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Stability region enlargement through anti-windup strategy for linear systems with dynamics restricted actuator

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This paper addresses the problem of determination of stability regions for linear systems with amplitude and successive dynamics restricted actuator through anti-windup strategies. Considering a linear dynamic output feedback designed with respect to the linear system (without saturation), an anti-windup design method is investigated to guarantee both the stability of the closed-loop system and the respect of the controlled output constraints for a region of admissible initial states as large as possible. Based on the modelling of the closed-loop system resulting from the controller plus the anti-windup loop as a linear system with dead-zone and dynamics restricted nonlinearities, LMI stability conditions are formulated. Numerical optimization procedures are discussed.

Keywords: Anti-windup scheme; Bounded controlled output; Amplitude and dynamics restricted actuator; Stability regions; LMI

1. Introduction

Physical, safety or technological constraints induce that the control actuators cannot provide unlimited amplitude signals neither unlimited speed of reaction. The control problems of combat aircraft prototypes and launchers offer interesting examples of the difficulties due to these major constraints. Neglecting both amplitude and dynamics actuator limitations can be a source of undesirable or even catastrophic behaviours for the closed-loop system (as the lost of the closed-loop stability) (Berg *et al.* 1996). For these reasons, the study of the control problem or analysis stability problem with respect to systems subject to both amplitude and rate actuator saturations has received the attention of many researchers in the last years (see, for example, Tyan and Bernstein (1997), Kapila and Grigoriadis (2002) and Gomes da Silva Jr *et al.* (2003)).

The anti-windup approach consists in taking into account the effect of saturations in a second step after a preliminary design performed disregarding the saturation terms. The idea is then to introduce control modifications in order to recover, as much as possible, the performance induced by the previous design carried out on the basis of the unsaturated system. In particular, anti-windup schemes have been successfully applied in order to avoid or minimise the windup of the integral action in PID controllers, largely applied in the industry. In this case, most of the related literature focuses on the performance improvement in the sense of avoiding large and oscillatory transient responses (see, among others, Fertik and Ross (1967), Åström and Rundqwist (1989)).

Then, special attention has been paid to the influence of the anti-windup schemes in the stability and the performances of the closed-loop system (see, for example, Kapoor *et al.* (1998), Kothare and Morari (1999), Barbu *et al.* (2000)). Several results on the anti-windup problem are concerned with achieving global stability properties. However, global results cannot be achieved for open-loop unstable linear systems in the presence of actuator saturation, for which local results have to be developed. In this context, a key issue concerns the determination of domains of stability for the closed-loop system. It is worth noting that the basin of attraction is modified by the anti-windup loop. If the resulting basin of attraction is not sufficiently large, the system can present a divergent behaviour depending on its initialization and the action of disturbances.

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More recently, in Cao *et al.* (2002) and Gomes da Silva Jr and Tarbourietch (2003) or in the ACC03 Workshop “T-1: Modern Anti-windup Synthesis”, some constructive conditions have been proposed both to determine suitable anti-windup gains and to quantify the closed-loop region of stability in the case of amplitude saturation actuator. Different from the references cited above, in the current paper, we focus our attention on linear systems with amplitude and successive dynamics restricted actuator and bounded controlled outputs. To the knowledge of the authors, this type of actuator has not been widely studied despite its practical aspect (see, for example, Tarbouriech *et al.* (2003, 2004) for some preliminary results). Results classically obtained for amplitude and rate saturation actuators (Kapila and Grigoriadis 2002) do not directly extend to this type of actuator representation, in which the main characteristics are that amplitude and dynamics cannot saturate simultaneously. We aim at designing suitable anti-windup gains in order to ensure the closed-loop stability for regions of admissible initial states as large as possible. Based on the modelling of the closed-loop system resulting from the controller plus the anti-windup loop as a linear system with dead-zone and dynamics restricted nonlinearities, original constructive stability conditions are directly formulated as LMI conditions.

The paper is organized as follows. Section 2 describes in detail the actuator and the plant under consideration. The anti-windup strategy is then presented and the resulting problem of designing the suitable anti-windup loops is summarised. Section 3 provides the main theoretical conditions allowing us to exhibit a solution to the problem studied. In section 4, some computational analysis issues are addressed. In particular, some discussion with respect to the tools and technique developed and the results published in the literature are pointed out. Finally, the optimization procedure to obtain an estimate of the basin of attraction of the closed-loop system as large as possible is described. In section 5, two numerical examples borrowed from the literature allow us to underline not only the potentialities of the method proposed but also the difficulties inherent to the actuator under study. Some concluding remarks, in which some future work is evoked, end the paper.

Notation: For any vector $x \in \Re^n$, $x \succeq 0$ means that all the components of x, denoted $x_{(i)}$, are non-negative. For two vectors x, y of $\Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, $\forall i = 1, \ldots, n$. $\mathbf{1}$ and $\mathbf{0}$ denote the identity matrix and the null matrix of appropriate dimensions, respectively. The elements of a matrix $A \in \Re^{m \times n}$ are denoted by $A_{(i,j)}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. $A_{(i)}$ denotes the ith row of matrix A. $|A|$ is the matrix constituted from the absolute value of each element of A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A. $1_m \triangleq [1 \ldots 1]' \in \Re^m$. For any vector u of $\Re^m$ one defines each component of $\mathrm{sat}_{u0}(u)$ by $\mathrm{sat}_{u_0}(u_{(i)}) = \mathrm{sign}(u_{(i)}) \min(u_{0(i)}, |u_{(i)}|)$, $i = 1, \ldots, m$.

2. Problem statement

In this paper, we consider a class of nonlinear systems obtained by cascading linear systems with actuator containing some nonlinearities of saturation type as shown in figure 1.

The actuator considered is a dynamic system containing amplitude and dynamics restrictions, that is, it is described via successive time-derivatives of the input of the plant. By setting

$$x_a = \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(q-1)} \end{bmatrix} \in \Re^{mq} \quad \text{and} \quad y_a = \mathrm{sat}_{u_0}(u) \in \Re^m, \qquad (1)$$

where $u^{(q)}$ denotes the q-order time-derivative of u, the model of the actuator reads as follows:

$$\begin{cases} \dot{x}_a(t) = A_a x_a(t) + B_{a0} \mathrm{sat}_{u_0}(C_a x_a(t)) \\ \qquad + \sum_{j=1}^{q-1} B_{aj} \mathrm{sat}_{u_j}\left(\mathrm{sat}_{u_0}\left(C_a x_a(t)\right)^{(j)}\right) + B_{aq} u_a \\ y_a(t) = \mathrm{sat}_{u_0}(C_a x_a(t)), \end{cases} \qquad (2)$$

where x_a is the state of the actuator, y_a is the output of the actuator and $u_a \in \Re^{n_{cp}}$ is the input of the actuator. Matrices $A_a \in \Re^{mq \times mq}$, $B_{aj} \in \Re^{mq \times m}$, $j = 0, \ldots, q-1$, $B_{aq} \in \Re^{mq \times n_{cp}}$ and $C_a \in \Re^{m \times mq}$ are defined by

$$A_a = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & & \ddots & & \vdots \\ \vdots & & \cdots & & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \cdots & & \cdots & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \cdots & & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}; \quad B_{aj} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \cdots \\ \cdots \\ \mathbf{0} \\ T_j \end{bmatrix};$$

$$C_a = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}. \qquad (3)$$

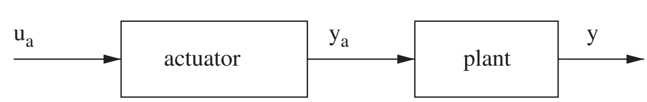

Figure 1. System under consideration.

