

STABILITY PROPERTIES OF GRAVITY THEORIES

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ABSTRACT

We study the stability properties of general relativity with a non-vanishing cosmological constant Λ by study of the energy. First, it is shown that there exists a suitable definition of energy in these models, for all metrics tending asymptotically to any background solution which has a timelike Killing symmetry, and that the energy has flux integral form. Stability is established for all systems tending asymptotically to anti-De Sitter space when $\Lambda < 0$, using supergravity techniques. Spinorial charges are defined which are also flux integrals and satisfy the global graded anti-De Sitter algebra. The latter then implies that the energy is always positive.

For $\Lambda > 0$, it is shown that small excitations about De Sitter space are stable, provided they occur within the event horizon intrinsic to this space. Outside the horizon an instability arises which signals the onset of Hawking radiation; it is shown to be universal to all systems. Semi-classical stability is also discussed for $\Lambda > 0$.

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1. INTRODUCTION

One of the open problems in current physics is the observed smallness of the cosmological constant Λ , or equivalently of the vacuum energy density of the Universe. From the point of view of particle physics, this is highly unnatural, requiring extreme fine tuning of parameters. One possible way to exclude the cosmological constant, already at the classical level, would be to show that it leads to some fundamental instabilities in the Einstein theory. This was the motivation for the present study, which has been carried out in collaboration with L.F. Abbott at CERN; details may be found in a forthcoming CERN preprint (TH.3136).

Stability of a bounded matter system in flat space is usually established by showing it to have positive energy, with respect to a lowest, vacuum, state. For gravity, (with $\Lambda = 0$) energy of any asymptotically flat solution is also perfectly definable with respect to flat space as vacuum. It turns out that this energy is always positive and that the theory (also in the presence of positive energy matter) is stable; quite general and rigorous results have been obtained in recent years¹⁾⁻⁴⁾. On the other hand, when $\Lambda \neq 0$, flat space is no longer an acceptable background (since it does not solve the Einstein equations), but must be replaced as vacuum by the "flattest", maximally symmetric solutions of the cosmological equations, namely De Sitter $O(4,1)$ or anti-De Sitter $O(3,2)$ space according to whether $\Lambda > 0$ or $\Lambda < 0$, respectively. At this point, a number of problems arise for the stability programme. First, can any reasonable physical substitute for energy be defined? Having lost asymptotic Poincaré invariance, one is left with the asymptotic De Sitter or anti-De Sitter algebra at infinity, for which P_μ^2 is no longer a Casimir operator, being replaced by the five-dimensional "rotations" $J_{ab} = -J_{ba}$ ($a, b = 0, \dots, 4$). One must therefore show that J_{04} , which becomes P_0 upon contraction ($\Lambda \rightarrow 0$), is acceptable and that this quantity is really definable as a flux integral at infinity so as to satisfy the asymptotic global algebra. This is accomplished in Section 2, through a general analysis of how to obtain and interpret conserved quantities with respect to a background which possesses symmetries. We will see that, associated to every such symmetry, there is a generator which is conserved and of flux integral form. The preferred one, which we call the Killing energy, is that which is connected with a timelike Killing vector or symmetry. Fortunately, for both signs of Λ , the maximally symmetric backgrounds have timelike symmetries, associated with J_{04} . The other nine generators are not timelike (nor are $(P_i, J_{\mu\nu})$ when $\Lambda = 0$). A brief review of the properties of De Sitter spaces in Section 3 will show that for $\Lambda > 0$, the necessary presence of an event horizon, where the timelike Killing vector becomes null and then spacelike requires a more careful analysis of stability. Within the horizon, however, and in all space for $\Lambda < 0$ (where there is no horizon) it will be seen in Section 4 that small oscillations about vacuum have positive energy. The possibility of negative

energy for excitations outside the horizon (for $\Lambda > 0$) is a reflection, in Hamiltonian form, of the generic features which lead to Hawking radiation⁵⁾. Next we shall discuss semi-classical stability, against quantum mechanical tunnelling, for $\Lambda > 0$, which would be violated in the presence of Euclidean "bounce" solutions⁶⁾⁻⁸⁾; no evidence for these is found.

