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Research Article

Stability of Trigonometric Functional Equations in Generalized Functions

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We consider the Hyers-Ulam stability of a class of trigonometric functional equations in the spaces of generalized functions such as Schwartz distributions, Fourier hyperfunctions, and Gelfand generalized functions.

1. Introduction

The Hyers-Ulam stability of functional equations go back to 1940 when Ulam proposed the following problem [1]:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d(f(xy), f(x)f(y)) \le \epsilon. \tag{1.1}$$

Then does there exist a group homomorphism L and $\delta_\epsilon > 0$ such that

$$d(f(x), L(x)) \le \delta_\epsilon \tag{1.2}$$

for all $x \in G_1$?

This problem was solved affirmatively by Hyers [2] under the assumption that G_2 is a Banach space. In 1949-1950, this result was generalized by the authors Bourgin [3, 4] and Aoki [5] and since then stability problems of many other functional equations have been investigated [2, 6–8, 8–19]. In 1990, Székelyhidi [18] has developed his idea of using invariant subspaces of functions defined on a group or semigroup in connection with stability questions for the sine and cosine functional equations. We refer the reader to [6–8, 12–14, 19] for Hyers-Ulam stability of functional equations of trigonometric type. In this paper, following the method of Székelyhidi [18] we consider a distributional analogue of the Hyers-Ulam stability problem of the trigonometric functional equations

$$f(x-y) = f(x)g(y) - g(x)f(y), \tag{1.3}$$

$$g(x-y) = g(x)g(y) + f(x)f(y), \tag{1.4}$$

where $f, g : \mathbb{R}^n \to \mathbb{C}$. Following the formulations as in [6, 20–22], we generalize the classical stability problems of above functional equations to the spaces of generalized functions u, v as

$$u \circ S - u \otimes v + v \otimes u \in L^\infty\left(\mathbb{R}^{2n}\right), \tag{1.5}$$

$$v \circ S - v \otimes v - u \otimes u \in L^\infty\left(\mathbb{R}^{2n}\right), \tag{1.6}$$

where $S(x,y) = x - y$, $x, y \in \mathbb{R}^n$, $\circ$ and $\otimes$ denote the pullback and the tensor product of generalized functions, respectively, and $L^\infty(\mathbb{R}^{2n})$ denotes the space of bounded measurable functions on $\mathbb{R}^{2n}$.

We prove as results that if generalized function (u,v) satisfies (1.5), then (u,v) satisfies one of the followings:

(i) $u \equiv 0$ and v is arbitrary;

(ii) u and v are bounded measurable functions;

(iii) $u = c \cdot x + r(x)$, $v = \lambda(c \cdot x + r(x)) + 1$;

(iv) $u = \lambda \sin(c \cdot x)$, $v = \cos(c \cdot x) + \lambda \sin(c \cdot x)$,

for some $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, and a bounded measurable function $r(x)$.

Also if generalized function (u,v) satisfies (1.6), then (u,v) satisfies one of the followings:

(i) u and v are bounded measurable functions,

(ii) $u = \cos(c \cdot x)$, $v = \sin(c \cdot x)$, $c \in \mathbb{C}^n$.

2. Generalized Functions

For the spaces of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, we refer the reader to [23–25]. Here we briefly introduce the spaces of Gelfand generalized functions and Fourier hyperfunctions. Here we use the following notations: $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = \partial/\partial x_j$.

Definition 2.1. For given $r, s \geq 0$, we denote by $\mathcal{S}_r^s$ or $\mathcal{S}_r^s(\mathbb{R}^n)$ the space of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ such that there exist positive constants A and B satisfying

$$\|\varphi\|_{h,k} := \sup_{x \in \mathbb{R}^n,\ \alpha,\beta \in \mathbb{N}_0^n} \frac{\left|x^\alpha \partial^\beta \varphi(x)\right|}{h^{|\alpha|} k^{|\beta|} \alpha!^r \beta!^s} < \infty. \tag{2.1}$$

The topology on the space $\mathcal{S}_r^s$ is defined by the seminorms $\|\cdot\|_{h,k}$ in the left-hand side of (2.1) and the elements of the dual space $\mathcal{S}'^s_r$ of $\mathcal{S}_r^s$ are called *Gelfand-Shilov generalized functions*. In particular, we denote $\mathcal{S}'^1_1$ by $\mathcal{F}'$ and call its elements *Fourier hyperfunctions*.