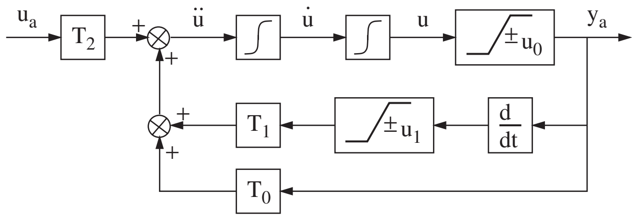

Figure 2. Actuator under consideration in the case $m=1$ and $q=2$.

Such a model is the type of actuator encountered in the control of launchers (see Langouët *et al.* (2002) in which $m=1$ and $q=2$). Indeed, it allows us to represent the limitations on the thruster angle of deflection and its time-derivative present in some phases of the flight path of launchers, in particular during the atmospheric phase. Thus, the actuator in the case $q=2$, $m=1$ can be represented in figure 2.

In (2), the positive vectors u_0 and u_j, $j=1,\ldots,q-1$, may be viewed as bounds on the position and the successive dynamics of the actuator state. Thus, it clearly appears that one cannot have simultaneously position and dynamics saturation. Indeed, if the ith component of the amplitude actuator saturation is effective (i.e., $|C_{a(i)}x_a| > u_{0(i)}$) then the corresponding component of the successive dynamics saturation does not affect the system (i.e., $\mathrm{sat}_{u_j}(\mathrm{sat}_{u_0}(C_{a(i)}x_a(t))^{(j)}) = \mathbf{0}$ and $(\mathrm{sat}_{u_0}(C_a x_a(t)) - C_a x_a(t))^{(j)} = 0$, $j=1,\ldots,q-1$).

Moreover, in order to protect the actuator from important input signal a saturation is added as follows:

$$u_a = \mathrm{sat}_{y_0}(y_c(t)), \tag{4}$$

where y_c is the output of the controller.

The plant controlled through the previous actuator is a continuous-time linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + By_a(t) \\ y(t) = Cx(t) \\ z(t) = C_2x(t) + D_2y_a(t), \end{cases} \tag{5}$$

where $x \in \Re^n$, $y \in \Re^p$ and $z \in \Re^l$ are the state, the measured output and the controlled output vectors, respectively. A, B, C, C_2 and D_2 are real constant matrices of appropriate dimensions. $y_a \in \Re^m$ is both the output of the actuator and the input of the plant.

Without saturation terms ($u_a = \mathrm{sat}_{y_0}(y_c) = y_c$, $\mathrm{sat}_{u_0}(C_a x_a) = C_a x_a = u$ and $\mathrm{sat}_{u_j}(\mathrm{sat}_{u_0}(C_a x_a)^{(j)}) = C_a x_a^{(j)} = u^{(j)}$, $j=1,\ldots,q-1$), system (2)–(5) is linear and reads

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \dot{x}_a(t) = (A_a + [B_{a0} \quad \cdots \quad B_{aq-1}])x_a(t) + B_{aq}y_c(t) \\ y(t) = Cx(t) \\ z(t) = C_2x(t) + D_2C_a x_a(t). \end{cases} \tag{6}$$

Under the (A, B)-controllability and (C, A)-observability assumptions, we assume that an n_c-order dynamic output stabilising controller has been determined to stabilize the linear system (6) and is described as

$$\begin{cases} \dot{\eta}(t) = A_c\eta(t) + B_c y(t) \\ y_c(t) = C_c\eta(t) + D_c y(t), \end{cases} \tag{7}$$

where $\eta \in \Re^{n_c}$ is the controller state, y is the controller input and $y_c \in \Re^{n_{cp}}$ is the controller output.

Furthermore, in the presence of the saturation terms, in order to mitigate the undesirable effects of windup, we want to consider an anti-windup strategy. For this, we assume that the variables $u = C_a x_a(t)$, $y_a = \mathrm{sat}_{u_0}(C_a x_a(t))$ (variables of the actuator), y_c and $u_a = \mathrm{sat}_{y_0}(y_c)$ (input of the actuator) can be measured (that is the case in the type of launchers studied in Langouët *et al.* (2002)). Hence, some terms based on the following differences (see Teel (1999), Cao *et al.* (2002) for anti-windup in amplitude saturation case)

$$\begin{aligned} &\big(\mathrm{sat}_{u_0}(C_a x_a(t)) - C_a x_a(t)\big) \\ &\big(\mathrm{sat}_{u_0}(C_a x_a(t)) - C_a x_a(t)\big)^{(j)}, \quad j=1,\ldots,q-1 \\ &\big(\mathrm{sat}_{y_0}(y_c(t)) - y_c(t)\big) \end{aligned}$$

can be added to the system, in particular to the controller, through adequate gains. The idea in a classical anti-windup strategy is to use the difference between the actuator output which can be saturate and a fictitious signal corresponding to the non-saturated one in order to reduce the effect of windup. This idea is directly extended here to the signal derivative which can saturate leading to the introduction of degrees of freedom (gains) associated with each saturated derivative. Thus, considering the dynamic controller and this anti-windup strategy, the closed-loop system reads

$$\begin{cases} \dot{x}(t) = Ax(t) + B\mathrm{sat}_{u_0}(C_a x_a(t)) \\ \dot{x}_a(t) = A_a x_a(t) + B_{a0}\mathrm{sat}_{u_0}(C_a x_a(t)) \\ \qquad + \sum_{j=1}^{q-1} B_{aj}\mathrm{sat}_{uj}\big(\mathrm{sat}_{u_0}(C_a x_a(t))^{(j)}\big) + B_{aq}\mathrm{sat}_{y_0}(y_c(t)) \\ \dot{\eta}(t) = A_c\eta(t) + B_cCx(t) + G_c(\mathrm{sat}_{y_0}(y_c(t)) - y_c(t)) \\ \qquad + E_c(\mathrm{sat}_{u_0}(C_a x_a(t)) - C_a x_a(t)) \\ y_c(t) = C_c\eta(t) + D_cCx(t) + F_c(\mathrm{sat}_{u_0}(C_a x_a(t)) \\ \qquad - C_a x_a(t)) + \sum_{j=1}^{q-1} H_{cj}(\mathrm{sat}_{u_0}(C_a x_a(t)) - C_a x_a(t))^{(j)} \\ z(t) = C_2x(t) + D_2\mathrm{sat}_{u_0}(C_a x_a(t)), \end{cases} \tag{8}$$

where E_c, F_c, G_c and H_{cj}, $j=1,\ldots,q-1$, are the anti-windup gains to be determined. It is worth noticing that, as mentioned above, in system (8) the added terms are only active when the saturation occurs.

Remark 1: Preliminary results had been obtained in Tarbouriech *et al.* (2004) for the same problem (actuator model, anti-windup strategy). However, the part due to H_{cj}, $j=1,\ldots,q-1$, was not taken into account. In that case, some numerical problems appeared when solving the first condition of Proposition 1 in (Tarbouriech *et al.* 2004): in fact this inequality appeared to be marginally feasible. Some difficulties could be encountered to have coherent numerical solutions with LMILab in MATLAB. The use of SEDUMI solver decreased them. Nevertheless, by studying carefully this condition, one could apprehend that this problem was structural (due to the form of different matrices of the complete system). In the current solution, this problem disappears thanks to the added terms through the gains H_{cj}, $j=1,\ldots,q-1$.