The following Section, 6, is devoted to a demonstration of stability for all asymptotically anti-De Sitter solutions when $\Lambda < 0$ by showing that the full energy is positive in that case. Here one uses methods of supergravity, parallel to those which were used to establish²⁾ positivity of the energy for $\Lambda = 0$. To accomplish this it is first necessary to define spinorial charges which are conserved and also have flux integral form and show the corresponding existence of appropriate Killing spinors, whose presence is implicit but rather trivial when $\Lambda = 0$.

The outcome of our analysis is thus the opposite of what had been hoped for, and we end up by praising, rather than burying, theories with $\Lambda \neq 0$, at least from stability considerations (spaces with $\Lambda \neq 0$ do have peculiarities⁹⁾, which we mention, but are not directly concerned with here). On the other hand, there emerges a rather coherent picture of the quantities basic to understanding Einstein theory and their properties, whatever the value of Λ , as well as a simple Hamiltonian way of understanding the effect of event horizons on stability.

2. CONSERVED QUANTITIES

Consider the physical system defined by the Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad (2.1)$$

together with a background metric $\bar{g}_{\mu\nu}$ which satisfies (2.1); we decompose the full metric $g_{\mu\nu}$ according to

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \quad (2.2)$$

where $h_{\mu\nu}$ is not necessarily small, but does obey the boundary condition that it vanishes asymptotically at some appropriate speed. We will construct conserved quantities from $(\bar{g}_{\mu\nu}, h_{\mu\nu})$ corresponding to the symmetries of the background. Although we are primarily concerned with background De Sitter or anti-De Sitter spaces, the method is completely general, and leads to flux integral expressions for these generators, which are constructed from the gravitational stress-tensor and the appropriate Killing vectors. Our conventions are that $R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu} = \partial_{\mu}\Gamma^{\alpha}_{\alpha\nu}$, signature $(+++ -)$ and all operations such as covariant differentiation (\bar{D}_{μ}) or index moving are with respect to $\bar{g}_{\mu\nu}$. We define the symmetric stress tensor $T^{\mu\nu}$ to be all terms of second and higher order in $h_{\mu\nu}$ when the decomposition (2.2) is inserted in (2.1):

$$G_L^{\mu\nu}(\bar{g}, h) + \Lambda h_{\mu\nu} = T^{\mu\nu} = T^{\nu\mu}. \quad (2.3)$$

The subscript L refers to terms linear in $h_{\mu\nu}$. Using the fact that $G_{\mu\nu}(\bar{g}) + \Lambda\bar{g}_{\mu\nu} = 0$ and that the left side of (2.3) therefore obeys the (exact) linearized Bianchi identity $\bar{D}_{\mu}(G_L^{\mu\nu} + \Lambda h^{\mu\nu}) = 0$, the field equations imply covariant conservation of $T^{\mu\nu}$:

$$\bar{D}_{\mu} T^{\mu\nu} = 0. \quad (2.4)$$

To turn covariant into ordinary conservation, we have to define a conserved (background) contravariant vector density from the tensor $T^{\mu\nu}$ since then, and only then, is $\bar{D}_{\mu} J^{\mu} \equiv \partial_{\mu} J^{\mu}$. When $\bar{g}_{\mu\nu}$ has a symmetry, there exists a Killing vector $\bar{\xi}_{\mu}$, obeying

$$\bar{D}_{\mu} \bar{\xi}_{\nu} + \bar{D}_{\nu} \bar{\xi}_{\mu} = 0. \quad (2.5)$$

