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Stability, reconstruction formula and regularization for an inverse source hyperbolic problem by a control method

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Abstract. Let $u(f)$ be the solution to a hyperbolic equation in a bounded domain $\Omega \subset \mathbb{R}^r$:

$$u''(x,t) = \Delta u(x,t) + \sigma(t)f(x) \qquad (x \in \Omega, 0 < t < T)$$
$$u(x,0) = u'(x,0) = 0 \qquad (x \in \Omega)$$
$$u(x,t) = 0 \qquad (x \in \partial\Omega,\ 0 < t < T).$$

We assume that $\sigma \in C^1[0,T]$ is a known function, $\sigma(0) \neq 0$, and $f \in L^2(\Omega)$ is unknown, and $\Gamma \subset \partial\Omega$ is given. We consider an inverse problem of determining $f(x)(x \in \Omega)$ from $[\partial u(f)/\partial n](x,t)(x \in \Gamma, 0 < t < T)$. For a sufficiently large $T > 0$, we will show the stability estimate of $\|f\|_{L^2(\Omega)}$ by $\|\partial u(f)/\partial n\|_{H^1(0,T;L^2(\Gamma))}$, a reconstruction formula of f from $\partial u(f)/\partial n$ and a Tikhonov regularization. Our methodology is based on exact boundary controllability and a Volterra integral equation of the first kind with kernel σ.

1. Introduction

We consider an initial/boundary value problem for a hyperbolic equation:

$$\begin{aligned} &u''(x,t) = \Delta u(x,t) + \sigma(t)f(x) \qquad (x \in \Omega, t > 0) \\ &u(x,0) = 0 \qquad u'(x,0) = 0 \qquad (x \in \Omega) \\ &u(x,t) = 0 \qquad (x \in \partial\Omega, t > 0). \end{aligned} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^r$ is a bounded domain with smooth boundary $\partial\Omega$, and we set $u'(x,t) = \partial u/\partial t(x,t)$, $u''(x,t) = \partial^2 u/\partial t^2(x,t)$, and Δ is the Laplacian. Let $L^2(\Omega)$ be the space of all real-valued square integrable functions with the inner product $(\cdot,\cdot)_{L^2(\Omega)}$ and the norm $\|\cdot\|_{L^2(\Omega)}$.

The term $\sigma(t)f(x)$ is considered to be an external force. External forces in this form of separation of variables are important in modelling vibrations. For example, if we set $\sigma(t) = \cos\omega t$ ($\omega \in \mathbb{R}$), then it describes a spatial force which varies harmonically. Moreover the system (1.1) is regarded as an approximation to a model for elastic waves from a point dislocation source (e.g. Aki and Richards (1980) ch 4).

We assume that σ is a known non-zero C^1-function and is independent of the space variable x, and $f \in L^2(\Omega)$ is unknown.

We consider the

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Inverse problem. Determine f from

$$\frac{\partial u}{\partial n}(x,t) \qquad (x \in \Gamma, 0 < t < T).$$

Here $\Gamma \subset \partial\Omega$ is given (see (2.3) below), $T > 0$ is an observation time and we set

$$\frac{\partial u}{\partial n}(x) = \sum_{i=1}^{r} \nu_i(x) \frac{\partial u}{\partial x_i}(x) \qquad (x \in \partial\Omega)$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_r(x))$ is the outward unit normal to $\partial\Omega$ at x.

Remark 1. Our methodology proposed in this paper is based on exact boundary controllability and is applicable to hyperbolic equations with variable coefficients:

$$\begin{aligned} &u''(x,t) = \sum_{i,j=1}^{r} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x,t)\right) + \sigma(t)f(x) \qquad (x \in \Omega, t > 0) \\ &u(x,0) = 0 \qquad u'(x,0) = 0 \qquad (x \in \Omega) \\ &u(x,t) = 0 \qquad (x \in \partial\Omega, t > 0) \end{aligned} \tag{1.1'}$$

where $\{a_{ij}(\cdot)\}_{1 \leqslant i,j \leqslant r}$ should satisfy some stricter positivity condition than the uniform ellipticity (Komornik 1989a).

$$\begin{aligned} &u''(x,t) = \Delta u(x,t) + b(x)u(x,t) + \sigma(t)f(x) \qquad (x \in \Omega, t > 0) \\ &u(x,0) = 0 \qquad u'(x,0) = 0 \qquad (x \in \Omega) \\ &u(x,t) = 0 \qquad (x \in \partial\Omega, t > 0) \end{aligned} \tag{1.1''}$$

where $b \in L^\infty(\Omega)$ (Komornik 1989b).

Our methodology is also applicable to other boundary conditions in (1.1) such as the zero Neumann one.

Remark 2. In (1.1) we can consider a more general external force $f(x)\widetilde{\sigma}(x,t)$ $(x \in \Omega, t > 0)$ where $f \in L^2(\Omega)$ is unknown and $\widetilde{\sigma}$ is known and depends also on x. The inverse hyperbolic problem with such $\widetilde{\sigma}$ is related to the determination of coefficients (e.g. the proof of theorem 3.7 in Isakov (1993) and theorem 4.7 in Klibanov (1992)). In the case where $\Gamma = \partial\Omega$ (that is, the observation boundary is the complete $\partial\Omega$), we can refer to Isakov (1990, 1993) and Klibanov (1992) to obtain the uniqueness of f. Moreover, in the case where Γ is a part of $\partial\Omega$, our methodology works in principle if we make serious modifications, so that stability estimates as well as uniqueness can be shown by taking rare exceptional cases into consideration (Puel and Yamamoto 1994a, b).

