\,\big(\|z(t_0+1)\|_D + \|z'(t_0+1)\|_{D_{A(0)}(\alpha,\infty)} + \|g\|_{C^\alpha_\omega([t_0+1,+\infty[;X)}\big)\end{aligned} \tag{1.21}$$

Recalling that $\|G(t,s)\|_{L(X_2(s);\,D)}$ is bounded in $\{(t,s)\in\mathbf{R}^2,\ s\ge t\}$, we get

$$\|z(t_0)\|_D \le \|x_1\| + \text{const}\sup_{t\ge t_0}\|e^{\omega(t-t_0)}g(t)\| \tag{1.22}$$

Then (1.15) follows from (1.19)–(1.22) and (ii) is proved. ∎

We consider now the problem of finding T-periodic solutions to

$$u'(t) = A(t)u(t) + f(t), \quad t\in\mathbf{R} \tag{1.23}$$

when $f:\mathbf{R}\to X$ is T-periodic and α-Hölder continuous. We replace assumption (1.10) by

$$1 \text{ belongs to the resolvent set of } G(T,0). \tag{1.24}$$

Also the following result is proved in [L2, prop. 3.14].

Proposition 1.2. *Let (1.5), (1.6), (1.24) hold. Then for any T-periodic and α-Hölder continuous function $f: R\to X$, problem (1.23) has a unique T-periodic strict solution u. Moreover u belongs to $C^\alpha(\mathbf{R};D)\cap C^{1,\alpha}(\mathbf{R};X)$ and there is $C_4>0$ such that*

$$\|u\|_{C^\alpha(\mathbf{R};D)} + \|u'\|_{C^\alpha(\mathbf{R};X)} \le C_4\|f\|_{C^\alpha(\mathbf{R};X)}\,. \tag{1.25}$$

2. Stability and instability of periodic solutions.

(A) The nonautonomous case. Let $f:\mathbf{R}\times D\to X$, $(t,x)\to f(t,x)$ be continuous and T-periodic with respect to time; let $t_0<t_1$. Consider the equation

$$u'(t) = f(t,u(t)), \quad t_0\le t\le t_1. \tag{2.1}$$

A function u belonging to $C([t_0,t_1];\,D)\cap C^1([t_0,t_1];\,X)$ and satisfying (2.1) is said to be a strict solution of (2.1). The existence of strict solutions of (2.1) may be proved under the following assumptions on f:

$$\begin{cases} \text{Regularity– for any } t\in\mathbf{R},\ f(t,\cdot) \text{ belongs to } C^2(D;X);\\ \text{there is } \alpha\in]0,1[\text{ such that } f(\cdot,x),\ f_x(\cdot,x),\ f_{xx}(\cdot,x) \text{ are}\\ \alpha\text{-Hölder continuous, locally uniformly with respect to } x; \end{cases} \tag{2.2}$$

$$\begin{cases} \text{Parabolicity– for any } t\in\mathbf{R} \text{ and } x\in D, \text{ the operator}\\ A=f_x(t,x) \text{ satisfies (1.2), and the graph norm of } A \text{ is}\\ \text{equivalent to the D-norm.} \end{cases} \tag{2.3}$$

Then the following result holds.

Proposition 2.1. *Under assumptions (2.2) and (2.3), for any $t_0 \in \mathbf{R}$ and $u_0 \in D$ such that the necessary compatibility condition*

$$f(t_0, u_0) \in \overline{D} \tag{2.4}$$

holds, equation (2.1) has a maximally defined strict solution $u = u(\cdot, u_0) : [t_0, t_0 + \tau[\to D$ ($\tau = \tau(t_0, u_0) > 0$) such that $u(t_0) = u_0$. u is the unique solution of the i.v.p. belonging to $z^\alpha([t_0, t_0 + \delta]; D)$ for any $\delta \in]0, \tau[$. Moreover, for any $\delta \in]0, \tau[$ there are ρ_0, $k_0 > 0$ such that:

(i) *if $u_1 \in D$, $f(t_0, u_1) \in \overline{D}$ and $\|u_0 - u_1\|_D = \rho \le \rho_0$ then $\tau(u_1) \ge \delta$ and $\|u(t, u_0) - u(t, u_1)\|_D \le \rho k_0$ for $t_0 \le t \le t_0 + \delta$.*

(ii) *if $w \in C([t_0, t_0 + \delta]; X) \cap C^1(]t_0, t_0 + \delta]; X) \cap Z^\alpha([t_0, t_0 + \delta]; D)$ is such that $w(t_0) = u_0$, $w'(t) = f(t, w(t))$ for $t_0 \le t \le t_0 + \delta$, and $\|w - u(\cdot, u_0)\|_{Z^\alpha([t_0, t_0+\delta]; D)} \le \rho_0$, then $w(t) = u(t, u_0)$ for $t_0 \le t \le t_0 + \delta$.*

Proof: The proposition was proved in [L3], except statement (ii), which we show now. Fix $t_1 \in]t_0, t_0+\delta]$ and $k > 0$, and set $Y = \{z \in Z^\alpha([t_0, t_1]; D); \|z(\cdot) - u_0\|_{L^\infty(t_0, t_1; D)} \le k\}$. If t_1 and k are sufficiently small, then for every $z \in Y$ the function $t \to f(t, z(t)) - Az(t)$ (where $A = f_x(t_0, u_0)$) belongs to $Z^\alpha([t_0, t_1]; X)$. Therefore, if $z \in C^1(]t_0, t_1]; X) \cap C([t_0, t_1]; X) \cap Y$ is a solution of $z'(t) = f(t, z(t))$, $t_0 \le t \le t_1$, $z(t_0) = u_0$, then z is a fixed point of the operator $\phi : Y \to Z^\alpha([t_0, t_1]; D)$, $\phi(z) = v$, where v is the solution of

$$\begin{cases} v'(t) = Av(t) + [f(t, z(t)) - Az(t)], & t_0 \le t \le t_1 \\ v(t_0) = u_0 \end{cases}$$

(v belongs to $Z^\alpha([t_0, t_1]; D) \cap C([t_0, t_1]; X) \cap C^1(]t_0, t_1]; X)$ thanks to prop. 1 of [L3]). In the proof of theorem 2 of [L3] it is shown that ϕ is a 1/2-contraction in the norm of $Z^\alpha([t_0, t_1], D)$, provided $t_1 - t_0$ and k are sufficiently small, so that ϕ has at most one fixed point in Y. Let $r = \|w - u(\cdot, u_0)\|_{Z^\alpha([t_0, t_0+\delta]; D)}$. Then, if r and $t_1 - t_0$ are sufficiently small, $w_{|[t_0, t_1]}$ belongs to Y. Actually, $\|w - u_0\|_{L^\infty(t_0, t_1; D)} \le r + \|u(\cdot, u_0) - u_0\|_{L^\infty(t_0, t_1; D)}$ and $u(\cdot, u_0)$ is continuous. Therefore both $w_{|[t_0, t_1]}$ and $u(\cdot, u_0)_{|[t_0, t_1]}$ are fixed points of ϕ belonging to Y, so that $w(t) = u(t, u_0)$ for $t_0 \le t \le t_1$. On the other hand, $w_{|[t_1, t_0+\delta]}$ belongs to $C^\alpha([t_1, t_0 + \delta]; D) \subset z^\alpha([t_1, t_0 + \delta]; D)$ and $f(t_1, w(t_1)) = w'(t_1)$ belongs to $\overline{D}$, so that, due to the first part of the theorem, w coincides with $u(\cdot, u_0)$ also in the interval $[t_1, t_0 + \delta]$. ∎

Assume now that eq. (2.1) has a T-periodic strict solution $\overline{u}$ belonging to $Z^\alpha([0, T]; D)$. Then, using prop. 2.1, it is easy to see that $\overline{u}$ belongs to $C^\alpha(\mathbf{R}; D) \cap C^{1,\alpha}(\mathbf{R}; X)$, so that the family of operators

$$A(t) \doteq f_x(t, \overline{u}(t)), \quad t \in \mathbf{R}, \tag{2.5}$$

satisfies (1.5) and (1.6). Therefore the associated evolution operator $G(t, s)$ is well defined, and so does the family $V(s) \doteq G(s + T, s)$, $s \in \mathbf{R}$ (see sect. 1).

We give now a result of exponential asymptotic stability.

Theorem 2.2. *Let $f : \mathbf{R} \times D \to X$ $(t,x) \to f(t,x)$ be T-periodic w.r. to time and satisfy (2.2), (2.3). Let $\overline{u}$ be a T-periodic strict solution of (2.1), belonging to $z^\alpha([0,T]; D)$. Define $A(t)$ by (2.5) and $V(s)$ by (1.9), and assume that*

$$\sup\{|\lambda|;\ \lambda \in \sigma(V(s)),\ s \in \mathbf{R}\} \doteq \rho_1 < 1 \tag{2.6}$$

Then $\overline{u}$ is exponentially asymptotically stable: more precisely, for any $\omega \in]0, -T^{-1}\log\rho_1[$ there are δ_0, $M > 0$ such that for each $t_0 \in \mathbf{R}$ and $u_0 \in D$ satisfying (2.4) and such that $\|u_0 - \overline{u}(t_0)\|_D = \delta \le \delta_0$, then $u(\cdot, u_0)$ is defined in $[t_0, +\infty[$ and $\|u(t,u_0) - \overline{u}(t)\|_D \le M\delta e^{-\omega(t-t_0)}$.

Proof: Let $t_0 \in \mathbf{R}$, $u_0 \in D$ be such that $f(t_0, u_0) \in \overline{D}$. Set $z(t) = u(t,u_0) - \overline{u}(t)$, $t \in [t_0,\ t_0 + \tau(t_0,u_0)[$. Then z satisfies

$$\begin{aligned} z'(t) &= f_x(t,\overline{u}(t))z(t) + [f(t,\overline{u}(t) + z(t)) - f(t,\overline{u}(t)) - f_x(t,\overline{u}(t))z(t)] \\ &\doteq A(t)z(t) + g(t,z(t)). \end{aligned} \tag{2.7}$$

We shall show that z may be continued in $[t_0 + \tau,\ +\infty[$, still satisfying (2.7), and that the continuation decays exponentially as $t \to +\infty$ provided $\|u_0 - \overline{u}(t_0)\|_D$ is sufficiently small. To this aim we define the mapping

$$\Gamma : B(0,r) \subset Z^\alpha_\omega([t_0,\ +\infty[;\ D) \to Z^\alpha_\omega([t_0,\ +\infty[;\ D), \quad \Gamma z = v \tag{2.8}$$

where v is the solution of

$$\begin{cases} v'(t) = A(t)v(t) + g(t,z(t)), & t > t_0 \\ v(t_0) = u_0 - \overline{u}(t_0) \end{cases} \tag{2.9}$$

We have to show that Γ is well defined, it maps $B(0,r)$ into itself and it is a contraction if r and $\|u_0 - \overline{u}(t_0)\|_D$ are small. The unique fixed point of Γ is then the desired extension of $u(\cdot,u_0) - \overline{u}(\cdot)$.

