Stability of the Poincaré Bundle

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Abstract

Let C be a nonsingular projective curve of genus $g \geq 2$ defined over the complex numbers, and let M_ξ denote the moduli space of stable bundles of rank n and determinant ξ on C, where ξ is a line bundle of degree d on C and n and d are coprime. It is shown that the universal bundle $\mathcal{U}_\xi$ on $C \times M_\xi$ is stable with respect to any polarisation on $C \times M_\xi$. It is shown further that the connected component of the moduli space of $\mathcal{U}_\xi$ containing $\mathcal{U}_\xi$ is isomorphic to the Jacobian of C.

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Introduction

In the study of moduli spaces of stable bundles on an algebraic curve C, various bundles on the moduli space or on the product of the moduli space with C arise in a natural way. An interesting question to ask about any such bundle is whether it is itself stable in some sense.

More precisely, let C be a nonsingular projective curve of genus $g \geq 2$ defined over the complex numbers, and let $M = M_{n,d}$ denote the moduli space of stable bundles of rank n and degree d on C, where n and d are coprime. For any line bundle ξ of $\deg d$ on C, let M_ξ denote the subvariety of M corresponding to bundles with determinant ξ. There exists on $C \times M$ a universal (or Poincaré) bundle $\mathcal{U}$ such that $\mathcal{U}\big|_{C\times\{m\}}$ is the bundle on C corresponding to m. Moreover the bundle $\mathcal{U}$ is determined up to tensoring with a line bundle lifted from M.

The direct image of $\mathcal{U}$ on M is called the Picard sheaf of $\mathcal{U}$; for $d > n(2g-2)$, this sheaf is a bundle. It was shown recently by Y. Li [Li] that, if $d > 2gn$, this bundle is stable with respect to the ample line bundle corresponding to the generalised theta divisor (cf. [DN]). (Recall here that, if H is an ample divisor on a projective variety X, the *degree* $\deg E$ of a torsion-free sheaf E on X is defined to be the intersection number $[c_1(E) \cdot H^{\dim X - 1}]$. E is said to be *stable* with respect to H (or H-*stable*) if, for every proper subsheaf F of E,

$$\frac{\deg F}{\operatorname{rank} F} < \frac{\deg E}{\operatorname{rank} E}.$$

The definition depends only on the polarisation defined by H.) This extends previously known results for the case $n = 1$ ([U, Ke1, EL]). We remark that the question of stability of the Picard sheaf of $\mathcal{U}_\xi$ is still open (cf. [BV]).

In this paper, we investigate the stability of the Poincaré bundle $\mathcal{U}$ and its restriction $\mathcal{U}_\xi$ to M_ξ using methods similar to those of [Li].

Our main results are

Theorem 1.5. $\mathcal{U}_\xi$ is stable with respect to any polarisation on $C \times M_\xi$.

Theorem 1.6. $\mathcal{U}$ is stable with respect to any polarisation of the form

$$a\alpha + b\Theta, \quad a, b > 0$$

where α is ample on C and Θ is the generalised theta divisor on M. (Note that C has a unique polarisation whereas M does not.)

These results are proved in §1. In §2 we discuss some properties of the bundles $\mathrm{End}\,\mathcal{U}_\xi$ and $\mathrm{ad}\,\mathcal{U}_\xi$. It is reasonable to conjecture that $\mathrm{ad}\,\mathcal{U}_\xi$ is also stable but we are able to prove this only in the case $n = 2$.

