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STABILITY OF THE RIEMANN SEMIGROUP WITH RESPECT TO THE KINETIC CONDITION

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Abstract. This note deals with systems of hyperbolic conservation laws that are endowed with a generalized kinetic relation and develop phase transitions. The $\mathbf{L^1}$-Lipschitzean continuous dependence of the solution from the kinetic relation is proved. Preliminarily, we rephrase several results known in the case of standard conservation laws to the case comprising phase boundaries.

1. Introduction. This paper deals with conservation laws in presence of phase transitions. More precisely, we deal with the system

$$\partial_t u + \partial_x [f(u)] = 0 \tag{1.1}$$

with $t \in [0, +\infty[$, $x \in \mathbf{R}$, $u \in \Omega$, $f \colon \Omega \mapsto \mathbf{R}^n$ and $\Omega \subset \mathbf{R}^n$, under the assumption that Ω be the disjoint union of two open sets, which we refer to as *phases*, i.e.,

$$\Omega = \Omega_0 \cup \Omega_1 \,. \tag{1.2}$$

A *phase transition* is a jump discontinuity in a solution u to 1.1 between states $u(t, x-)$ and $u(t, x+)$ belonging to different phases.

Physical models leading to this setting are provided by liquid - vapor phase transitions, elastodynamics, or combustion models; see [2, 7, 8, 9, 19, 20] and the references therein. Typically, in the case 1.2 the Riemann problem for 1.1 turns out to be underdetermined and further conditions need to be supplemented. Physically, various criteria have been devised: viscosity [20], viscocapillarity [19], or other kinetic conditions [2]. From an

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analytical point of view, the above criteria can be described through the generalized kinetic condition

$$\Psi\left(u(t,x-),u(t,x+)\right)=0 \tag{1.3}$$

for a given smooth function Ψ having a suitable number of components.

When no phase transitions develop, 1.1 generates the so-called *Standard Riemann Semigroup*, or SRS for short; see [5, 15] and the references therein. Phase transitions, when present, depend on the particular admissibility criterion 1.3 chosen. Hence, the solution operator generated by 1.1–1.3 is referred to as the *Ψ-Riemann Semigroup*, or ΨRS; see [7]. For the sake of completeness, we note here that 1.3 can be substituted by constraints on the *structure* of the solution, as in [8].

The aim of this paper is first (in Sec. 2) to extend several results obtained in the case of the SRS to the case of systems endowed with a generalized kinetic relation. Secondly, in Sec. 3, we study the dependence of the solution u to 1.1–1.3 on the flow f and, in particular, on the function Ψ. We shall prove that the solution u in $\mathbf{L^1}$ is a Lipschitz continuous function of f and Ψ in $\mathbf{C^1}$. The last section is devoted to examples of possible applications of these results.

2. Notations and Preliminary Results.

On the system (1.1) we require that

(1): f is of class $\mathbf{C^3}$, the $n\times n$ matrix $Df(u)$ is strictly hyperbolic both in Ω_0 and in Ω_1; i.e., Df has n real distinct eigenvalues and each characteristic field is either genuinely nonlinear or linearly degenerate.

For $i=1,\ldots,n$ and $u\in\Omega$, denote by $\lambda_i(u)$ and $r_i(u)$ the i-th eigenvalue and the corresponding right eigenvector of the $n\times n$ matrix $A(u)=Df(u)$. The indexes are chosen so that $\lambda_{i-1}(u)<\lambda_i(u)$ for all u and i. If the i-th characteristic field is genuinely nonlinear, the eigenvector r_i is normalized so that $\nabla\lambda_i(u)\cdot r_i(u)\equiv 1$. Denote by $\hat\lambda$ an upper bound for $|\lambda_i(u)|$, for all $i=1,\ldots,n$ and $u\in\Omega$. We refer to [5, 11] for the basic definitions related to conservation laws. In particular, below we mean *entropic* in the sense specified by Lax inequalities [5, Formula (4.38)].

Let $u\colon [0,+\infty[\,\times\mathbf{R}\mapsto\Omega$ be a weak solution to 1.1, entropic both in Ω_0 and in Ω_1 and such that $u(t,\cdot)\in\mathbf{BV}$ for all t. A Lipschitz-continuous curve $x=p(t)$ is a *phase boundary* if the traces

$$u\left(t,p(t)-\right)=\lim_{x\to p(t)-}u(t,x) \quad\text{and}\quad u\left(t,p(t)+\right)=\lim_{x\to p(t)+}u(t,x)$$

are in different phases. The phase boundary $x=p(t)$ is of type (j,h) [13] at time t if

$$\begin{array}{rcccl}\lambda_{j-1}\left(u\left(t,p(t)-\right)\right) & < & \dot p(t) & < & \lambda_j\left(u\left(t,p(t)-\right)\right),\\ \lambda_h\left(u\left(t,p(t)+\right)\right) & < & \dot p(t) & < & \lambda_{h+1}\left(u\left(t,p(t)+\right)\right).\end{array} \tag{2.1}$$

The above inequalities mean that the characteristics entering into the phase boundary are precisely those numbered by $j,j+1,\ldots,n$ on the left and $1,2,\ldots,h$ on the right. The usual Lax shocks of the k-th characteristic family behave as a (k,k) phase boundary. The stability of $(2,1)$ phase boundaries is considered in [7] in the framework of elastodynamics and liquid-vapor systems.