If $\sigma \in C^1[0,T]$, then for any $f \in L^2(\Omega)$, there exists a unique solution

$$u = u(f) \in C^1([0,T]; H_0^1(\Omega)) \cap C^2([0,T]; L^2(\Omega))$$

to (1.1) and

$$\frac{\partial u(f)}{\partial n} \in H^1(0,T; L^2(\partial\Omega)).$$

Moreover we have

$$\|u(f)\|_{L^\infty(0,T,H_0^1(\Omega))} + \|u(f)'\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C_1\|f\|_{L^2(\Omega)} \qquad (f \in L^2(\Omega)) \tag{1.2}$$

(lemma 3.6 (p 39) in Lions (1988b)) and

$$\left\|\frac{\partial u(f)}{\partial n}\right\|_{H^1(0,T;L^2(\partial\Omega))} \leqslant C_1\|f\|_{L^2(\Omega)} \qquad (f \in L^2(\Omega)) \tag{1.3}$$

for some constant $C_1 > 0$ which is independent of f. The estimate (1.3) can be proved by theorem 4.1 (p 44) in Lions (1988b) and, for completeness, we will give a proof in the appendix.

Henceforth for a measurable subset Γ of $\partial\Omega$, we set

$$(u, v)_{H^1(0,T;L^2(\Gamma))} = \int_0^T \int_\Gamma \left(u(x,t)v(x,t) + \frac{\partial u}{\partial t}(x,t)\frac{\partial v}{\partial t}(x,t)\right) \mathrm{d}S_x\, \mathrm{d}t$$

and

$$\|u\|_{H^1(0,T;L^2(\Gamma))} = (u,u)^{1/2}_{H^1(0,T;L^2(\Gamma))}$$

for $u, v \in H^1(0,T;L^2(\Gamma))$, and $H^s(\Omega)(s > 0)$ denotes the Sobolev space (e.g. Lions and Magenes 1972).

In this paper, we propose the application of a control method for the following three topics in our inverse problem.

(A) (Stability). We shall estimate $\|f\|_{L^2(\Omega)}$ by a suitable norm of $\partial u(f)/\partial n (x \in \Gamma, 0 < t < T)$.

(B) (Reconstruction formula). We shall give a reconstruction formula of f in terms of $\partial u(f)/\partial n$. In particular we shall give the Fourier coefficients of f by $\partial u(f)/\partial n$.

(C) (Convergence rates of regularized solutions). We shall determine the range $\mathcal{R}(G^*) = \{G^*v;\, v \in L^2(\Gamma \times (0,T))\}$.

Here the operator $G : L^2(\Omega) \longrightarrow L^2(\Gamma \times (0,T))$ is defined by

$$Gf = \frac{\partial u(f)}{\partial n} \tag{1.4}$$

and G^* is the adjoint operator of G.

In Yamamoto (1995a), problem B is solved and a characterization of the range $\{\partial u(f)/\partial n;\, f \in L^2(\Omega)\}$ is given by which problem A is discussed. On the other hand, in this paper, we will give a direct proof of A, and the Fourier coefficients of f in terms of $\partial u(f)/\partial n$. Problem C is essential for obtaining convergence rates of regularized solutions which are obtained by Groetsch's theory (Groetsch 1984).

The purposes of this paper are to clarify that a control method (namely the Hilbert uniqueness method) offers very unified solutions for the above three problems (A, B and C). The Hilbert uniqueness method is widely applicable to various equations (Komornik 1992, Lagnese 1991, Lions 1988a, b, Zuazua 1987, 1993). In this paper, in order to explain the essential features for applying the Hilbert uniqueness method to inverse source problems, we mainly consider a wave equation (1.1). Applications of our methodology to inverse source problems for other types of partial differential equation will be discussed in succeeding papers.

We conclude this section with reference to Belishev and Kurylev (1991) where the Dirichlet-to-Neumann map approach (e.g. Isakov (1993)) is discussed in terms of boundary control techniques.

This paper is composed of six sections. In section 2, we will precisely formulate the three problems in our inverse problem and state the main results in theorems 1, 2 and 3 respectively for problems A, B and C. In section 3 we will apply theorem 3 for *obtaining a convergence rate of regularized solutions toward the exact solution* by a variant of Tikhonov's regularization. In sections 4 to 6, we will prove respectively theorems 1–3.

2. Formulation and main results

Throughout this paper, for an arbitrarily fixed $x_0 \in \mathbb{R}^r$, we set

$$\Gamma_+(x_0) = \{x \in \partial\Omega;\ (x - x_0, \nu(x))_{\mathbb{R}^r} > 0\} \tag{2.1}$$

$$R_0 = R_0(x_0) = \sup\{|x - x_0|_{\mathbb{R}^r};\ x \in \partial\Omega\}$$

and, for an observation time $T > 0$ and a part Γ of $\partial\Omega$ where $\partial u(f)/\partial n$ is observed, we assume

$$T > 2R_0 \tag{2.2}$$

$$\Gamma \supset \Gamma_+(x_0). \tag{2.3}$$

Furthermore let σ satisfy

$$\sigma(0) \neq 0 \qquad \sigma \in C^1[0, T]. \tag{2.4}$$

First we can state the answer to problem A.

Theorem 1 (Stability). Under the assumptions (2.2)–(2.4), there exists a constant $C = C(\Omega, \Gamma, T, x_0) > 0$ such that

$$C^{-1}\left\|\frac{\partial u(f)}{\partial n}\right\|_{H^1(0,T;L^2(\Gamma))} \leqslant \|f\|_{L^2(\Omega)} \leqslant C\left\|\frac{\partial u(f)}{\partial n}\right\|_{H^1(0,T;L^2(\Gamma))} \tag{2.5}$$

for any $f \in L^2(\Omega)$.

Remark 3. Our governing equation in (1.1) is hyperbolic so that, for uniqueness and stability, we have to choose a large observation time T satisfying (2.2). The restriction on the geometry of Γ arises from the assumption for a result concerning the exact controllability by the Hilbert uniqueness method (e.g. ch I sections 7 and 10 in Lions (1988b)). In many cases, it turns out that (2.2) requires that T should be greater than the diameter of Ω.

Second we proceed to discussion of the second problem B stated in section 1. For this, we define three operators A, Π and Φ.