The function $g(t,x) : \mathbf{R} \times D \to X$ has the following properties:

$$\begin{cases} \text{(i)} \quad g(t,x) = g(t+T,x), \quad t \in \mathbf{R},\ x \in D \\ \text{(ii)} \quad \sup\limits_{t\in\mathbf{R},\, x\in B(0,r)\subset D} \|g_x(t,x)\|_{L(D,X)} \doteq k_1(r) \to 0 \text{ as } r \to 0 \\ \text{(iii)} \quad \sup\limits_{x\in B(0,r)\subset D} [g(\cdot,x)]_{C^\alpha(\mathbf{R};\,L(D,X))} \doteq k_2(r) \to 0 \text{ as } r \to 0 \end{cases} \tag{2.10}$$

Let $r > 0$ be so small that

$$\sup_{t\in\mathbf{R},\, x\in B(0,r)\subset D} \|g_{xx}(t,x)\|_{L(D,\,L(D,X))} \doteq k_3(r) < +\infty \tag{2.11}$$

Then for any z_1, $z_2 \in B(0,r) \subset Z^\alpha_\omega([t_0,\ +\infty[;\ D)$ we have

$$e^{\omega(t-t_0)}\|g(t,\ z_1(t)) - g(t,\ z_2(t))\| \le k_1(r)e^{\omega(t-t_0)}\|z_1(t) - z_2(t)\|_D\,,\ t \ge t_0 \tag{2.12}$$

and, for $t \geq s \geq t_0$:

$$
\begin{aligned}
&\left\|e^{\omega(t-t_0)}[g(t,z_1(t)) - g(t,z_2(t))] - e^{\omega(s-t_0)}[g(s,z_1(s)) - g(s,z_2(s))]\right\| = \\
&\quad \Big\| \int_0^1 \big[g_x\big(t, \sigma z_1(t) + (1-\sigma) z_2(t)\big) e^{\omega(t-t_0)}(z_1(t) - z_2(t)) \\
&\qquad - g_x\big(s, \sigma z_1(s) + (1-\sigma)z_2(s)\big) e^{\omega(s-t_0)}(z_1(s) - z_2(s))\big]\, d\sigma \Big\| \leq \\
&\quad \Big\| \int_0^1 \big[g_x\big(t, \sigma z_1(t) + (1-\sigma)z_2(t)\big) - g_x\big(s, \sigma z_1(s) + (1-\sigma)z_2(s)\big)\big] \\
&\qquad \cdot e^{\omega(t-t_0)}(z_1(t) - z_2(t))\, d\sigma \Big\| + \Big\| \int_0^1 g_x\big(s, \sigma z_1(s) + (1-\sigma)z_2(s)\big) \\
&\qquad \cdot \big[e^{\omega(t-t_0)}(z_1(t) - z_2(t)) - e^{\omega(s-t_0)}(z_1(s) - z_2(s))\big]\, d\sigma \Big\| \leq \\
&\quad \Big[k_2(r)(t-s)^\alpha + k_3(r) \int_0^1 [\sigma \|z_1(t) - z_1(s)\|_D + (1-\sigma)\|z_2(t) - z_2(s)\|_D]\, d\sigma\Big] \\
&\quad \|e^{\omega(t-t_0)}(z_1(t) - z_2(t))\| + k_1(r)\left\|e^{\omega(t-t_0)}(z_1(t) - z_2(t)) - e^{\omega(s-t_0)}(z_1(s) - z_2(s))\right\|
\end{aligned}
\tag{2.13}
$$

so that $t \to e^{\omega(t-t_0)}[g(t, z_1(t)) - g(t, z_2(t))]$ belongs to $C^\alpha\big([t_0+1, +\infty[; X\big)$ and

$$
\begin{aligned}
&\big[e^{\omega(t-t_0)}\big(g(t, z_1(t)) - g(t, z_2(t))\big)\big]_{C^\alpha\left([t_0+1,\, +\infty[;\, X\right)} \leq \\
&\big[k_2(r) + \big(\omega^\alpha(1-\alpha)^{1-\alpha} + 1\big) r k_3(r)\big] \sup_{s \geq t_0+1} \left\|e^{\omega(s-t_0)}(z_1(s) - z_2(s))\right\|_D \\
&\quad + k_1(r)[z_1 - z_2]_{C^\alpha_\omega\left([t_0+1,\, +\infty[;\, D\right)}
\end{aligned}
\tag{2.14}
$$

Analogously, setting $\omega = 0$ in 2.13, we get, for $t_0 + \epsilon/2 \leq s < t \leq t_0 + \epsilon$, $0 < \epsilon \leq 1$:

$$
\begin{aligned}
&\epsilon^\alpha \left\|g(t, z_1(t)) - g(t, z_2(t)) - g(s, z_1(s)) + g(s, z_2(s))\right\| \\
&\quad \leq \Big\{[k_1(r) + r k_3(r)] \sup_{t_0 \leq s \leq t_0+1} \|z_1(s) - z_2(s)\|_D \\
&\qquad + k_1(r)[z_1 - z_2]_{C^\alpha\left([t_0+\epsilon/2,\, t_0+\epsilon];\, D\right)}\Big\} (t-s)^\alpha
\end{aligned}
\tag{2.15}
$$

so that by (2.12), (2.14), (2.15) we have:

$$
\begin{aligned}
&\|g(\cdot, z_1(\cdot)) - g(\cdot, z_2(\cdot))\|_{Z^\alpha_\omega\left([t_0,\, +\infty[;\, X\right)} \\
&\quad \leq \big[k_1(r) + k_2(r) + \big(\omega^\alpha(1-\alpha)^{1-\alpha} + 1\big) r k_3(r)\big] \|z_1 - z_2\|_{Z^\alpha_\omega\left([t_0,\, +\infty[;\, D\right)}
\end{aligned}
\tag{2.16}
$$

Let now $r_0 > 0$ be such that

$$
k_1(r_0) + k_2(r_0) + \big(\omega^\alpha(1-\alpha)^{1-\alpha} + 1\big) r_0 k_3(r_0) \leq 1/2C_1 \tag{2.17}
$$

where C_1 is given in (1.15). Then, using (2.16), (2.6), (1.15) and the definition (2.8) of Γ we find that:

(i) for any $r \leq r_0$, Γ is a contraction with constant $1/2$ in $B(0,r) \subset Z^\alpha_\omega\big([t_0, +\infty[; D\big)$

(ii) for any $r \leq r_0$, if $\|u_0 - \overline{u}(t_0)\|_D \leq r/2C_1$, then Γ maps $B(0,r) \subset Z^\alpha_\omega\big([t_0, +\infty[; D\big)$ into itself.

The statement of the theorem follows now easily, with $\delta_0 = r_0/2C_1$ and $M = 2C_1$. ∎

Let us give now an instability result.

Theorem 2.3. *Assume the same hypotheses as in Theorem 2.2, except (2.6) which is replaced by*

$$\begin{cases} \sigma_2 \doteq \{\lambda \in \sigma(V(s)), \quad |\lambda| > 1, \ s \in \mathbf{R}\} \neq \emptyset \\ \inf\{|\lambda|; \ \lambda \in \sigma_2\} \doteq \rho_2 > 1 . \end{cases} \tag{2.18}$$

Then $\overline{u}$ is unstable. More precisely, there is $\epsilon > 0$ such that for any $t_0 \in \mathbf{R}$ there exists a sequence $\{x_n\} \subset D$ with $\lim_{n\to+\infty} x_n = \overline{u}(t_0)$ but $\sup_{t\geq t_0} \|u(t,x_n) - \overline{u}(t)\| \geq \epsilon$.

Proof:: Every $\rho \in]1, \rho_2[$ obviously satisfies (1.10). Let $P_1(s)$, $P_2(s)$ $(s \in \mathbf{R})$ be defined by (1.13). We shall show that for any $t_0 \in \mathbf{R}$, there is a backward solution of (2.1), $v :]-\infty, t_0] \to D$, $v \neq \overline{u}$, such that $\lim_{t\to-\infty} \|\overline{u}(t) - v(t)\|_D = 0$. We remark that if such a v exists, then the difference $z(t) = v(t) - \overline{u}(t)$ satisfies equation (2.7) in $]-\infty, t_0]$. Therefore, in view of Proposition 1.1, we have to solve the integral equation

$$z(t) = G(t,t_0)x_2 + \int_{t_0}^t G(t,s)P_2(s)g(s,z(s))\,ds + \int_{-\infty}^t G(t,s)P_1(s)g(s,z(s))\,ds \quad t \leq t_0 \tag{2.19}$$

with $x_2 \in X_2(t_0)$ fixed. As in Theorem 2.2, we may solve (2.19) in a set of exponentially decaying functions:

$$Y = B(0,r) \subset C^\alpha_\omega(]-\infty, t_0]; D) \tag{2.20}$$

with $\omega \in]0, T^{-1}\log\rho_2[$. Here we do not need to set our problem in the space $Z^\alpha_\omega(]-\infty, t_0]; D)$ because $X_2(t_0) \subset D_{A(0)}(\alpha+1, \infty)$, so that $G(\cdot, t_0)x_2$ belongs to $C^\alpha_\omega(]-\infty, t_0]; D)$ for any $x_2 \in X_2$.

Here we use the notations of Theorem 2.2. Since g satisfies (2.10), estimate (2.13) holds for any $z \in Y$ and $s \leq t \leq t_0$. Proceeding as in Theorem 2.2, it is easy to show that (2.19) has a unique solution in Y provided r and $\|x_2\|$ are sufficiently small. Using again estimate (2.13) it is possible to see that if $x_2 \neq 0$ then z does not vanish. More precisely, we find $z(t_0) = x_2 + P_1(t_0)z(t_0)$ and $\|P_1(t_0)z(t_0)\| \leq h(r)$, with $h(r)$ independent on t_0 and $\lim_{r\to0} h(r) = 0$. The statement of the theorem follows, with $\epsilon = \|x_2\|/2$ and $x_n = \overline{u}(t_0 - nT) + z(t_0 - nT) = \overline{u}(t_0) + z(t_0 - nT)$. ∎

(B) The autonomous case. Here we use the same notations as in subsection 2A, with obvious modifications. We assume that $f : D \to X$ is a C^3 function and we study the stability properties of the periodic solutions of

$$u'(t) = f(u(t)). \tag{2.21}$$

The parabolicity condition (2.3) is assumed to hold. By Proposition 2.1, any T-periodic solution $\overline{u} \in Z^\alpha([0,T]; D)$ of (2.21) belongs to $C^\alpha(\mathbf{R}; D) \cap C^{1,\alpha}(\mathbf{R}, X)$; in fact, it belongs to $C^{1,\beta}(\mathbf{R}; D) \cap C^{2,\beta}(\mathbf{R}; X)$ for any $\beta \in]0,1[$ due to [L4]. Therefore, the derivative $v(t) = \overline{u}'(t)$ satisfies

$$v'(t) = f'(\overline{u}(t))v(t), \quad t \in \mathbf{R},$$

and it has the same period of $\overline{u}$, so that 1 is an eigenvalue of $V(s)$ for each $s \in \mathbf{R}$, and the stability theorem of subsection 2A cannot be used. Nevertheless, the following stability result holds.

