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STABILITY REGIONS AND SPECTRA OF DISCRETE THIRD-ORDER AUTOREGRESSIVE TIME-SERIES

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ABSTRACT: This paper presents an analysis of third order autoregressive time-series. The parameter regions, in the three dimensional parameter space, that produce the six separate types of power spectral density are analyzed. The study reveals that when a particular two dimensional cross section of the three dimensional parameter space is taken, the region of stability is always triangular. Within each triangular stability region in this two dimensional space, subregions which produce the six possible types of spectral shape are indicated. From these subregions it is possible to approximately choose the parameters necessary to model a process whose power spectral density contains at most two critical frequencies (maxima and minima).

1. INTRODUCTION

Autoregressive discrete time-series, $X_t-\mu = a_1(X_{t-1}-\mu) + a_2(X_{t-2}-\mu) + \ldots + a_p(X_{t-p}-\mu) + e_t$, are often utilized by the simulation analyst for modeling physical and economic processes which are stochastic in nature (Fishman 1969a). In particular, if the output time series of a discrete event simulation may be assumed to be an autoregressive process, then useful statistical techniques have been developed for efficiently estimating the variance of the simulation output's sample mean (Fishman 1978b). Burg has shown that when the first m autocovariances of a weakly-stationary time-series are estimated, the spectrum estimate, f(w), that is consistent with the m autocovariance estimates and that maximizes the entropy, $\int_{-\pi}^{\pi} \log f(w)dw$, is the spectrum associated with the above autoregressive processes (Lacoss). Thus, the autoregressive structure is important in modeling and analyzing weakly-stationary time-series such as the outputs of many discrete event simulations. In the above recurrence relation, X_t is a random variable whose value represents the state of the process at epoch t, μ is the mean of the process, a_i are real constants, and e_t is a sequence of independent, identically distributed normal variates with mean zero and variance σ_e^2. When the process to be modeled is covariance stationary, the roots of the characteristic equation,

$$1-a_1z^{-1}-a_2z^{-2}\ldots-a_pz^{-p} = 0$$

must all be within the unit circle in the z-plane (Jenkins 1968).

The spectrum of the above p^{th} order autoregressive time-series will display at most p-1 critical frequencies corresponding to peaks and troughs. Since the spectrum of many simulation output time-series possess two critical frequencies, a detailed study has been made of the stability regions and spectra of third-order autoregressive processes in terms of the parameters a_1, a_2, and a_3. Analytical solutions were not used because of the size and complexities of the expressions involved. A numerical search procedure was employed both to find the region of stability and the sub-regions within the stability region that produce distinct spectral shapes. The results of this study are presented below.

In order that the process be covariance stationary, the roots of the characteristic equation $Z^3-a_1Z^2-a_2Z-a_3 = 0$ must have modulus less than one (be within the unit circle in the z-plane).

This study reveals that when an a_1, a_2 cross section of the a_1 a_2, a_3 space is taken, the stability region is always triangular.

In Fig. 1 through 9 the stability triangles are shown when the parameter a_3 equals 1.00, 0.75, 0.50, 0.25, 0.00, -0.25, -0.50, -0.75, -1.00, respectively. The parameter a_3 affects the stability triangle by inducing rotation and reduction in its area as a_3 is increased in absolute magnitude from zero to one.

The stability triangle boundaries correspond to parameter values that yield one or more roots to the characteristic equation lying on the unit circle in the z-plane with the remaining roots within the unit circle.

2. SPECTRAL SUB-REGIONS

The power spectral density of the third-order autoregressive process, assuming that $\mu = 0$ and the sampling period is 1, is given by (Bloomfield 1976):

$$S(f) = \sigma_e^2/(1+a_1^2+a_2^2+a_3^2+2(a_2-a_1a_3)-2(a_1-a_2(a_1+a_3)-3a_3)\cos 2\pi f-$$