Consequently,

$$\bar{D}_{\mu}(T^{\mu\nu} \bar{\xi}_{\nu}) = (\bar{D}_{\mu} T^{\mu\nu}) \bar{\xi}_{\nu} + \frac{1}{2} T^{\mu\nu} (\bar{D}_{\mu} \bar{\xi}_{\nu} + \bar{D}_{\nu} \bar{\xi}_{\mu}) = 0 \quad (2.6)$$

and $\sqrt{-\bar{g}} T^{\mu\nu} \bar{\xi}_\nu$ is the desired contravariant tensor density, for which true conservation holds:

$$\bar{D}_\mu (\sqrt{-\bar{g}} T^{\mu\nu} \bar{\xi}_\nu) \equiv \partial_\mu (\sqrt{-\bar{g}} T^{\mu\nu} \bar{\xi}_\nu) = 0. \quad (2.7)$$

So to every Killing vector, there is associated a conserved generator

$$E(\bar{\xi}) = 1/8\pi G \int d^3x \sqrt{-\bar{g}} T^{\mu\nu} \bar{\xi}_\nu \quad (2.8)$$

In particular, if $\bar{\xi}_\nu$ is timelike, this is the Killing energy. Despite the fact that $\Lambda \neq 0$, we now show that $E(\bar{\xi})$ can be written in flux integral form, just as for $\Lambda = 0$. From (2.3) it follows that

$$T^{\mu\nu} \equiv \bar{D}_\alpha \bar{D}_\beta K^{\mu\alpha\nu\beta} + \frac{1}{2} (\bar{R}^{\mu\alpha\beta\nu} H^{\alpha\beta} + \Lambda H^{\mu\nu}) \quad (2.9)$$

where the superpotential K is defined by

$$\begin{aligned} 2K^{\mu\alpha\nu\beta} &\equiv \bar{g}^{\mu\beta} H^{\nu\alpha} + \bar{g}^{\nu\alpha} H^{\mu\beta} - \bar{g}^{\mu\alpha} H^{\nu\beta} - \bar{g}^{\nu\beta} H^{\mu\alpha}, \\ H^{\mu\nu} &\equiv h^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} h^\alpha{}_\alpha. \end{aligned} \quad (2.10)$$

It has the algebraic symmetries of the Riemann tensor:

$$K^{\mu\alpha\nu\beta} = -K^{\alpha\mu\nu\beta} = K^{\nu\beta\mu\alpha} = K^{\beta\alpha\nu\mu}. \quad (2.11)$$

Symmetry and conservation of $T^{\mu\nu}$ in (2.9) can easily be checked using the background field equations and its derivative consequences,

$$\bar{D}_\beta \bar{R}_{\mu\nu\alpha}{}^\beta \equiv \bar{D}_\nu \bar{R}_{\alpha\mu} - \bar{D}_\mu \bar{R}_{\alpha\nu} = 0, \quad \bar{D}_\beta \bar{R}^{\mu\nu} = 0$$

but without any assumptions on the full Riemann tensor.

Next, we form $T^{\mu\nu} \bar{\xi}_\nu$, and recast it into the expression

$$\begin{aligned} \sqrt{-\bar{g}} T^{\mu\nu} \bar{\xi}_\nu &= \sqrt{-\bar{g}} \bar{D}_\alpha [(\bar{D}_\beta K^{\mu\alpha\nu\beta}) \bar{\xi}_\nu - K^{\mu\alpha\nu\beta} \bar{D}_\beta \bar{\xi}_\nu] \\ &\equiv \bar{D}_\alpha F^{\mu\alpha}. \end{aligned} \quad (2.12)$$