In order to deal with system (8), let us define the q nonlinearities ϕ_0 and ϕ_j, $j=1,\ldots,q-1$

$$\phi_0(C_a x_a(t)) = y_a(t) - C_a x_a(t) = \mathrm{sat}_{u_0}(C_a x_a(t)) - C_a x_a(t) \tag{9}$$

$$\begin{aligned}\phi_j(C_a x_a(t)) &= \mathrm{sat}_{u_j}\big(y_a^{(j)}(t)\big) - y_a^{(j)}(t)\\ &= \mathrm{sat}_{u_j}\big(\mathrm{sat}_{u_0}(C_a x_a(t))^{(j)}\big)\\ &\quad - sat_{u_0}(C_a x_a(t))^{(j)}.\end{aligned} \tag{10}$$

From the definition of ϕ_0, one gets

$$\mathrm{sat}_{u_0}(C_a x_a(t))^{(j)} = \phi_0^{(j)}(C_a x_a(t)) + C_a x_a^{(j)}(t). \tag{11}$$

By the same way, one can define the nonlinearity ϕ_c

$$\begin{aligned}\phi_c(y_c(t)) &= \mathrm{sat}_{y_0}(y_c(t)) - y_c(t)\\ &= \mathrm{sat}_{y_0}\Bigg(C_c\eta(t) + D_c C x(t) + F_c\phi_0(C_a x_a(t))\\ &\qquad + \sum_{j=1}^{q-1} H_{cj}\phi_0^{(j)}(C_a x_a(t))\Bigg)\\ &\quad - \Bigg(C_c\eta(t) + D_c C x(t) + F_c\phi_0(C_a x_a(t))\\ &\qquad + \sum_{j=1}^{q-1} H_{cj}\phi_0^{(j)}(C_a x_a(t))\Bigg).\end{aligned} \tag{12}$$

Therefore, the system (8) can be written in a compact form. For this, define the extended state vector

$$\xi(t) = \begin{bmatrix} x(t)' & x_a(t)' & \eta(t)' \end{bmatrix}' \in \Re^{n+mq+n_c} \tag{13}$$

and the following matrices of appropriate dimensions

$$\left.\begin{aligned}
\mathbb{A} &= \begin{bmatrix} A & BC_a & \mathbf{0} \\ B_{aq}D_cC & (A_a + [B_{a0} \ \cdots \ B_{aq-1}]) & B_{aq}C_c \\ B_cC & \mathbf{0} & A_c \end{bmatrix}\\
&\quad \in \Re^{(n+mq+n_c)\times(n+mq+n_c)}\\
\mathbb{B}_0 &= \begin{bmatrix} B \\ B_{a0} \\ \mathbf{0} \end{bmatrix} \quad \mathbb{B}_j = \begin{bmatrix} \mathbf{0} \\ B_{aj} \\ \mathbf{0} \end{bmatrix} \in \Re^{(n+mq+n_c)\times m}\\
\mathbb{B}_q &= \begin{bmatrix} \mathbf{0} \\ B_{aq} \\ \mathbf{0} \end{bmatrix} \in \Re^{(n+mq+n_c)\times n_{cp}}\\
\mathbb{R} &= \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \in \Re^{(n+mq+n_c)\times n_c}\\
\mathbb{K} &= \begin{bmatrix} \mathbf{0} & C_a & \mathbf{0} \end{bmatrix} \in \Re^{m\times(n+mq+n_c)}\\
\mathbb{C}_2 &= \begin{bmatrix} C_2 & D_2C_a & \mathbf{0} \end{bmatrix} \in \Re^{l\times(n+mq+n_c)}\\
\mathbb{C}_0 &= \begin{bmatrix} D_cC & \mathbf{0} & C_c \end{bmatrix} \in \Re^{n_{cp}\times(n+mq+n_c)}
\end{aligned}\right\} \tag{14}$$

for $l=1,\ldots,q$. Thus, the closed-loop system reads

$$\left\{\begin{aligned}
\dot{\xi}(t) &= \mathbb{A}\xi(t) + (\mathbb{B}_0 + \mathbb{R}E_c + \mathbb{B}_qF_c)\phi_0(\mathbb{K}\xi(t))\\
&\quad + \sum_{j=1}^{q-1}\big(\mathbb{B}_j\phi_j(\mathbb{K}\xi(t)) + (\mathbb{B}_j + \mathbb{B}_qH_{cj})\phi_0^{(j)}(\mathbb{K}\xi(t))\big)\big)\\
&\quad + (\mathbb{R}G_c + \mathbb{B}_q)\phi_c(y_c(t))\\
z(t) &= \mathbb{C}_2\xi(t) + D_2\phi_0(\mathbb{K}\xi(t)).
\end{aligned}\right. \tag{15}$$

For simplicity, $\phi_0(\mathbb{K}\xi(\mathrm{t}))$, $\phi_j(\mathbb{K}\xi(\mathrm{t}))$, $\phi_0^{(j)}(\mathbb{K}\xi(t))$, $j=1,\ldots,q-1$ and $\phi_c(y_c(t))$ will be denoted ϕ_0, ϕ_j, $\phi_0^{(j)}$and ϕ_c. Note that in the absence of saturation one gets $\phi_0=\mathbf{0}$, $\phi_j=\mathbf{0}$, $\phi_0^{(j)}=\mathbf{0}$, $\phi_c=\mathbf{0}$ and system (15) becomes a linear system in which the matrix $\mathbb{A}$, defined in (14), is Hurwitz.

The problem we intend to solve is summarized as follows:

Problem 1: Determine anti-windup gains E_c, F_c, G_c, H_{cj}, $j=1,\ldots,q-1$, and a set S_0 such that

1. (**Stability**) The asymptotic stability of the closed-loop system (8) is ensured for any $[x(0)' \ x_a(0)' \ \eta(0)']' \in S_0$, where S_0 is as large as possible.
2. (**Performance**) For any $[x(0)' \ x_a(0)' \ \eta(0)']' \in S_0$ the controlled output z takes values in the set Z_0 defined by:

$$Z_0 = \{z \in \Re^l;\ -z_0 \preceq z \preceq z_0, z_{0(i)} > 0\}. \tag{16}$$

The implicit objective in Problem 1 is to compute E_c, F_c, G_c and H_{cj}, $j=1,\ldots,q-1$, for enlarging the basin of attraction of the closed-loop system, whereas the amplitude of the controlled output does not become too large.

3. Theoretical anti-windup gains design conditions

3.1 *Preliminaries*

Let us consider the generic nonlinearity $\varphi(v) = \mathrm{sat}_{v_0}(v) - v$, $\varphi(v) \in \Re^m$ and define the following set:

$$S(v_0) = \{v \in \Re^m, w \in \Re^m; -v_0 \preceq v - w \preceq v_0\}. \quad (17)$$

Lemma 1: Tarbouriech *et al.* (2004)**:** *If v and w are elements of* $S(v_0)$ *then the nonlinearity* $\phi(v)$ *satisfies the following inequality*:

$$\varphi(v)' T(\varphi(v) + w) \leq 0 \quad (18)$$

for any diagonal positive definite matrix $T \in \Re^{m\times m}$.

Note that in the case of classical sector condition the nonlinearity $\phi(v)$, with $v = \mathbb{K}\xi$, satisfies inequality (18) with $w = -\Lambda\mathbb{K}\xi$ where Λ is a positive diagonal matrix ($\mathbf{0} \leq \Lambda < \mathbf{1}$). By using such a choice for w, the conditions of stability are expressed through BMI conditions (Tarbouriech *et al.* 2003).

Moreover, Lemma 1 applies to nonlinearities defined in (9), (10) and (12). For this, it suffices to define the sets $S(u_0)$, $S(u_j)$, $j = 1, \ldots, q-1$, and $S(y_0)$ associated to the bounds u_0, u_j, $j = 1, \ldots, q-1$, and y_0 with adequate vectors v and w. This will be detailed in the sequel.