$$4(a_2-a_1a_3)\cos^2 2\pi f-8a_3\cos^3 2\pi f), -.5 \leq f \leq .5 \qquad (1)$$

There exist six possible types of spectra from equation 1, namely, low frequency, high frequency, single peak, single trough, trough-peak, and peak-trough. Peaks and troughs are obtained, from the calculus, by obtaining the local maxima and minima of equation 1 on the frequency interval (0.0, 0.5). The frequencies of peaks and troughs are obtained from:

$$\cos 2\pi f = (a_1a_3-a_2 \pm (a_2^2+a_3^2(a_1^2+3a_2+9)+a_1a_3(a_2-3))^{\frac{1}{2}})/6a_3 \qquad (2)$$

when the right side of equation 2 is real, $a_3 \neq 0$, and $|\cos 2\pi f| < 1$ for either or both roots. When $a_3 = 0$, only one critical frequency may occur and is obtained from:

$$\cos 2\pi f = -a_1(1-a_2)/4a_2 \qquad (3)$$

when $a_2 \neq 0$ and $|\cos 2\pi f| < 1$. Positive values of equations 2 and 3 correspond to frequencies on the interval (0.0, 0.25) while negative values correspond to frequencies on the interval (0.25, 0.5).

Referring to equation 2, it is empirically observed that the positive root corresponds to peak frequencies while the negative root corresponds to trough frequencies. When two critical frequencies are present, the sign of a_3 determines whether a peak preceeds a trough or visa versa. When $a_3 > 0$, the peak preceeds the trough and when $a_3 < 0$, the trough preceeds the peak. When the value of a_3 is close to 1, high frequency spectra are not generated. When the value of a_3 is close to -1, low frequency spectra are not generated.

When both peak and trough frequencies are present the region is subdivided such that in one area $\cos 2\pi f_1$ and $\cos 2\pi f_2$ have the same sign and in the other they have opposite signs. Opposite signs correspond to one frequency on the interval (0.0, 0.25) and one frequency on the interval (0.25, 0.50) since $\cos 2\pi f$ changes sign at $f = \frac{1}{4}$. When $a_3 < 0$, the same sign, negative, corresponds to both frequencies on the interval (0.25, 0.5).

In Fig. 1 through 9 the stability triangles are decomposed into spectral sub-regions. In Fig. 10 through 15, typical single trough, single peak, trough-peak, peak-trough, high frequency and low frequency spectra are depicted for $\sigma_e^2 = 1$. From these figures it is possible to locate bounds for the parameters a_1, a_2, and a_3 as a first step in modeling a process whose spectral shape has at most two critical frequencies.

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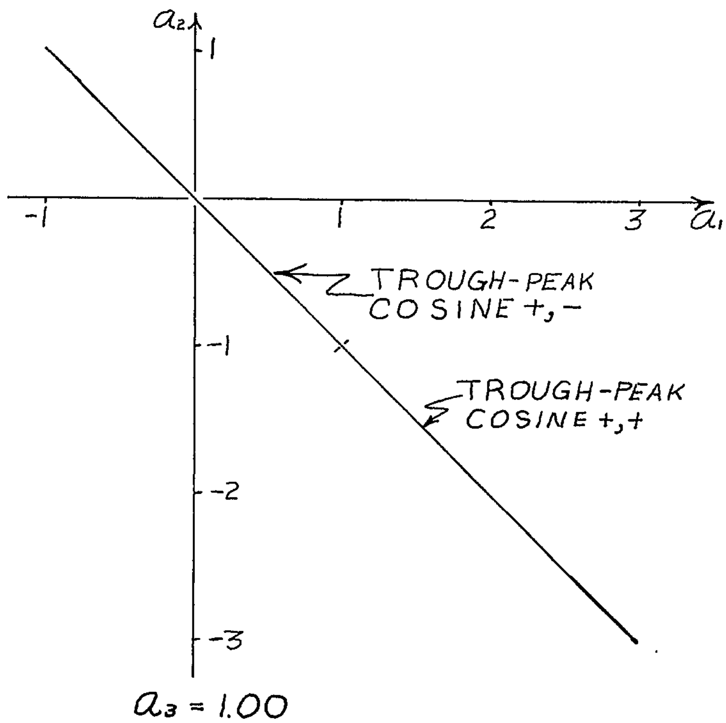