It may be checked that all additional terms "miraculously" vanish upon use of the Killing identity $\bar{D}_\beta \bar{D}_\alpha \bar{\xi}_{\nu} + R^\lambda_{\beta\alpha} \bar{\xi}_{\nu} \equiv 0$. Furthermore, the quantity $F^{\mu\alpha}$ is (almost obviously) an antisymmetric tensor density, and therefore its divergence is an ordinary one $\bar{D}_\alpha F^{\mu\alpha} \equiv \partial_\alpha F^{\mu\alpha}$. Hence the desired result:

$$8\pi G E(\bar{\xi}) = \int d^3x \sqrt{-\bar{g}} T^{\nu\mu} \bar{\xi}_\nu = \oint dS_i F_i^{\mu\alpha} \quad (2.13)$$

As a check, when $\Lambda = 0$ and the background is chosen to be flat, introduction of Cartesian co-ordinates simplifies (2.13) to be the usual expression for the Poincaré generators. In particular, when $\bar{\xi}_\mu = (1, \vec{0})$ we obtain the standard energy formula

$$16\pi G E = \oint dS_i (h_{ji,j} - h_{ij,i}) \quad (2.14)$$

which correctly reproduces the mass of any asymptotically Schwarzschild metric. Note incidentally that the ten Poincaré generators automatically obey the (global) Poincaré algebra.

When $\Lambda \neq 0$, with $\bar{g}_{\mu\nu}$ a De Sitter or anti-De Sitter metric, we would obtain the 10 Killing generators corresponding to the background De Sitter or anti-De Sitter symmetries, and they automatically satisfy the appropriate global algebra. In particular, we get the timelike $E(\bar{\xi})$ expression^{*}). We also mention that this whole procedure could also have been carried out in first order form (used in Section 4) to yield $E(\bar{\xi})$ there as well.

We complete this section with a treatment of the graded algebra which can be introduced when $\Lambda < 0$ (but not $\Lambda > 0$!), in terms of spinorial charges Q . The resulting local supersymmetry is that of supergravity with a cosmological term¹⁰⁾ and a spin 3/2 "mass" term¹¹⁾. This will be used in Section 6 to show that the supergravity energy operator is positive and from this establish stability for classical gravity with $\Lambda < 0$. First, however, we must show that the spinor charges can be written as surface integrals as in (2.13) so as to satisfy the graded global algebra at infinity. The spinorial charge density is

$$Q^\mu = \varepsilon^{\mu\alpha\beta\nu} \delta_5 \bar{\delta}_\alpha \tilde{D}_\beta \psi_\nu. \quad (2.15)$$

Its origin may be understood, just like that of $T^{\mu\nu}$, in terms of a decomposition of the full Rarita-Schwinger equation into a "linear" part and a remainder. Here ψ_ν

^{*}) A check here is to verify that the equivalent of the Schwarzschild solution for $\Lambda \neq 0$ has energy m . For $\Lambda > 0$, there are corrections because one must stay within the unavoidable event horizon (rather than go to infinity) in calculating the flux integral; apart from these, the correct result $E = m$ emerges.

is the spin 3/2 field, $\bar{\gamma}_\alpha$ are the background covariant γ matrices with respect to the background vierbein and the modified covariant derivative on a spinor, \tilde{D}_β , defined by

$$\tilde{D}_\beta \equiv \bar{D}_\beta + \frac{1}{2} m \bar{\gamma}_\beta \quad , \quad m^2 \equiv \frac{1}{3} |A| \quad (2.16)$$

has the basic property that $[\tilde{D}_\beta, \tilde{D}_\alpha] = 0$ for a background anti-De Sitter space. The current Q^μ satisfies $\tilde{D}_\mu Q^\mu = 0$, and to convert this to an ordinary conservation law, we introduce Killing spinors obeying $\tilde{D}_\mu \alpha = 0$ (consistent with $[\tilde{D}_\nu, \tilde{D}_\mu] \alpha = 0$). The quantity $\bar{\alpha} Q^\mu$ is easily seen to take the form

$$\bar{\alpha} Q^\mu = \bar{D}_\rho (\bar{\alpha} \varepsilon^{\mu\alpha\beta\nu} \gamma_5 \bar{\gamma}_\alpha \psi_\nu) \equiv \partial_\rho (\bar{\alpha} \varepsilon^{\mu\alpha\beta\nu} \gamma_5 \bar{\gamma}_\alpha \psi_\nu) \quad (2.17)$$