Finally, in §3 we consider the deformation theory of $\mathcal{U}_\xi$ using the results in §2. The main result we prove is that the only deformations of $\mathcal{U}_\xi$ are those of the form $\mathcal{U}_\xi \otimes p_C^* L$, where L is a line bundle of degree 0 on C. More precisely, let H be any ample divisor on $C \times M_\xi$ and let $M(\mathcal{U}_\xi)$ denote the moduli space of H-stable bundles with the same numerical invariants as $\mathcal{U}_\xi$ on $C \times M_\xi$; then

Theorem 3.1. The connected component $M(\mathcal{U}_\xi)_0$ of $M(\mathcal{U}_\xi)$ containing $\{\mathcal{U}_\xi\}$ is isomorphic to the Jacobian $J(C)$, the isomorphism $J(C) \longrightarrow M(\mathcal{U}_\xi)_0$ being given by

$$L \longmapsto \mathcal{U}_\xi \otimes p_C^* L,$$

where $p_C : C \times M_\xi \longrightarrow C$ is the projection.

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§1 Stability of $\mathcal{U}$

We begin with some lemmas which are probably well known, but which we could not find in the literature.

Lemma 1.1. *Let X and Y be smooth projective varieties of the same dimension m. Let $f : X - - \to Y$ be a dominant rational map defined outside a subset $Z \subset X$ with $\operatorname{codim}_X Z \geq 2$. Suppose that D_X and D_Y are ample divisors on X and Y such that $f^*D_Y|_{X-Z} \simeq D_X|_{X-Z}$.*

Let E be a vector bundle on Y such that f^*E extends to a vector bundle F on X. If F is D_X-semi-stable (resp. stable) on X, then E is D_Y-semi-stable (resp. stable) on Y.

Proof. The proof is fairly straight forward (cf. [Li, pp.548, 549]). Let rank $E = n$. Suppose that V is a torsion-free quotient of E,

$$E \longrightarrow V \longrightarrow 0 \tag{1}$$

with rank $V = r < n$. When F is D_X-semi-stable, we need to check the inequality:

$$\frac{\deg V}{r} \geq \frac{\deg E}{n}. \tag{2}$$

From (1), we have

$$f^*E \longrightarrow f^*V \longrightarrow 0.$$

If $G = f^*V/(torsion)$, then

$$\deg f^*V \geq \deg G. \tag{3}$$

(Note that $\deg f^*V = [c_1(f^*V) \cdot D_X^{m-1}]$, where $c_1(f^*V)$ makes sense since f^*V is defined outside a subset of codimension 2.)

Moreover we have

$$0 \longrightarrow G^* \longrightarrow f^*(E)^*$$

on X-Z. Let $\widetilde{G}$ be a subsheaf of F^* extending G^*. Since $\operatorname{codim}_X Z \geq 2$, we have $\deg \widetilde{G} = \deg G^* = -\deg G$.

By semi-stability of F^*,

$$\frac{\deg \widetilde{G}}{r} \leq \frac{\deg F^*}{n}. \tag{4}$$

Note that

$$\begin{aligned} \deg E &= [c_1(E).D_Y^{m-1}] \\ &= [c_1(f^*E).D_X^{m-1}](\deg f)^{-1} \\ &= (\deg f^*E)(\deg f)^{-1}. \end{aligned}$$

The same applies to V, so we have

$$\begin{aligned} n \deg V - r \deg E &= (n \deg f^*V - r \deg f^*E)(\deg f)^{-1} \\ &\geq (n \deg G - r \deg F)(\deg f)^{-1} \quad \text{by (3)} \\ &= (-n \deg \widetilde{G} + r \deg F^*)(\deg f)^{-1} \\ &\geq 0 \quad \text{by (4)}. \end{aligned}$$

This proves (2).

The proof in the stable case is similar.

Lemma 1.2. *Let X^m and Y^n be smooth projective varieties, and D_X and D_Y ample divisors on X and Y. Let $\eta = aD_X + bD_Y$, $a, b > 0$. Suppose that E is a vector bundle on $X \times Y$, such that for generic $x \in X$, $y \in Y$, $E_x \simeq E|_{\{x\}\times Y}$ and $E_y \simeq E|_{X\times\{y\}}$ are respectively D_Y-semi-stable and D_X-semi-stable. Then E is η-semi-stable.*

Further, if either E_x or E_y is stable, then E is stable.