Fig. 1 Stability Region and Spectral Sub-Regions for a_3 = 1.00

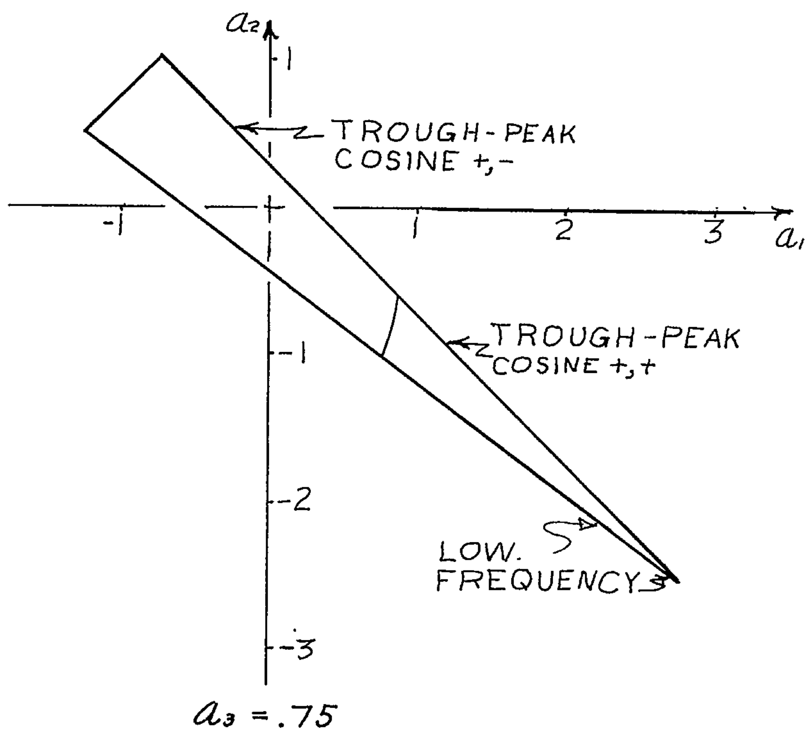

Fig. 2 Stability Region and Spectral Sub-Regions for a_3 = .75

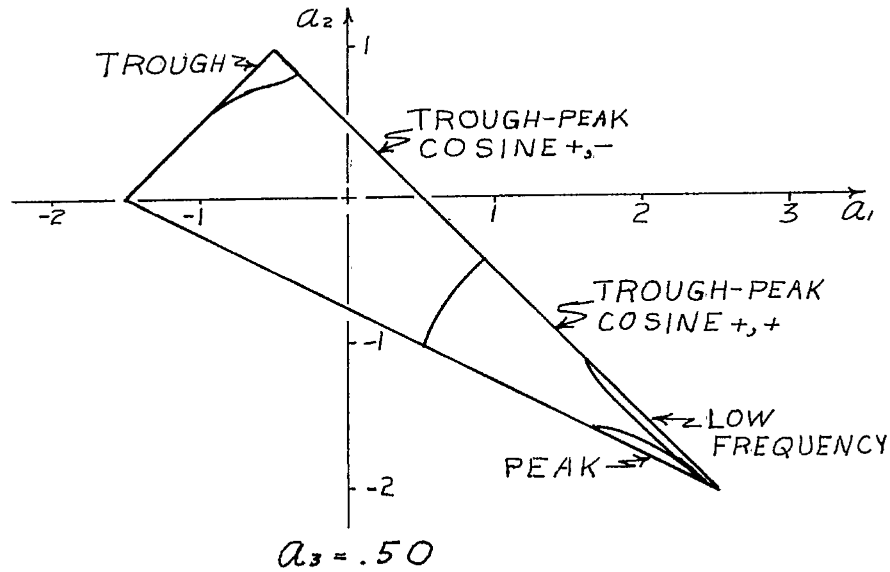

Fig. 3 Stability Region and Spectral Sub-Regions for a_3 = .50