Definition of the operator A in $L^2(\Omega)$. Let A be the realization of $-\Delta$ in $L^2(\Omega)$ with Dirichlet boundary condition $Au(x) = -\Delta u(x)$ and $\mathcal{D}(A) = \{u \in H^2(\Omega); u|_{\partial\Omega} = 0\}$. Let us number the eigenvalues of A repeatedly according to their multiplicities:

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \dots. \tag{2.6}$$

That is, if the multiplicity of λ_i is m, then λ_i appears in (2.6) m times. Let ϕ_k be an eigenfunction for the eigenvalue λ_k of A $(k \geqslant 1)$. We can choose $\{\phi_k\}_{k\geqslant 1}$ such that

$$(\phi_k, \phi_l)_{L^2(\Omega)} = \delta_{kl}. \tag{2.7}$$

Here we set $\delta_{kl} = 1(k = l), = 0$ (otherwise).

Definition of the operator Π: $L^2(\Omega) \longrightarrow L^2(\Gamma \times (0, T))$. We show:

Lemma 1 Exact controllability. (Lions 1988a, b). On the assumptions (2.2) and (2.3), to each $\phi_0 \in L^2(\Omega)$ we can construct a unique $v = v(\phi_0) \in L^2(\Gamma \times (0, T))$ in such a way that the following properties hold.
(i) The weak solution $\phi = \phi(v) \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ to

$$\begin{aligned}
&\phi''(x,t) = \Delta\phi(x,t) \qquad (x \in \Omega, 0 < t < T)\\
&\phi(x,0) = \phi_0(x) \qquad \phi'(x,0) = 0 \qquad (x \in \Omega)\\
&\phi(x,t) = v(x,t) \qquad (x \in \Gamma, 0 < t < T)\\
&\phi(x,t) = 0 \qquad (x \in \partial\Omega \setminus \Gamma, 0 < t < T)
\end{aligned} \tag{2.8}$$

satisfies

$$\phi(x,T) = 0 \qquad \phi'(x,T) = 0 \qquad (x \in \Omega). \tag{2.9}$$

(ii) There exists a constant $C_2 = C_2(\Omega, T, x_0) > 0$ such that

$$\|v(\phi_0)\|_{L^2(\Gamma\times(0,T))} \leqslant C_2\|\phi_0\|_{L^2(\Omega)} \tag{2.10}$$

for any $\phi_0 \in L^2(\Omega)$.

Here $H^{-1}(\Omega)$ denotes the dual of $H_0^1(\Omega) \equiv \{u \in H^1(\Omega); u|_{\partial\Omega} = 0\}$.

By lemma 1, we can define a bounded linear operator $\Pi : L^2(\Omega) \longrightarrow L^2(\Gamma \times (0, T))$ by

$$\Pi\phi_0 = v(\phi_0) \qquad (\phi_0 \in L^2(\Omega)). \tag{2.11}$$

Remark 4. For the unique existence of a weak solution to (2.8), we can refer, for example, to theorem 4.2 (pp 46–7) in Lions (1988b).

Definition of the operator Φ*:* $L^2(\Gamma \times (0,T)) \longrightarrow H^1(0,T;L^2(\Gamma))$. Let us consider a Volterra equation of the second kind

$$\begin{aligned} &\sigma(0)\theta'(x,t) + \int_t^T (\sigma'(\xi - t)\theta'(x,\xi) + \sigma(\xi - t)\theta(x,\xi))\,\mathrm{d}\xi = \eta(x,t) \\ &\quad (x \in \Gamma, 0 < t < T) \\ &\theta(x,0) = 0 \qquad (x \in \Gamma). \end{aligned} \tag{2.12}$$

This is uniquely solvable in $\theta \in H^1(0,T;L^2(\Gamma))$ for any $\eta \in L^2(\Gamma \times (0,T))$ by using the resolvent kernel and we have

$$\|\theta\|_{H^1(0,T;L^2(\Gamma))} \leqslant C_3\|\eta\|_{L^2(\Gamma\times(0,T))} \qquad (\eta \in L^2(\Gamma \times (0,T)))$$

with some constant $C_3 > 0$ (e.g. Tricomi (1985)). Then we can define a bounded linear operator $\Phi : L^2(\Gamma \times (0,T)) \longrightarrow H^1(0,T;L^2(\Gamma))$ by

$$\theta = \Phi\eta \qquad \eta \in L^2(\Gamma \times (0,T)). \tag{2.13}$$

Now we are ready to state a formula for reconstruction of f in terms of $\partial u(f)/\partial n$.

Theorem 2 (Reconstruction formula). We assume (2.2)–(2.4), and set

$$\theta_k = -\Phi\Pi\phi_k \qquad (k \geqslant 1). \tag{2.14}$$

Then we have

$$(f, \phi_k)_{L^2(\Omega)} = \left(\frac{\partial u(f)}{\partial n}, \theta_k\right)_{H^1(0,T;L^2(\Gamma))} \tag{2.15}$$

for $k \geqslant 1$. In particular,

$$f = \sum_{k=1}^{\infty} \left(\frac{\partial u(f)}{\partial n}, \theta_k\right)_{H^1(0,T;L^2(\Gamma))} \phi_k.$$

Remark 5. Since $\{\theta_k\}_{k\geqslant1}$ can be constructed only by Ω, T, Γ, and is independent of $f \in L^2(\Omega)$, conclusion (2.15) means that the kth Fourier coefficient of an unknown f is given in terms of datum $\partial u(f)/\partial n$.

Remark 6. For $\{\theta_k\}_{k\geqslant1}$, we can further show (theorem 4 in Yamamoto (1995a)) that the system $\{\theta_k\}_{k\geqslant1}$ is a Riesz basis (e.g. Gohberg and Kreĭn (1969)) in $H^1(0,T;L^2(\Gamma))$ under the assumptions (2.2)–(2.4).

Finally we will discuss problem C.

Theorem 3 (Range of G^**).* Under (2.2)–(2.4), we get

$$\mathcal{R}(G^*) \equiv \{G^*y;\, y \in L^2(\Gamma \times (0,T))\} \supset H_0^1(\Omega). \tag{2.16}$$

Furthermore we can prove that $\mathcal{R}(G^*) \subset H^1(\Omega)$ by an argument on regularity of solutions of a hyperbolic equation with non-homogeneous boundary values (cf Lions (1988b)). This theorem is proved by

(i) the range of the adjoint of an observation map G coinciding with a reachable set of a control system associated with (1.1) (e.g. Engl *et al* (1995)); and

(ii) a reachable set is $H_0^1(\Omega)$ by the Hilbert uniqueness method (Lions 1988a, b).