Proof. Let $F \subset E$ be a subsheaf. Since $\operatorname{Sing} F$ has codimension ≥ 2, we can choose $x \in X$ and $y \in Y$ such that $\operatorname{Sing} F_x$ and $\operatorname{Sing} F_y$ also have codimension

≥ 2. Thus any torsion in F_x or F_y is supported in codimension ≥ 2 and does not contribute to $c_1(F_x)$ or $c_1(F_y)$. Let $\text{rank}\,(E) = n$, $\text{rank}\,(F) = r$. Then we need to show that

$$\frac{\deg F}{r} \leq \frac{\deg E}{n} \tag{5}$$

assuming E_x and E_y are semi-stable. Now,

$$\begin{aligned}
\deg E &= c_1(E) \cdot [aD_X + bD_Y]^{m+n-1} \\
&= c_1(E)[\lambda D_X^m \cdot D_Y^{n-1} + \mu D_X^{m-1} \cdot D_Y^n] \quad \text{for some} \quad \lambda, \mu > 0 \\
&= [c_1(E_x) + c_{1,1}(E) + c_1(E_y)] \cdot [\lambda D_X^m \cdot D_Y^{n-1} + \mu D_X^{m-1} \cdot D_Y^n] \\
&= [c_1(E_x) \cdot D_Y^{n-1}] \cdot \lambda D_X^m + [c_1(E_y) \cdot D_X^{m-1}] \cdot \mu D_Y^n. \qquad (6)
\end{aligned}$$

We have a similar expression for $\deg F$,

$$\deg F = [c_1(F_x) \cdot D_Y^{n-1}] \cdot \lambda D_X^m + [c_1(F_y) \cdot D_X^{n-1}] \cdot \mu D_Y^n. \tag{7}$$

(5) follows trivially by comparing the terms in (6) and (7) and using semi-stability of E_x and E_y. The rest of the lemma follows in a similar fashion.

Before stating the next lemma, we recall very briefly some facts on spectral curves. For details see [BNR] and [Li].

Let $K = K_C$ be the canonical bundle and let $W = \oplus_{i=1}^n H^0(C, K^i)$. Let $s = (s_1, \cdots, s_n) \in W$, and let C_s be the associated spectral curve. Then we have a morphism

$$\pi : C_s \longrightarrow C$$

of degree n, such that for $x \in C$, the fibre $\pi^{-1}(x)$ is given by points $y \in K_x$ which are zeros of the polynomial

$$f(y) = y^n + s_1(x) \cdot y^{n-1} + \cdots + s_n(x).$$

The condition that x be unramified is that the resultant $R(f, f')$ of f and its derivative f' be non-zero at the point $(s_1(x), \cdots, s_n(x))$. Note that $R(f, f')$ is a polynomial in the $s_i(x)$, $i = 1, \cdots, n$.

Lemma 1.3. *Given $x \in C$, there exists a smooth spectral curve C_s such that the covering map $\pi : C_s \longrightarrow C$ is unramified at x.*

Proof. Note first that, if $x \in C$, there exists $s = (s_1, \cdots, s_n) \in W$ such that

$$R(f, f')(s_1(x), \cdots, s_n(x)) \neq 0.$$

Indeed, since $|K^i|$ has no base points, given any $(\alpha_1, \cdots, \alpha_n) \in \bigoplus_{i=1}^n K_x^i$, there exist $s_i \in H^0(C, K^i)$ such that $s_i(x) = \alpha_i$, $i = 1, \cdots, n$.

Observe that this is clearly an open condition on W.

Further, the subset $\{s \in W \mid C_s \text{ is smooth}\}$ is a non-empty open subset of W [BNR, Remark 3.5] and the lemma follows.

Let M_ξ and $\mathcal{U}_\xi$ be as in the introduction, and let Θ_ξ denote the restriction of the generalized theta divisor to M_ξ.

Proposition 1.4. *Let $\mathcal{U}_\xi$ be the Poincaré bundle on $C \times M_\xi$ and $x \in C$. Then the bundle*

$$\mathcal{U}_{\xi,x} \cong \mathcal{U}_\xi|_{\{x\} \times M_\xi}$$

is Θ_ξ-semi-stable on M_ξ.