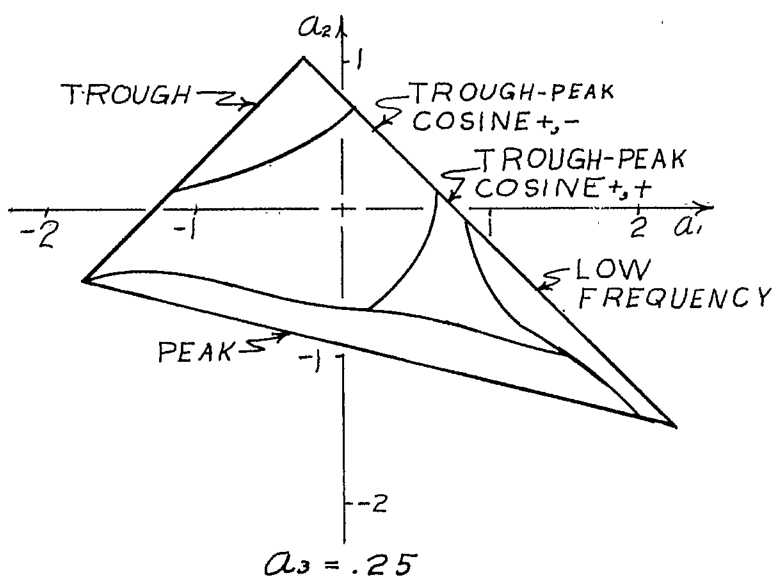

Fig. 4 Stability Region and Sub-Regions for a_3 = .25

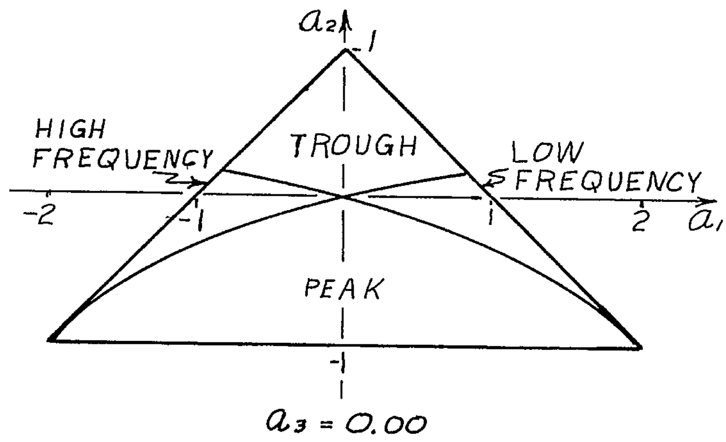

Fig. 5 Stability Region and Spectral Sub-Regions for a_3 = ~~0~~.00

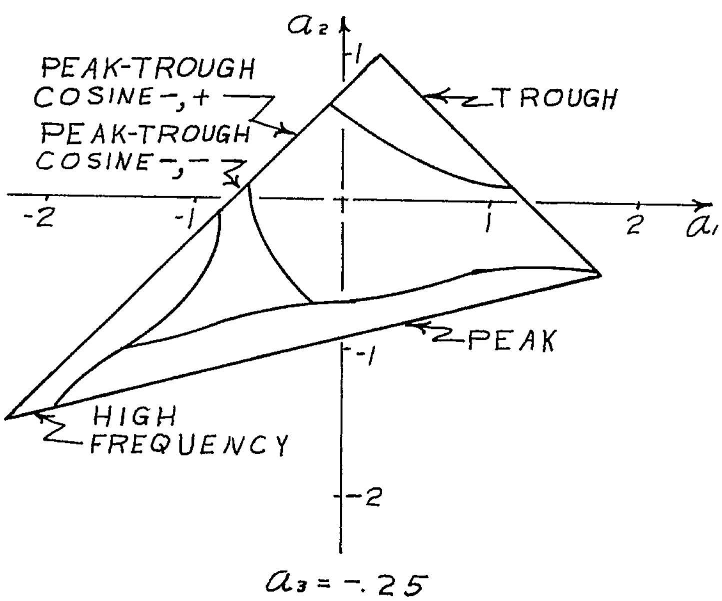