The last equality follows because the quantity in parentheses is an antisymmetric tensor density. But since $\bar{\alpha} Q^\mu$ is a contravariant vector, we see immediately that its ordinary divergence vanishes identically, $\partial_\mu (\bar{\alpha} Q^\mu) \equiv 0$, so that the spinor charge is both conserved and has the flux form

$$Q(\alpha) \equiv \int d^3x \bar{\alpha} Q^0 \equiv \oint dS_j \bar{\alpha} \varepsilon^{0ijk} \gamma_5 \bar{\gamma}_i \psi_k \quad (2.18)$$

This is the required analogue of (2.13) for the bosonic generators. [The analogy actually goes further, in that only one "Coulomb" component of ψ_k enters in (2.18), corresponding to the "Coulomb" part of the metric in (the appropriate generalization of) Eq. (2.14)].

Each of the four independent Killing spinors $\alpha_{\beta'}^{(\beta)}$, where β' is the spinor index and (β) is the label of each spinor, defines a fermionic charge $Q_{(\beta)}$. These then satisfy the graded relation

$$\{Q_{(\beta)}, Q_{(\beta')}\} = \frac{1}{2} \left(\gamma^{(\mu)} \right)_{(\beta\beta')} J_{(\mu\nu)} + \left(\sigma^{(\mu\nu)} \right)_{(\beta\beta')} J_{(\mu\nu)} \quad (2.19)$$

if we take the α 's to commute (for convenience). Having defined the required conserved quantities, we turn next to the choice of Killing vectors.

3. DE SITTER SPACES

We give a brief review of the symmetries of "vacuum" spaces when $\Lambda \neq 0$. De Sitter space corresponds to a four-surface $z_{\mu}^2 + z_4^2 = 3/\Lambda$, $\Lambda > 0$, in flat five-space. Among the rotations of the embedding space are the boosts mixing z_0 with (z_1, z_4) . For example $\xi_a = (z_4, 0, 0, 0, z_0)$ is a timelike Killing vector when $|z_4| > |z_0|$, but signals the existence of an event horizon at $|z_4| = |z_0|$, where stability must be discussed separately, since $E(\xi)$ no longer acts like an energy beyond it. Of course, an observer will only interact with events inside the horizon, which means that $E(\xi)$ tests stability to excitations visible to the observer. It is illuminating to apply these ideas to a simple model, namely a scalar field in De Sitter space. Representing the metric in the form

$$ds^2 = -dt^2 + f^2(t)(dx^2 + dy^2 + dz^2), \quad f(t) \equiv \exp \sqrt{\frac{\Lambda}{3}} t \quad (3.1)$$

with the "timelike" vector

$$\bar{\xi}^\mu = (-1, \sqrt{\frac{\Lambda}{3}} \vec{x}), \quad \bar{\xi}^2 = -1 + \frac{\Lambda}{3} |\vec{x} f|^2 \quad (3.2)$$

we see that the horizon appears (at any given time) for distances such that $|\sqrt{\Lambda/3} \vec{x} f| = 1$. The action and energy momentum densities for this theory are

$$\begin{aligned} I &= \int d^4x f^3 \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} f^{-2} (\nabla\phi)^2 - V(\phi) \right] \\ -\mathcal{E}_0^0 &= \frac{1}{2} f^{-3} \pi^2 + \frac{1}{2} f (\nabla\phi)^2 + f^3 V(\phi) \\ -\mathcal{E}_i^0 &= \pi \partial_i \phi \end{aligned} \quad (3.3)$$

where $\pi \equiv f^3 \dot{\phi}$. The energy density is positive [if $V(\phi)$ is] but time-dependent. The conserved energy (which is of course not a flux integral here) is still $E(\bar{\xi}) = \int d^3x \bar{\xi}^{\mu 0} \mathcal{E}_\mu$. The integrand has the form

$$\bar{\xi}^{\mu 0} \mathcal{E}_\mu = \left\{ \frac{1}{2} [f^{-3} \pi^2 + f (\nabla\phi)^2] - \sqrt{\frac{\Lambda}{3}} \vec{x} \cdot \pi \vec{\nabla}\phi \right\} + f^3 V(\phi). \quad (3.4)$$