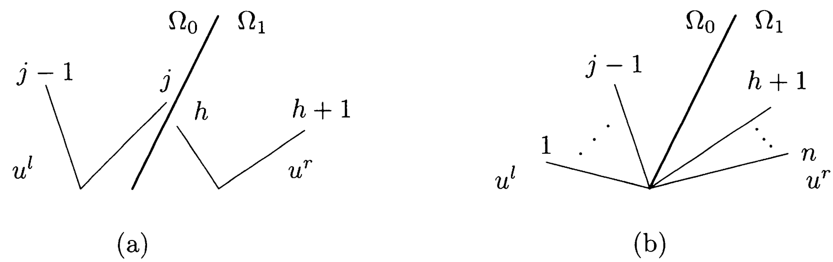

FIG. 1. (a) Characteristics and a (j,h) phase boundary. (b) Solution to a Riemann problem. The phase boundary is represented by a thick line, the other waves by thin lines.

The requirement that a solution u with a phase transition between the states $u^l \in \Omega_0$ and $u^r \in \Omega_1$ be a weak solution to 1.1, implies that the Rankine-Hugoniot conditions [5, §4.2] between u^l, u^r and the speed $\dot{p}$ of the transition be satisfied.

In 2.1 we neglect the *sonic* case in which one of the inequalities is replaced by an equality. This situation can be treated, for example, as in [8, 9].

2.1. *The Riemann Problem.* Let $\underline{u}^l \in \Omega_0$ and $\underline{u}^r \in \Omega_1$ and assume that the Riemann problem

$$\begin{cases} \partial_t u + \partial_x [f(u)] = 0 \\ u(0,x) = \begin{cases} \underline{u}^l & \text{if } x < 0 \\ \underline{u}^r & \text{if } x > 0 \end{cases} \end{cases} \tag{2.2}$$

admits a weak solution consisting of a (j,h) phase boundary $x = \underline{\Lambda} t$ that satisfies the Rankine - Hugoniot condition

$$f(\underline{u}^l) - f(\underline{u}^r) = \underline{\Lambda} \cdot (\underline{u}^l - \underline{u}^r)\,. \tag{2.3}$$

If $j > h$, attempting to solve any small perturbation of 2.2 leads to an underdetermined problem; see [13]. Indeed, further $j-h$ conditions need to be supplemented through the introduction of an admissibility function $\Psi\colon (\Omega_0 \times \Omega_1) \cup (\Omega_1 \times \Omega_0) \mapsto \mathbf{R}^{j-h}$. We assume throughout that Ψ is of class $\mathbf{C^2}$ and, in order to respect the $x \longrightarrow -x$ symmetry of 1.1, we also require that

$$\Psi(u^l, u^r) = \Psi(u^r, u^l)\,. \tag{2.4}$$

DEFINITION 2.1. Given problem 1.1 together with an admissibility function $\Psi\colon (\Omega_0 \times \Omega_1) \cup (\Omega_1 \times \Omega_0) \mapsto \mathbf{R}^{j-h}$, a phase transition in a weak entropic solution u to 1.1 is Ψ-admissible if 1.3 is satisfied at almost every point of the phase boundary. A *Ψ-admissible solution* to the Riemann problem

$$\begin{cases} \partial_t u + \partial_x [f(u)] = 0 \\ u(0,x) = \begin{cases} u^l & \text{if } x < 0 \\ u^r & \text{if } x > 0 \end{cases} \end{cases} \tag{2.5}$$

with data in different phases is a self-similar weak solution to 1.1 consisting, from left to right,

(1) if $u^l \in \Omega_0$ and $u^r \in \Omega_1$, of $j-1$ Lax waves in Ω_0, a Ψ-admissible phase boundary, and $n-h$ Lax waves in Ω_1;

(2) if $u^l \in \Omega_1$ and $u^r \in \Omega_0$, of $n-h$ Lax waves in Ω_1, a Ψ-admissible phase boundary, and $j-1$ Lax waves in Ω_0.

Above, by *Lax waves* we mean the usual (possibly null) simple waves that constitute the Lax [16] solution to Riemann problems.