3. Application of theorem 3 to a regularization method

Let us recall that the operator $G : L^2(\Omega) \longrightarrow L^2(\Gamma \times (0, T))$ is defined by (1.4). As is proved in Yamamoto (1995a), the operator G is compact from $L^2(\Omega)$ to $L^2(\Gamma \times (0, T))$, so that the problem of solving

$$y = Gf \tag{3.1}$$

with respect to $f \in L^2(\Omega)$ for a given $y \in L^2(\Gamma \times (0, T))$ is ill-posed.

In this section, assuming that $y_0 = Gf_0$, we consider reconstruction of f_0 from inexact available data y_δ:

$$\|y_\delta - y_0\|_{L^2(\Gamma\times(0,T))} \leqslant \delta \tag{3.2}$$

where $\delta \geqslant 0$ is a noise level. The reconstruction problem is to find reasonable approximations f_δs for f_0 by using data y_δ. By the reasonable approximations f_δs, we mean that we can stably construct f_δ from y_δ and that $\lim_{\delta\to 0}\|f_\delta - f_0\|_{L^2(\Omega)} = 0$. Since G is compact and $\mathcal{R}(G)$ is a proper closed subset of $L^2(\Gamma \times (0, T))$ (Yamamoto 1995b), we must be concerned with the following difficulties:

(i) $y_\delta \in \mathcal{R}(G)$ does not necessarily hold no matther how small $\delta > 0$ is.

(ii) $G^{-1} : \mathcal{R}(G) \subset L^2(\Gamma\times(0, T)) \longrightarrow L^2(\Omega)$ is not continuous, although G is injective by theorem 1 in section 2.

For overcoming these difficulties, various regularization techniques have been proposed (e.g. Baumeister (1987), Groetsch (1984, 1993), Hofmann (1986), Tikhonov and Arsenin (1977)). In this section, according to Groetsch (1984), ch 3, we consider the following regularization.

Regularization. Let $\alpha > 0$ be a parameter. Minimize the functional

$$F_\alpha(f) = \|Gf - y_\delta\|^2_{L^2(\Gamma\times(0,T))} + \alpha\|f\|^2_{L^2(\Omega)} \tag{3.3}$$

over $f \in L^2(\Omega)$.

Then, by Groetsch (1984), for an arbitrarily fixed $\alpha > 0$, there exists a unique minimizer f_α^δ for a given $y_\delta \in L^2(\Gamma \times (0, T))$, and if f_α^δ and $\widetilde{f_\alpha^\delta}$ are minimizers of F_α respectively with y_δ and $\widetilde{y_\delta}$, then

$$\|f_\alpha^\delta - \widetilde{f_\alpha^\delta}\|_{L^2(\Omega)} \leqslant C_3\|y_\delta - \widetilde{y_\delta}\|_{L^2(\Gamma\times(0,T))}$$

for a constant $C_3 > 0$ which is independent of y_δ, $\widetilde{y_\delta}$. Here we call f_α^δ a *regularized solution.* Moreover, if we choose parameters $\alpha = \alpha(\delta)$ such that

$$\lim_{\delta\to 0}\frac{\delta^2}{\alpha(\delta)} = 0 \tag{3.4}$$

then

$$\lim_{\delta\to 0}\|f_\alpha^\delta - f_0\|_{L^2(\Omega)} = 0. \tag{3.5}$$

However, the choice (3.4) of regularizing parameters does not guarantee any concrete convergence rates in (3.5). If we can pose an additional assumption on f_0, then we can derive a concrete rate under suitable choices of regularizing parameters. That is, *Theorem 0 (Corollary 3.1.3 (p 35) in Groetsch (1984)).* If

$$f_0 \in \mathcal{R}(G^*) \tag{3.6}$$

and

$$\alpha = \alpha(\delta) = c_4\delta \tag{3.7}$$

for an arbitrarily given constant $c_4 > 0$, then

$$\|f_\alpha^\delta - f_0\|_{L^2(\Omega)} = O(\sqrt{\delta})$$

as $\delta \longrightarrow 0$.

Thus we can combine theorem 3 with this theorem and we have:

Theorem 4 (Convergence rates of regularized solutions). If $f_0 \in H_0^1(\Omega)$ and we choose $\alpha = \alpha(\delta)$ satisfying (3.7), then

$$\|f_\alpha^\delta - f_0\|_{L^2(\Omega)} = O(\sqrt{\delta})$$

as $\delta \longrightarrow 0$.

Remark 7. For another regularization to our inverse problem, we refer to Yamamoto (1995b), where asymptotic behaviour for the singular values of G is given.

4. Proof of theorem 1

Let us consider another initial value/boundary value problem for a hyperbolic equation:

$$\begin{aligned} &w''(x,t) = \Delta w(x,t) \qquad (x \in \Omega, t > 0)\\ &w(x,0) = 0 \qquad w'(x,0) = f(x) \qquad (x \in \Omega)\\ &w(x,t) = 0 \qquad (x \in \partial\Omega, t > 0). \end{aligned} \tag{4.1}$$

For any $f \in L^2(\Omega)$, to (4.1), there exists a unique solution $w = w(f) \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ (e.g. lemma 3.6 (p 39) in Lions (1988b)) and

$$\|w(f)\|_{L^\infty(0,T;H_0^1(\Omega))} + \|w(f)'\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C_5\|f\|_{L^2(\Omega)} \tag{4.2}$$

for some constant $C_5 > 0$. First we show:

Lemma 2 (Komornik 1989a, Lions 1988b). We assume (2.2) and (2.3). Then there exists a constant $C = C(\Omega, \Gamma, T, x_0) > 0$ such that

$$C^{-1}\left\|\frac{\partial w(f)}{\partial n}\right\|_{L^2(\Gamma\times(0,T))} \leqslant \|f\|_{L^2(\Omega)} \leqslant C\left\|\frac{\partial w(f)}{\partial n}\right\|_{L^2(\Gamma\times(0,T))}$$

for any $f \in L^2(\Omega)$.