Theorem 2.4. *Let $\overline{u}$ be a T-periodic strict solution of (2.21), belonging to $Z^\alpha([0,T];\, D)$ for some $\alpha \in]0,1[$. Let $A(t) = f'(\overline{u}(t))$, $t \in \mathbf{R}$, and $G(t,s)$ be the corresponding evolution operator, and let $V(s) = G(s+T,\, s)$, $s \in \mathbf{R}$. Assume that*

$$\begin{cases} (i) & \text{1 is a simple eigenvalue of } V(s), \quad s \in \mathbf{R} \\ (ii) & \sup\{|\lambda|;\ \lambda \in \sigma(V(s))\backslash\{1\},\ \ s \in \mathbf{R}\} \doteq \rho_1 < 1 \end{cases} \tag{2.22}$$

Then $\overline{u}$ is orbitally asymptotically stable with asymptotic phase. More precisely, for $\omega \in]0, -T^{-1}\log\rho_1[$, denoting by Γ the orbit $\{\overline{u}(t),\ t \in \mathbf{R}\}$, there are $\rho_0 > 0$, $M > 0$ such that if $u_0 \in D$, $f(u_0) \in \overline{D}$, and $\operatorname{dist}(u_0, \Gamma) = \rho \le \rho_0$, then the solution $u(\cdot, u_0)$ of $u'(t) = f(u(t))$, $t \ge 0$; $u(0) = u_0$, is defined in $[0, +\infty[$ and there is $\theta = \theta(u_0) \in \mathbf{R}$ such that

$$\|u(t,u_0) - \overline{u}(t+\theta)\|_D \le M\,\rho\, e^{-\omega t}, \quad t \ge 0. \tag{2.23}$$

Proof: Let $P_1(s)$, $P_2(s)$ $(s \in \mathbf{R})$ be defined by (1.13), with any $\rho \in]\rho_1, 1[$. Let $u_0 \in D$ be close to Γ and be such that $f(u_0) \in \overline{D}$; let $u(\cdot, u_0)$ be the solution of (2.21) given by Proposition 2.1. We may assume (replacing possibly $\overline{u}(t)$ by $\overline{u}(t+t_0)$, $0 < t_0 < T$) that u_0 is close to $\overline{u}(0)$. For any $\theta \in \mathbf{R}$ the difference $z(t,\theta) = u(t,u_0) - \overline{u}(t+\theta)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} z(t,\theta) &= f'(\overline{u}(t))z(t,\theta) + \left[f\big(\overline{u}(t+\theta) + z(t,\theta)\big) - f(\overline{u}(t+\theta)) - f'(\overline{u}(t))z(t,\theta)\right] \\ &\doteq A(t)z(t,\theta) + g(t, z(t,\theta), \theta), \qquad 0 \le t < \tau. \end{aligned}$$

Therefore (see Prop. 1.1 (ii)), we look for a solution of

$$\begin{aligned} z(t) &= G(t,0)x_1 + \int_0^t G(t,s)P_1(s)g(s,z(s),\theta)\,ds - \int_t^{+\infty} G(t,s)P_2(s)g(s,z(s),\theta)\,ds \\ &\doteq (F_\theta\, z)(t) \end{aligned} \tag{2.24}$$

where $x_1 \in D \cap X_1(0)$ is fixed in a ball $B(0,r) \subset Z^\alpha_\omega([0, +\infty[;\, D)$, $0 < \alpha < 1$. Since $\overline{u}$ belongs to $C^1(\mathbf{R}; D)$, it is not difficult to show that $g : \mathbf{R} \times D \times \mathbf{R} \to X$ satisfies

$$\begin{cases} (i) & g(t+T,\, x, \theta) = g(t,x,\theta), \ \ g(t,0,0) = 0 \\ (ii) & \sup\limits_{t\in\mathbf{R},\, x \in B(0,r)\subset D} \left\|g_x(t,x,\theta)\right\|_{L(D,X)} \doteq K_1(r,\theta) \to 0 \ \text{ as } r,\, \theta \to 0 \\ (iii) & \sup\limits_{x \in B(0,r)\subset D} \left[g_x(\cdot,x,\theta)\right]_{C^\alpha(\mathbf{R};\, L(D,X))} \doteq K_2(r,\theta) \to 0 \ \text{ as } r,\, \theta \to 0 \\ (iv) & \sup\limits_{t,\theta\in\mathbf{R},\, x \in B(0,r)\subset D} \left\|g_\theta(t,x,\theta)\right\| \doteq K_3(r) \to 0 \ \text{ as } r \to 0 \\ (v) & \sup\limits_{\theta\in\mathbf{R}} \left[g_\theta(\cdot,x,\theta)\right]_{C^\alpha(\mathbf{R};X)} \le K_4(r)\|x\|_D\,, \ \ x \in B(0,r) \subset D. \end{cases} \tag{2.25}$$

For $r > 0$ define

$$\begin{cases} K_5(r) \doteq \sup\limits_{t,\theta\in\mathbf{R},\, x\in B(0,r)\subset D} \|g_{xx}(t,x,\theta)\|_{L(D,L(D,X))} \\ K_6(r) \doteq \sup\limits_{t,\theta\in\mathbf{R},\, x\in B(0,r)\subset D} \|g_{\theta x}(t,x,\theta)\|_{L(D,X)}. \end{cases} \tag{2.26}$$

Then, using estimates (2.12), (2.14), (2.15), and (2.16), we find that F_θ is Lipschitz continuous, with Lipschitz constant $\Phi(r,\theta)$, where

$$\Phi(r,\theta) \doteq K_1(r,\theta) + K_2(r,\theta) + \big(\omega^\alpha(1-\alpha)^{1-\alpha}+1\big)r\,K_5(r). \tag{2.27}$$

Therefore, if $r \le r_0$ and $|\theta| \le \theta_0$, where r_0, $\theta_0 > 0$ are so small that

$$\Phi(r_0,\theta_0) \le 1/2C_1, \tag{2.28}$$

where C_1 is given in (1.15), F_θ is a contraction with constant 1/2. Due to (1.15), we find also that if $\|x_1\|_D \le r/2C_1$, then F_θ maps $B(0,r) \subset Z^\alpha_\omega\big([0,+\infty[;\, D\big)$ into itself, so that it has a unique fixed point $z = z(\cdot, x_1, \theta)$ in $B(0,r)$, and z satisfies the inequality

$$\big\|z(\cdot,x_1,\theta)\big\|_{Z^\alpha_\omega\left([0,\,+\infty[;\,D\right)} \le 2C_1\|x_1\|_D \tag{2.29}$$

Now, we want to show that, if u_0 is sufficiently close to $\overline{u}(0)$, then there is $x_1 \in B(0, r_0/2C_1) \subset D$ and $\theta \in \mathbf{R}$ near 0 such that

$$u(t,u_0) = \overline{u}(t+\theta) + z(t,x_1,\theta) \quad \text{in } [0,\overline{t}], \quad \text{for some } \overline{t} > 0. \tag{2.30}$$

This will prove the theorem, since (2.30) defines an extension of $u(\cdot, u_0)$ in $[0,+\infty[$ satisfying estimate (2.23) with $M = 2C_1\|P_1(0)\|_{L(D)}$, thanks to (2.29) and to the equality $x_1 = P_1(0)z(0,x_1,\theta)$.

Equality (2.30) is equivalent to

$$u_0 = \overline{u}(\theta) + z(0,x_1,\theta)\,. \tag{2.31}$$

Actually, since $\|u(\cdot,u_0) - u_0\|_{Z^{\alpha/2}\left([0,\overline{t}];\,D\right)}$ and $\|\overline{u}(\cdot+\theta) - \overline{u}(0)\|_{Z^{\alpha/2}\left([0,\overline{t}];\,D\right)}$ converge to 0 as $\overline{t} \to 0$, then both members of (2.30) are solutions of (2.21) belonging to $Y = \big\{w \in C\big([0,\overline{t}];\, X\big) \cap C^1\big(]0,\overline{t}];\, X\big) \cap Z^{\alpha/2}\big([0,\overline{t}];\, D\big);\ w(0) = u_0,\ \|w(\cdot) - u_0\|_{Z^{\alpha/2}\left([0,\overline{t}];\,D\right)} \le \rho_0\big\}$, where ρ_0 is given by Proposition 2.1, with α replaced by $\alpha/2$, provided $\overline{t}$, r, θ, and $\|u_0 - \overline{u}(0)\|_D$ are sufficiently small. Then equality (2.30) follows from (2.31), due to Proposition 2.1 (ii).

To solve (2.31), let us write it in a convenient way. Let $\phi \in X^*$ be such that $P_2(0)x = \langle x,\phi\rangle\overline{u}'(0)$ for any $x \in X$. Recall that for any $t \in \mathbf{R}$, $X_2(t)$ is spanned by $\overline{u}'(t)$. Then (2.31) is equivalent to equation $G(x_1,\theta) = (x_1,\theta)$, where

$$\begin{cases} G : \big(X_1(0) \cap B(0,\epsilon) \subset D\big) \times [-\delta,\delta] \to X_1(0) \cap D \times \mathbf{R} \\ G(x_1,\theta) = \big(P_1(0)(u_0 - \overline{u}(\theta)),\ \langle u_0 - \overline{u}(\theta) - z(0,x_1,\theta) + \theta\overline{u}'(0), \phi\rangle\big) \end{cases} \tag{2.32}$$

and $\epsilon \in]0,\, r_0/2C_1]$, $\delta \in]0,\theta_0]$. Recall that r_0 and θ_0 are such that (2.28) holds. Now we prove that G is a contraction and maps $(X_1(0) \cap B(0,\epsilon)) \times [-\delta,\delta]$ into itself, provided ϵ, δ, and $\|u_0 - \overline{u}(t_0)\|_D$ are sufficiently small; $B(0,\epsilon) \times [-\delta,\delta]$ has the usual product norm. We have

$$\begin{aligned} G(x_1,\theta_1) - G(x_2,\theta_2) = \Big(&- P_1(0)\big(\overline{u}(\theta_1) - \overline{u}(\theta_2)\big)\,,\ \langle -\overline{u}(\theta_1) + \overline{u}(\theta_2) + (\theta_1-\theta_2)\overline{u}'(0) \\ &- z(0,x_1,\theta_1) + z_1(0,x_2,\theta_2),\ \phi\rangle\Big) \end{aligned}$$

and

$$
\begin{aligned}
&\left\|P_1(0)\big(\overline{u}(\theta_1)-\overline{u}(\theta_2)\big)\right\|_D\\
&=\left\|P_1(0)\int_0^1\left[\overline{u}'\big(\sigma\theta_1+(1-\sigma)\theta_2\big)-\overline{u}'(0)\right]d\sigma\,(\theta_1-\theta_2)\right\|_D\\
&\le \|P_1(0)\|_{L(D)}[\overline{u}']_{C^\alpha(\mathbf{R};D)}\,2^\alpha\,\delta^\alpha\,|\theta_1-\theta_2|,
\end{aligned}
\tag{2.33}
$$

$$
\begin{aligned}
&\left|\langle-\overline{u}(\theta_1)+\overline{u}(\theta_2)+(\theta_1-\theta_2)\overline{u}'(0),\,\phi\rangle\right|\\
&=\left|\int_0^1\langle\overline{u}'(0)-\overline{u}'\big(\sigma\theta_1+(1-\sigma)\theta_2\big),\,\phi\rangle\,d\sigma(\theta_1-\theta_2)\right|\\
&\le \|\phi\|_{X^*}\,[\overline{u}']_{C^\alpha(\mathbf{R};X)}\,2^\alpha\,\delta^\alpha\,|\theta_1-\theta_2|,
\end{aligned}
\tag{2.34}
$$

$$
\left|\langle z(0,x_2,\theta_2)-z(0,x_1,\theta_1),\,\phi\rangle\right|\le\|\overline{u}'(0)\|^{-1}\,\left\|P_2(0)\big[z(0,x_2,\theta_2)-z(0,x_1,\theta_1)\big]\right\| \tag{2.35}
$$