The local well-posedness of the Riemann problem 2.5 near $\underline{u}^l$ and $\underline{u}^r$ requires suitable compatibility conditions between f and Ψ. A sufficient condition, obtained in [10], is

(2): $\Psi\colon (\Omega_0 \times \Omega_1) \cup (\Omega_1 \times \Omega_0) \mapsto \mathbf{R}^{j-h}$ is of class $\mathbf{C^1}$, satisfies 2.4 and the matrix

$$\begin{bmatrix} (\underline{\Lambda} - \underline{\lambda}_1)\underline{r}_1 & \dots & (\underline{\Lambda} - \underline{\lambda}_{j-1})\underline{r}_{j-1} & [\underline{u}] & (\underline{\Lambda} - \underline{\lambda}_{h+1})\underline{r}_{h+1} & \dots & (\underline{\Lambda} - \underline{\lambda}_n)\underline{r}_n \\ D_1\Psi\underline{r}_1 & \dots & D_1\Psi\underline{r}_{j-1} & 0 & -D_2\Psi\underline{r}_{h+1} & \dots & -D_2\Psi\underline{r}_n \end{bmatrix}$$

is invertible.

Above, $D_1\Psi$ (resp. $D_2\Psi$) is the $(j-h)\times n$ matrix of the partial derivative of Ψ with respect to the first (resp. second) argument $\underline{u}^l$ (resp. $\underline{u}^r$), evaluated at $(\underline{u}^l, \underline{u}^r)$. Similarly, $\lambda_1, \dots, \lambda_{j-1}$ and the corresponding eigenvectors are computed at $\underline{u}^l$, while $\lambda_{h+1}, \dots, \lambda_n$ as well as their related eigenvectors are evaluated at $\underline{u}^r$.

A direct application of the implicit function theorem leads to the following proposition.

PROPOSITION 2.2. Let assumption **(1)** hold. Fix two states $\underline{u}^l \in \Omega_0$ and $\underline{u}^r \in \Omega_1$ such that **(2)** holds. Then, for all u^l, u^r in suitable neighborhoods of $\underline{u}^l$ and $\underline{u}^r$, the Riemann problem 2.5 admits a unique Ψ-admissible solution in the sense of Definition 2.1.

When $\underline{u}^l$ and $\underline{u}^r$ are in the same phase, a Lax solution to 2.2 may not necessarily exist, for the jump $\|\underline{u}^l - \underline{u}^r\|$ may well be large. It is then natural to look for a solution to 2.2 containing two different phase transitions. Such a solution models the *nucleation* of two phase boundaries. A solution to 2.2, and hence also of its perturbation 2.5, is obtained by gluing solutions of type (1) and (2) above. Note that the two phase boundaries need not be of the same type (j, h); hence, a wide variety of cases may appear. The extension of the Proposition above to the case of nucleation (and even to the case of several phase boundaries) follows easily, simply assuming the stability condition (2) on each of the phase boundaries in the solution to the unperturbed problem 2.2; see [7, 17, 18].

2.2. *The Ψ-Riemann Semigroup.* In this first part, most of the proofs are omitted, for they usually follow from the corresponding ones related to the SRS and full references are provided. In this section we consider (j, h) phase boundaries, for fixed j and h, $j > h$.

DEFINITION 2.3. System 1.1 and condition 1.3, with the admissibility function $\Psi\colon (\Omega_0 \times \Omega_1) \cup (\Omega_1 \times \Omega_0) \mapsto \mathbf{R}^{j-h}$ satisfying 2.4, generate a *Ψ-Riemann Semigroup* (ΨRS) $S\colon [0, +\infty[\times \mathcal{D} \mapsto \mathcal{D}$ if the following holds:

(1) $\mathcal{D}$ is a non-trivial domain in $\mathbf{BV}(\mathbf{R})$;

(2) S is a semigroup: $S_0 = \mathrm{Id}$ and $S_t \circ S_s = S_{t+s}$;

(3) S is $\mathbf{L^1}$-Lipschitzean: there exists a positive L such that for all $u, w \in \mathcal{D}$,

$$\|S_t u - S_s w\|_{\mathbf{L}^1} \le L \cdot (\|u - w\|_{\mathbf{L}^1} + |t-s|) \ ;$$

(4) if $u \in \mathcal{D}$ is piecewise constant with jumps at, say, $\bar{x}_j$, $j = 1, \dots, m$, then for t small, $S_t u$ coincides with the gluing of the Ψ-admissible solutions to the Riemann problem 2.5 with $u^l = u(\bar{x}_j-)$ and $u^r = u(\bar{x}_j+)$.

In the case of the Standard Riemann Semigroup (SRS) (see [5]), (1) above amounts to asking that $\mathcal{D}$ contains all functions with suitably small total variation. In [7], this condition is replaced by the assumption that $\mathcal{D}$ contains, at least, all **BV**-small perturbations of the Riemann data 2.2. Note also that, due to the uniform continuity implied by (3), S can be uniquely extended to the $\mathbf{L}^1$ closure of $\mathcal{D}$.

A first result towards the construction of a ΨRS was obtained in [7] in the case $n = 2$. In the general case $n \geq 2$, the techniques used in [7, 17, 18] allow to prove the following result. Denote by $\mathcal{M}$ the set of smooth increasing diffeomorphisms $\mathbf{R} \mapsto \mathbf{R}$.