Moreover we show:

Lemma 3. We denote an operator $K : L^2(\Gamma \times (0,T)) \longrightarrow H^1(0,T;L^2(\Gamma))$ by

$$(Kg)(x,t) = \int_0^t \sigma(t-s)g(x,s)\,\mathrm{d}s \qquad (x \in \Gamma, 0 < t < T). \tag{4.3}$$

Then there exists a constant $C_6 = C_6(\Omega, T) > 0$ such that

$$C_6^{-1}\|Kg\|_{H^1(0,T;L^2(\Gamma))} \leqslant \|g\|_{L^2(\Gamma\times(0,T))} \leqslant C_6\|Kg\|_{H^1(0,T;L^2(\Gamma))} \tag{4.4}$$

for any $g \in L^2(\Gamma \times (0,T))$.

Proof of lemma 3. By $\sigma \in C^1[0,T]$, taking time derivatives in (4.3), we get

$$\frac{\partial(Kg)}{\partial t}(x,t) = \sigma(0)g(x,t)+\int_0^t \sigma'(t-s)g(x,s)\,\mathrm{d}s \qquad (x \in \Gamma, 0 < t < T).$$

We have $\sigma(0) \neq 0$ by the assumption (2.4), so that this is a Volterra equation of the second kind with respect to g. Consequently we obtain

$$C_6'^{-1}\left\|\frac{\partial(Kg)}{\partial t}\right\|_{L^2(\Gamma\times(0,T))} \leqslant \|g\|_{L^2(\Gamma\times(0,T))} \leqslant C_6'\left\|\frac{\partial(Kg)}{\partial t}\right\|_{L^2(\Gamma\times(0,T))} \tag{4.5}$$

(e.g. Tricomi (1985)). Moreover, directly from (4.3), we get

$$\|Kg\|_{L^2(\Gamma\times(0,T))} \leqslant C_6'\|g\|_{L^2(\Gamma\times(0,T))}. \tag{4.6}$$

Combining (4.5) with (4.6), we reach (4.4). □

Next we can show a key lemma which connects our inverse problem with an exact controllability problem.

Lemma 4. For any $f \in C_0^\infty(\Omega)$, we have

$$u(f)(x,t) = (Kw(f))(x,t) = \int_0^t \sigma(s)w(f)(x,t-s)\,\mathrm{d}s \qquad (x \in \Omega, t > 0). \tag{4.7}$$

Remark 8. The relation (4.7) holds for any $f \in L^2(\Omega)$. In fact, we can prove (4.7) for $f \in L^2(\Omega)$ by approximating $f \in L^2(\Omega)$ by functions in $C_0^\infty(\Omega)$ and using estimates (1.2) and (4.2).

Proof of lemma 4. This lemma is seen directly from Duhamel's principle. For convenience, we give the proof here. Let us set the right-hand side of (4.7) by $\widetilde{u}(x,t)$. It is sufficient to prove that $\widetilde{u} \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ satisfies (1.1). By $f \in C_0^\infty(\Omega)$, the solutions $u(f)$ and $w(f)$ are smooth on $\overline{\Omega} \times [0,T]$. In particular, $\widetilde{u} \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ and, moreover, by direct computations we see that $\widetilde{u}$ satisfies (1.1). By the uniqueness of solutions to (1.1), we get $\widetilde{u}(x,t) = u(f)(x,t) (x \in \Omega, t > 0)$. This completes the proof of lemma 4. □

Now let us complete the proof of theorem 1. It is sufficient to prove (2.5) for $f \in C_0^\infty(\Omega)$. In fact, let us assume that (2.5) holds for $f \in C_0^\infty(\Omega)$. Let $f \in L^2(\Omega)$ be given arbitrarily. Then since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, we can take $f_n \in C_0^\infty(\Omega)$ such that $\|f_n - f\|_{L^2(\Omega)} \longrightarrow 0$ as $n \longrightarrow \infty$. By our present assumption, we can get

$$C^{-1}\left\|\frac{\partial u(f_n)}{\partial n}\right\|_{H^1(0,T;L^2(\Gamma))} \leqslant \|f_n\|_{L^2(\Omega)} \leqslant C\left\|\frac{\partial u(f_n)}{\partial n}\right\|_{H^1(0,T;L^2(\Gamma))} \qquad (n \geqslant 1).$$

On the other hand, by (1.3), we have

$$\left\|\frac{\partial u(f_n)}{\partial n}\right\|_{H^1(0,T;L^2(\Gamma))} \longrightarrow \left\|\frac{\partial u(f)}{\partial n}\right\|_{H^1(0,T;L^2(\Gamma))}.$$

Thus we can reach (2.5) for any $f \in L^2(\Omega)$.

Finally let us proceed to the proof of (2.5) for $f \in C_0^\infty(\Omega)$. Since $w(f)$ is sufficiently smooth on $\overline{\Omega} \times [0, T]$, we have

$$\frac{\partial}{\partial n}\int_0^t \sigma(s) w(f)(x, t-s)\,\mathrm{d}s = \int_0^t \sigma(s)\frac{\partial w(f)}{\partial n}(x, t-s)\,\mathrm{d}s \qquad (x \in \partial\Omega, 0 < t < T).$$

Therefore we get by (4.7)

$$\frac{\partial u(f)}{\partial n}(x,t) = \int_0^t \sigma(s)\frac{\partial w(f)}{\partial n}(x,t-s)\,\mathrm{d}s = K\left(\frac{\partial w(f)}{\partial n}\right)(x,t) \qquad (x \in \Gamma, 0 < t < T). \tag{4.8}$$

Consequently by lemma 3, we obtain

$$C_6^{-1}\left\|\frac{\partial u(f)}{\partial n}\right\|_{H^1(0,T;L^2(\Gamma))} \leqslant \left\|\frac{\partial w(f)}{\partial n}\right\|_{L^2(\Gamma\times(0,T))} \leqslant C_6\left\|\frac{\partial u(f)}{\partial n}\right\|_{H^1(0,T;L^2(\Gamma))}$$
$$(f \in C_0^\infty(\Omega)).$$