Using (2.24) and (1.15) we find, noting that $z(\cdot,x_i,\theta_i), i=1,2$, belongs to $B(0,r)\subset Z^\alpha_\omega\big([0,+\infty[;\,D\big)$ and $r=2C_1\,\epsilon\le r_0$, $|\theta_i|\le\theta_0$,

$$
\begin{aligned}
&\left\|z(\cdot,x_2,\theta_2)-z(\cdot,x_1,\theta_1)\right\|_{Z^\alpha_\omega([0,+\infty[;\,D)}\\
&\le C_1\Big(\|x_2-x_1\|_D+\|g(\cdot,z(\cdot,x_2,\theta_2),\theta_2)-g(\cdot,z(\cdot,x_1,\theta_1),\theta_1)\|_{Z^\alpha_\omega([0,+\infty[;\,X)}\Big)\\
&\le C_1\|x_2-x_1\|_D+\frac{1}{2}\|z(\cdot,x_2,\theta_2)-z(\cdot,x_1,\theta_1)\|_{Z^\alpha_\omega([0,+\infty[;\,D)}\\
&\quad+C_1\|g(\cdot,z(\cdot,x_2,\theta_2),\theta_2)-g(\cdot,z(\cdot,x_2,\theta_2),\theta_1)\|_{Z^\alpha_\omega([0,+\infty[;\,X)}
\end{aligned}
\tag{2.36}
$$

By (2.25) (iv) (v) and (2.26), we have:

$$
\begin{aligned}
&\left\|g\big(\cdot,z(\cdot,x_2,\theta_2),\theta_2\big)-g\big(\cdot,z(\cdot,x_2,\theta_2),\theta_1\big)\right\|_{Z^\alpha_\omega([0,+\infty[;\,X)}\le\\
&\quad K_3(r)|\theta_2-\theta_1|+\left[g\big(\cdot,z(\cdot,x_2,\theta_2),\theta_2\big)-g\big(\cdot,z(\cdot,x_2,\theta_2),\theta_1\big)\right]_{Z^\alpha([0,1];\,X)}\\
&\quad+\left[g\big(\cdot,z(\cdot,x_2,\theta_2),\theta_2\big)-g\big(\cdot,z(\cdot,x_2,\theta_2),\theta_1\big)\right]_{C^\alpha_\omega([1,+\infty[;\,X)}\\
&\le K_3(r)|\theta_2-\theta_1|+r\big(K_4(r)+K_6(r)\big)|\theta_2-\theta_1|\\
&+r\big(\omega K_3(r)+K_4(r)+K_6(r)(\omega^\alpha(1-\alpha)^{1-\alpha}+1)\big)|\theta_2-\theta_1|\\
&\doteq K_7(r)|\theta_2-\theta_1|
\end{aligned}
\tag{2.37}
$$

Then, by (2.36) and (2.37) we have

$$
\left\|z(\cdot,x_2,\theta_2)-z(\cdot,x_1,\theta_1)\right\|_{Z^\alpha_\omega([0,+\infty[;\,D)}\le 2C_1\big(\|x_2-x_1\|_D+K_7(r)|\theta_2-\theta_1|\big) \tag{2.38}
$$

so that, using again (2.24) and (1.15), (2.29), (2.37), (2.38), we find

$$
\begin{aligned}
&\left\|P_2(0)(z(0,x_2,\theta_2)-z(0,x_1,\theta_1)\right\|\\
&\le C_1\left\|g\big(\cdot,z(\cdot,x_2,\theta_2),\theta_2\big)-g\big(\cdot,z(\cdot,x_1,\theta_1),\theta_1\big)\right\|_{Z^\alpha_\omega([0,+\infty[;\,D)}\\
&\le C_1\left[\Phi(2C_1\epsilon,\,\delta)\big(2C_1\|x_2-x_1\|_D+K_7(r)|\theta_2-\theta_1|\big)+K_7(r)|\theta_2-\theta_1|\right]
\end{aligned}
\tag{2.39}
$$

where Φ is defined in (2.27). Using (2.33)–(2.35), (2.39) and recalling that $\lim_{r,\delta\to 0}\Phi(r,\delta) = 0$, $\lim_{r\to 0} K_7(r) = 0$, we find that G is a 1/2 contraction provided ϵ and δ are sufficiently small. Fixed such ϵ and δ, since

$$G(0,0) = \big(P_1(0)(u_0 - \overline{u}(0)),\, \langle u_0 - \overline{u}(0),\, \phi\rangle\big)$$

then, by (2.30), G maps $(B(0,\epsilon)\cap X_1(0)) \times [-\delta,\delta]$ into itself, provided

$$\|u_0 - \overline{u}(0)\|_D \le \max\left\{\big(2\|P_1(0)\|_{L(D)}\big)^{-1}\epsilon,\ \big(2\|\phi\|_{X^*}\big)^{-1}\delta\right\}.$$

In this case, G has a unique fixed point in $\big(B(0,\epsilon)\cap X_1(0)\big) \times [-\delta,\delta]$, and the proof is complete. ∎

Now we give a result of orbital instability, which is stronger than the one of Theorem 2.3. The proof is very similar to the one of [H, Theorem 8.2.4], so it is only sketched.

Theorem 2.5. *Assume the same hypotheses as in Theorem 2.4, except (2.22) which is replaced by*

$$\begin{cases} (i) & \sigma_2 \doteq \big\{\lambda \in \sigma(V(s)),\ |\lambda| > 1,\ s \in \mathbf{R}\big\} \ne \emptyset \\ (ii) & \inf\big\{|\lambda|;\ \lambda\in\sigma_2\big\} \doteq \rho_2 > 1 \end{cases} \tag{2.40}$$

Then $\overline{u}$ is orbitally unstable, i.e., there are $\delta > 0$ and a sequence $\{u_n\} \subset D$ such that $f(u_n) \in \overline{D}$, $dist(u_n,\Gamma) \to 0$ as $n \to +\infty$, but

$$\sup_{t\in[0,\,\tau(u_n)[} dist\big(u(t,u_n),\Gamma\big) \ge \delta.$$

Proof: Let $P_1(s)$, $P_2(s)$, and $X_1(s)$, $s \in \mathbf{R}$, be defined by (1.13), with any $\rho \in]1,\rho_2[$. Arguing as in Theorem 2.4 and using estimate (1.17) instead of (1.15), for every $x_2 \in X_2(0)$ sufficiently small, i.e., $\|x_2\|_D \le r/2C_3$, we can find a solution w of

$$w(t) = G(t,0)x_2 + \int_{-\infty}^{t} G(t,s)P_1(s)g(s,w(s),0)\,ds + \int_0^t G(t,s)P_2(s)g(s,w(s),0)\,ds, \quad t \le 0 \tag{2.41}$$

belonging to $B(0,r) \subset C^\alpha_\omega\big(]-\infty,0];\ D\big)$, provided r is sufficiently small (g is the same as in Theorem 2.4). Moreover, arguing as in the proof of (2.39), one can see that if $\|x_2\| \le r/2C_3$, then

$$\|w(0) - x_2\|_D \le K(r)\|x_2\|\,; \quad \lim_{r\to 0} K(r) = 0 \tag{2.42}$$

Now set

$$u(t) \doteq \overline{u}(t) + w(t)\,, \quad t \le 0 \tag{2.43}$$

Then u is a solution of (2.21) in $]-\infty,0]$ which converges exponentially to $\overline{u}$ as $t \to -\infty$. Let us show that the distance between $u(0)$ and Γ is positive if r is small. For any $t \in [0,T]$ we have

$$\begin{aligned}\|u(0) - \overline{u}(t)\|_D &= \|\overline{u}(0) + w(0) - \overline{u}(t)\|_D \\ &\ge \|x_2 - \overline{u}'(0)t\|_D - \|w(0) - x_2\|_D - \|\overline{u}(t) - \overline{u}(0) - \overline{u}'(0)t\|_D\end{aligned}$$

so that, using (2.42) and recalling that $\|x_2-\overline{u}'(0)t\|_D > 0$ because $x_2 \in X_2(0), \overline{u}'(0) \in X_1(0)$, there are ϵ, δ_1, r_1 such that if $r \leq r_1$ and $t \in [0,\epsilon] \cup [T-\epsilon, T]$, then $\|u(0)-\overline{u}(t)\|_D \geq \delta_1$. If t belongs to $[\epsilon, T-\epsilon]$, we have

$$\|u(0)-\overline{u}(t)\|_D = \|\overline{u}(0)+w(0)-\overline{u}(t)\|_D \geq \|\overline{u}(0)-\overline{u}(t)\|_D - \|w(0)\|_D \geq \|\overline{u}(0)-\overline{u}(t)\|_D - r$$

so that there are r_2, δ_2 such that if $r \leq r_2$, then $\|u(0)-\overline{u}(t)\|_D \geq \delta_2$ for $\epsilon \leq t \leq T-\epsilon$. Therefore $\text{dist}(u(0),\Gamma) \geq \delta = \min\{\delta_1,\delta_2\}$ if r is small. The statement of the theorem follows now taking $u_n = u(-n)$, $\delta = \min\{\delta_1,\delta_2\}$. ∎

Remark 2.6. We assumed that $f(t,x)$ in subsection A, $f(x)$ in subsection B, are defined for any $x \in D$ and satisfy (2.2), (2.3) in the whole space D. This was done just to simplify notations. In fact we could assume that they are defined and satisfy (2.2), (2.3) only for x belonging to some neighborhood of the orbit $\Gamma = \{\overline{u}(t),\ t \in \mathbf{R}\}$.

3. Examples and Applications In this section we give examples of fully nonlinear abstract evolution equations having periodic solutions, whose stability properties are studied using the results of Section 2. The usual bifurcation methods, both in the autonomous and in the nonautonomous case are employed. These methods work in spite of the strong nonlinearities thanks to maximal regularity properties stated in §1.