THEOREM 2.4. Assume that $f\colon \Omega_0 \cup \Omega_1 \mapsto \mathbf{R}^n$ satisfies **(1)**. Fix $j, h \in \{1, \ldots, n\}$ with $j > h$ and let $\Psi\colon (\Omega_0 \times \Omega_1) \cup (\Omega_1 \times \Omega_0) \mapsto \mathbf{R}^{j-h}$, $\underline{u}^l \in \Omega_0$ and $\underline{u}^r \in \Omega_1$ such that **(2)** holds. Let $\underline{u}$ be the Ψ-admissible solution to 2.2.

Then, the problem 1.1–1.3 generates a ΨRS $S\colon\ [0, +\infty[\, \times \mathcal{D} \mapsto \mathcal{D}$ with the following properties:

(1) $t \mapsto S_t u$ is a weak solution to 1.1–1.3;

(2) there exists a $\delta > 0$ such that $\mathcal{D}$ contains all $u\colon \mathbf{R} \mapsto \Omega$, for which $\exists \mu \in \mathcal{M}$ with

$$\|u(\cdot) - \underline{u}(1, \mu(\cdot))\|_{\mathbf{L}^1} < \infty, \quad \mathrm{TV}\,\{u(\cdot) - \underline{u}(1, \mu(\cdot))\} \leq \delta\,; \tag{2.6}$$

(3) for every $u \in \mathcal{D}$, $S_t u$ is the limit of front tracking approximations;

(4) there exists a map $p\colon\ [0, +\infty[\, \mapsto \mathbf{R}$ whose graph $x = p(t)$ supports the phase boundary, p is Lipschitzean, and $\dot{p}$ has bounded variation.

We remark that the above theorem also ensures the structural stability of the phase boundary with respect to all perturbations having suitably small total variation.

Note also that in the case of nucleation, i.e., of two phase boundaries in the solution to 2.2, Theorem 2.4 still holds, provided the so-called *strong non-resonance conditions* are imposed on f, Ψ, and on the data $\underline{u}^l$, $\underline{u}^r$; see [6, Formula (2.13)], [7, Formula (2.12)], [17, Formulas (1.11)–(1.12)], and [18, Formulas (2.12)–(2.13)].

Essentially, as in the case of the SRS, when Theorem 2.4 applies, then the ΨRS is unique and its orbits yield solution to 1.1–1.3.

THEOREM 2.5. With the same assumptions of Theorem 2.4 above, call $S\colon\ [0, +\infty[\, \times \mathcal{D} \mapsto \mathcal{D}$ the ΨRS constructed therein. Call $\bar{S}\colon\ [0, +\infty[\, \times \bar{\mathcal{D}} \mapsto \bar{\mathcal{D}}$ another ΨRS, with $\bar{\mathcal{D}} \supseteq \mathcal{D}$. Then, for all $u \in \mathcal{D}$,

$$\bar{S}_t u = S_t u \quad \text{for all } t \geq 0\,.$$

The proof follows the lines of [5, Theorem 9.1].

Once a ΨRS is constructed as limit of wave front tracking approximations, the problem of characterizing the ΨRS can also be solved, again as in the case of the SRS. Indeed, consider a trajectory $t \mapsto S_t u$ of the ΨRS S. Fix a point (τ, ξ) and denote $u^{\pm} = \lim_{x \to \xi\pm} u(\tau, x)$. We define:

- the function $U^{\sharp}_{(u;\tau,\xi)}$ is the Ψ-admissible solution to the Riemann problem with initial data (u^-, u^+);

- the function $U^{\flat}_{(u;\tau,\xi)}$ is the solution to the linear initial-value problem

$$\begin{cases} \partial_t w + A\left(u(\tau,\xi)\right)\partial_x w = 0 \\ w(\tau,x) = u(\tau,x)\,. \end{cases}$$

THEOREM 2.6. Assume that 1.1 generates a ΨRS $S\colon [0,+\infty[\,\times\mathcal{D}\mapsto\mathcal{D}$. Then, the orbits $t\mapsto S_t u$ are admissible solutions to 1.1. Moreover, for all $\bar u\in\mathcal{D}$, the function $u(t,x)=S_t\bar u$ satisfies the following estimates:
(A) for every $\tau\geq 0$, $\xi\in\mathbf{R}$ and ρ sufficiently small, there exists a constant $C>0$ such that

$$\frac{1}{h}\int_{\xi-\rho+h\hat\lambda}^{\xi+\rho-h\hat\lambda}\left\|u(\tau+h,x)-U^{\sharp}_{(u;\tau,\xi)}(h,x-\xi)\right\|dx \\ \leq C\cdot\mathrm{TV}\left\{u(\tau);\,]\xi-\rho,\xi[\,\cup\,]\xi,\xi+\rho[\right\}\,;$$