It will be positive provided the bracketed quantity is. The triangle inequality

$$\frac{1}{2} (\vec{A}^2 + \vec{B}^2) > \sqrt{\frac{\Lambda}{3}} f |\vec{x}| \vec{A} \cdot \vec{B}, \quad \vec{A} \equiv f^{-3/2} \pi \hat{x}, \quad \vec{B} \equiv f^{1/2} \vec{\nabla}\phi \quad (3.5)$$

makes it easy to see that the positivity condition corresponds to $\bar{\xi}^2 < 0$ in (3.2), i.e., to excitations within the horizon. This correlation between the event horizon and positivity is an expression in Hamiltonian form of Hawking radiation⁵⁾, and will be seen later to be universal.

Anti-De Sitter space is the covering space for the surface $z_\mu^2 - z_4^2 = 3/\Lambda$, $\Lambda < 0$. Here there is a global timelike Killing vector corresponding to (z_0, z_4) rotation. It is $\bar{\xi}_a = (z_4, 0, 0, 0, -z_0)$, $\bar{\xi}^2 = -(z_4^2 + z_0^2) < 0$, since $z_4 = z_0 = 0$ is excluded. There is one peculiar feature of anti-De Sitter space which should be mentioned. Specification of initial data on a complete spacelike surface does not lead to a unique prediction of the future state of a system (including gravity itself). Radiation, not specified by the initial conditions can propagate in from infinity at a later time. Unlike the usual case where initial boundary conditions exclude incoming radiation thereafter, one is "safe" here only within ever more restricted regions of space at later times. Therefore, although the initial energy is perfectly well defined by the initial data, one can only extend the integration volume to all space at a later time if further (timelike) boundary conditions at infinity are imposed. It is in this sense that our stability results are to be understood, although the proof that energy is positive holds formally on any complete initial surface with no incoming radiation.

4. SMALL EXCITATIONS

We now apply the Killing energy together with the Killing vectors defined in the last two sections to discuss small oscillations about De Sitter or anti-De Sitter backgrounds. For this purpose, a canonical approach¹²⁾ is most useful to discuss the $\mathcal{O}(h^2)$ part of $E(\bar{\xi})$. Indeed T_V^0 was derived to this order long ago by Nariai and Kimura¹³⁾. We will skip all details here, only noting that the excitations can be parametrized by the h_{ij} and their conjugate momenta p^{ij} , both being transverse traceless with respect to \bar{g} .

For $\Lambda > 0$, the Hamiltonian density can be cast into the form

$$\begin{aligned} -\mathcal{L}^0_0 &= \frac{1}{2} \left(f^{-3} (p^{ij})^2 + f (\nabla Q_{ij})^2 \right) \\ -\mathcal{L}^0_i &= p^{jk} \partial_i Q_{jk} \end{aligned} \tag{4.1}$$

exactly as for the scalar field of (3.2), in terms of a canonically transformed set (Q, P) . Not surprisingly, all the other features of the scalar model follow as well: $\int d^3x \mathcal{L}^0_{\bar{\xi}_V}$ is conserved and is positive within the event horizon $\bar{\xi}^2 < 0$. However, outside, when $\bar{\xi}^2 > 0$, the energy is no longer positive, because the triangle inequality no longer applies, as with (3.5). One would expect all physical systems to behave in this way: the free part of the energy is always $\sim \frac{1}{2} \int \{\pi^2 + (\nabla\phi)^2\}$ while the momentum density is $\sim \nabla\phi$. For physical matter, the non-linear parts of the energy are, like $V(\phi)$ in the scalar case, positive so the critical condition arises primarily at the free field level, where excitations beyond the horizon can give negative contributions. In particular, if the higher terms in $T^0_0(h)$ are effectively positive (as in the $\Lambda = 0$ case), then the only De Sitter instability would be that due to the horizon.