(A) The nonautonomous case. We consider a family of nonautonomous equations, depending on a real parameter λ, under nonresonance assumptions.

$$u'(t) = f(\lambda, t, u(t)), \tag{3.1}$$

where $f : [-1,1] \times \mathbf{R} \times D \to X$, $(\lambda,t,x) \to f(\lambda,t,x)$ satisfies the following:

$$\begin{cases} (a) & \text{(Regularity) } f(\cdot,t,\cdot) \text{ belongs to } C^3\big([-1,1]\times D; X\big) \text{ for any} \\ & t \in \mathbf{R}, \text{ and all the partial derivatives of } f(\cdot,t,\cdot) \text{ up to} \\ & \text{order 3 are } \alpha\text{-Hölder continuous in } t, \text{ locally uniformly} \\ & \text{with respect to the other variables} \\ (b) & \text{(Periodicity) There is } T>0 \text{ such that } f(\lambda,t,x) = f(\lambda,t+T,x) \\ & \text{for any } \lambda \in [-1,1],\ t \in \mathbf{R},\ x \in D \\ (c) & \text{(Parabolicity and nonresonance) The family } \{B(t);\ t \in \mathbf{R}\}, \\ & B(t) = f_x(0,t,0) \text{ satisfies the assumptions of Proposition 1.2} \\ (d) & f(0,t,0) = 0 \text{ for any } t \in \mathbf{R} \end{cases} \tag{3.2}$$

Theorem 3.1. *Under the assumptions (3.2), there exist $\lambda_0 > 0$ and $r_0 > 0$ such that for any $\lambda \in [-\lambda_0, \lambda_0]$ equation (3.1) has a unique T-periodic strict solution u satisfying the inequality*

$$\|u\|_{C^\alpha(\mathbf{R};D)} + \|u'\|_{C^\alpha(\mathbf{R};X)} \leq r_0 \tag{3.3}$$

Proof: Consider the Banach spaces

$$\begin{cases} Y = \{u \in C^\alpha(\mathbf{R};D) \cap C^{1,\alpha}(\mathbf{R};X);\ \ u(t) = u(t+T) \quad \text{for any } \ t \in \mathbf{R}\} \\ Z = \{v \in C^\alpha(\mathbf{R};X);\ \ v(t) = v(t+T) \quad \text{for any } \ t \in \mathbf{R}\} \end{cases}$$

and define a mapping $F : [-1,1] \times Y \to Z$ by

$$F(\lambda, u) = u' - f(\lambda, \cdot, u(\cdot)) \tag{3.4}$$

Obviously, a function $u \in Y$ is a solution of (3.1) if and only if $F(\lambda, u) = 0$. Since $F(0,0) = 0$ by (3.2)(d), we have only to check that

(i) F belongs to $C^1([-1,1] \times Y;\, Z)$

(ii) $F_u(0,0)$ is an isomorphism from Y onto Z.

Concerning (i), one can easily prove that there exist the Gateaux derivatives

$$F_\lambda(\lambda, u) = -f_\lambda(\lambda, \cdot, u(\cdot)) \tag{3.5}$$

$$F_u(\lambda, u)v = v' - f_x(\lambda, \cdot, u(\cdot))v \tag{3.6}$$

and they are, in addition, continuous in $[-1,1] \times Y$. The proof follows by straightforward (but tedious) computations, involving assumption (3.2)(a). Let us check, for instance, (3.6). To this end it is sufficient to prove that the Gateaux derivative of $u \to f(\lambda, \cdot, u(\cdot))$ at the point u is the linear mapping $v \to f_x(\lambda, \cdot, u(\cdot))v$. Actually, we have

$$\begin{aligned} &h^{-1}\big[f\big(\lambda, t, u(t) + hv(t)\big) - f\big(\lambda, t, u(t)\big)\big] - f_x\big(\lambda, t, u(t)\big)v(t) \\ &= h \int_0^1 d\tau \int_0^1 d\sigma \big[\sigma f_{xx}\big(\lambda, t, u(t) + h\tau\sigma v(t)\big)\big(v(t), u(t)\big)\big] \\ &\doteq \phi_h(t) \end{aligned}$$

and ϕ_h converges to 0 in the C^α norm as $h \to 0$, thanks to assumption (3.2)(a).

Let us show (ii). By (3.6) we have $F_u(0,0)v = v' - A(\cdot)v$, where $A(\cdot) = f_x(0, \cdot, 0)$ satisfies (3.2)(c). Then (ii) follows from Proposition 1.2. ∎

Example 3.2. Consider the problem of existence and stability of periodic solutions to

$$\begin{cases} u_t(t,x) = \phi\left(\Delta u(t,x)\right) + \psi(t)u(t,x) + \lambda\eta(t,x), & t \in \mathbf{R},\ x \in \overline{\Omega} \\ u(t,x) = 0, \quad t \in \mathbf{R},\ x \in \partial\Omega \end{cases} \tag{3.7}$$

where $\lambda \in \mathbf{R}$, Ω is a bounded open set in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, and there are $\alpha \in]0,1[$, $T > 0$ such that

$$\begin{cases} (i) & \phi \in C^3(\mathbf{R}),\ \phi(0) = 0,\ \phi'(0) > 0 \\ (ii) & \psi \in C^{\alpha/2}(\mathbf{R}),\ \psi(t) = \psi(t+T) \quad \text{for any } t \in \mathbf{R} \\ (iii) & \eta \in C^{\alpha/2,\alpha}(\mathbf{R} \times \overline{\Omega}),\ \eta(t,x) = \eta(t+T,x) \text{ for any } t \in \mathbf{R},\ x \in \overline{\Omega} \end{cases} \tag{3.8}$$

We recall that $\eta \in C^{\alpha/2,\alpha}(\mathbf{R} \times \overline{\Omega})$ means that

$$\sup\left\{\left(|t-s|^{-\alpha/2} + |x-y|^{-\alpha}\right)|\eta(t,x) - \eta(s,y)|,\ t,s \in \mathbf{R},\ t \neq s,\ x,y \in \overline{\Omega},\ x \neq y\right\} < +\infty.$$

Applying theorems 2.9 and 3.1 gives the following results.

Proposition 3.3. *Under hypotheses (3.8), assume in addition that*

$$\int_0^T \psi(s)\,ds + T\phi'(0)\mu \neq 0 \tag{3.9}$$

for any eigenvalue μ of the Laplace operator Δ with Dirichlet boundary conditions. Then there are $\lambda_0 > 0$, $r_0 > 0$ such that for any $\lambda \in [-\lambda_0, \lambda_0]$ there is a unique solution u of (3.7) such that

$$\begin{cases} \overline{u}(t,x) = \overline{u}(t+T,x), \quad t \in \mathbf{R}, \quad x \in \overline{\Omega} \\ \overline{u}(t,\cdot) \in C^{2,\alpha}(\overline{\Omega}) \quad \text{uniformly for } t \in \mathbf{R} \\ \overline{u}(\cdot,x) \in C^{1,\alpha/2}(\mathbf{R}) \quad \text{uniformly for } x \in \overline{\Omega} \\ \sup\limits_{x\in\overline{\Omega}} \|u(\cdot,x)\|_{C^\alpha(\mathbf{R})} + \sup\limits_{x\in\overline{\Omega}} \|\Delta u(\cdot,x)\|_{C^\alpha(\mathbf{R})} \leq r_0 \end{cases} \tag{3.10}$$

Moreover, if $\psi(t) \leq 0$ for any t, then $\overline{u}$ is exponentially asymptotically stable, in the sense that there exist $M > 0$, $r > 0$ such that if there are $t_0 \in \mathbf{R}$, $\lambda \in [-\lambda_0, \lambda_0]$, $u_0 \in C^{2,\alpha}(\overline{\Omega})$ with

$$\begin{cases} (a) \quad u_0(x) = \phi(\Delta u_0(x)) + \psi(t_0)u_0(x) + \lambda\eta(t_0,x) = 0 \quad \text{for any } x \in \partial\Omega \\ (b) \quad \sup\limits_{x\in\overline{\Omega}} |u_0(x) - \overline{u}(\lambda,t_0,x)| \leq r, \quad \sup\limits_{x\in\overline{\Omega}} |\Delta u_0(x) - \Delta\overline{u}(\lambda,t_0,x)| \leq r \end{cases} \tag{3.11}$$

then problem (3.7) has a unique solution $u : [t_0, +\infty[\times\overline{\Omega} \to \mathbf{R}$ such that

$$\begin{cases} u(t_0,x) = u_0(x), \quad x \in \overline{\Omega} \\ u(t,\cdot) \in C^{2,\alpha}(\overline{\Omega}) \quad \text{uniformly for } t \in \mathbf{R} \\ u(\cdot,x) \in C^{1,\alpha/2}(\mathbf{R}) \quad \text{uniformly for } x \in \overline{\Omega} \end{cases} \tag{3.12}$$

and moreover

$$\begin{aligned} &\sup_{x\in\overline{\Omega}} |u(t,x) - \overline{u}(\lambda,t,x)| + \sup_{x\in\overline{\Omega}} |\Delta u(t,x) - \Delta\overline{u}(\lambda,t,x)| \\ &\quad \leq Me^{-\omega(t-t_0)}\Big(\sup_{x\in\overline{\Omega}} |u_0(x) - \overline{u}(\lambda,t_0,x)| + \sup_{x\in\overline{\Omega}} |\Delta u_0(x) - \Delta\overline{u}(\lambda,t_0,x)|\Big) \end{aligned} \tag{3.13}$$

Proof: Let $X = C(\overline{\Omega})$ be endowed with the sup norm $\|\cdot\|_\infty$, and let $D = \{\xi \in C(\overline{\Omega});\ \Delta\xi \in C(\overline{\Omega}),\ \xi_{|\partial\Omega} = 0\}$ be endowed with the graph norm $\|\xi\|_D \doteq \|\xi\|_\infty + \|\Delta\|_\infty$ (the Laplace operator Δ is in the distributional sense). Set

$$f(\lambda,t,\xi) = \phi(\Delta\xi(\cdot)) + \psi(t)\xi(\cdot) + \lambda\eta(t,\cdot), \ \lambda \in [-1,1], \ t \in \mathbf{R}, \ \xi \in D \tag{3.14}$$

Then $f : [-1,1] \times \mathbf{R} \times D \to X$ satisfies (3.2)(a), and

$$f_\xi(\lambda,t,\xi)\varsigma = \phi'(\Delta\xi(\cdot))\varsigma(\cdot) + \psi(t)\varsigma(\cdot), \ \lambda \in [-1,1], t \in \mathbf{R}, \ \xi,\varsigma \in D \tag{3.15}$$

For any $\lambda \in [-1,1]$, $t \in \mathbf{R}$ and small $\xi \in D$, $f_\xi(\lambda,t,\xi)$ is an elliptic operator, thanks to (3.8)(i), with continuous coefficients, so that (see [St]) its complexification satisfies (1.5). Setting $B(t) : D \to X$, $B(t) = f_\xi(0,t,0)$, then the function $\mathbf{R} \to L(D,X)$, $t \to B(t)$

is T-periodic and $\alpha/2$-Hölder continuous thanks to (3.8)(ii). To fulfill the assumptions of Theorem 3.1, it remains to check that 1 belongs to the resolvent set of $J(T,0)$, where $J(t,s)$ is the evolution operator associated to the family $\{B(t);\ t\in\mathbf{R}\}$. Denoting by $B: D\to X$ the Laplace operator Δ, and donoting by e^{tB} the semigroup generated by B in X, we have

$$J(T,0)=e^{\int_0^T\psi(s)\,ds}e^{T\phi'(0)B}.$$

Then the condition $1\in\rho(J(T,0))$ is satisfied if and only if (3.9) holds. Existence and uniqueness of a small T-periodic solution $\overline{u}(\lambda,t,x)$ of (3.7) follows now by Theorem 3.1. Concerning the regularity properties of $\overline{u}$, it is sufficient to remark that, since $v(t)=\overline{u}(\lambda,t,\cdot)$ belongs to $C^{1,\alpha/2}(\mathbf{R};X)\cap C^{\alpha/2}(\mathbf{R};D)$, then $v'(t)=\overline{u}_t(\lambda,t,\cdot)$ is bounded with values in $D_{B(0)}(\alpha/2,\infty)$, (see lemma 1.1 of [L1]). In our case we have $D_{B(0)}(\alpha/2,\infty)=D_\Delta(\alpha/2,\infty)=\{\xi\in C^\alpha(\overline{\Omega});\ \xi_{|\partial\Omega}=0\}$ (see [L5]). This implies, if r_0 is sufficiently small, i.e., such that $\phi'(x)\neq 0$ for $-r_0\le x\le r_0$, that $\Delta\overline{u}(\lambda,t,\cdot)$ is bounded in $C^{2,\alpha}(\overline{\Omega})$, and, by Schauder's Theorem, that $\overline{u}(\lambda,t,\cdot)$ belongs to $C^{2,\alpha}(\overline{\Omega})$ and $\sup_{t\in\mathbf{R}}\|\overline{u}(\lambda,t,\cdot)\|_{C^{2,\alpha}(\overline{\Omega})}<+\infty$.