It is known that if $r > 0$ and $0 \le s < 1$, the space $\mathcal{S}_r^s(\mathbb{R}^n)$ consists of all infinitely differentiable functions $\varphi(x)$ on $\mathbb{R}^n$ that can be continued to an entire function on $\mathbb{C}^n$ satisfying

$$\left|\varphi(x+iy)\right| \le C\exp\left(-a|x|^{1/r} + b\left|y\right|^{1/(1-s)}\right) \tag{2.2}$$

for some $a, b > 0$.

It is well known that the following topological inclusions hold:

$$\mathcal{S}_{1/2}^{1/2} \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{S}, \qquad \mathcal{S}' \hookrightarrow \mathcal{F}' \hookrightarrow \mathcal{S}'^{1/2}_{1/2}. \tag{2.3}$$

We refer the reader to [24, chapter V-VI], for tensor products and pullbacks of generalized functions.

3. Stability of the Equations

In view of (2.2), it is easy to see that the n-dimensional heat kernel

$$E_t(x) = (4\pi t)^{-n/2}\exp\left(-\frac{|x|^2}{4t}\right), \quad t > 0, \tag{3.1}$$

belongs to the Gelfand-Shilov space $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^n)$ for each $t > 0$. Thus the convolution $(u * E_t)(x) := \langle u_y, E_t(x-y)\rangle$ is well defined for all $u \in \mathcal{S}'^{1/2}_{1/2}$. It is well known that $U(x,t) = (u * E_t)(x)$ is a smooth solution of the heat equation $(\partial/\partial_t - \Delta)U = 0$ in $\{(x,t) : x \in \mathbb{R}^n,\ t > 0\}$ and $(u * E_t)(x) \to u$ as $t \to 0^+$ in the sense of generalized functions, that is, for every $\varphi \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^n)$,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+}\int (u * E_t)(x)\varphi(x)dx. \tag{3.2}$$

We call $(u * E_t)(x)$ *the Gauss transform* of u. Let $\langle G, +\rangle$ be a semigroup and $\mathbb{C}$ be the field of complex numbers. A function $l : G \to \mathbb{C}$ is said to be *additive* provided $l(x+y) = l(x) + l(y)$, and $m : G \to \mathbb{C}$ is said to be *exponential* provided $m(x+y) = m(x)m(y)$.

We first discuss the solutions of the corresponding trigonometric functional equations in the space $\mathcal{S}'^{1/2}_{1/2}$ of Gelfand generalized functions. As a consequence of the result [6, 26], we have the following.

Lemma 3.1. *The solutions $u, v \in \mathcal{S}'^{1/2}_{1/2}$ of the equation*

$$\begin{aligned} &u \circ S - u \otimes v + v \otimes u = 0, \\ &v \circ S - v \otimes v - u \otimes u = 0, \end{aligned} \tag{3.3}$$

are equal, respectively, to the smooth solution f, g of corresponding classical functional equations (1.3) *and* (1.4).

Remark 3.2. We refer the reader to Aczél [27, page 180] and Aczél and Dhombres [28, pages 209–217] for the general solutions and measurable solutions of (1.3) and (1.4).

For the proof of the stability of (1.5), we need the following lemma.

Lemma 3.3. *Let $U, V : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ be continuous functions satisfying the inequality; there exists a positive constant M such that*

$$\left|U(x-y, t+s) - U(x,t)V(y,s) + V(x,t)U(y,s)\right| \leq M \tag{3.4}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Then either there exist $\lambda, \nu \in \mathbb{C}$, not both zero, and $L > 0$ such that

$$|\lambda U(x,t) - \nu V(x,t)| \leq L, \tag{3.5}$$

or else

$$U(x-y, t+s) - U(x,t)V(y,s) + V(x,t)U(y,s) = 0 \tag{3.6}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$.