Now by applying these inequalities in lemma 2, the estimate (2.5) for $f \in C_0^\infty(\Omega)$ is straightforward. Thus the proof of theorem 1 is complete. □

5. Proof of theorem 2

It suffices to prove theorem 2 for $f \in C_0^\infty(\Omega)$ by the estimate (1.3) and the denseness of $C_0^\infty(\Omega)$ in $L^2(\Omega)$. Henceforth let $f \in C_0^\infty(\Omega)$. Then, using an eigenfunction expansion of $w(f)$, we can see

$$\frac{\partial w(f)}{\partial n}(x,t) = \sum_{k=1}^\infty (f, \phi_k)_{L^2(\Omega)}\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}\frac{\partial\phi_k}{\partial n}(x) \qquad (x \in \Gamma, 0 < t < T) \tag{5.1}$$

where the series is convergent in $H^1(0, T; L^2(\Gamma))$.

Next we prove:

Lemma 5. Under the assumptions (2.2) and (2.3), we have

$$\left(\frac{\sin\sqrt{\lambda_l}t}{\sqrt{\lambda_l}}\frac{\partial\phi_l}{\partial n}, -\Pi\phi_k\right)_{L^2(\Gamma\times(0,T))} = \delta_{kl} \equiv \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l. \end{cases}$$

For the proof, first we show:

Lemma 6. Let $p(x, t)$ and $q(x, t)$ be sufficiently smooth and satisfy

$$\begin{aligned} &p''(x,t) = \Delta p(x,t) \qquad (x \in \Omega, 0 < t < T) \\ &p(x,T) = 0 \qquad p'(x,T) = 0 \qquad (x \in \Omega) \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} &q''(x,t) = \Delta q(x,t) \qquad (x \in \Omega, 0 < t < T) \\ &q(x,0) = 0 \qquad q'(x,0) = f(x) \qquad (x \in \Omega) \\ &q(x,t) = 0 \qquad (x \in \partial\Omega, 0 < t < T). \end{aligned} \tag{5.3}$$

Then

$$\int_\Omega p(x,0) f(x)\,\mathrm{d}x = -\int_0^T \int_{\partial\Omega} p(x,t)\frac{\partial q}{\partial n}(x,t)\,\mathrm{d}S_x\,\mathrm{d}t. \tag{5.4}$$

Proof of lemma 6. Since p and q are sufficiently smooth, the following calculations are justified.

$$\begin{aligned} \int_0^T \int_\Omega \Delta p(x,t) q(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \int_\Omega \left(\int_0^T p''(x,t) q(x,t)\,\mathrm{d}t\right)\mathrm{d}x \qquad \text{(by (5.2))} \\ &= \int_\Omega \left([p'(x,t)q(x,t)]_0^T - \int_0^T p'(x,t)q'(x,t)\,\mathrm{d}t\right)\mathrm{d}x \\ &\quad \text{(by integration by parts)} \\ &= -\int_\Omega \int_0^T p'(x,t)q'(x,t)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad \text{(by } p'(x,T) = q(x,0) = 0) \\ &= \int_\Omega \int_0^T p(x,t)q''(x,t)\,\mathrm{d}t\,\mathrm{d}x + \int_\Omega p(x,0)f(x)\,\mathrm{d}x \\ &\quad \text{(by integration by parts and } p(x,T) = 0,\ q'(x,0) = f(x)) \\ &= \int_0^T \int_\Omega p(x,t)\Delta q(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega p(x,0)f(x)\,\mathrm{d}x \\ &\quad \text{(by (5.3))} \end{aligned}$$

namely

$$\int_0^T \int_\Omega (q(x,t)\Delta p(x,t) - p(x,t)\Delta q(x,t))\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega p(x,0)f(x)\,\mathrm{d}x.$$

Applying the Green formula, and $q(x,t) = 0 (x \in \partial\Omega, 0 < t < T)$, we see (5.4). □

Proof of lemma 5. Let ψ be the solution to

$$\psi''(x,t) = \Delta\psi(x,t) \qquad (x \in \Omega, 0 < t < T)$$
$$\psi(x,T) = 0 \qquad \psi'(x,T) = 0 \qquad (x \in \Omega)$$
$$\psi(x,t) = v(x,t) \qquad (x \in \Gamma, 0 < t < T)$$
$$\psi(x,t) = 0 \qquad (x \in \partial\Omega \setminus \Gamma, 0 < t < T)$$

for $v \in L^2(\Gamma \times (0,T))$. For any $v \in L^2(\Gamma \times (0,T))$, we can define the weak solution ψ and we can prove (theorem 4.2 (pp 46–7) in Lions (1988b)) that there exists a unique solution $\psi = \psi(v) \in C^0([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ and, moreover,

$$\|\psi(v)\|_{L^\infty(0,T;L^2(\Omega))} + \|\psi(v)'\|_{L^\infty(0,T;H^{-1}(\Omega))} \leqslant C_1\|v\|_{L^2(\Gamma\times(0,T))}. \tag{5.6}$$

For $\psi(v)$, the following equation holds:

$$\int_\Omega \psi(v)(x,0)\phi_l(x)\,\mathrm{d}x = -\int_0^T\int_\Gamma v(x,t)\frac{\sin\sqrt{\lambda_l}t}{\sqrt{\lambda_l}}\frac{\partial\phi_l}{\partial n}(x)\,\mathrm{d}S_x\,\mathrm{d}t \tag{5.7}$$

for any $v \in L^2(\Gamma \times (0,T))$ and $l \geqslant 1$.

Proof of (5.7). First let us assume that $v \in C_0^\infty(\Gamma \times (0,T))$. Then the solution $\psi = \psi(v)$ is so smooth that we can set $p(x,t) = \psi(v)(x,t)$ and $q(x,t) = (\sin\sqrt{\lambda_l}t/\sqrt{\lambda_l})\phi_l(x)$ in lemma 6. Therefore noting the boundary condition of $\psi(v)$ in (5.5), we can obtain (5.7) for any $v \in C_0^\infty(\Gamma \times (0,T))$.