Proof. For the point $x \in C$ above, choose a spectral curve C_s by Lemma 1.3, so that

$$\pi : C_s \longrightarrow C$$

is unramified at x. Let $\pi^{-1}(x) = \{y_1, \cdots, y_n\}$, y_i being distinct points in C_s.

Let $J^\delta(C_s)$ denote the variety of line bundles of degree

$$\delta = d - \deg \pi_*(\mathcal{O}_{C_s})$$

on C_s, and let P_s denote the subvariety of $J^\delta(C_s)$ consisting of those line bundles L for which the vector bundle $\pi_* L$ has determinant ξ. (P_s is a translate of the Prym variety of π.) Let $\mathcal{L}$ denote the restriction of the Poincaré bundle on $C_s \times J^\delta(C_s)$ to $C_s \times P_s$. Then [BNR, proof of Proposition 5.7], we have a dominant rational map defined on an open subset T_s of P_s such that $\operatorname{codim}(P_s - T_s) \geq 2$, and

$$\phi : T_s \longrightarrow M_\xi$$

is generically finite. The morphism ϕ on T_s is defined by the family $(\pi \times 1)_* \mathcal{L}$ on $C_s \times T_s$; so, by the universal property of $\mathcal{U}_\xi$, we have

$$(\pi \times 1)_* \mathcal{L} \simeq (1 \times \phi)^* \mathcal{U}_\xi \otimes p_T^* L_0 \tag{8}$$

for some line bundle L_0 on T_s. (Here $p_T : C_s \times T_s \longrightarrow T_s$ is the projection.) By (8) we have

$$\phi^* \mathcal{U}_{\xi,x} \simeq [(\pi \times 1)_* \mathcal{L}]_x \otimes L_0^{-1} \text{ on } T_s.$$

But $[(\pi \times 1)_* \mathcal{L}]_x \simeq \oplus_{i=1}^n \mathcal{L}_{y_i}$ on P_s. Hence

$$\phi^* \mathcal{U}_{\xi,x} \simeq \oplus_{i=1}^n (\mathcal{L}_{y_i} \otimes L_0^{-1}) \text{ on } T_s. \tag{9}$$

We observe that the $\mathcal{L}_{y_i} \otimes L_0^{-1}$ are the restrictions to T_s of algebraically equivalent line bundles on P_s. Further one knows [Li, Theorem 4.3] that $\phi^* \Theta_\xi$ is a multiple of the restriction of the usual theta divisor on $J(C_s)$ to T_s.

Now we are in the setting of Lemma 1.1 and we conclude that $\mathcal{U}_{\xi,x}$ is semi-stable with respect to Θ_ξ on M_ξ.

Theorem 1.5. *The Poincaré bundle $\mathcal{U}_\xi$ on $C \times M_\xi$ is stable with respect to any polarisation.*

Proof. Since $\operatorname{Pic} M_\xi = \mathbf{Z}$, $\operatorname{Pic}(C \times M_\xi) = \operatorname{Pic} C \oplus \operatorname{Pic} M_\xi$. Thus, any polarisation η on $C \times M_\xi$ can be expressed in the form

$$\eta = a\alpha + b\Theta_\xi, \quad a, b > 0.$$

for some ample divisor α on C.

By Proposition 1.4, $\mathcal{U}_{\xi,x}$ is semi-stable with respect to Θ_ξ for all $x \in C$ and by definition $\mathcal{U}_\xi|_{C\times\{m\}}$ is stable with respect to any polarisation on C. Hence by Lemma 1.2, $\mathcal{U}_\xi$ is stable with respect to η on $C \times M_\xi$.