Let us define the vectors Φ_1, U_1 and Φ_0^d

$$\Phi_1 = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{q-1} \end{bmatrix} \in \Re^{m(q-1)}; \quad U_1 = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{q-1} \end{bmatrix} \in \Re^{m(q-1)};$$

$$\Phi_0^d = \begin{bmatrix} \dot{\phi}_0 \\ \phi_0^{(2)} \\ \vdots \\ \phi_0^{(q-1)} \end{bmatrix} \in \Re^{m(q-1)} \quad (19)$$

and the augmented matrices

$$\mathcal{B}_1 = \begin{bmatrix} \mathbb{B}_1 & \cdots & \mathbb{B}_{q-1} \end{bmatrix} \in \Re^{(n+mq+n_c)\times m(q-1)};$$

$$\mathbb{D} = \begin{bmatrix} \mathbb{K}\mathbb{A} \\ \vdots \\ \mathbb{K}\mathbb{A}^{q-1} \end{bmatrix} \in \Re^{m(q-1)\times(n+mq+n_c)}. \quad (20)$$

The following proposition provides a solution to Problem 1 by using Lemma 1, the S-procedure and a quadratic Lyapunov function.

Proposition 1: *If there exist a symmetric positive definite matrix* W, *matrices* X, Y, Z, N, $\mathbb{Q}_0$, $\mathbb{Q}_1$, $\mathbb{Q}_2$, $\mathbb{Q}_3$, *diagonal positive matrices* $\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2$ *and a block-diagonal positive definite matrix* $\mathbb{S}_3$ *of appropriate dimensions satisfying* (*the symbol * stands for symmetric blocks*):

$$\begin{bmatrix} W\mathbb{A}' + \mathbb{A}W & * & * & * & * & * \\ \mathbb{S}_0\mathbb{B}_0' + Y'\mathbb{R}' + Z'\mathbb{B}_q' - \mathbb{Q}_0 & -2\mathbb{S}_0 & * & * & * & * \\ \mathbb{S}_1\mathcal{B}_1' - \mathbb{Q}_1 & \mathbf{0} & -2\mathbb{S}_1 & * & * & * \\ \mathbb{S}_2\mathbb{B}_q' + X'\mathbb{R}' - \mathbb{Q}_2 & -\mathbb{Q}_3 & \mathbf{0} & -2\mathbb{S}_2 & * & * \\ \mathbb{D}W & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbb{S}_3 & * \\ \mathbb{S}_3\mathcal{B}_1' + N'\mathbb{B}_q' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbb{S}_3 \end{bmatrix} < \mathbf{0} \quad (21)$$

$$\begin{bmatrix} W & W\mathbb{K}_{(i)}' - \mathbb{Q}_{0(i)}' \\ * & u_{0(i)}^2 \end{bmatrix} \geq \mathbf{0}, \quad i = 1, \ldots, m \quad (22)$$

$$\begin{bmatrix} W & W\mathbb{D}_{(i)}' - Q_{1(i)}' \\ * & U_{1(i)}^2 \end{bmatrix} \geq \mathbf{0}, \quad i = 1, \ldots, m(q-1) \quad (23)$$

$$\begin{bmatrix} W & \mathbb{Q}_0' & W\mathbb{C}_{0(i)}' - Q_{2(i)}' \\ * & 2\mathbb{S}_0 & Z_{(i)}' - \mathbb{Q}_{3(i)}' \\ * & * & y_{0(i)}^2 \end{bmatrix} \geq \mathbf{0}, \quad i = 1, \ldots, n_{cp} \quad (24)$$

$$\begin{bmatrix} W & \mathbb{Q}_0' & W\mathbb{C}_{2(i)}' \\ * & 2\mathbb{S}_0 & \mathbb{S}_0 D_{2(i)}' \\ * & * & z_{0(i)}^2 \end{bmatrix} \geq \mathbf{0}, \quad i = 1, \ldots, l \quad (25)$$

then the gains $E_c = Y\mathbb{S}_0^{-1}$, $F_c = Z\mathbb{S}_0^{-1}$, $G_c = X\mathbb{S}_2^{-1}$, $[H_{c1} \ \ldots \ H_{cq-1}] = N\mathbb{S}_3^{-1}$ *and the set* $\mathbb{B}(W^{-1}, 1) = \{\xi \in \Re^{n+mq+n_c}; \xi' W^{-1}\xi \leq 1\}$ *are solutions to Problem* 1.

Proof: First, note that due to the structure of the system, one gets the following property: $\mathbb{K}\xi^{(j)} = \mathbb{K}\mathbb{A}^j\xi$, $j = 1, \ldots, q-1$. According to the nonlinearities ϕ_0, ϕ_j and $\phi_0^{(j)}$, $j = 1, \ldots, q-1$, it follows:

- For ϕ_0. Lemma 1 applies by considering $v = \mathbb{K}\xi$, $w = \mathbb{E}_0\xi$ and $v_0 = u_0$. Indeed, the corresponding relation (18) writes

$$\phi_0' T_0(\phi_0 + \mathbb{E}_0\xi) \le 0. \qquad (26)$$

- For ϕ_j. Lemma 1 applies by considering $v = \mathbb{K}\xi^{(j)} = \mathbb{K}\mathbb{A}^j\xi$, $w = \mathbb{E}_j\xi$ and $v_0 = u_j$, $j = 1, \ldots, q-1$. Indeed, the corresponding relation (18) writes

$$\phi_j' T_j(\phi_j + \mathbb{E}_j\xi) \le 0, \quad j = 1, \ldots, q-1. \qquad (27)$$

- Indeed, by using the augmented vectors Φ_1 and U_1, and by considering the above item, Lemma 1 globally applies with $v = \mathbb{D}\xi$, $w = \mathbb{G}_1\xi$ and $v_0 = U_1$, where

$$\mathbb{G}_1 = \begin{bmatrix} \mathbb{E}_1' & \mathbb{E}_2' & \cdots & \mathbb{E}_{q-1}' \end{bmatrix}'.$$

- For ϕ_c. Lemma 1 applies by considering $v = \mathbb{C}_0\xi + F_c\phi_0 + \Sigma_{j=1}^{q-1} H_{cj}\phi_0^{(j)}$, $w = \mathbb{F}_0\xi + \mathbb{F}_1\phi_0 + \Sigma_{j=1}^{q-1}\mathbb{F}_2\phi_0^{(j)}$ and $v_0 = y_0$. Indeed, the corresponding relation (18) writes

$$\phi_c' T_c\left(\phi_c + \mathbb{F}_0\xi + \mathbb{F}_1\phi_0 + \sum_{j=1}^{q-1}\mathbb{F}_{2j}\phi_0^{(j)}\right) \le 0. \qquad (28)$$

Consider $\mathbb{E}_0 = \mathbb{Q}_0 W^{-1}$ and $\mathbb{G}_1 = \mathbb{Q}_1 W^{-1}$. Therefore the satisfaction of relation (22) implies that the set $\mathcal{E}(W^{-1}, 1)$, with $P = W^{-1}$, is included in $S(u_0)$. In the same way, the satisfaction of relation (23) means that set $\mathcal{E}(W^{-1}, 1)$ is included in $\cap_{j=1}^{q-1} S(u_j)$. Hence, the non-linearities ϕ_0 and ϕ_j, $j = 1, \ldots, q-1$ satisfy the sector conditions (26) and (27), respectively.