For $\Lambda < 0$, the small excitations are straightforwardly treated; this time only T^{00} is required, since we can pick a co-ordinate system which is static and in which $\bar{\xi}_i = 0$, and there is no horizon. The energy density is positive,

$$\mathcal{L}^0_0 \sim \left[p_{ij}^2 + \frac{1}{4} (\nabla h_{ij})^2 + \frac{|\Lambda|}{8} h_{ij}^2 \right], \tag{4.2}$$

and the system is stable. The masslike term in (4.2) is an artifact just like that of the spin 3/2 field¹¹⁾ in cosmological supergravity; the gravitons have only two degrees of freedom since h_{ij} is transverse-traceless.

5. SEMI-CLASSICAL STABILITY FOR $\Lambda > 0$

Having shown that $\Lambda > 0$ solutions are stable to small fluctuations about the vacuum, at least within the horizon, one may make the further test of semi-classical stability in this case, i.e., look for Euclidean "bounce" solutions whose presence would signal quantum tunnelling instability. Of course even better would be proof that the total energy is positive, which will be given for $\Lambda < 0$ in the next section; lacking this for $\Lambda > 0$, we make some comments on bounces there. In general, unusual topologies can give stability problems in gravity¹⁴⁾ and semi-classical instability has been found in other gravitational contexts^{7),8)}.

A bounce solution here would be a metric which is asymptotically De Sitter and solves the Euclideanized Einstein equations. For example, the Euclidean continuation of the Schwarzschild-De Sitter metric would be a candidate. However, it is impossible to remove both the Schwarzschild and horizon singularities of this metric by the usual periodicity trick¹⁵⁾. In terms of Hawking radiation, De Sitter space contains radiation at a temperature fixed by the value of Λ . If a black hole could form in this space with an intrinsic temperature less than this, it would grow forever by accretion. However, for the Schwarzschild-De Sitter black hole, the black hole temperature is always larger⁵⁾ than that of the exterior and the space is stable against this catastrophe.

Although it is doubtful on general grounds that bounce solutions exist for $\Lambda > 0$, it would clearly be desirable to extend the proofs of their absence^{4),16)} for $\Lambda = 0$ to this domain as well.

6. STABILITY FOR $\Lambda < 0$

We now show that the Killing energy is positive for all excitations about the anti-De Sitter vacuum which vanish at infinity, and thereby establish stability in the $\Lambda < 0$ sector. We have already noted in Section 2 that all the generators of the graded anti-De Sitter algebra in supergravity are expressed as flux integrals, with the result that they obey the global algebra relations, in particular that of Eq. (2.19):

$$\{Q_{(\beta)}, \bar{Q}_{(\beta')}\} = \frac{1}{2} \gamma_{(\beta\beta')}^{(\mu)} J_{(\mu\nu)} + \sigma_{(\beta\beta')}^{(\mu\nu)} J_{(\mu\nu)}.$$