Also the inequality (3.5) *together with* (3.4) *implies one of the followings;*

(i) *$U = 0$, V: arbitrary;*

(ii) *U and V are bounded functions;*

(iii) *$U(x,t) = c \cdot x e^{-bt} + R(x,t)$ and $V(x,t) = \mu U(x,t) + e^{-bt}$ where $c \in \mathbb{C}^n$, b, $\mu \in \mathbb{C}$ with $\Re b \geq 0$ and R is a bounded function.*

Proof. Following the approach as in [29, page 139, Lemma 6.8], we first prove that (3.6) is satisfied if the condition (3.5) fails. Assume that $|\lambda U(x,t) - \rho V(y,s)| \leq L$ for some $L > 0$ implies $\lambda = \rho = 0$. Let

$$F(x,y,t,s) = U(x+y, t+s) - U(x,t)V(-y,s) + V(x,t)U(-y,s). \tag{3.7}$$

Then we can choose y_1 and s_1 satisfying $U(-y_1, s_1) \neq 0$. It is easy to show that

$$V(x,t) = \lambda_0 U(x,t) + \lambda_1 U(x+y_1, t+s_1) - \lambda_1 F(x, y_1, t, s_1), \tag{3.8}$$

where $\lambda_0 = V(-y_1, s_1)/U(-y_1, s_1)$, and $\lambda_1 = -1/U(-y_1, s_1)$.

Using (3.8), we have the following two equations:

$$\begin{aligned}
&U((x+y)+z,(t+s)+r)\\
&=U(x+y,t+s)V(-z,r)-V(x+y,t+s)U(-z,r)+F(x+y,z,t+s,r)\\
&=(U(x,t)V(-y,s)-V(x,t)U(-y,s)+F(x,y,t,s))V(-z,r)\\
&\quad-(\lambda_0U(x+y,t+s)+\lambda_1U(x+y+y_1,t+s+s_1)-\lambda_1F(x+y,y_1,t+s,s_1))U(-z,r)\\
&\quad+F(x+y,z,t+s,r)\\
&=(U(x,t)V(-y,s)-V(x,t)U(-y,s)+F(x,y,t,s))V(-z,r)\\
&\quad-\lambda_0(U(x,t)V(-y,s)-V(x,t)U(-y,s)+F(x,y,t,s))U(-z,r)\\
&\quad-\lambda_1(U(x,t)V(-y-y_1,s+s_1)-V(x,t)U(-y-y_1,s+s_1)+F(x,y+y_1,t,s+s_1))U(-z,r)\\
&\quad+\lambda_1F(x+y,y_1,t+s,s_1)U(-z,r)+F(x+y,z,t+s,r),\\
&V(x+(y+z),t+(s+r))=U(x,t)V(-y-z,s+r)-V(x,t)U(-y-z,s+r)+F(x,y+z,t,s+r).
\end{aligned}\tag{3.9}$$

From (3.9), we have

$$\begin{aligned}
&U(x,t)(V(-y,s)V(-z,r)-\lambda_0V(-y,s)U(-z,r)-\lambda_1V(-y-y_1,s+s_1)U(-z,r)-V(-y-z,s+r))\\
&\quad+V(x,t)(-U(-y,s)V(-z,r)+\lambda_0U(-y,s)U(-z,r)\\
&\qquad\qquad+\lambda_1U(-y-y_1,s+s_1)U(-z,r)+U(-y-z,s+r))\\
&=-F(x,y,t,s)V(-z,r)+\lambda_0F(x,y,t,s)U(-z,r)+\lambda_1F(x,y+y_1,t,s+s_1)U(-z,r)\\
&\quad-\lambda_1F(x+y,y_1,t+s,s_1)U(-z,r)-F(x+y,z,t+s,r)+F(x,y+z,t,s+r).
\end{aligned}\tag{3.10}$$

When y, s, z, r are fixed, the right-hand side of the above equation is bounded, so we have the following.

$$\begin{aligned}
&F(x,y+z,t,s+r)-F(x+y,z,t+s,r)\\
&\quad=F(x,y,t,s)V(-z,r)\\
&\qquad+(-\lambda_0F(x,y,t,s)-\lambda_1F(x,y+y_1,t,s+s_1)+\lambda_1F(x+y,y_1,t+s,s_1))U(-z,r).
\end{aligned}\tag{3.11}$$

Again considering (3.11) as a function of z and r for all fixed x, y, t, s, we have $F(x,y,t,s)\equiv 0$.