(B) for every $\tau\geq 0$, there exists a constant C such that, for every $\xi\in\,]a,b[$ and $h\in\left]0,(b-a)/2\hat\lambda\right[$

$$\frac{1}{h}\int_{a+h\hat\lambda}^{b-h\hat\lambda}\left\|u(\tau+h,x)-U^{\flat}_{(u;\tau,\xi)}(\tau+h,x)\right\|dx\leq C\cdot\mathrm{TV}\left\{u(\tau);\,]a,b[\right\}^2\,.$$

Conversely, if $u\colon[0,T]\mapsto\mathcal{D}$ is $\mathbf{L^1}$-Lipschitz continuous and satisfies **(B)** together with **(C)** for every $\tau\geq 0$ and $\xi\in\mathbf{R}$

$$\lim_{h\to 0+}\frac{1}{h}\int_{\xi-h\hat\lambda}^{\xi+h\hat\lambda}\left\|u(\tau+h,x)-U^{\sharp}_{(u;\tau,\xi)}(h,x-\xi)\right\|dx=0\,,$$

then u coincides with a trajectory of the ΨRS, i.e., $u(t,\cdot)=S_t u(0,\cdot)$.

The proof is analogous to the one given in [5, Theorem 9.2] and is here omitted.

3. The Stability of the ΨRS. This section, inspired by [4], studies the dependence of a ΨRS on the flow f in 1.1 and on the admissibility function Ψ in 1.3.

Let $\mathrm{Hyp}(\Omega_0\cup\Omega_1)$ be the set of couples (f,Ψ) with $f\colon\Omega_0\cup\Omega_1\mapsto\mathbf{R}^n$ and $\Psi\colon(\Omega_0\times\Omega_1)\cup(\Omega_1\times\Omega_0)\mapsto\mathbf{R}^{j-h}$ that generate a ΨRS $S^{f,\Psi}\colon[0,+\infty[\,\times\mathcal{D}^{f,\Psi}\mapsto\mathcal{D}^{f,\Psi}$. Call $\mathcal{P}^{f,\Psi}$ the set of pairs (u^l,u^r) such that the Riemann data $u^l\chi_{]-\infty,0]}+u^r\chi_{]0,+\infty[}$ is in $\mathcal{D}^{f,\Psi}$. Moreover, $\mathcal{R}^{f,\Psi}\colon\mathcal{P}^{f,\Psi}\mapsto\mathcal{D}^{f,\Psi}$ is the Riemann solver defined by f and Ψ; that is, $x\mapsto\mathcal{R}^{f,\Psi}(u^l,u^r)$ is the Ψ-admissible solution to the Riemann problem 2.5 evaluated at, say, time $t=1$.

Let (g_1,Φ_1) and (g_2,Φ_2) be in $\mathrm{Hyp}(\Omega)$ with $\mathcal{P}^{g_1,\Phi_1}\supseteq\mathcal{P}^{g_2,\Phi_2}$. Define

$$\hat d\left((g_1,\Phi_1),(g_2,\Phi_2)\right)=\sup_{\mathcal{P}^{g_2,\Phi_2}}\frac{\left\|\mathcal{R}^{g_1,\Phi_1}(u^l,u^r)-\mathcal{R}^{g_2,\Phi_2}(u^l,u^r)\right\|_{\mathbf{L^1}}}{\|u^+-u^-\|}\,. \tag{3.1}$$

A map of this kind was first introduced in [3]. Note that $\hat d$ is *not* a distance because $\hat d\left((g_1,\Phi_1),(g_2,\Phi_2)\right)=\hat d\left((g_1+C_1,K_1\Phi_1),(g_2+C_2,K_2\Phi_2)\right)$, for all constants C_1, C_2, K_1, and K_2.

The following result is proved as in [4, Theorem 2.1].

THEOREM 3.1. Let $(g_i, \Phi_i) \in \mathrm{Hyp}(\Omega)$ for $i = 1, 2$. Denote for brevity S^i, $\mathcal{D}^i$, and L_i the respective $\Phi_i RS$'s, the domains, and Lipschitz constants. If $\mathcal{D}^1 \supseteq \mathcal{D}^2$ then, for every $u \in \mathcal{D}^2$,

$$\left\| S_t^1 u - S_t^2 u \right\|_{\mathbf{L}^1} \leq L_1 \cdot \hat{d}\left((g_1, \Phi_1), (g_2, \Phi_2)\right) \cdot \int_0^t \mathrm{TV}(S_t^2 u)\, dt\,. \tag{3.2}$$

The above estimate 3.2 does not allow an immediate understanding of which norms of f and Ψ have a role in bounding the dependence of S^i on (g_i, Φ_i). We therefore provide the following more explicit result. Below, if $\phi\colon \Omega \mapsto \mathbf{R}^n$, we denote $\|\phi\|_{\mathbf{C}^1} = \sup_{u\in\Omega} \left(\|\phi(u)\| + \|D\phi(u)\|\right)$. The closed sphere in $\mathbf{R}^n$ centered at u with radius δ is denoted by $B(u, \delta)$.