Let us prove now the stability property of $\overline{u}$ in the case $\psi(t)\le 0$. Setting $A(t)\xi=\phi'(\Delta\overline{u}(\lambda,t,\cdot))\Delta\xi+\psi(t)\xi$ for $t\in\mathbf{R}$, $\xi\in D$ (see (3.15)), let $G(t,s)$ $(t\ge s)$ be the evolution operator associated to the family $\{A(t);\ t\in\mathbf{R}\}$, and let $V(s)=G(s+T,s)$. Since the inclusion $D\hookrightarrow X$ is compact, then for every $s\in\mathbf{R}$, $V(s)$ is a compact operator, so that its spectrum, except the point 0, consists of eigenvalues, not depending on s. If ξ is an eigenvector of $V(0)$ with eigenvalue μ, then $V(0)\xi=\mu\xi=v(T,\cdot)$, where $v(t,x)$ is the classical solution of

$$\begin{cases} v_t(t,x)=\phi'(\Delta\overline{u}(\lambda,t,x))\Delta v(t,x)+\psi(t)v(t,x), & t\ge 0,\ x\in\overline{\Omega}\\ v(0,x)=\xi(x), & x\in\overline{\Omega}\\ v(t,x)=0, & t\ge 0,\ x\in\partial\Omega\end{cases}$$

Since $\phi'(\Delta\overline{u}(\lambda,t,x))>0$, if $\psi(t)\le 0$ for any t, then the parabolic maximum principle holds, so that $\|v(T,\cdot)\|_\infty\le\|\xi\|_\infty$ and $v(T,\cdot)$ cannot coincide with ξ. Using again the maximum principle, we can show that $V(0)$ is a positive operator, so that its spectral radius is an eigenvalue. But 1 is not an eigenvalue, so that $\rho_1\doteq\sup\{|\lambda|;\ \lambda\in\sigma(V(s)),\ s\in\mathbf{R}\}<1$, and Theorem 2.2 may be applied to find (3.13). We recall that $\overline{D}=\{\xi\in C(\overline{\Omega});\ \xi_{|\partial\Omega}=0\}$, so that condition (3.11)(a) implies $f(\lambda,t_0,u_0)\in\overline{D}$.

The regularity properties (3.12) of u may be shown as the corresponding ones of $\overline{u}(\lambda,t,x)$.

(B) The autonomous case. We recall here a Hopf bifurcation theorem for fully nonlinear evolution equations

$$u'(t)=f(\lambda,u(t)) \tag{3.16}$$

where

$$\begin{cases} (a) & f\in C^\infty([-1,1]\times D;\ X);\ \ f(\lambda,0)=0\ \text{ for }\ -1\le\lambda\le 1\\ (b) & \text{for any } \lambda\in[-1,1], \text{ the linear operator } A(\lambda): D\to X,\\ & A(\lambda)\doteq f_x(\lambda,0)\ \text{ satisfies (1.5).}\end{cases} \tag{3.17}$$

If X and D are real Banach spaces, we denote by $\tilde{D} = \{x + iy;\ x, y \in D\}$ and $\tilde{X} = \{x + iy;\ x, y \in X\}$ their complexifications, and we denote by $\tilde{A}(\lambda) : \tilde{D} \to \tilde{X}$ the complexification of $A(\lambda)$, defined by $\tilde{A}(\lambda)(x+iy) = A(\lambda)x + iA(\lambda)y$. We assume the Hopf bifurcation hypotheses:

$$\begin{cases} (a) & \pm i \text{ are simple isolated eigenvalues of } \tilde{A}(0) \\ (b) & 1 \text{ is a semisimple isolated eigenvalue of } e^{2\pi\tilde{A}(0)} \\ & \text{with algebraic multiplicity 2.} \end{cases} \tag{3.18}$$

Due to assumption (3.18)(a), there are $\lambda_0 \in]0,1]$ and two C^∞ functions $[-\lambda_0, \lambda_0] \to \mathbf{R}$, $\lambda \to \alpha(\lambda)$, $\lambda \to \beta(\lambda)$, such that

$$\begin{cases} \alpha(0) = 0, \quad \beta(0) = 1 \\ \alpha(\lambda) \pm i\beta(\lambda) \text{ are simple isolated eigenvalues of } \tilde{A}(\lambda) \end{cases} \tag{3.19}$$

The usual transversality assumption is

$$\alpha'(0) \neq 0 \tag{3.20}$$

The following theorem has been proved in [DPL].

Theorem 3.4. *Let (3.17), (3.18), and (3.20) hold, and fix $\gamma \in]0,1[$. Then there exist $\sigma_0 > 0$ and $\lambda : [-\sigma_0, \sigma_0] \to \mathbf{R}$, $\sigma \to \lambda(\sigma)$; $\rho : [-\sigma_0, \sigma_0] \to \mathbf{R}$, $\sigma \to \rho(\sigma)$; $u : [-\sigma_0, \sigma_0] \to C^\gamma(\mathbf{R}; D) \cap C^{1,\gamma}(\mathbf{R}; X)$, $\sigma \to u(\sigma)(\cdot)$, such that*

$$\begin{cases} \lambda(0) = 0, \ \rho(0) = 1, \ u(0)(t) = 0 \text{ for any } t \in \mathbf{R} \\ \text{for } \sigma \neq 0, \ u(\sigma)(\cdot) \text{ is a } 2\pi\rho(\sigma)\text{-periodic and} \\ \text{nonconstant solution of (3.16) with } \lambda = \lambda(\sigma) \end{cases} \tag{3.21}$$

Moreover there is $\epsilon_0 > 0$ such that if $\overline{\lambda} \in [-1,1]$, $\overline{\rho} \in \mathbf{R}$ and $\overline{u} \in C^\gamma(\mathbf{R}; D) \cap C^{1,\gamma}(\mathbf{R}; X)$ is a $2\pi\overline{\rho}$-periodic function verifying

$$\begin{cases} \overline{u}'(t) = f(\overline{\lambda}, \overline{u}(t)), \quad t \in \mathbf{R} \\ |\overline{\lambda}| \leq \epsilon_0, \ |1 - \overline{\rho}| \leq \epsilon_0, \ \|\overline{u}\|_{C^\gamma(\mathbf{R};D)} + \|\overline{u}'\|_{C^\gamma(\mathbf{R};X)} \leq \epsilon_0 \end{cases} \tag{3.22}$$

then there exist $\theta \in \mathbf{R}$, $\sigma \in [-\sigma_0, \sigma_0]$ such that

$$\overline{\lambda} = \lambda(\sigma), \ \overline{\rho} = \rho(\sigma), \ \overline{u}(t) = u(\sigma)(t + \theta), \ t \in \mathbf{R}. \quad \blacksquare \tag{3.23}$$

To recognize the stability properties of $u(\sigma)$, we have to know the spectrum of $V(s) = G(s + 2\pi\rho(\sigma), s)$, where $G(t,s)$ is the evolution operator associated to the family $A(t) = f_x(\lambda(\sigma), u(\sigma)(t))$, $t \in \mathbf{R}$. If $V(s)$ is a compact operator; in particular, if the inclusion $D \hookrightarrow X$ is compact, then its spectrum, except at most the point 0, consists of eigenvalues not depending on s. It is easy to see that $z \neq 0$ is an eigenvalue of $\tilde{V}(0)$ if and only if the problem

$$w'(t) = f_x\big(\lambda(\sigma), u(\sigma)(t)\big)w(t) - kw(t), \quad t \in \mathbf{R} \tag{3.24}$$

has a nontrivial $2\pi\rho(\sigma)$-periodic solution for $k = (2\pi\rho(\sigma))^{-1}\log z$. Such a k is called a Floquet exponent. It may be shown that, for σ small, the spectrum of $\tilde{V}(0)$ is close to the spectrum of $e^{2\pi\tilde{A}(0)}$. In particular, if we assume that $\sigma(e^{2\pi\tilde{A}(0)})\backslash\{1\}$ is far from the unit circle, then, if $|\sigma|$ is sufficiently small, $\tilde{V}(0)$ has two eigenvalues near 1, and the other elements are far from the unit circle. As we have already seen, 1 is an eigenvalue of $V(0)$, and hence of $\tilde{V}(0)$. We can solve our stability problem if we know that the other eigenvalue has modulus less than or more than 1, or equivalently, if the corresponding Floquet exponent $k = k(\sigma)$ has positive or negative real part. The following lemma states that $k(\sigma)$ is real for $|\sigma|$ small, and gives information about the sign of $k(\sigma)$.

Lemma 3.5. *Under the assumptions of Theorem 3.4 there are $\sigma_1 > 0$ and a continuous function $[-\sigma_1, \sigma_1] \to \mathbf{R}$, $\sigma \to k(\sigma)$, such that*

$$\begin{cases} (i) \quad k(0) = 0 \\ (ii) \quad \text{for any } \sigma \in [-\sigma_1, \sigma_1], \text{ problem (3.24) has a nontrivial} \\ \qquad 2\pi\rho(\sigma)\text{-periodic solution with } k = k(\sigma) \end{cases} \tag{3.25}$$

and

$$\left|k(\sigma) + \alpha'(0)\sigma\lambda'(\sigma)\right| = \gamma(\sigma)|\sigma\lambda'(\sigma)| \tag{3.26}$$

with $\lim_{\sigma\to 0}\gamma(\sigma) = 0$. ∎

The proof is very similar to the corresponding one in the semilinear case (see [CR]), so it will be only sketched in the appendix. Formula (3.25) implies that there is a neighborhood of $\sigma = 0$ in which $k(\sigma)$ and $\sigma\lambda'(\sigma)$ have the same zeroes, and in which $-k(\sigma)$ and $\alpha'(0)\sigma\lambda'(\sigma)$ have the same sign, if they do not vanish. Computing the sign of $\alpha'(0)\sigma\lambda'(\sigma)$ is tedious but not difficult, as we will see in example 3.7.

First we give an example of unstable periodic solutions of fully nonlinear parabolic equations.