Note that Proposition 1.4 remains true if we replace $M_\xi, \mathcal{U}_\xi$ and Θ_ξ by M, $\mathcal{U}$ and Θ. (The key point is that [Li, Theorem 4.3] is valid in this context). We deduce at once

Theorem 1.6. *$\mathcal{U}$ is stable with respect to any polarisation of the form*

$$a\alpha + b\Theta, \quad a, b > 0,$$

where α is ample on C and Θ is the generalized theta divisor on M.

Remark 1.7. Since $C \times M$ is a Kähler manifold, then by a theorem of Donaldson-Uhlenbeck-Yau, $\mathcal{U}$ admits an Hermitian-Einstein metric. One can expect that the restriction of this metric to each factor is precisely the metric on the factor. It would be interesting to know an explicit description of the metric on $\mathcal{U}$. Note that [Ke2] contains such a description for the Picard sheaf in the case $g = 1$, $n = 1$.

§2 Some properties of End $\mathcal{U}_\xi$

Our first object in this section is to calculate the dimension of some of the cohomology spaces of $\operatorname{End}\mathcal{U}_\xi$. We denote by $p : C \times M_\xi \longrightarrow M_\xi$ and $p_C : C \times M_\xi \longrightarrow C$ the projections.

Proposition 2.1. *Let $h^i(\operatorname{End}\mathcal{U}_\xi) = \dim H^i(\operatorname{End}\mathcal{U}_\xi)$. Then,*

$$h^0(\operatorname{End}\mathcal{U}_\xi) = 1, \quad h^1(\operatorname{End}\mathcal{U}_\xi) = g, \quad h^2(\operatorname{End}\mathcal{U}_\xi) = 3g - 3.$$

Proof. We can write $\operatorname{End}\mathcal{U}_\xi \cong \mathcal{O} \oplus \operatorname{ad}\mathcal{U}_\xi$; hence

$$H^i(C \times M_\xi, \operatorname{End}\mathcal{U}_\xi) = H^i(C \times M_\xi, \mathcal{O}) \oplus H^i(C \times M_\xi, \operatorname{ad}\mathcal{U}_\xi).$$

Since $\mathcal{U}_\xi\Big|_{C\times\{m\}}$ is always stable, we have $R^0_p(\operatorname{ad}\mathcal{U}_\xi) = 0$. So by the Leray spectral sequence and the fact that $R^1_p(\operatorname{ad}\mathcal{U}_\xi)$ is the tangent bundle TM_ξ of M_ξ, we have

$$\begin{aligned} H^i(C \times M_\xi, \operatorname{ad}\mathcal{U}_\xi) &\cong H^{i-1}(M_\xi, R^1_p(\operatorname{ad}\mathcal{U}_\xi)) \\ &\cong H^{i-1}(M_\xi, TM_\xi) \end{aligned}$$

By [NR, Theorem 1], this space is 0 if $i \neq 2$, and has dimension $3g-3$ if $i = 2$.

On the other hand, since M_ξ is unirational, it follows from the Künneth formula that

$$H^i(C \times M_\xi, \mathcal{O}) = H^i(C, \mathcal{O}_C).$$

This space has dimension 1 if $i = 0$, g if $i = 1$ and 0 otherwise. The proposition follows.

Remark 2.2. In fact the proof shows that

$$h^i(\operatorname{End}\mathcal{U}_\xi) = 0 \text{ if } i > 2.$$

Lemma 2.3. *Let $L \in J(C)$ and suppose that $E \cong E \otimes L$ for all $E \in M_\xi$. Then $\mathcal{U}_\xi \cong \mathcal{U}_\xi \otimes p_C^* L$.*

Proof. Since $E \cong E \otimes L$ and $\mathcal{U}_\xi \otimes p_C^* L$ is a family of stable bundles, there is a line bundle L_1 over M_ξ such that

$$\mathcal{U}_\xi \cong \mathcal{U}_\xi \otimes p_C^* L \otimes p^* L_1.$$

Fix $x \in C$, then $\mathcal{U}_{\xi,x} \cong \mathcal{U}_{\xi,x} \otimes L_1$ over M_ξ. Hence $c_1(\mathcal{U}_{\xi,x}) = c_1(\mathcal{U}_{\xi,x}) + nc_1(L_1)$ so $nc_1(L_1) = 0$. But $\operatorname{Pic} \mathcal{M}_\xi \cong \mathbf{Z}$ (see [DN]); so $c_1(L_1) = 0$, which implies that L_1 is the trivial bundle.