COROLLARY 3.2. Assume that $f\colon \Omega_0 \cup \Omega_1 \mapsto \mathbf{R}^n$ satisfies **(1)**. Fix $j, h \in \{1, \ldots, n\}$ with $j > h$, $K > 0$ and choose $\Psi\colon (\Omega_0 \times \Omega_1) \cup (\Omega_1 \times \Omega_0) \mapsto \mathbf{R}^{j-h}$, $\underline{u}^l$ in Ω_0, and $\underline{u}^r$ in Ω_1 such that **(2)** holds. Then, there exist positive C, M, and δ such that:

(1) $(f, \Psi) \in \mathrm{Hyp}(\Omega_0 \cup \Omega_1)$ and if $u \in \mathcal{D}^{f,\Psi}$ with $u(\mathbf{R}) \subseteq B(\underline{u}^l, \delta) \cup B(\underline{u}^r, \delta)$ and $\mathrm{TV}(u) \leq K$, then for a.e. $t \geq 0$, $S_t^{f,\Psi} u(\mathbf{R}) \subseteq B(\underline{u}^l, \delta) \cup B(\underline{u}^r, \delta)$ and $\mathrm{TV}(S_t^{f,\Psi} u) \leq M$. Moreover, $\mathcal{P}^{f,\Psi} \supseteq B(\underline{u}^l, \delta) \times B(\underline{u}^r, \delta)$;

(2) if (g, Φ) is such that $\|g - f\|_{\mathbf{C}^1} < \delta$ and $\|\Phi - \Psi\|_{\mathbf{C}^1} < \delta$, then $(g, \Phi) \in \mathrm{Hyp}(\Omega)$ and $\mathcal{P}^{g,\Phi} \supseteq B(\underline{u}^l, \delta) \times B(\underline{u}^r, \delta)$;

(3) if, for $i = 1, 2$, (g_i, Φ_i) are such that $\|g_i - f\|_{\mathbf{C}^1} < \delta$, $\|\Phi_i - \Psi\|_{\mathbf{C}^1} < \delta$, and $\mathcal{D}^1 \supseteq \mathcal{D}^2$, then for all $u \in \mathcal{D}^2$,

$$\left\| S_t^1 u - S_t^2 u \right\|_{\mathbf{L}^1} \leq C \cdot \left(\|g_1 - g_2\|_{\mathbf{C}^1} + \|\Phi_1 - \Phi_2\|_{\mathbf{C}^1}\right) \cdot t\,. \tag{3.3}$$

Proof. (1) and (2) follow from Theorem 2.4 and the implicit function theorem. According to Theorem 3.1, we need to prove that

$$\hat{d}\left((g_1, \Phi_1), (g_2, \Phi_2)\right) \leq C \cdot \left(\|g_1 - g_2\|_{\mathbf{C}^1} + \|\Phi_1 - \Phi_2\|_{\mathbf{C}^1}\right)\,.$$

In turn, the latter estimate holds provided

$$\begin{aligned} &\left\| \mathcal{R}^2(u^l, u^r) - \mathcal{R}^1(u^l, u^r) \right\|_{\mathbf{L}^1} \\ &\leq C \cdot \left(\|g_1 - g_2\|_{\mathbf{C}^1} + \|\Phi_1 - \Phi_2\|_{\mathbf{C}^1}\right) \cdot \left\| u^r - u^l \right\| \end{aligned} \tag{3.4}$$

for all $(u^l, u^r) \in \mathcal{P}^2$. It is sufficient to verify 3.4 in the case that $\mathcal{R}^2(u^l, u^r)$ has a single jump. If this jump is a Lax shock, then the same computations in [4] apply. Therefore, we assume that $\mathcal{R}^2(u^l, u^r)$ consists of a Φ_2-admissible phase boundary.

Let $x = \Lambda_2 t$ be the support of the Φ_2-admissible phase boundary in $u^2 = \mathcal{R}^2(u^l, u^r)$. With reference to the pair (g_1, Φ_1), let $\sigma \mapsto \mathcal{L}_\ell^+(u_o, \sigma)$ be the Lax shock-rarefaction curve of the ℓ-th family exiting u_o parametrized by, say, arc length. Similarly, $\sigma \mapsto \mathcal{L}_\ell^-(u_o, \sigma)$ is the ℓ-th Lax curve entering u_o, that is, if $u^l = \mathcal{L}_\ell^-(u_o, \sigma)$ and $u^r = u_o$, then 2.5 is solved by a single Lax ℓ wave. Then, $\mathcal{R}^1(u^l, u^r)$ attains the states

$$\text{in } \Omega_0 \quad \begin{array}{rcr} v_0 &=& u^l \\ v_1 &=& \mathcal{L}_1^+(v_0, \sigma_1) \\ \cdots && \cdots \\ v_{j-1} &=& \mathcal{L}_{j-1}^+(v_{j-2}, \sigma_{j-1}) \end{array} \qquad \text{in } \Omega_1 \quad \begin{array}{rcr} w_h &=& \mathcal{L}_{h+1}^-(w_{h+1}, \sigma_{h+1}) \\ \cdots && \cdots \\ w_{n-1} &=& \mathcal{L}_n^-(w_n, \sigma_n) \\ w_n &=& u^r \end{array}$$

with, moreover,

$$\begin{aligned}\Lambda_1 \cdot (w_h - v_{j-1}) &= f(w_h) - f(v_{j-1}) \\ \Phi_1(v_{j-1}, w_h) &= 0\,.\end{aligned}$$