Furthermore, by defining $\mathbb{F}_0 = \mathbb{Q}_2 W^{-1}$, $\mathbb{F}_1 = \mathbb{Q}_3\mathbb{S}_0^{-1}$ and $\mathbb{F}_{2j} = H_{cj}$, $j = 1, \ldots, q-1$, one has to prove that $\mathcal{E}(W^{-1}, 1)$ is included in $S(y_0)$. That corresponds to satisfy, $\forall i = 1, \ldots, n_{cp}$:

$$\begin{bmatrix} \xi' & \phi_0' \end{bmatrix} \begin{bmatrix} \mathbb{C}_{0(i)} - \mathbb{F}_{0(i)}' \\ F_{c(i)}' - \mathbb{F}_{1(i)}' \end{bmatrix} \begin{bmatrix} \mathbb{C}_{0(i)} - \mathbb{F}_{0(i)} & F_{c(i)}' - \mathbb{F}_{1(i)}' \end{bmatrix} \begin{bmatrix} \xi \\ \phi_0 \end{bmatrix} \le y_{0(i)}^2 .$$

$$\text{for } \xi \text{ and } \phi_0 \text{ such that } \begin{cases} \xi' W^{-1}\xi \le 1 \\ \phi_0' T_0(\phi_0 + \mathbb{E}_0\xi) \le 0. \end{cases}$$

Indeed, the satisfaction of relation (24) ensures that the above condition is satisfied and therefore that $\mathcal{E}(W^{-1}, 1)$ is included in $S(y_0)$.

Consider the quadratic Lyapunov function $V(\xi) = \xi' P\xi$, with $P = P' > \mathbf{0}$. The time-derivative of $V(\xi)$ along the trajectories of system (15) reads:

$$\begin{aligned} \dot{V}(\xi) &= \xi'(\mathbb{A}'P + P\mathbb{A})\xi + 2\xi' P(\mathbb{B}_0 + \mathbb{R}E_c + \mathbb{B}_q F_c)\phi_0 \\ &\quad + 2\xi' P\sum_{j=1}^{q-1}\left(\mathbb{B}_j\phi_j + \left(\mathbb{B}_j + \mathbb{B}_q H_{cj}\right)\phi_0^{(j)}\right) \\ &\quad + 2\xi' P(\mathbb{B}_q + \mathbb{R}G_c)\phi_c. \end{aligned}$$

Thus, by using the sector conditions (26), (27) and (28), it follows:

$$\begin{aligned} \dot{V}(\xi) &\le \xi'(\mathbb{A}'P + P\mathbb{A})\xi + 2\xi' P(\mathbb{B}_0 + \mathbb{R}E_c + \mathbb{B}_q F_c)\phi_0 \\ &\quad + 2\xi' P\sum_{j=1}^{q-1}\left(\mathbb{B}_j\phi_j + \left(\mathbb{B}_j + \mathbb{B}_q H_{cj}\right)\phi_0^{(j)}\right) \\ &\quad + 2\xi' P(\mathbb{B}_q + \mathbb{R}G_c)\phi_c - 2\phi_0' T_0(\phi_0 + \mathbb{E}_0\xi) \\ &\quad - 2\sum_{j=1}^{q-1}\phi_j' T_j(\phi_j + \mathbb{E}_j\xi) \\ &\quad - 2\phi_c' T_c\left(\phi_c + \mathbb{F}_0\xi + \mathbb{F}_1\phi_0 + \sum_{j=1}^{q-1}\mathbb{F}_{2j}\phi_0^{(j)}\right) \end{aligned}$$

for all $\xi \in \mathcal{E}(W^{-1}, 1)$.

By using the definition of $\phi_0^{(j)}$, it appears that $(\phi_0^{(j)})'\phi_0^{(j)} \le \xi'\mathbb{A}^{j\prime}\mathbb{K}'\mathbb{K}\mathbb{A}^j\xi$. Then, one can write

$$\begin{aligned} &2\xi' P\sum_{j=1}^{q-1}(\mathbb{B}_j + \mathbb{B}_q H_{cj})\phi_0^{(j)} \\ &\le \sum_{j=1}^{q-1}\Big[\xi' P(\mathbb{B}_j + \mathbb{B}_q H_{cj})L_j \\ &\quad \times(\mathbb{B}_j + \mathbb{B}_q H_{cj})' P\xi + (\phi_0^{(j)})' L_j^{-1}\phi_0^{(j)}\Big] \\ &\le \sum_{j=1}^{q-1}\Big[\xi' P(\mathbb{B}_j + \mathbb{B}_q H_{cj})L_j \\ &\quad \times(\mathbb{B}_j + \mathbb{B}_q H_{cj})' P\xi + \xi'\mathbb{A}^{j\prime}\mathbb{K}' L_j^{-1}\mathbb{K}\mathbb{A}^j\xi\Big]. \end{aligned}$$

Hence, by using the augmented vectors Φ_1 and Φ_0^d defined in (19), and this above inequality, the right term of the expression of $\dot{V}(\xi)$ writes

$$\zeta' \begin{bmatrix} \mathbb{A}'P + P\mathbb{A} & * & * & * & * & * \\ \mathbb{B}_0' P + \mathbb{E}_c'\mathbb{R}' P + F_c'\mathbb{B}_q' P - T_0\mathbb{E}_0 & -2T_0 & * & * & * & * \\ \mathcal{B}' P_1 - \mathbb{T}_1\mathbb{G}_1 & \mathbf{0} & -2\mathbb{T}_1 & * & * & * \\ \mathbb{B}_q' P + G_c'\mathbb{R}' P - T_c\mathbb{F}_0 & -T_c\mathbb{F}_1 & \mathbf{0} & -2T_c & * & * \\ \mathbb{D} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbb{S}_3 & * \\ \mathbb{S}_3\mathcal{B}_1' P + \mathbb{S}_3\mathcal{H}_c'\mathbb{B}_q' P & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbb{S}_3 \end{bmatrix} \zeta$$

with $\zeta = [\xi' \quad \phi_0' \quad \Phi_1' \quad \phi_c' \quad \Phi_0^{d'}]'$, $\mathbb{T}_1 = \text{diag}(T_j)$, $\mathbb{S}_3 = \text{diag}(L_j)$, $j = 1, \ldots, q-1$, and $\mathcal{H}_c = [H_{c1} \ \ldots \ H_{cq-1}]$.

By denoting $W = P^{-1}$, $\mathbb{S}_0 = T_0^{-1}$, $\mathbb{S}_1 = \mathbb{T}_1^{-1}$, $\mathbb{S}_2 = T_c^{-1}$, one can prove that if relation (21) is satisfied one has $\dot{V}(\xi) < 0$ with the gains $E_c = Y\mathbb{S}_0^{-1}$, $F_x = Z\mathbb{S}_0^{-1}$, $G_c = X\mathbb{S}_2^{-1}$ and H_{cj}, $j = 1, \ldots, q-1$, such that $[H_{c1} \ \ldots \ H_{cq-1}] = N\mathbb{S}_3^{-1}$. Since this reasoning is valid for all $\xi \in \mathcal{E}(W^{-1}, 1)$, $\xi \neq \mathbf{0}$, one can conclude that $\mathcal{E}(W^{-1}, 1)$ is a set of stability for the saturated closed-loop system. Moreover, the satisfaction of relation (25) means that for all $\xi \in \mathcal{E}(W^{-1}, 1)$ the resulting controlled output z remains bounded in its set $\mathbb{Z}_0$. Thus, the satisfaction of relations of Proposition 1 implies that the anti-windup gains $E_c = Y\mathbb{S}_0^{-1}$, $F_c = Z\mathbb{S}_0^{-1}$, $G_c = X\mathbb{S}_3^{-1}$, $[H_{c1} \ \ldots \ H_{cq-1}] = N\mathbb{S}_3^{-1}$ and the set $\mathcal{E}(W^{-1}, 1) = \{\xi \in \Re^{n+mq+n_c}; \xi' W^{-1} \xi \leq 1\}$ are solutions to Problem 1. □

Proposition 1 states a local stability condition for the closed-loop saturated system (8). In the absence of controlled output constraints (i.e., $z_0 \to +\infty$), the global asymptotic stability of the closed-loop system can be considered as follows, provided that the open-loop system has the required stability assumption.