Now we consider the case that the inequality (3.5) holds. If $U=0$, V is arbitrary, which is the case (i). If U is a nontrivial bounded function, in view of (3.4) V is also bounded, which is the case (ii). If U is unbounded, it follows from (3.5) that $\nu \neq 0$ and

$$V(x,t) = \mu U(x,t) + B(x,t) \tag{3.12}$$

for some $\mu \in \mathbb{C}$ and a bounded function B. Put (3.12) in (3.4) to get

$$\left|U(x-y,t+s) - U(x,t)B(y,s) + B(x,t)U(y,s)\right| \leq M. \tag{3.13}$$

Replacing (x,t) by (y,s) and (y,s) by (x,t) and using the triangle inequality, we have

$$|U(x,t) + U(-x,t)| \leq 2M \tag{3.14}$$

for all $x \in \mathbb{R}^n,\ t > 0$. Replacing x by $-x$, y by $-y$ and using the inequality (3.14), we have for some $M^* > 0$,

$$\left|U(-x+y,t+s) + U(x,t)B(-y,s) - B(-x,t)U(y,s)\right| \leq M^*. \tag{3.15}$$

Using (3.13), (3.14), (3.15), and the triangle inequality, we have

$$\left|U(x,t)(B(y,s) - B(-y,s)) - U(y,s)(B(x,t) - B(-x,t))\right| \leq M^* + 3M. \tag{3.16}$$

Since U is unbounded, it follows from (3.16) that $B(y,s) = B(-y,s)$ for all $y \in \mathbb{R}^n,\ s > 0$. Also, in view of (3.13), for fixed $y \in \mathbb{R}^n$ and $s > 0$, $x \to U(x+y,t+s) - U(x,t)B(-y,s)$ is a bounded function of x and t. Thus it follows from [24, page 104, Theorem 5.2] that $B(-y,s) = B(y,s)$ is an exponential function. Given the continuity of U, V, we have $B(x,t) = e^{-bt}$ for some $b \in \mathbb{C}$ with $\Re b \geq 0$. Replacing y by $-y$ in (3.13) and using (3.14), we have

$$\left|U(x+y,t+s) - U(x,t)e^{-bs} - U(y,s)e^{-bt}\right| \leq 3M. \tag{3.17}$$

Now we consider the stability of (3.17). From (3.17) and the continuity of U, it is easy to see that

$$\limsup_{t \to 0^+} U(x,t) := f(x) \tag{3.18}$$

exists. Putting $y = 0$ and letting $t \to 0^+$ so that $U(x,t) \to f(x)$ in (3.17) and using the triangle inequality and (3.14), we have

$$\left|U(x,s) - f(x)e^{-bs}\right| \leq 3M + |U(0,s)| \leq 4M. \tag{3.19}$$

It follows from (3.17), (3.19), and the triangle inequality that

$$\left|f(x+y) - f(x) - f(y)\right| \leq 15Me^{b(t+s)} \tag{3.20}$$

for all $x, y \in \mathbb{R}^n,\ t, s > 0$. Letting $t, s \to 0^+$ in (3.20), we have

$$\left|f(x+y) - f(x) - f(y)\right| \leq 15M \tag{3.21}$$

for all $x, y \in \mathbb{R}^n$. Thus it follows from the Hyer-Ulam stability theorem [2] and the continuity of f that there exists $c \in \mathbb{C}^n$ such that

$$\left|f(x) - c \cdot x\right| \leq 15M \tag{3.22}$$

for all $x \in \mathbb{R}^n$. Finally, from (3.19) and (3.22), we have

$$\left|U(x,t) - c \cdot x e^{-bt}\right| \leq 19M \tag{3.23}$$

for all $x \in \mathbb{R}^n$. From (3.12) and (3.23), we have (iii). This completes the proof. □

Remark 3.4. In particular, if U and V are solutions of the heat equation the case (iii) of the abovelemma is reduced as

$$U(x,t) = c \cdot x + R(x,t), \qquad V(x,t) = \mu U(x,t) + 1. \tag{3.24}$$

for some $\mu \in \mathbb{C}$ and bounded solution $R(x,t)$ of the heat equation.

Theorem 3.5. *Let $u, v \in \mathcal{S}'^{1/2}_{1/2}$ satisfy* (1.5). *Then u and v satisfy one of the followings*:

(i) $u \equiv 0$ *and v is arbitrary;*

(ii) *u and v are bounded measurable functions;*

(iii) $u = c \cdot x + r(x)$, $v = \lambda(c \cdot x + r(x)) + 1$;

(iv) $u = \lambda \sin(c \cdot x)$, $v = \cos(c \cdot x) + \lambda \sin(c \cdot x)$,

for some $c \in \mathbb{C}^n,\ \lambda \in \mathbb{C}$, and a bounded measurable function $r(x)$.