Then

$$\begin{aligned}&\left\|\mathcal{R}^2(u^l,u^r) - \mathcal{R}^1(u^l,u^r)\right\| \\ &\leq \int_{-\infty}^{\min\{\Lambda_1 t,\Lambda_2 t\}} \left\|\mathcal{R}^2(u^l,u^r)(t) - \mathcal{R}^1(u^l,u^r)(t)\right\| dt \\ &\quad + \left|\int_{\Lambda_1 t}^{\Lambda_2 t} \left\|\mathcal{R}^2(u^l,u^r)(t) - \mathcal{R}^1(u^l,u^r)(t)\right\| dt\right| \\ &\quad + \int_{\max\{\Lambda_1 t,\Lambda_2 t\}}^{+\infty} \left\|\mathcal{R}^2(u^l,u^r)(t) - \mathcal{R}^1(u^l,u^r)(t)\right\| dt \\ &\leq \emptyset \sum_{\ell=1}^{j-1} \left\|v_\ell - u^l\right\| \\ &\quad + \mathcal{O}(1)\left(\|g_2 - g_1\|_{\mathbf{C}^1} + \|\Phi_2 - \Phi_1\|_{\mathbf{C}^1}\right) \\ &\qquad \cdot \left(\sum_{\ell=1}^{j-1} \left\|v_\ell - u^l\right\| + \sum_{\ell=h}^{n-1} \|w_\ell - u^r\|\right) \\ &\quad + \mathcal{O}(1) \sum_{\ell=h}^{n-1} \|w_\ell - u^r\| \\ &\leq \mathcal{O}(1)\left(\|g_2 - g_1\|_{\mathbf{C}^1} + \|\Phi_2 - \Phi_1\|_{\mathbf{C}^1} + 1\right)\left(\sum_{\ell=1}^{j-1} |\sigma_\ell| + \sum_{\ell=h+1}^{n} |\sigma_\ell|\right).\end{aligned}$$

Let $E^i(u^l,u^r) = (\sigma_1, \ldots, \sigma_{j-1}, \sigma_{h+1}, \ldots, \sigma_n)$ denote the vector of the wave sizes. Note that E^i is implicitly defined through g^i and Φ^i. Moreover, $E^2(u^l,u^r) = (0, \ldots, 0)$, so that

$$\begin{aligned}&\left\|\mathcal{R}^2(u^l,u^r) - \mathcal{R}^1(u^l,u^r)\right\| \\ &\leq \mathcal{O}(1)\left(\|g_2 - g_1\|_{\mathbf{C}^1} + \|\Phi_2 - \Phi_1\|_{\mathbf{C}^1} + 1\right)\left(E^1(u^l,u^r) - E^2(u^l,u^r)\right) \\ &\leq \mathcal{O}(1)\left(\|g_2 - g_1\|_{\mathbf{C}^1} + \|\Phi_2 - \Phi_1\|_{\mathbf{C}^1} + 1\right) \\ &\qquad \cdot \sup_{\mathcal{P}^2} \left\|E^1(u^l,u^r) - E^2(u^l,u^r)\right\|_{\mathbf{C}^0} \\ &\leq \mathcal{O}(1)\left(\|g_2 - g_1\|_{\mathbf{C}^1} + \|\Phi_2 - \Phi_1\|_{\mathbf{C}^1}\right),\end{aligned}$$

where the last estimate follows from the Lipschitzean dependence of the implicit function upon the defining function. Now, 3.4 directly follows choosing δ sufficiently small also with respect to $\left\|\underline{u}^l - \underline{u}^r\right\|$. □

Note that the bound 3.3 differs from the analogous one in [4]. Indeed, consider the case $\Phi_1 = \Psi = \Phi_2$, that is, the admissibility function is the same for both flows. The estimate obtained in [4] is

$$\left\|S_t^1 u - S_t^2 u\right\|_{\mathbf{L}^1} \leq C \cdot \left(\|Dg_1 - Dg_2\|_{\mathbf{C}^0}\right) \cdot t\,. \tag{3.5}$$

The estimate 3.3 implies 3.5 as soon as Ω is connected, thanks to an elementary argument. However, an estimate like 3.5 may not hold in the present case where Ω is the *disjoint* union of two phases. In fact, consider the case

$$g_1 = f, \quad g_2 = \begin{cases} f & \text{in } \Omega_0 \\ f + c & \text{in } \Omega_1 \end{cases}$$

for a constant $c \neq 0$. The semigroup generated by (g_1, Ψ) is *different* from the semigroup generated by (g_2, Ψ), since the Rankine-Hugoniot conditions differ on all phase transitions. Hence, estimate 3.5 may not hold and needs to be replaced by 3.3.