Proposition 2: *If there exist a symmetric positive definite matrix W, matrices X, Y, Z, N, three diagonal positive matrices $\mathbb{S}_0$, $\mathbb{S}_1$, $\mathbb{S}_2$ and a block-diagonal matrix positive definite $\mathbb{S}_3$, of appropriate dimensions satisfying*

$$\begin{bmatrix} W\mathbb{A}' + \mathbb{A}W & * & * & * & * & * \\ \mathbb{S}_0'\mathbb{B}_0' + Y'\mathbb{R}' + Z'\mathbb{B}_q' - \mathbb{K}W & -2\mathbb{S}_0 & * & * & * & * \\ \mathbb{S}_1'\mathcal{B}_1' - \mathbb{D}W & \mathbf{0} & -2\mathbb{S}_1 & * & * & * \\ \mathbb{S}_2'\mathbb{B}_q' + X'\mathbb{R}' - \mathbb{C}_0 W & -Z & \mathbf{0} & -2\mathbb{S}_2 & * & \\ \mathbb{D}W & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbb{S}_3 & * \\ \mathbb{S}_3\mathcal{B}_1' + N'\mathbb{B}_q' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbb{S}_3 \end{bmatrix} < \mathbf{0} \tag{29}$$

then the gains $E_c = Y\mathbb{S}_0^{-1}$, $F_c = Z\mathbb{S}_0^{-1}$, $G_c = X\mathbb{S}_2^{-1}$ and $[H_{c1} \ \ldots \ H_{cq-1}] = N\mathbb{S}_3^{-1}$ are such that the closed-loop system (8) *is globally asymptotically stable.*

Proof: By considering $\mathbb{E}_0 = \mathbb{K}, \mathbb{E}_j = \mathbb{K}\mathbb{A}^j$, $j = 1, \ldots, q-1$, that is, $\mathbb{G}_1 = \mathbb{D}, \mathbb{F}_0 = \mathbb{C}_0, \mathbb{F}_1 = F_c$ and $\mathbb{F}_{2j} = H_{cj}$, $j = 1, \ldots, q-1$, the sector conditions (26), (27) and (28) are globally satisfied, for any $\xi \in S(u_0) = \Re^{n+mq+n_c}$, $\xi \in S(u_j) = \Re^{n+mq+n_c}$, $\xi \in S(y_0)$. Then the satisfaction of (29) allows us to satisfy $\dot{V}(\xi) < 0$, for any $\xi \in \Re^{n+mq+n_c}$ along the trajectories of the closed-loop system, which ensures the global asymptotic stability of the closed-loop system. □

4. Numerical anti-windup gains design

4.1 *Computational analysis*

Conditions to be satisfied in Proposition 1 are under LMI form in the decision variables, as a consequence of the use of Lemma 1 and therefore of model (8). This represents a main advantage with what would be obtained by using the classical nonlinear sector condition as in Gomes da Silva Jr *et al.* (2002) or a polytopic model as in Cao *et al.* (2002), for which BMI conditions have been obtained in the case of amplitude actuator limitation only (that is without constraints on the dynamics of the actuator). In the same way, the current actuator model has been studied in Tarbouriech *et al.* (2003) albeit without controller output saturation (that is $\phi_c = \mathbf{0}$). Classical sector conditions used with respect to nonlinearities ϕ_0 and ϕ_j lead to BMI conditions. In such BMI cases, to obtain anti-windup gains maximising the estimate of the basin of attraction of the closed-loop system, one requires to use some relaxation schemes, which are very sensitive to the initial considered guesses. In the current case, the solution does not need neither initial guesses nor iterative schemes.

Moreover, it is important to underline that sufficient conditions for the global stability of the closed-loop system can be proposed (as stated in Proposition 2). If this type of condition can be obtained with the current modified nonlinear sector condition as well as with the classical one, it is however not possible by using a polytopic model to represent the saturated closed-loop system.

Suppose now that for the considered actuator the following assumption holds:

Assumption 1: *Dimensions m and n_{cp} are such that: $n_{cp} \geq m$ and* $\text{rank}(B_{aq}) = \text{rank}(T_q) = m$, *which means that $T_q \in \Re^{m \times n_{cp}}$ is full row rank.*

This assumption means that the right pseudo inverse to T_q, denoted $T_q^{\#}$, exists. Note that in the case of launchers this assumption is verified. When Assumption 1 is satisfied, some modified conditions with a reduced number of LMIs and decision variables can be provided to solve Problem 1. This point is stated in the following corollary.

Corollary 1: *If there exist a symmetric positive definite matrix* W*, matrices* X, Y, Z, $\mathbb{Q}_0, \mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$, H_{cj}, $j = 1, \ldots, q-1$, *and diagonal positive matrices* $\mathbb{S}_0, \mathbb{S}_1$ *and* $\mathbb{S}_2$ *of appropriate dimensions satisfying relations* (22), (23), (24), (25) *and*

$$\begin{bmatrix} W\mathbb{A}' + \mathbb{A}W & * & * & * \\ \mathbb{S}_0\mathbb{B}_0' + Y'\mathbb{R}' + Z'\mathbb{B}_q' - \mathbb{Q}_0 & -2\mathbb{S}_0 & * & * \\ \mathbb{S}_1\mathcal{B}_1' - \mathbb{Q}_1 & \mathbf{0} & -2\mathbb{S}_1 & * \\ \mathbb{S}_2\mathbb{B}_q' + X'\mathbb{R}' - \mathbb{Q}_2 & -\mathbb{Q}_3 & \mathbf{0} & -2\mathbb{S}_2 \end{bmatrix} < \mathbf{0} \tag{30}$$

$$T_j + T_q H_{cj} = \mathbf{0}, \quad j = 1, \ldots, q-1 \tag{31}$$

then the gains $E_c = Y\mathbb{S}_0^{-1}$, $F_c = Z\mathbb{S}_0^{-1}$, $G_c = X\mathbb{S}_2^{-1}$, $H_{cj} = -T_q^{\#}T_j$, $j = 1, \ldots, q-1$ *and the set* $\mathcal{E}(W^{-1}, 1) = \{\xi \in \Re^{n+mq+n_c}; \xi' W^{-1}\xi \le 1\}$ *are solutions to Problem* 1.

Proof: The proof follows the same lines as that one of Proposition 1 and by using Assumption 1. The satisfaction of relation (31) means that the term $2\xi' P(\mathbb{B}_j + \mathbb{B}_q H_{cj})\phi_0^{(j)} = 0$. It is important to note that since from Assumption 1 one gets $n_{cp} \ge m$, relation (31) always admits a solution: in particular $H_{cj} = -T_q^{\#}T_j$, $j = 1, \ldots, q-1$. □

Proposition 2 can be modified by the same way.

4.2 *Optimisation Issues*

Proposition 1 provides feasibility conditions. We can then consider a set Ξ_0 with a given shape and a scaling factor β. For example, let Ξ_0 be defined as a polyhedral set described by its vertices: $\Xi_0 = Co\{v_r; r = 1, \ldots, n_r, v_r \in \Re^{n+mq+n_c}\}$. Thus, we want to satisfy $\beta\,\Xi_0 \subset \mathcal{E}(W^{-1}, 1)$. For a given β, this problem reduces to a feasibility problem. The control problem may be also recasted into a problem of maximizing β, which corresponds to define through Ξ_0 the directions in which we want to maximize $\mathcal{E}(W^{-1}, 1)$.