Fig. 6 Stability Region and Spectral Sub-Regions for a_3 = –.25

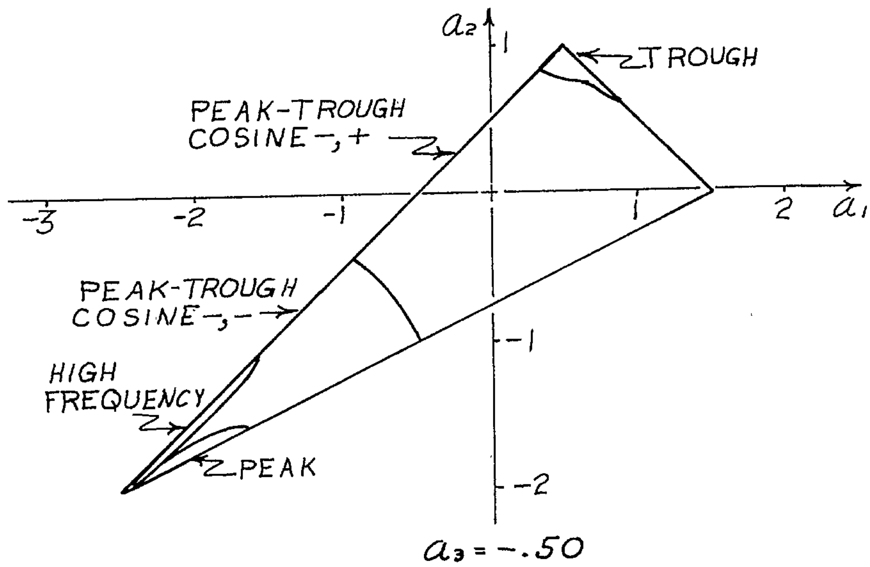

Fig. 7 Stability Region and Spectral Sub-Region for $a_3 = -.50$

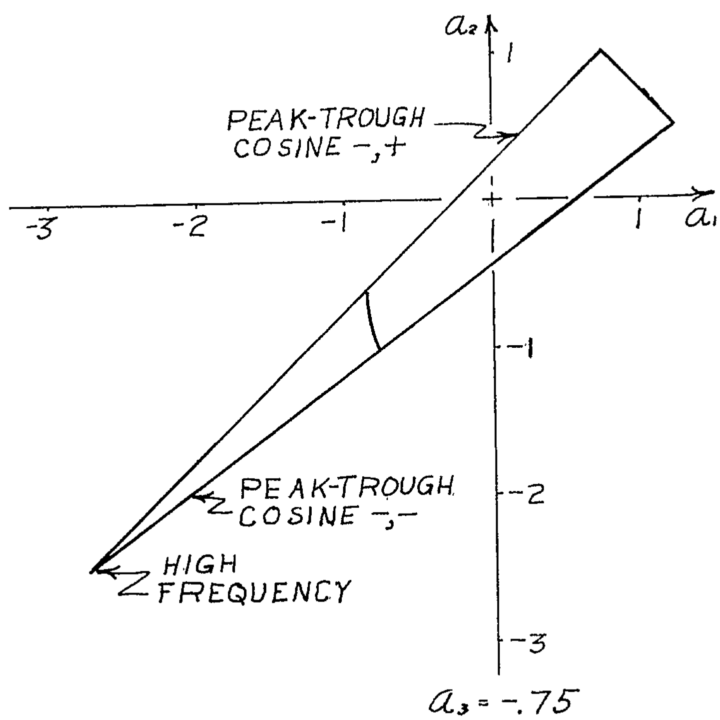