4. Applications. In this section we show some applications of Corollary 3.2. The first example concerns the system

$$\begin{cases} \partial_t v - \partial_x \sigma(w) = 0 \\ \partial_t w - \partial_x v = 0. \end{cases} \tag{4.1}$$

This is a standard model [11] for longitudinal motions of an elastic bar with unit cross-sectional area; here v is the particle velocity, w is the strain, and $\sigma = \sigma(w)$ is the stress function. In order to model phase transitions the function σ must be chosen non-monotone; we assume that it is the cubic stress

$$\sigma(w) = \sigma_m + k(w - a)\left(w - \frac{a+b}{2}\right)(w - b),$$

with $a < b$ and σ_m, k real numbers; see [22]. In this case $\Omega_0 = \mathbf{R} \times]-\infty, \alpha[$, $\Omega_1 = \mathbf{R} \times]\beta, +\infty[$, where α and β are respectively the maximum and minimum of σ. An explicit kinetic condition for $(2, 1)$ phase boundaries is obtained in [22] by looking at travelling waves of an enlarged system where viscosity and a strain-gradient term are present. This condition is 1.3 with

$$\Phi_\omega(w^-, w^+) = 3\left(1 - \tfrac{12}{\omega^2}\right)\left(\tfrac{w^+ + w^-}{b-a} - \tfrac{a+b}{b-a}\right)^2 + \left(\tfrac{w^+ - w^-}{b-a}\right)^2 - 1\,. \tag{4.2}$$

Above $\omega = \eta/\sqrt{\varepsilon}$, where η (resp. ε) is the viscosity (strain-gradient term) coefficient. A straightforward computation shows that

$$\|\Phi_{\omega_1} - \Phi_{\omega_2}\|_{\mathbf{C}^1} = \frac{36}{(b-a)^2} \cdot \left(C_\delta^2 + 2\sqrt{2}C_\delta\right) \cdot \left|\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right|$$

where $C_\delta = \sup\{|w^+ + w^- - (a+b)|;\ |w^\pm - \underline{w}^\pm| \leq \delta\}$. A semigroup for a large class of systems including 4.1 was constructed in [7], under generalized kinetic relations comprising 4.2. Then 3.3 applies.

Also in the case that the stress function σ is trilinear, an explicit kinetic condition is at disposal, [1]. An estimate analogous to that above, though a bit more complicated, can be given as well.

Finally we consider weak deflagrations within the following standard [8, 9, 12, 14] combustion model. At time $t = 0$, burnt gas covers the half-line $x < 0$, while unburnt gas fills $x > 0$. In Eulerian coordinates,

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p) = 0 \\ \partial_t E + \partial_x(Ev + pv) = 0 \end{cases} \tag{4.3}$$

where ρ is the density, v is the velocity, $m = \rho v$ is the momentum, p is the pressure, $E = \rho\varepsilon + \rho v^2/2$ is the total energy, and ε is the total internal energy. We assume that

$$\varepsilon = \begin{cases} e & \text{for burnt gas} \\ e + q & \text{for unburnt gas} \end{cases}$$

where $e = \frac{1}{\gamma-1}\frac{p}{\rho}$ is the internal nonchemical energy. We assume $q > 0$; that is, the reaction is exothermic. Explicit values for q can be found, for instance, in [12]. The boundary between the two gases, namely, the reaction front, can be considered as a $(3,2)$ phase boundary in the case of weak deflagrations; see [14, Figure 4.7.2]. The Riemann problem in this case is 1-underdetermined and a kinetic condition was proposed in [21]. This condition is of the form $v^+ + K(T^+)^Q - \Lambda = 0$ where v^+ and T^+ are the velocity and the temperature of the unburnt gas, Λ the propagation speed of the front, K and Q constants. We assume for simplicity $Q = 1/2$ (laminary flames). Under a γ-law for the pressure, the kinetic condition above reads

$$\Phi_\omega(u^-, u^+) = \frac{m^+}{\rho^+} + \omega\sqrt{e^+} - \Lambda(u^-, u^+)$$

where $\omega = K\sqrt{\gamma - 1}$; see [9]. Then

$$\|\Phi_{\omega_1} - \Phi_{\omega_2}\|_{\mathbf{C}^1} = \left(\sup\sqrt{e^+} + \sup\left\|\nabla\sqrt{e^+}\right\|\right)\cdot(\omega_1 - \omega_2). \tag{4.4}$$

Therefore Corollary 3.2 applies and 3.3 measures the distance between two solutions. Notice that since the difference in 4.4 is computed only for the states on the right, the remark after the end of the proof of Corollary 3.2 applies and the term $\sup\sqrt{e^+}$ can be estimated by $\sup\left\|\nabla\sqrt{e^+}\right\|$.

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