The problem of maximizing β can be solved by using the following convex optimization scheme

$$\begin{gathered} \min \mu \\ \text{subject to relations (21), (22), (23), (24), (25)} \\ \begin{bmatrix} \mu & v_r' \\ v_r & W \end{bmatrix} \ge \mathbf{0} \quad r = 1, \ldots, n_r. \end{gathered} \tag{32}$$

Considering $\beta = 1/\sqrt{\mu}$, the minimization of μ implies the maximization of β.

Other criteria associated to the size of $\mathcal{E}(W^{-1}, 1)$, (e.g. the volume or the size of the minor axis) can be adopted in order to maximize the stability region. Moreover, some structural or norm constraints on the anti-windup gains E_c, F_c and G_c could be considered (see Gomes da Silva and Tarbouriech (2003)).

Regarding the bounds on controlled output z, we can formulate the following comments.

1. The amplitude limitation $z_{0(i)}$, $i = 1, \ldots, l$, can be a priori fixed or can be a decision variable. Hence, by replacing $z_{0(i)}$, $i = 1, \ldots, l$ by δ (a unique variable for simplicity) in relation (25) of Proposition 1, we can try to minimize δ. In this case the optimization problem (32) can be modified as follows:

$$\begin{gathered} \min \eta_0\mu + \eta_1\delta \\ \text{subject to relations (21), (22), (23), (24)} \\ \begin{bmatrix} \mu & v_r' \\ v_r & W \end{bmatrix} \ge \mathbf{0} \quad r = 1, \ldots, n_r \\ \begin{bmatrix} W & \mathbb{Q}_0' & W\mathbb{C}_2' \\ * & 2\mathbb{S}_0 & \mathbb{S}_0 D_2' \\ * & * & \delta 1 \end{bmatrix} \ge \mathbf{0} \end{gathered} \tag{33}$$

   where η_0 and η_1 are tuning parameters. δ corresponds to the smallest upper bound on the output peak using the fact that the ellipsoid $\mathcal{E}(W^{-1}, 1)$ is an invariant set in which the closed-loop trajectories remain confined (see also Boyd *et al.* (1994)).
2. If we want to guarantee some bound on controlled output energy, we can remove relation (25) and replace relation (21) in Proposition 1 by

$$\begin{bmatrix} M & L' \\ L & -\gamma\mathbf{1} \end{bmatrix} < \mathbf{0} \tag{34}$$

   with M is the matrix of relation (21) and $L = [\mathbb{C}_2 W \;\; D_2\mathbb{S}_0 \;\; \mathbf{0} \;\; \mathbf{0}]$. The satisfaction of relation (34) turns out to satisfy

$$\dot{V}(\xi) + \frac{1}{\gamma} z'z < \mathbf{0} \tag{35}$$

   with the quadratic function $V(\xi) = \xi' W^{-1}\xi$. From (35), one gets

$$V(\xi(T)) = V(\xi(0)) < -\frac{1}{\gamma}\int_0^T z'z\,dt, \quad \forall T > 0, \tag{36}$$

or still

$$\int_0^T z'z\,dt < \gamma\mu, \quad \forall T > 0, \tag{37}$$

for all $\xi(0) \in (1/\sqrt{\mu})\Xi_0 \subset \mathcal{E}(W^{-1}, 1)$. In this case, we want to minimise $\gamma\mu$, which corresponds to the smallest upper bound on the output energy for all $\xi(0) \in 1/\sqrt{\mu}\Xi_0 \subset \mathcal{E}(W^{-1}, 1)$. Hence, the optimization problem (32) can be modified as follows:

$$\begin{aligned} &\min\ \eta_0\mu + \eta_1\gamma \\ &\text{subject to relations (34), (22), (23), (24)} \\ &\begin{bmatrix} \mu & \nu_r' \\ \nu_r & W \end{bmatrix} \geq \mathbf{0} \quad r = 1, \ldots, n_r \end{aligned} \tag{38}$$

where η_0 and η_1 are tuning parameters.

In both cases above, a compromise should be managed between the size of the region of closed-loop stability (through the value of μ) and the smallest upper bound on the output (in the output peak case through δ and in the output energy through γ).

5 Numerical examples

5.1 *Example 1*

Let us consider the following open-loop unstable system borrowed from Gomes da Silva Jr and Tarbouriech (2003)

$$A = 0.1; \quad B = 1; \quad C = 1; \quad C_2 = 1; \quad D_2 = 0.$$

The actuator (2) is set with $q = 2$, $T_0 = -25$, $T_1 = -10$, $T_q = 25$, and with bounds given as $u_0 = 2$, $u_1 = 50$ and $y_0 = 3$. For this system, a dynamic stabilizing PI controller is given by

$$A_c = 0; \quad B_c = -0.2; \quad C_c = 1; \quad D_c = -2.$$

The polyhedral set Ξ_0 is defined by

$$\Xi_0 = C_0\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\},$$

that is, we are interested in enlarging the set of admissible initial states of the open-loop system. Using the optimization problem (33) and the tuning parameters $\eta_0 = 100$, $\eta_1 = 1$, we obtain the following results:

$$\begin{aligned} &\beta = \frac{1}{\sqrt{\mu}} = 3.0919, \quad \delta = 10.2499 \\ &E_c = 0.1417, \quad F_c = 3.2280, \\ &G_c = 0.0711, \quad H_{c1} = 0.4000. \end{aligned}$$

Note that H_{c1} obtained from Proposition 1 is exactly equal to the solution which would be obtained by numerical implementation of Corollary 1, that is, $H_{c1} = -T_q^{\#}T_1 = 0.4$. Simulations are plotted on figure 3 to compare the results with (solid line) and without (dashed line) anti-windup, from an initial state $\xi(0) = [3\ \ 0\ \ 0\ \ 0]'$ belonging to $\mathcal{E}(W^{-1}, 1)$.

One can check that, without anti-windup, the trajectory initiated from $\xi(0)$ does not converge towards the origin. On the contrary, the anti-windup solution preserves the stability of the closed-loop system. Finally, the time evolution of the additional inputs of the controller provided by the anti-windup strategy are plotted in figure 4, i.e. ϕ_c (solid line), ϕ_0 (dashed line) and $\dot{\phi}_0$ (dotted line). It can be seen that the anti-windup action only occurs during the first instants when the saturations are active.

5.2 *Example 2*

Let us now consider the linearised model of F/A-18 HARV aircraft lateral dynamics (Shewchun and Feron 1999), in horizontal stabilized flight at Mach 0.7 and altitude 20000 ft

$$A = \begin{bmatrix} -2.3142 & 0.5305 & -15.5763 & 0 \\ -0.0160 & -0.1287 & 3.0081 & 0 \\ 0.0490 & -0.9980 & -0.1703 & 0.0440 \\ 1.0000 & 0.0491 & 0 & 0 \end{bmatrix};$$

$$B = \begin{bmatrix} 23.3987 & 21.4333 & 3.2993 \\ -0.1644 & 0.3313 & -1.9836 \\ -0.0069 & -0.0153 & 0.0380 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}; \quad D_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

Governs are limited in amplitude $\mu_0 = [25\ \ 10.53\ \ 0]'$ and rate $\mu_1 = [100\ \ 40\ \ 82]'$. The controller output saturation is set to $y_0 = [30\ \ 20\ \ 35]'$. The actuator is then defined