We emphasize that in this expression, all indices are labels of particular Killing vectors or spinors. The explicit relations between the two are quite analogous to those holding in the Poincaré case, and indeed can be essentially reduced to it because the \tilde{D}_μ commute; there exists¹⁷⁾ a transformation $\alpha = S\eta$ which reduces the equation $\tilde{D}_\mu \alpha = 0$ to $\partial_\mu \eta = 0$. A basis for the latter is given by, e.g., $\eta_\beta = \delta_\beta^{(\beta')}$. In any case, we may now simply treat the spinor "labels" (β) , (β') in (2.19), which refer to the particular Killing spinor defining the corresponding charge $Q_{(\beta)}$, as normal flat space spinor indices. Multiplying (2.19) by the numerical matrix $\gamma_{\beta\beta'}^{(0)}$ and tracing gives the positivity relation for the operator $J_{(04)}$:

$$J_{(04)} = \sum_{\beta=1}^4 Q_{(\beta)} Q_{(\beta)} \geq 0 \quad (6.1)$$

since the $Q_{(\beta)}$ are real Majorana spinor operators. Now we just proceed as in the $\Lambda = 0$ case²⁾, taking matrix elements of (6.1) with no on-shell fermions and go to the tree limit, $\hbar \rightarrow 0$. This implies that $E(\vec{\xi})$, which is just this limit of $J_{(04)}$, is positive for classical $\Lambda < 0$ gravity.

We also believe, although we have not carried out the details, that Witten's recent purely classical proof⁴⁾ that energy is positive for $\Lambda = 0$ gravity can also be applied here. His proof, inspired by the supergravity argument, is based on considering solutions of the Dirac equation $\nabla_\mu \epsilon \equiv \gamma^i D_i \epsilon = 0$ in an external metric satisfying $G_{0\mu} = 0$. From the relations

$$0 = \epsilon^* \not{\partial}^2 \epsilon \equiv \epsilon^* (D^2 + G_{0\mu} \gamma^0 \gamma^\mu) \epsilon = \epsilon^* D^2 \epsilon,$$

it follows upon integration that

$$\oint dS_i \epsilon^* D^i \epsilon = \int |\nabla \epsilon|^2 d^3x \geq 0. \quad (6.2)$$

The surface integral is then separately shown to be proportional to E , which establishes positivity of the latter. The same reasoning should apply here with D_i

replaced by \tilde{D}_i , and the metric now satisfying $G_{0\mu} + \Lambda g_{0\mu} = 0$ provided, as is likely, the surface integral is again proportional to E . Similarly, it would be of interest to generalize the classical geometrical proof of Schoen and Yan³⁾ to the $\Lambda < 0$ case. It may even be possible to establish full non-linear stability in the $\Lambda > 0$ case for excitations lying within the horizon by analytic continuation from $\Lambda < 0$, using the static form of the $O(4,1)$ metric which covers the interior region only.

7. CONCLUSIONS

We have seen first that it is possible to parallel all the arguments of $\Lambda = 0$ gravity in a background flat metric in order to establish a satisfactory energy expression in the general $\Lambda \neq 0$ case which is conserved and of flux integral form, as long as the background metric has a timelike symmetry. When applied to the stability problem, these expressions enable one to show that the energy is positive for all asymptotically anti-De Sitter metrics in the $\Lambda < 0$ sector, using methods of supergravity, analogous to the $\Lambda = 0$ case, for grading the algebra. When $\Lambda > 0$, stability of small excitations (about the De Sitter vacuum) which lie within the event horizon was demonstrated. A clear and universal relation between event horizon and Hawking radiation "instability" was then obtained in terms of the general property of any free fields that $T^{00} \geq |T^{0i}|$, together with the simple facts that $|\dot{\xi}_0| < |\dot{\xi}|$ beyond the horizon and that $E(\dot{\xi}) = \int (T^{00}\dot{\xi}_0 + T^{0i}\dot{\xi}_i) d^3x$ is the relevant energy. Consequently, contributions to $E(\dot{\xi})$ from beyond the horizon are no longer positive. Semi-classical stability for $\Lambda > 0$ also seems likely, as well as general energy positivity for excitations within the horizon.

We conclude that at least classically, there is no instability argument to rule out the cosmological extensions of Einstein theory, and that they are much like the $\Lambda = 0$ model in this respect.

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