Example 3.6. Let us consider the equation treated in [DPL]

$$u_t(t,x) = \phi\big(\lambda, u(t,x),\, u_x(t,x),\, u_{xx}(t,x)\big) \tag{3.27}$$

We are looking for periodic solutions of (3.27), both with respect to time and with respect to space. To this aim we choose

$$\begin{cases} X = \{\xi : \mathbf{R} \to \mathbf{R} \text{ continuous; } \xi(x) = \xi(x+2\pi) \text{ for } x \in \mathbf{R}\} \\ D = C^2(\mathbf{R}) \cap X \end{cases} \tag{3.28}$$

We assume that $(\lambda, p) \to \phi(\lambda, p)$ belongs to $C^\infty([-1,1] \times \mathbf{R}^3; \mathbf{R})$ and $\phi(\lambda, 0) = 0$, so that the function $f(\lambda, u) = (\lambda, u, u', u'')$ belongs to $C^\infty([-1,1] \times D;\, X)$ and (3.17)(a) is satisfied. Moreover, we assume the parabolicity condition

$$\phi_{p_3}(\lambda, 0) > 0 \tag{3.29}$$

and the existence of $h \in \mathbf{Z}$ such that

$$\begin{cases} (a) \quad \phi_{p_1}(0,0) = h^2\phi_{p_3}(0,0), \quad h\phi_{p_2}(0,0) = 1 \\ (b) \quad \phi_{p_1\lambda}(0,0) \neq h^2\phi_{p_3\lambda}(0,0) \end{cases} \tag{3.30}$$

Setting $A(\lambda) = f_u(\lambda, 0)$, we have

$$A(\lambda)\xi = \phi_{p_1}(\lambda,0)\xi + \phi_{p_2}(\lambda,0)\xi' + \phi_{p_3}(\lambda,0)\xi''$$

for any $\xi \in D$, so that the spectrum of $\tilde{A}(\lambda)$ consists of the eigenvalues

$$\mu_n = \phi_{p_1}(\lambda,0) + in\phi_{p_2}(\lambda,0) - n^2\phi_{p_3}(\lambda,0), \quad n \in \mathbf{Z} \tag{3.31}$$

By (3.30)(a), $\tilde{A}(0)$ has a couple of conjugate eigenvalues on the imaginary axis, and no other purely imaginary eigenvalue. Since $(\tilde{A}(0))^{-1}$ is a compact operator and $A(0)$ generates an analytic semigroup thanks to (3.29), condition (3.18)(b) holds. Then Theorem 3.5 is applicable, so that equation (3.27) has small periodic solutions for suitable values of λ near 0. But these solutions are orbitally unstable, because (3.29) and (3.30)(a) imply $\phi_{p_1}(0,0) > 0$, so that $\tilde{A}(0)$ has a positive eigenvalue for $n = 0$. This implies that $e^{tA(0)}$ has an eigenvalue with modulus greater than 1 for any $t > 0$, and so does $V(t)$, if λ is sufficiently close to 0, (we use the notation of Theorem 2.5). Then Theorem 2.5 applies. ∎

We finish the paper with an example of a nonlinear parabolic system having stable time-periodic solutions.

Example 3.7. Let $\Phi : [-1,1]\times\mathbf{R}^3 \to \mathbf{R}$, $(\lambda, p_1, p_2, p_3) \to \Phi(\lambda, p_1, p_2, p_3)$, $\Psi : [-1,1]\times\mathbf{R}^2 \to \mathbf{R}$, $(\lambda, p_1, p_2) \to \Psi(\lambda, p_1, p_2)$ be C^∞ functions, and consider the system

$$\begin{cases} u_t(t,x) = \Phi\big(\lambda, u(t,x), v(t,x), u_{xx}(t,x)\big), & t \in \mathbf{R},\ 0 \le x \le \pi \\ v_t(t,x) = \Psi\big(\lambda, u(t,x), v(t,x)\big), & t \in \mathbf{R},\ 0 \le x \le \pi \\ u(t,0) = u(t,\pi) = 0, & t \in \mathbf{R} \end{cases} \tag{3.32}$$

We assume that (3.32) has the stationary solution $(u,v) = (0,0)$ and that (3.32) is parabolic near $(u,v) = (0,0)$; i.e.,

$$\Phi(\lambda,0) = \Psi(\lambda,0) = 0; \quad \Phi_{p_3}(\lambda,0) > 0, \quad -1 \le \lambda \le 1 \tag{3.33}$$

We choose $X = C\big([0,\pi]; \mathbf{R}\big) \times C\big([0,\pi]; \mathbf{R}\big)$ and $D = \big\{(\xi,\eta) \in C^2\big([0,\pi]; \mathbf{R}\big) \times C\big([0,\pi]; \mathbf{R}\big);\ \xi(0) = \xi(\pi) = 0\big\}$; X and D are endowed with the product norm. The function $f : [-1,1]\times D \to X$ is defined by $f(\lambda,(\xi,\eta))(x) = (\Phi(\lambda,\xi(x),\eta(x),\xi''(x)),\ \Psi(\lambda,\xi(x),\eta(x)))$ and belongs obviously to $C^\infty([-1,1]\times D; X)$. An easy computation shows that the spectrum of $\tilde{A}(\lambda) = \tilde{f}_{(\xi,\eta)}(\lambda,0)$ consists of the point $\mu_0(\lambda) = \Psi_{p_2}(\lambda,0)$ and of the eigenvalues $\mu_{\pm n}$, $n \in \mathbf{N}$, given by

$$\begin{aligned} \mu_{\pm n}(\lambda) = \frac{1}{2}\Big\{\Psi_{p_2} + \Phi_{p_1} - n^2\Phi_{p_3} \pm \big[\big(\Psi_{p_2} + \Phi_{p_1} - n^2\Phi_{p_3}\big)^2 \\ - 4\big(\Phi_{p_1}\Psi_{p_2} - \Phi_{p_2}\Psi_{p_1} - n^2\Phi_{p_3}\Psi_{p_2}\big)\big]^{1/2}\Big\} \end{aligned} \tag{3.34}$$

where the derivatives Φ_{p_j}, Ψ_{p_j} are evaluated at $(\lambda,0)$. Thanks to assumption $\Phi_{p_3}(\lambda,0) > 0$, it is easy to check that each $\tilde{A}(\lambda)$ satisfies (1.5). The Hopf bifurcation assumptions (3.18) are satisfied if there is $h \in \mathbf{N}$ such that

$$\begin{cases} (a) & \Psi_2 + \Phi_1 = h^2\Phi_3;\quad \Phi_1\Psi_2 - \Phi_2\Psi_1 = h^2\Phi_3\Psi_1 + 1 \\ (b) & \Psi_2 \ne 0 \end{cases} \tag{3.35}$$

Here and in the following we set for brevity $\Phi_j = \Phi_{p_j}(0,0)$, $\Phi_{j\lambda} = \Phi_{p_j\lambda}(0,0)$, $j = 1,2,3$, and so on. Actually, (3.35)(a) implies obviously (3.18)(a), and (3.35)(b) implies that $\sigma(\tilde{A}(0))\backslash\{+i,-i\}$ is far from the imaginary axis. Set now $\tilde{X} = X_0 \oplus X_1 \oplus X_2$, where $X_j = P_j(\tilde{X})$, $j = 0,1,2$, and

$$P_0 = \frac{1}{2\pi i}\int_{\gamma_0}(z-\tilde{A}(0))^{-1}\,dz,\;\; P_2 = \frac{1}{2\pi i}\int_{\gamma_2}(z-\tilde{A}(0))^{-1}\,dz,\;\; P_1 = 1-P_0-P_2 \qquad (3.36)$$

γ_0 and γ_2 are suitable paths around $\{+i,-i\}$ and $\sigma(\tilde{A}(0))\cap\{z\in\mathbf{C};\ \text{Re}\ z>0\}$ respectively. Then $\tilde{A}(0) = A_0 \oplus A_1 \oplus A_2$, where $A_j = \tilde{A}(0)P_j$. Since $\sup\{\text{Re}\ z;\ z\in\sigma(A_1)\}<0$ and $\inf\{\text{Re}\ z;\ z\in\sigma(A_2)\}>0$, then there are $M,\omega>0$ such that $\|e^{tA_1}\|_{L(X)}\le Me^{-\omega t}$ and (if $X_2\neq\{0\}$) $\|e^{tA_2}\|_{L(X)}\ge Me^{\omega t}$ for any $t>0$. In particular, the spectral radius of $e^{2\pi A_1}$ (respectively $e^{2\pi A_2}$) is less than 1 (respectively greater than 1), so that 1 belongs to the resolvent set of both A_1 and A_2. Since 1 is a semisimple eigenvalue of A_0 with multiplicity 2, (3.18)(b) follows. The transversality condition (3.20) holds if

$$\Psi_{2\lambda}+\Phi_{1\lambda}\neq h^2\Phi_{3\lambda} \qquad (3.37)$$

Therefore, under assumptions (3.33), (3.35), (3.37), the hypotheses of Theorem 3.4 are satisfied, and system (3.32) has small time periodic solutions (u,v) for suitable values of λ near 0. We do not expect they are stable if h (given in (3.35)) is greater than 1, because in this case we get Re $\mu_1>0$. (μ_1 is given in (3.34), see also example 3.6). Then from now on we assume

$$\begin{cases} (a) & \Psi_2+\Phi_1=\Phi_3,\quad \Phi_2\Psi_1+\Psi_2^2+1=0 \\ (b) & \Psi_2<0 \\ (c) & \Psi_{2\lambda}+\Phi_{1\lambda}\neq\Phi_{3\lambda}\end{cases} \qquad (3.38)$$

so that the spectrum of $e^{2\pi\tilde{A}(0)}$, except the point 1, is strictly inside the unit circle. Our goal is to compute the sign of $\alpha'(0)\sigma\lambda'(\sigma)$ for σ near 0, see Lemma 3.5. The linear space X_0 is spanned by $a_0\pm ib_0$, with

$$a_0(x)=\big(\sin x,\ -\Psi_1\Psi_2/(1+\Psi_2^2)\sin x\big),\ \ b_0(x)=\big(0,\ -\Psi_1/(1+\Psi_2^2)\sin x\big) \qquad (3.38)$$

In [DPL] it is shown that the solution $u(\sigma)$ of (3.15) is given by

$$u(\sigma)(t)=\sigma\big[v(\sigma)(t/\rho(\sigma))+\exp(t/\rho(\sigma)A(0))a_0\big] \qquad (3.40)$$

where $\lambda(\sigma)$, $\rho(\sigma)$, $v(\sigma)$ are the solutions, near $\sigma=0$, $\lambda=0$, $\rho=1$, $v=0$, of $G(\sigma,\lambda,\rho,v)=0$, with

$$\begin{cases} G:[-1,1]\times[-1,1]\times[0,2]\times V\to\{v\in C^\gamma(\mathbf{R};X);\ v(t)=v(t+2\pi),\ t\in\mathbf{R}\} \\ G(\sigma,\lambda,\rho,v)=\begin{cases}\frac{1}{\sigma}\big[\sigma\frac{d}{dt}(e^{t\tilde{A}(0)}a_0+v)-\rho f\big(\lambda,\sigma(e^{t\tilde{A}(0)}a_0+v)\big)\big] & \text{if } \sigma\neq0 \\ \frac{d}{dt}(e^{t\tilde{A}(0)}a_0+v)-\rho f_u(\lambda,0)(e^{t\tilde{A}(0)}a_0+v) & \text{if } \sigma=0\end{cases}\end{cases} \qquad (3.41)$$

and

$$V = \Big\{ v \in C^\gamma(\mathbf{R};\, D) \cap C^{1,\gamma}(\mathbf{R};\, X);\quad v(t) = v(t+2\pi),$$
$$\int_0^{2\pi} \langle e^{(2\pi - s)A(0)}\, v(s),\, a_0^*\rangle\, ds = \int_0^{2\pi} \langle e^{(2\pi - s)A(0)}\, v(s),\, b_0^*\rangle\, ds = 0 \Big\} \tag{3.42}$$