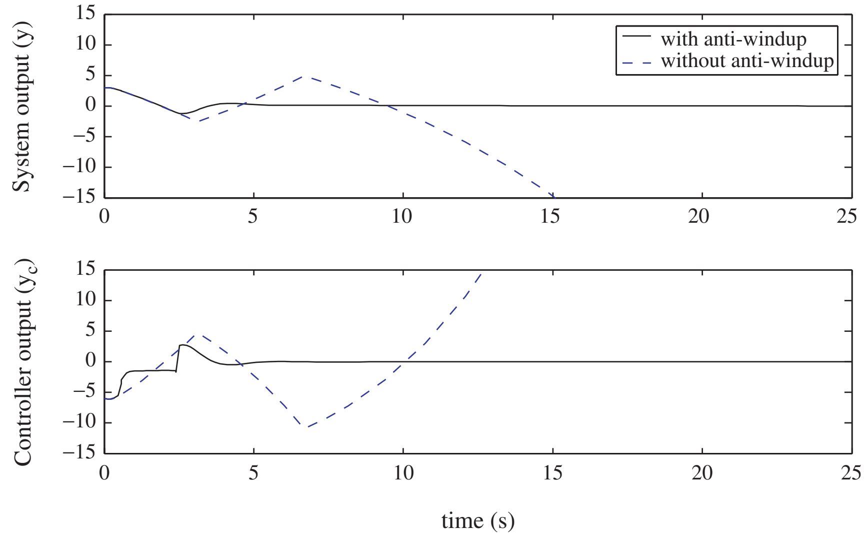

Figure 3. Evolution of the system output y and controller output y_c with (solid line) and without (dashed line) anti-windup strategy.

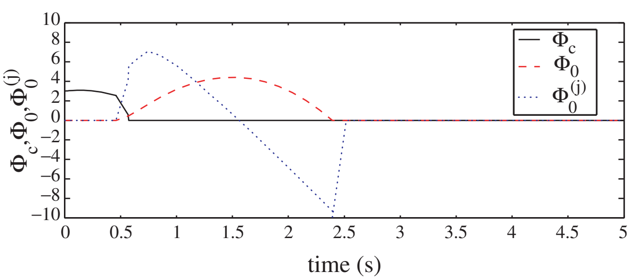

Figure 4. Evolution of the additional anti-windup inputs ϕ_c (solid line), ϕ_0 (dashed line) and $\dot{\phi}_0$ (dotted line).

with $q=2$, for which we consider the following dynamics:

$$T_0 = \begin{bmatrix} -25 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & -25 \end{bmatrix}; \quad T_1 = \begin{bmatrix} -20 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -20 \end{bmatrix};$$

$$T_q = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}.$$

Let us consider the dynamic compensator evaluated in Pittet (1998) such that the closed-loop poles are placed in a pre-specified region of the complex plane

$$A_c = \begin{bmatrix} -0.98 & 0.05 & -0.03 & -1.84 \\ 32.55 & -4.09 & 0.42 & -16.22 \\ 65.56 & -2.90 & -6.85 & -9.77 \\ 10.91 & 0.20 & -0.05 & -9.92 \end{bmatrix};$$

$$B_c = \begin{bmatrix} 0.24 & -0.03 \\ 0.205 & -0.2897 \\ -46.23 & 0.89 \\ 1.59 & -0.14 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 32.55 & -0.00 & -0.63 & -10.57 \\ 20.11 & 0.18 & -0.26 & -7.73 \\ -1.61 & -0.73 & -0.47 & 5.40 \end{bmatrix};$$

$$D_c = \begin{bmatrix} -2.77 & -0.1 \\ -0.64 & -0.11 \\ -4.22 & 0.19 \end{bmatrix}$$

and a polyhedral set of initial condition Ξ_0 given by its vertices with components 1 or -1 of the first four components corresponding to the aircraft plane. Note that the extended state vector is of dimension 14, which does not prevent numerical evaluation of the anti-windup gains. The solution to the anti-windup design procedure (33) then is:

$$E_c = \begin{bmatrix} 67.8126 & 61.0215 & 11.9013 \\ 343.3460 & 277.8643 & 164.9775 \\ 813.1080 & 750.8726 & 75.2629 \\ 141.9346 & 126.2512 & 29.9107 \end{bmatrix};$$

$$G_c = \begin{bmatrix} -2.1675 & -0.8090 & 0.2992 \\ -33.2143 & 0.8957 & 15.7241 \\ -12.4687 & -12.4196 & 1.4458 \\ -5.5404 & -1.2901 & 1.0108 \end{bmatrix}$$

$$F_c = \begin{bmatrix} -17.9843 & -17.5551 & -2.5392 \\ -2.8298 & -1.3228 & -0.3978 \\ -4.1877 & -3.9160 & 0.6091 \end{bmatrix};$$

$$H_{c1} = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}$$

for which $\beta = (1/\sqrt{\mu}) = 0.9839$ and $\delta = 105.03$.

Finally, to illustrate the trade-off between the size of the set of admissible initial states and the performance, one can see in table 1 different solutions relative to different weights.

Let us now consider the same aircraft model and controller but with an actuator of order two restricted dynamics, that is, limited in amplitude, rate and acceleration. The actuator is then defined with $q = 3$ where, in plus of matrices T_0, T_1 and T_q previously defined (recall that in the first part of the example T_q is set for $q = 2$ although in this second part of the example T_q is set for $q = 3$), one considers

$$T_2 = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

and the acceleration limitation is set to $u_2 = [200\ \ 200\ \ 200]'$. In that case, the extended state vector becomes that of dimension 17. The solution to the anti-windup design procedure (33) is

$$E_c = \begin{bmatrix} -72.2388 & -63.9756 & -17.2953 \\ -1915.6343 & -1700.6605 & -37.01849 \\ -224.7423 & -188.5632 & -118.4952 \\ -213.6312 & -189.6355 & -46.6333 \end{bmatrix};$$

$$G_c = \begin{bmatrix} -1.9020 & -1.8563 & 1.1083 \\ -17.1957 & -19.9843 & 32.9455 \\ -16.9507 & -16.2244 & 3.9918 \\ -4.2785 & -4.2620 & 3.3720 \end{bmatrix}$$

Table 1. Illustration of the trade-off between the size of the stability domain (the larger β is, the larger the domain of stability is) and the performance (the smaller δ, is the best performances are).

η_0, η_1	1, 1	100, 1	10^5, 1	10^6, 1	1, 0
β	0.3110	0.9839	5.5602	9.8203	43.6270
$\sqrt{\delta} = \max\{z_{0(i)}\}$	3.2388	10.2485	58.0266	102.6002	2664.3

$$F_c = \begin{bmatrix} 0.0037 & -3.9722 & 0.4191 \\ 13.6919 & 16.5391 & 3.1088 \\ -39.5226 & -33.3721 & -8.9584 \end{bmatrix};$$

$$H_{c1} = \begin{bmatrix} 0.8 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0.2 \end{bmatrix}$$

for which $\beta = (1/\sqrt{\mu}) = 0.8183$ and $\delta = 152.2761$. This shows the reduction of admissible initial conditions, related to β, and the reduction of the performance of the controlled output (increase of the bound δ) when limitations on high order actuator dynamics are involved.

6. Conclusion

We have addressed the problem of designing anti-windup gains in order to obtain a region of stability, as large as possible, for linear systems with amplitude and dynamics restricted actuator. The strategy developed consisted in adding the part due to the saturation of the output actuator both in the state evolution (via E_c) and in the output of the controller (via F_c), in adding the part due to the output controller (or input actuator) in the state evolution (via G_c) and in adding the successive time-derivative of the saturation of the output actuator in the output of the controller (via H_{cj}, $j = 1, \ldots, q-1$). To do this, sector modified conditions have been used in order to obtain directly LMI conditions.

In a future work, the problem of measured variables used to build anti-windup loops should be considered. Hence, it would be coherent to consider the computation of some observer in order to effectively access the states of the actuator.

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