Finally we have to prove (5.7) for any $v \in L^2(\Gamma \times (0,T))$. Since $C_0^\infty(\Gamma \times (0,T))$ is dense in $L^2(\Gamma \times (0,T))$, we can choose $v_n \in C_0^\infty(\Gamma \times (0,T))$ $(n \geqslant 1)$ such that

$$\|v_n - v\|_{L^2(\Gamma\times(0,T))} \longrightarrow 0$$

as $n \longrightarrow \infty$. As is already proved, we have (5.7) for $v_n (n \geqslant 1)$. By (5.6) with $v - v_n$, we can make n tend to ∞ in (5.7), so that we complete the proof of (5.7).

Let us complete the proof of lemma 5. In lemma 1, for $k \geqslant 1$, we take $\phi_0(x) = -\phi_k(x)(x \in \Omega)$ and set $v = -\Pi\phi_k \in L^2(\Gamma \times (0,T))(k \geqslant 1)$. Then $\phi(v)$ satisfies (2.8) with $\phi_0 = -\phi_k$ and (2.9). By uniqueness of weak solutions to (5.5), we see $\psi(v) = \phi(v)$. Therefore, applying this v in (5.7), we obtain

$$\left(\frac{\sin\sqrt{\lambda_l}t}{\sqrt{\lambda_l}}\frac{\partial\phi_l}{\partial n}, -\Pi\phi_k\right)_{L^2(\Gamma\times(0,T))} \equiv \int_0^T\int_\Gamma -\Pi\phi_k(x,t)\frac{\sin\sqrt{\lambda_l}t}{\sqrt{\lambda_l}}\frac{\partial\phi_l}{\partial n}(x)\,\mathrm{d}S_x\,\mathrm{d}t$$
$$= \int_\Omega \phi_k(x)\phi_l(x)\,\mathrm{d}x = \delta_{kl}$$

by the orthonormality (2.7) of $\{\phi_k\}_{k\geqslant1}$. Thus the proof of lemma 5 is complete. □

Let $K^* : \mathcal{R}(K) \longrightarrow L^2(\Gamma\times(0,T))$ be the adjoint of the operator $K : L^2(\Gamma\times(0,T)) \longrightarrow H^1(0,T : L^2(\Gamma))$. Then we can directly verify

$$(K^*h)(x,t) = \sigma(0)h'(x,t) + \int_t^T (\sigma'(\xi - t)h'(x,\xi) + \sigma(\xi - t)h(x,\xi))\,\mathrm{d}\xi$$
$$(x \in \Gamma, 0 < t < T). \tag{5.8}$$

Therefore by definition (2.13) of Φ, the equality

$$K^*\Phi\eta = \eta \qquad (\eta \in L^2(\Gamma \times (0,T))) \tag{5.9}$$

holds.

We are ready to complete the proof of theorem 2. In fact, we have

$$\begin{aligned}
\left(\frac{\partial u(f)}{\partial n}, \theta_k\right)_{H^1(0,T;L^2(\Gamma))} &= \left(K\left(\frac{\partial w(f)}{\partial n}\right), \Phi(-\Pi\phi_k)\right)_{H^1(0,T;L^2(\Gamma))} \\
&\quad \text{(by (2.14) and (4.8))} \\
&= \left(\frac{\partial w(f)}{\partial n}, (K^*\Phi)(-\Pi\phi_k)\right)_{L^2(\Gamma\times(0,T))} \\
&= \left(\frac{\partial w(f)}{\partial n}, -\Pi\phi_k\right)_{L^2(\Gamma\times(0,T))} \qquad \text{(by (5.9))} \\
&= \sum_{l=1}^{\infty} (f, \phi_l)_{L^2(\Omega)} \left(\frac{\sin\sqrt{\lambda_l}t}{\sqrt{\lambda_l}}\frac{\partial \phi_l}{\partial n}, -\Pi\phi_k\right)_{L^2(\Gamma\times(0,T))} \\
&\quad \text{(by (5.1))} \\
&= \sum_{l=1}^{\infty} (f, \phi_l)_{L^2(\Omega)}\delta_{kl} \qquad \text{(by lemma 5)} \\
&= (f, \phi_k)_{L^2(\Omega)}.
\end{aligned}$$

Thus the proof of theorem 2 is complete. □

6. Proof of theorem 3

We define an operator $L : L^2(\Omega) \longrightarrow L^2(\Gamma \times (0,T))$ by

$$Lf = \frac{\partial w(f)}{\partial n} \tag{6.1}$$

where $w(f)$ is the solution to (4.1). By lemma 2, the operator L is bounded from $L^2(\Omega)$ to $L^2(\Gamma \times (0,T))$. By (4.8), we decompose G as

$$Gf = K_{L^2}Lf \qquad (f \in L^2(\Omega)) \tag{6.2}$$

where we regard K as an operator from $L^2(0,T; L^2(\Gamma))$ to itself and we set $K = K_{L^2}$. Therefore we get $G^* = L^*K^*_{L^2}$, so that we have

$$\mathcal{R}(G^*) = \{L^*v;\, v \in \mathcal{R}(K^*_{L^2})\}.$$

On the other hand. we directly see $(K^*_{L^2}\eta)(x,t) = \int_t^T \sigma(\xi - t)\eta(x,\xi)\,d\xi$ $(x \in \Gamma, 0 < t < T)$, so that $\mathcal{R}(K^*_{L^2}) = \{v \in H^1(0,T; L^2(\Gamma));\, v(\cdot, T) = 0\}$ by (2.4). Consequently we obtain

$$\mathcal{R}(G^*) = \{L^*v;\, v \in H^1(0,T; L^2(\Gamma)),\, v(\cdot,T) = 0\}. \tag{6.3}$$

Thus, for the proof, we have only to determine $\mathcal{R}(L^*)$. For this, we need

$$-\int_0^T \int_\Gamma v(x,t)\frac{\partial w(f)}{\partial n}(x,t)\,\mathrm{d}S_x\,\mathrm{d}t = \int_\Omega \psi(v)(x,0)f(x)\,\mathrm{d}x \qquad (6.4)$$

for any $v \in L^2(\Gamma\times(0,T))$ and $f \in L^2(\Omega)$. Here we recall that $\psi(v) \in C^0([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ is the weak solution to (5.5). At the end of this section, we will prove (6.4).