$\langle \cdot, \cdot \rangle$ is the duality between X and X^*, and a_0^*, $b_0^* \in X^*$ are such that

$$\langle a_0\, ,\, a_0^* \rangle = \langle b_0\, ,\, b_0^* \rangle = 1, \quad \langle a_0\, ,\, b_0^* \rangle = \langle b_0\, ,\, a_0^* \rangle = 0.$$

Differentiating (3.41) twice, we find

$$\lambda''(0) = -\frac{1}{6\pi\alpha'(0)} \int_0^{2\pi} \langle e^{(2\pi - s)A(0)}\, f_{uuu}(0,0)(e^{sA(0)} a_0)^3\, ,\, a_0^* \rangle\, ds \tag{3.43}$$

In our case we have $e^{tA(0)} a_0 = a_0 \cos t - b_0 \sin t$, $e^{tA(0)} b_0 = a_0 \sin t + b_0 \cos t$, and

$$\langle (\xi, \eta), a_0^* \rangle = \frac{2}{\pi} \int_0^\pi \xi(x) \sin x\, dx$$
$$\langle (\xi, \eta), b_0^* \rangle = \frac{2}{\pi} \int_0^\pi \big[-\Psi_2 \xi(x) - \Psi_1^{-1}(1 + \Psi_2^2)\eta(x) \big] \sin x\, dx.$$

Therefore, using (3.42), we can compute $\lambda''(0)$, finding $\lambda''(0) = \Lambda(\Phi, \Psi)$, with

$$\begin{aligned} \Lambda(\Phi, \Psi) \doteq & -\frac{1}{4}(\Phi_{1\lambda} + \Psi_{2\lambda} - \Phi_{3\lambda})^{-1} \big[\Phi_{111} - 3\Phi_{113} + 3\Phi_{133} - \Phi_{333} \\ & - \Psi_1\Psi_2(1 + \Psi_2^2)^{-1}(2\Phi_{112} + 2\Phi_{233} - 4\Phi_{123}) + \Psi_1^2(1 + \Psi_2^2)^{-1}(\Phi_{122} - \Phi_{223}) \\ & - \Psi_1(1 + \Psi_2^2)^{-1}\Psi_{112} + 2\Psi_1^2\Psi_2(1 + \Psi_2^2)^{-2}\Psi_{122} - \Psi_1^3(1 + \Psi_2^2)^{-2}\Psi_{222} \big] \end{aligned} \tag{3.44}$$

We have established the following result:

Proposition 3.8. *Let $\Phi : [-1,1] \times \mathbf{R}^3 \to \mathbf{R}$, $\Psi : [-1,1] \times \mathbf{R}^2 \to \mathbf{R}$ be C^∞ functions satisfying (3.33) and (3.38). Assume that $\Lambda(\Phi, \Psi) \neq 0$, $\Lambda(\Phi, \Psi)$ is given in (3.44). Then there is $\lambda_0 > 0$ such that if $\Lambda(\Phi, \Psi) > 0$ (respectively $\Lambda(\Phi, \Psi) < 0$), for any $\lambda \in]0, \lambda_0]$ (respectively $[-\lambda_0, 0[$), problem (3.32) has a small nonconstant periodic solution $(\overline{u}_\lambda(t), \overline{v}_\lambda(t))$ with period approaching 2π as λ approaches 0. $(\overline{u}_\lambda, \overline{v}_\lambda)$ is orbitally stable if $(\Phi_{1\lambda} + \Psi_{2\lambda} - \Phi_{3\lambda})\Lambda(\Phi, \Psi) > 0$, it is orbitally unstable if $(\Phi_{1\lambda} + \Psi_{2\lambda} - \Phi_{3\lambda})\Lambda(\Phi, \Psi) < 0$. We mean smallness and stability with respect to the norm $\|(\xi, \eta)\| = \|\xi\|_{C^2([0,\pi])} + \|\eta\|_{C([0,\pi])}$.* ∎

Appendix

Sketch of the proof of Lemma 3.5. We use the notation of Theorem 3.4. Let $F : [-\sigma_0, \sigma_0] \times \mathbf{R} \times \mathbf{R} \times V \to C^\gamma(\mathbf{R}; X) \doteq \{ v \in C^\gamma(\mathbf{R}; X);\ v(t) = v(t + 2\pi),\ t \in \mathbf{R} \}$ be defined by

$$\begin{aligned} F(\sigma, k, \eta, z)(t) = & \frac{d}{dt}\big(e^{tA(0)} a_0 + z(t)\big) \\ & - \rho(\sigma) f_u\big(\lambda(\sigma), u(\sigma)(\rho(\sigma)t)\big)\big(e^{tA(0)} a_0 + z(t)\big) \\ & + k\big(e^{tA(0)} a_0 + z(t)\big) - \eta\sigma^{-1} \frac{d}{dt}(u(\sigma)(\rho(\sigma)t)) \end{aligned} \tag{A.1}$$

where $V \subset C^\gamma(\mathbf{R}; D)$ is defined in (3.42), $a_0 \pm ib_0$ are the generators of X_0, defined in (3.38). Then $F(0,0,0,0) = 0$, and the derivative of F with respect to (k, η, z), evaluated at $(0,0,0,0)$, is the linear operator

$$(\hat{k}, \hat{\eta}, \hat{z}) \to (e^{tA(0)}a_0)\hat{k} + (e^{tA(0)}b_0)\hat{\eta} + \hat{z}'(t) - A(0)\hat{z}(t)$$

which is an isomorphism from $\mathbf{R} \times \mathbf{R} \times V$ onto $C^\gamma(\mathbf{R}; X)$. Therefore for any σ sufficiently close to 0 there are $k(\sigma)$, $\eta(\sigma)$, $z(\sigma)$ such that

$$F(\sigma, k(\sigma), \eta(\sigma), z(\sigma)) = 0 \tag{A.2}$$

If $k(\sigma) = 0$ for some σ, then (A.1) and (A.2) imply that both

$$w_1(t) = \exp\big(t/\rho(\sigma)A(0)\big)a_0 + z(\sigma)(t/\rho(\sigma)) \quad \text{and} \quad w_2(t) = d/dt\big(u(\sigma)(t)\big)$$

are solutions of (3.24). By (3.40), w_1 and w_2 are linearly independent, so that 0 is a double eigenvalue of $V(0)$. If $k(\sigma) \neq 0$, we can easily check taht the function $w(\sigma)$, defined by

$$w(\sigma)(t) = \exp\big(t/\rho(\sigma)A(0)\big)a_0 + z(\sigma)(t/\rho(\sigma)) + \eta(\sigma)(k(\sigma))^{-1}\sigma^{-1}\frac{d}{dt}(u(\sigma)(t)) \tag{A.3}$$

is a nontrivial $2\pi\rho(\sigma)$-periodic solution of (3.24). Therefore (3.25) is proved. To show (3.26), let us differentiate with respect to σ the equality

$$\frac{d}{dt}\big[\sigma(v(\sigma)(t) + e^{tA(0)}a_0)\big] - \rho(\sigma)f\big(\lambda(\sigma), \sigma(v(\sigma)(t) + e^{tA(0)}a_0)\big) = 0$$

where $u(\sigma)$ is given by (3.40). Now, subtract (A.2) from the resulting equality, to get

$$\begin{aligned} &- k(\sigma)\big[e^{tA(0)}a_0 + z(\sigma)(t)\big] + \big(\eta(\sigma) - \sigma\rho'(\sigma)/\rho(\sigma)\big)\big[-e^{tA(0)}b_0 + \frac{d}{dt}v(\sigma)(t)\big] \\ &\quad + \sigma\lambda'(\sigma)\big[-\rho(\sigma)\sigma^{-1}f_\lambda\big(\lambda(\sigma), \sigma(v(\sigma)(t) + e^{tA(0)}a_0)\big)\big] + \Lambda(\sigma)\big(\frac{d}{d\sigma}(\sigma v(\sigma)) + z(\sigma)\big) \\ &= 0 \end{aligned} \tag{A.4}$$

where $\Lambda(\sigma) : C^\gamma(\mathbf{R}; D) \cap C^{1,\gamma}(\mathbf{R}; X) \to C^\gamma(\mathbf{R}; X)$ is the linear operator defined by

$$(\Lambda(\sigma)y)(t) = y'(t) - \rho(\sigma)f_u\big(\lambda(\sigma), \sigma(v(\sigma) + e^{tA(0)}a_0)\big)y(t), \ \ t \in \mathbf{R} \tag{A.5}$$

The linear mapping $(h, \xi, y) \to -h\big(e^{tA(0)}a_0 + z(\sigma)(t)\big) + \xi\big(-e^{tA(0)}b_0 + (d/dt)v(\sigma)(t)\big) + \Lambda(\sigma)y(t)$ is an isomorphism from $\mathbf{R} \times \mathbf{R} \times V$ onto $C^\gamma(\mathbf{R}; X)$ for $\sigma = 0$, and it depends continuously on σ. Moreover, we have

$$\lim_{\sigma \to 0} -\rho(\sigma)\sigma^{-1}f_\lambda\big(\lambda(\sigma), \sigma(v(\sigma)(t) + e^{tA(0)}a_0)\big) = \frac{\partial^2 f}{\partial\lambda\partial u}(0,0)e^{tA(0)}a_0 \tag{A.6}$$

The limit is in the $C^\gamma(\mathbf{R}; X)$ topology. Therefore there is a constant $C > 0$ such that for σ sufficiently small we have

$$|k(\sigma)| + |\eta(\sigma) - \sigma\rho'(\sigma)/\rho(\sigma)| + \|\frac{d}{d\sigma}(\sigma v(\sigma)) + z(\sigma)\|_{C^\gamma(\mathbf{R};D)} \leq C|\sigma\lambda'(\sigma)| \tag{A.7}$$

Set now for $|\sigma|$ small and $t \in \mathbf{R}$

$$g(\sigma)(t) = -k(\sigma)e^{tA(0)}a_0 + \big(\eta(\sigma) - \sigma\rho'(\sigma)/\rho(\sigma)\big)(-e^{tA(0)}b_0)$$
$$+ \sigma\lambda'(\sigma)\big[-\rho(\sigma)\sigma^{-1}f_\lambda\big(\lambda(\sigma), \sigma(v(\sigma)(t) + e^{tA(0)}a_0)\big)\big]$$

By (A.4) and (A.6) we have, since $v(0) = z(0) = 0$,

$$\|g(\sigma) - \Lambda(\sigma)\big(\frac{d}{d\sigma}(\sigma v(\sigma)) + z(\sigma)\big)\|_{C^\gamma(\mathbf{R};X)} = \mathrm{o}(1)|\sigma\lambda'(\sigma)|$$

when $\sigma \to 0$. Therefore, using (3.42), we get

$$\Big|\int_0^{2\pi} \langle e^{(2\pi-s)A(0)}g(\sigma)(s)\,,\,a_0^*\rangle\,ds\Big| = \mathrm{o}(1)|\sigma\lambda'(\sigma)|.$$

Computing the integral, (3.26) follows.

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