Proof. Convolving in (1.5) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels, we have

$$\begin{aligned} \left[(u \circ A) * (E_t(\xi)E_s(\eta))\right](x,y) &= \left\langle u_\xi, \int E_t(x-\xi+\eta)E_s(y-\eta)\,d\eta \right\rangle \\ &= \left\langle u_\xi, (E_t * E_s)(x+y-\xi)\right\rangle \\ &= \left\langle u_\xi, (E_{t+s})(x+y-\xi)\right\rangle \\ &= U(x+y, t+s). \end{aligned} \tag{3.25}$$

Similarly, we have

$$\begin{aligned} \left[(u \otimes v) * (E_t(\xi)E_s(\eta))\right](x,y) &= U(x,t)V(y,s), \\ \left[(v \otimes u) * (E_t(\xi)E_s(\eta))\right](x,y) &= V(x,t)U(y,s), \end{aligned} \tag{3.26}$$

where $U(x,t), V(x,t)$ are the Gauss transforms of u, v, respectively. Thus U, V satisfies the inequality (3.4). Now we can apply Lemma 3.3. First we assume that (3.5) holds and consider the cases (i), (ii), (iii) of Lemma 3.3. The case (i) implies (i) of our theorem. For the case (ii); it follows from [30, Page 61, Theorem 1] the initial values u, v of $U(x,t), V(x,t)$ as $t \to 0^+$ are bounded measurable functions, respectively. For the case (iii); using the remark running after Lemma 3.3 and [30, Page 61, Theorem 1], letting $t \longrightarrow 0^+$ the case (iii) of our theorem follows. Finally, if U, V satisfy the (3.6), letting $t, s \longrightarrow 0^+$ we have

$$u \circ S - u \otimes v + v \otimes u = 0. \tag{3.27}$$

The nontrivial solutions of (3.27) are given by (iv) or $u = c \cdot x$, $v = 1 + \lambda c \cdot x$ which is included in case (iii). This completes the proof. □

Now we prove the stability of (1.6).

Lemma 3.6. *Let $U, V : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ satisfy the inequality; there exists a positive constant M such that*

$$\left| V(x-y, t+s) - V(x,t)V(y,s) - U(x,t)U(y,s) \right| \leq M \tag{3.28}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$. Then either there exist $\lambda, \nu \in \mathbb{C}$, not both zero, and $L > 0$ such that

$$|\lambda U(x,t) - \nu V(x,t)| \leq L, \tag{3.29}$$

or else

$$V(x-y, t+s) - V(x,t)V(y,s) - U(x,t)U(y,s) = 0 \tag{3.30}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$.

Also the inequality (3.29) *together with* (3.28) *implies one of the followings*:

(i) *U and V are bounded functions;*

(ii) *U is bounded; V is an exponential;*

(iii) *$U = \pm i(V - m)$ for some bounded exponential m;*

(iv) *$U = \mu(m - R)/(\mu^2 + 1)$, $V = \mu^2 m + R/(\mu^2 + 1)$, where $\mu \in \mathbb{C}$, m is a bounded exponential and R is a bounded function.*

Proof. Suppose that, for $L > 0$, $|\lambda U(x,t) - \nu V(y,s)| \leq L$ does not hold unless $\lambda = \nu = 0$. Note that both U and V are unbounded. Let

$$F(x, -y, t, s) = V(x-y, t+s) - V(x,t)V(y,s) - U(x,t)U(y,s). \tag{3.31}$$

Just for convenience, we consider the following equation which is equivalent to (3.31).

$$F(x, y, t, s) = V(x+y, t+s) - V(x,t)V(-y,s) - U(x,t)U(-y,s). \tag{3.32}$$

Since U is nonconstant, we can choose y_1 and s_1 satisfying $U(-y_1, s_1) \neq 0$. It is easy to show that

$$U(x,t) = \lambda_0 V(x,t) + \lambda_1 V(x + y_1, t + s_1) - \lambda_1 F(x, y_1, t, s_1), \tag{3.33}$$

where $\lambda_0 = -V(-y_1, s_1)/U(-y_1, s_1)$ and $\lambda_1 = 1/U(-y_1, s_1)$.