The next lemma will also be required in §3.

Lemma 2.4. *If $\mathcal{U}_\xi \cong \mathcal{U}_\xi \otimes p_C^* L$ then $L \cong \mathcal{O}_C$.*

Proof. If $\mathcal{U}_\xi \cong \mathcal{U}_\xi \otimes p_C^* L$ then

$$\begin{aligned} \mathcal{O} \oplus \operatorname{ad}\mathcal{U}_\xi &\cong \operatorname{End}\mathcal{U}_\xi \\ &\cong \operatorname{End}\mathcal{U}_\xi \otimes p_C^* L \\ &\cong p_C^* L \oplus \operatorname{ad}\mathcal{U}_\xi \otimes p_C^* L. \end{aligned}$$

Hence $H^0(C \times M_\xi, p_C^* L)$ and $H^0(C \times M_\xi, \operatorname{ad}\mathcal{U}_\xi \otimes p_C^* L)$ cannot both be zero. Suppose there is a non-zero section $\phi : \mathcal{O} \longrightarrow \operatorname{ad}\mathcal{U}_\xi \otimes p_C^* L$. For some $x \in C$, the restriction of ϕ to $\{x\} \times M_\xi$ will define a non-zero section of $\operatorname{ad}\mathcal{U}_{\xi,x}$, which is a contradiction since $H^0(M_\xi, \operatorname{ad}\mathcal{U}_{\xi,x}) = 0$ (see [NR, Theorem 2]). Hence $H^0(C \times M_\xi, \operatorname{ad}\mathcal{U}_\xi \otimes p_C^* L) = 0$.

Therefore $H^0(C \times M_\xi, p_C^* L) \neq 0$. Since $\deg L = 0$, this implies $L \cong \mathcal{O}_C$.

Remark 2.5. The proof of Lemma 2.4 fails when $g = 1$ since [NR, Theorem 2] is not then valid. In fact, Lemma 2.4 and the remaining results of this section are false for $g = 1$.

We show next that a general stable bundle E is not isomorphic to $E \otimes L$ unless $L \cong \mathcal{O}_C$.

Proposition 2.6. *There exists a proper closed subvariety S of M_ξ such that, if $E \notin S$, then*

$$E \cong E \otimes L \Longrightarrow L \cong \mathcal{O}_C.$$

Proof. For any L, the subset $S_L = \{E \in M_\xi | E \cong E \otimes L\}$ is a closed subvariety of M_ξ. If $L \ncong \mathcal{O}_C$, then, by Lemmas 2.3 and 2.4, S_L is a proper subvariety. On the other hand, S_L can only be non-empty if $L^n \cong \mathcal{O}_C$; so only finitely many of the S_L are non-empty. Since M_ξ is irreducible, the union $S = \bigcup\{S_L | L \ncong \mathcal{O}_C\}$is a proper subvariety of M_ξ as required.

Remark 2.7. It follows at once from Proposition 2.6 that the action of $J(C)$ on M defined by $E \longmapsto E \otimes L$ is faithful. Another proof of this fact has been given in [Li, Theorem 1.2 and Proposition 1.6]. As the following proposition shows, our set S is analogous to the set S of [Li, Theorem 1.2].

Proposition 2.8. *Let S be as above and $E \in M_\xi$. Then $E \in S$ if and only if* $\operatorname{ad} E$ *has a line sub-bundle of degree zero.*

Proof. The trivial bundle cannot be a subbundle of $\operatorname{ad} E$.