Relation (6.4) implies

$$(Lf, v)_{L^2(\Gamma\times(0,T))} = (f, -\psi(v)(\cdot,0))_{L^2(\Omega)}$$

namely

$$L^*v = -\psi(v)(\cdot,0) \qquad v \in L^2(\Gamma\times(0,T)). \qquad (6.5)$$

By (6.3) we have

$$\mathcal{R}(G^*) = \{\psi(v)(\cdot,0); v \in H^1(0,T; L^2(\Gamma)), v(\cdot,T) = 0\}. \qquad (6.6)$$

This relation is the first point (i) mentioned just after the statement of theorem 3. To complete the proof of theorem 3, as the second point (ii) mentioned after the statement of theorem 3, we show an exact controllability result by the Hilbert uniqueness method.

Lemma 7 (Theorem 6.4 (p 75) in Lions (1988b)). We assume (2.2) and (2.3). For any $(\psi_0, \psi_1) \in H_0^1(\Omega)\times L^2(\Omega)$, there exists a $v \in H_0^1(0,T; L^2(\Gamma))$ such that

$$\psi(v)(x,0) = \psi_0(x) \qquad \psi(v)'(x,0) = \psi_1(x) \qquad (x \in \Omega).$$

Therefore from (6.6) we can reach $\mathcal{R}(G^*) \supset H_0^1(\Omega)$, the conclusion of theorem 3.

Proof of (6.4). First let us assume that $v \in C_0^\infty(\Gamma\times(0,T))$ and $f \in C_0^\infty(\Omega)$. Then $\psi(v)$ and $w(f)$ are sufficiently smooth, so that lemma 6 and the boundary condition in (5.5) imply

$$\begin{aligned}\int_\Omega \psi(v)(x,0)f(x)\,\mathrm{d}x &= -\int_0^T \int_{\partial\Omega} \psi(v)(x,t)\frac{\partial w(f)}{\partial n}(x,t)\,\mathrm{d}S_x\,\mathrm{d}t \\ &= -\int_0^T \int_\Gamma v(x,t)\frac{\partial w(f)}{\partial n}(x,t)\,\mathrm{d}S_x\,\mathrm{d}t. \qquad (6.7)\end{aligned}$$

Therefore we see (6.4) for $v \in C_0^\infty(\Gamma\times(0,T))$ and $f \in C_0^\infty(\Omega)$.

Finally let $v \in L^2(\Gamma\times(0,T))$ and $f \in L^2(\Omega)$. Since $C_0^\infty(\Gamma\times(0,T))$ and $C_0^\infty(\Omega)$ are dense respectively in $L^2(\Gamma\times(0,T))$ and $L^2(\Omega)$, there exist $v_n \in C_0^\infty(\Gamma\times(0,T))$ and $f_n \in C_0^\infty(\Omega)$ $(n \geqslant 1)$ such that $\|v_n - v\|_{L^2(\Gamma\times(0,T))} \longrightarrow 0$, $\|f_n - f\|_{L^2(\Omega)} \longrightarrow 0$ as $n \longrightarrow \infty$. By (5.6) and lemma 2, we see

$$\|\psi(v_n)(\cdot,0) - \psi(v)(\cdot,0)\|_{L^2(\Omega)} \longrightarrow 0$$

and

$$\left\|\frac{\partial w(f_n)}{\partial n} - \frac{\partial w(f)}{\partial n}\right\|_{L^2(\Gamma\times(0,T))} \longrightarrow 0$$

as $n \longrightarrow 0$. Therefore we can let n tend to ∞ in (6.4) with $v = v_n$ and $f = f_n$, so that we obtain (6.4) for any $v \in L^2(\Gamma\times(0,T))$ and any $f \in L^2(\Omega)$.

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Appendix. Proof of (1.3)

First by theorem 4.1 (p 44) in Lions (1988b), we see

$$\left\| \frac{\partial u(f)}{\partial n} \right\|_{L^2(\Gamma \times (0,T))} \leqslant C_1' \| f \|_{L^2(\Omega)} \tag{A1}$$

for any $f \in L^2(\Omega)$.

Let z be the weak solution to

$$z''(x,t) = \Delta z(x,t) + \sigma'(t) f(x) \qquad (x \in \Omega, 0 < t < T)$$

$$z(x,0) = 0 \qquad z'(x,0) = \sigma(0) f(x) \qquad (x \in \Omega)$$

$$z(x,t) = 0 \qquad (x \in \partial\Omega, 0 < t < T).$$

Since $\sigma' f \in L^1(0,T; L^2(\Omega))$ and $\sigma(0) f \in L^2(\Omega)$, it follows from lemma 3.6 (p 39) and theorem 4.1 (p 44) in Lions (1988b) that

$$z \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \tag{A2}$$

and

$$\left\| \frac{\partial z}{\partial n} \right\|_{L^2(\Gamma \times (0,T))} \leqslant C_1' \left(\| \sigma' f \|_{L^1(0,T;L^2(\Omega))} + \| \sigma(0) f \|_{L^2(\Omega)} \right)$$

$$\leqslant C_1' \| f \|_{L^2(\Omega)}$$

for any $f \in L^2(\Omega)$. By (2) we can set

$$\widetilde{u}(x,t) = \int_0^t z(x,s)\, \mathrm{d}s \qquad (x \in \Omega, 0 < t < T)$$

so that $\widetilde{u} \in C^1([0,T]; H_0^1(\Omega)) \cap C^2([0,T]; L^2(\Omega))$and $\widetilde{u}$ satisfies (1.1). By uniqueness of weak solution (lemma 3.6 (p 39) in Lions (1988b)), we get

$$u(f)(x,t) = \widetilde{u}(x,t) \qquad (x \in \Omega, 0 < t < T)$$

and

$$z(x,t) = u(f)'(x,t) \qquad (x \in \Omega, 0 < t < T).$$

Therefore (A1)–(A3) imply (1.3).

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