By the definition of F and the use of (3.33), we have the following equations

$$\begin{aligned}
&V((x+y)+z,(t+s)+r) \\
&= V(x+y,t+s)V(-z,r) + U(x+y,t+s)U(-z,r) + F(x+y,z,t+s,r) \\
&= (V(x,t)V(-y,s) + U(x,t)U(-y,s) + F(x,y,t,s))V(-z,r) \\
&\quad + (\lambda_0 V(x+y,t+s) + \lambda_1 V(x+y+y_1,t+s+s_1) - \lambda_1 F(x+y,y_1,t+s,s_1))U(-z,r) \\
&\quad + F(x+y,z,t+s,r) \\
&= (V(x,t)V(-y,s) + U(x,t)U(-y,s) + F(x,y,t,s))V(-z,r) \\
&\quad + \lambda_0(V(x,t)V(-y,s) + U(x,t)U(-y,s) + F(x,y,t,s))U(-z,r) \\
&\quad + \lambda_1(V(x,t)V(-y-y_1,s+s_1) + U(x,t)U(-y-y_1,s+s_1) + F(x,y+y_1,t,s+s_1))U(-z,r) \\
&\quad - \lambda_1 F(x+y,y_1,t+s,s_1)U(-z,r) + F(x+y,z,t+s,r), \\
&V(x+(y+z),t+(s+r)) = V(x,t)V(-y-z,s+r) + U(x,t)U(-y-z,s+r) + F(x,y+z,t,s+r).
\end{aligned} \tag{3.34}$$

By equating (3.34), we have

$$\begin{aligned}
&V(x,t)(V(-y,s)V(-z,r) + \lambda_0 V(-y,s)U(-z,r) + \lambda_1 V(-y-y_1,s+s_1)U(-z,r) - V(-y-z,s+r)) \\
&\quad + U(x,t)(U(-y,s)V(-z,r) + \lambda_0 U(-y,s)U(-z,r) \\
&\qquad + \lambda_1 U(-y-y_1,s+s_1)U(-z,r) - U(-y-z,s+r)) \\
&= -F(x,y,t,s)V(-z,r) - \lambda_0 F(x,y,t,s)U(-z,r) - \lambda_1 F(x,y+y_1,t,s+s_1)U(-z,r) \\
&\quad + \lambda_1 F(x+y,y_1,t+s,s_1)U(-z,r) - F(x+y,z,t+s,r) + F(x,y+z,t,s+r).
\end{aligned} \tag{3.35}$$

When y, s, z, r are fixed, the right-hand side of the above equality is bounded, so we have

$$\begin{aligned}
&F(x,y+z,t,s+r) - F(x+y,z,t+s,r) \\
&\quad = F(x,y,t,s)V(-z,r) \\
&\qquad + (\lambda_0 F(x,y,t,s) + \lambda_1 F(x,y+y_1,t,s+s_1) - \lambda_1 F(x+y,y_1,t+s,s_1))U(-z,r).
\end{aligned} \tag{3.36}$$

Again considering (3.36) as a function of z and r for all fixed x, y, t, s, we have $F(x,y,t,s) \equiv 0$ which is equivalent to (3.30). Now we consider the case when (3.29) holds. If U is bounded, then $V(x+y,t+s)-V(x,t)V(-y,s)$ is also bounded by the inequality (3.28). It follows from [29, Theorem 5.2] that V is bounded or exponential which gives the cases (i) and (ii). If U is unbounded, then V is also unbounded by the inequality (3.28), hence $\lambda \neq 0$ and $\nu \neq 0$. Now let $V=\mu U+B$ for some bounded function B and $\mu=\lambda/\nu$. Then the inequality (3.28) becomes

$$\begin{aligned}&\big|\mu U(x+y,t+s)+B(x+y,t+s)-(\mu U(x,t)+B(x,t))(\mu U(-y,s)+B(-y,s))\\&\quad+U(x,t)U(-y,s)\big|\leq M.\end{aligned} \tag{3.37}$$

Since B is bounded, we find that

$$\begin{aligned}&\Big|U(x+y,t+s)-\mu^{-1}\Big(\Big(\mu^2+1\Big)U(-y,s)+\mu B(-y,s)\Big)U(x,t)\Big|\\&\qquad\text{is bounded for fixed } y\in\mathbb{R}^n,\ s>0.\end{aligned} \tag{3.38}$$

Thus it follows from [29, page 104, Theorem 5.2] that

$$\mu^{-1}\Big(\Big(\mu^2+1\Big)U(y,s)+\mu B(y,s)\Big)=m(y,s) \tag{3.39}$$

for some exponential m. Thus if $\mu^2=-1$, we have $m=B$ and (iii) follows. If $\mu^2\neq -1$,

$$U=\frac{\mu(m-B)}{\mu^2+1},\qquad V=\frac{\mu^2 m+B}{\mu^2+1}, \tag{3.40}$$

which gives (iv). This completes the proof. □

Theorem 3.7. *Let $u,v\in\mathcal{S}'^{1/2}_{1/2}$ satisfy* (1.6). *Then u and v satisfy one of the followings*:

(i) *u and v are bounded measurable functions;*

(ii) $u=\cos(c\cdot x)$, $v=\sin(c\cdot x)$,

where $c\in\mathbb{C}^n$.