If $L \in J(C)$ is a subbundle of $\operatorname{ad} E$, so is it of $\operatorname{End} E$, therefore

$$H^0(C, \operatorname{End} E \otimes L^*) \neq 0.$$

Hence, there is a non-zero map $\phi : E \otimes L \longrightarrow E$, which is an isomorphism since $E \otimes L$ and E are stable bundles of the same slope. Hence $E \in S$.

Conversely, suppose $E \cong E \otimes L$ with $L \ncong \mathcal{O}_C$. The isomorphism $\mathcal{O}_C \oplus \operatorname{ad} E \cong L \oplus \operatorname{ad} E \otimes L$ implies that $\operatorname{ad} E \otimes L$ has a section, i.e. there is a non-zero map $\phi : L^* \longrightarrow \operatorname{ad} E$. Since L^* and $\operatorname{ad} E$ have the same slope and $\operatorname{ad} E$ is semi-stable (being a subbundle of a semi-stable bundle $\operatorname{End} E$ with the same slope), ϕ is an inclusion.

Corollary 2.9. *If E is a general stable bundle of rank* 2 *and determinant ξ, then* $\operatorname{ad} E$ *is stable.*

Proof. Note that $\mathrm{ad}\, E$ has rank 3 and degree 0 and is semi-stable. By the Proposition, $\mathrm{ad}\, E$ has no line subbundle of degree 0. On the other hand $\mathrm{ad}\, E$ is self-dual, so it cannot have a quotient line bundle of degree 0.

Remark 2.10. It would be interesting to know if $\mathrm{ad}\, E$ is stable for a general stable bundle E of rank greater than 2. It is certainly true that $\mathrm{ad}\, E$ is semistable and also that it is stable as an orthogonal bundle [R].

Theorem 2.11. *If $n = 2$, $\mathrm{ad}\mathcal{U}_\xi$ is stable with respect to any polarisation on $C \times M_\xi$.*

Proof. In view of Corollary 2.9 and Lemma 1.2, we need only prove that $\mathrm{ad}\mathcal{U}_{\xi,x}$ is semi-stable for some $x \in C$. The argument is the same as for Proposition 1.4; indeed (9) shows at once that $\phi^*\mathrm{ad}\mathcal{U}_{\xi,x}$ can be expressed as a direct sum of restrictions to T_s of algebraically equivalent line bundles on P_s.

For $n > 2$, we can show similarly that $\mathrm{ad}\mathcal{U}_\xi$ is semi-stable.

§3 Deformations

As in the introduction, let H be any ample divisor on $C \times M_\xi$, let $M(\mathcal{U}_\xi)$ denote the moduli space of H-stable bundles with the same numerical invariants as $\mathcal{U}_\xi$ on $C \times M_\xi$, and let $M(\mathcal{U}_\xi)_0$ denote the connected component of $M(\mathcal{U}_\xi)$ which contains $\mathcal{U}_\xi$. One can define a morphism

$$\beta : J(C) \longrightarrow M(\mathcal{U}_\xi)_0$$

by

$$\beta(L) = \mathcal{U}_\xi \otimes p_C^* L.$$

Our object in this section is to prove

Theorem 3.1. *β is an isomorphism.*

Remark 3.2. Note that this implies in particular that $M(\mathcal{U}_\xi)_0$ is independent of the choice of H, and is a smooth projective variety of dimension g. Since $h^2(\mathrm{End}\,\mathcal{U}_\xi) \neq 0$, there is no a priori reason why this should be so.

Proof of Theorem 3.1. By Lemma 2.4, β is injective. Moreover the Zariski tangent space to $M(\mathcal{U}_\xi)_0$ at $\mathcal{U}_\xi \otimes p_C^* L$ can be identified with $H^1(C \times M_\xi, \mathrm{End}\,\mathcal{U}_\xi)$, which has dimension g by Proposition 2.1. It follows that, at any point of Im β, $M(\mathcal{U}_\xi)_0$ has dimension precisely g and is smooth. Hence by Zariski's Main Theorem, β is an open immersion. Since $J(C)$ is complete, it follows that β is an isomorphism.

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