Proof. Similarly as in the proof of Theorem 3.5 convolving in (1.6) the tensor product $E_t(x)E_s(y)$, we obtain the inequality (3.28) where $U(x,t)$, $V(x,t)$ are the Gauss transforms of u,v, respectively. Now we can apply Lemma 3.6. First assume that (3.29) holds and consider the cases (i), (ii), (iii), and (iv) of Lemma 3.6. For the case (i), it follows from [30, Page 61, Theorem 1] that the initial values u, v of $U(x,t)$, $V(x,t)$ as $t\to 0^+$ are bounded measurable functions, respectively. For the case (ii), by the continuity of V, we have

$$V(x,t)=e^{c\cdot x+bt} \tag{3.41}$$

for some $c\in\mathbb{C}^n$, $b\in\mathbb{C}$. Putting (3.41) in (3.28) and letting $y=x$, $s=t$, and using the triangle inequality, we have

$$\Big|e^{2bt}(e^{c\cdot x}-1)\Big|\leq M \tag{3.42}$$

for some $M > 0$. In view of (3.42), we have $c = ia$, $a \in \mathbb{R}^n$. Thus $V(x,t)$ is bounded in $\mathbb{R}^n \times (0,1)$. Using [25, page 61, Theorem 1], the initial values u, v of $U(x,t)$, $V(x,t)$ as $t \to 0^+$ in (3.41) are bounded measurable functions. For the case (iii) putting $U = \pm i(V - m)$ in the inequality (3.28), we have

$$\left|V(x-y,t+s) - V(x,t)m(y,s) - V(y,s)m(x,t)\right| \leq M \tag{3.43}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where m is a bounded exponential. Using the continuity of V, it follows from (3.43) that $V(x,t)$ is bounded in $\mathbb{R}^n \times (0,1)$ and so is $U(x,t)$, which implies that both u and v are bounded measurable functions. For the case (iv) since $U = \lambda(m-B)/(\lambda^2+1)$, $V = (\lambda^2 m+B)/(\lambda^2+1)$ are unbounded continuous, m is unbounded continuous, and $m(x,t) = e^{c\cdot x+bt}$. On the other hand, it follows from (3.28) that $|V(x,t) - V(-x,t)| \leq 2M$, which occurs only when $c = 0$. Thus both $U(x,t)$ and $V(x,t)$ are bounded in $\mathbb{R}^n \times (0,1)$ and u, v are bounded measurable functions.

Secondly, assume that (3.30) holds. Letting $t, s \to 0^+$ in (3.30), we have

$$v \circ S - v \otimes v - u \otimes u = 0. \tag{3.44}$$

By Lemma 3.1, the nonconstant solution of (3.44) is given by $u = \cos(c \cdot x)$, $v = \sin(c \cdot x)$ for some $c \in \mathbb{C}^n$. This completes the proof. □

Remark 3.8. Taking the growth of $u = e^{c\cdot x}$ as $|x| \to \infty$ into account, $u \in \mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ only when $c = ia$ for some $a \in \mathbb{R}^n$. Thus the Theorems 3.5 and 3.7 are reduced to the followings.

Corollary 3.9. *Let $u, v \in \mathcal{S}'$ or $\mathcal{F}'$ satisfy* (1.5). *Then u and v satisfy one of the followings*:

(i) $u = 0$ *and v is arbitrary;*

(ii) *u and v are bounded measurable functions;*

(iii) $u = c \cdot x + r(x)$, *only* $v = \lambda c \cdot x + r(x) + 1$,

for some $c \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, and a bounded measurable function $r(x)$.

Corollary 3.10. *Let $u, v \in \mathcal{S}'$ or $\mathcal{F}'$ satisfy* (1.6). *Then u and v are bounded measurable functions.*

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