Fig. 8 Stability Region and Spectral Sub-Regions for $a_3 = -.75$

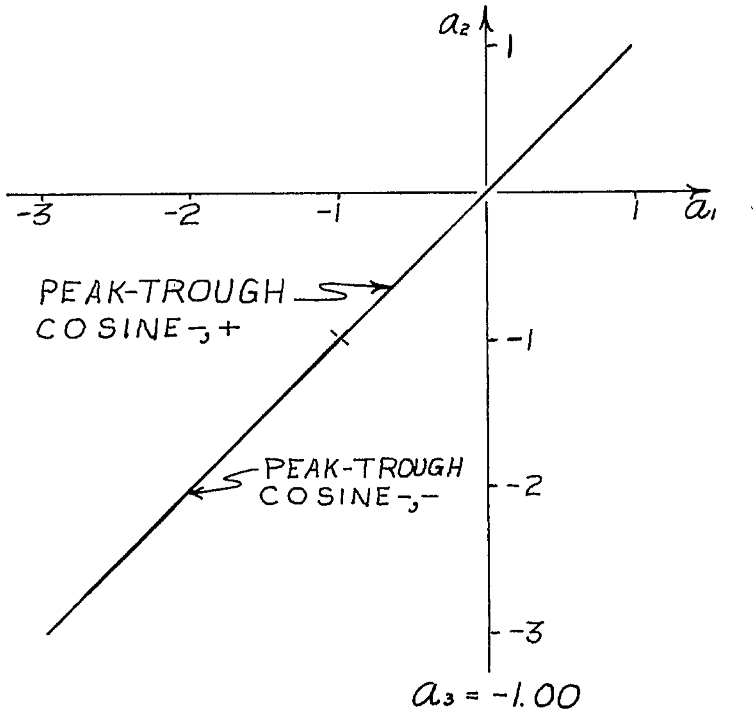

Fig. 9 Stability Region and Spectral Sub-Region for $a_3 = -1.00$

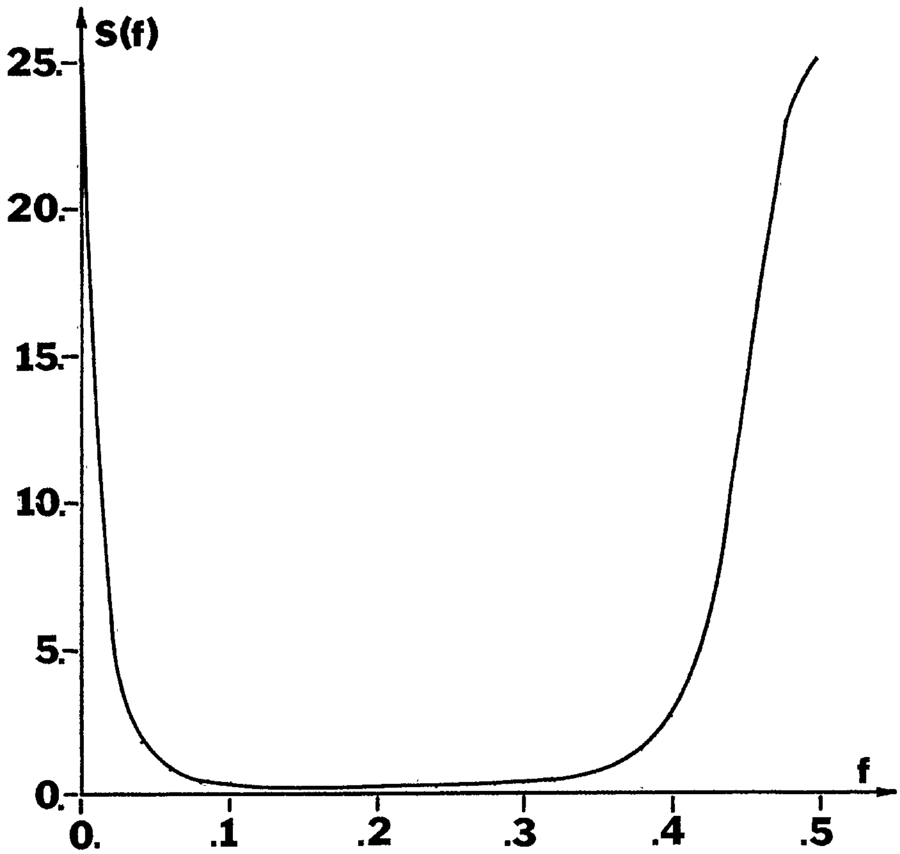

Fig. 10 Trough Spectra for $a_1 = -.50$, $a_2 = .80$, $a_3 = .50$

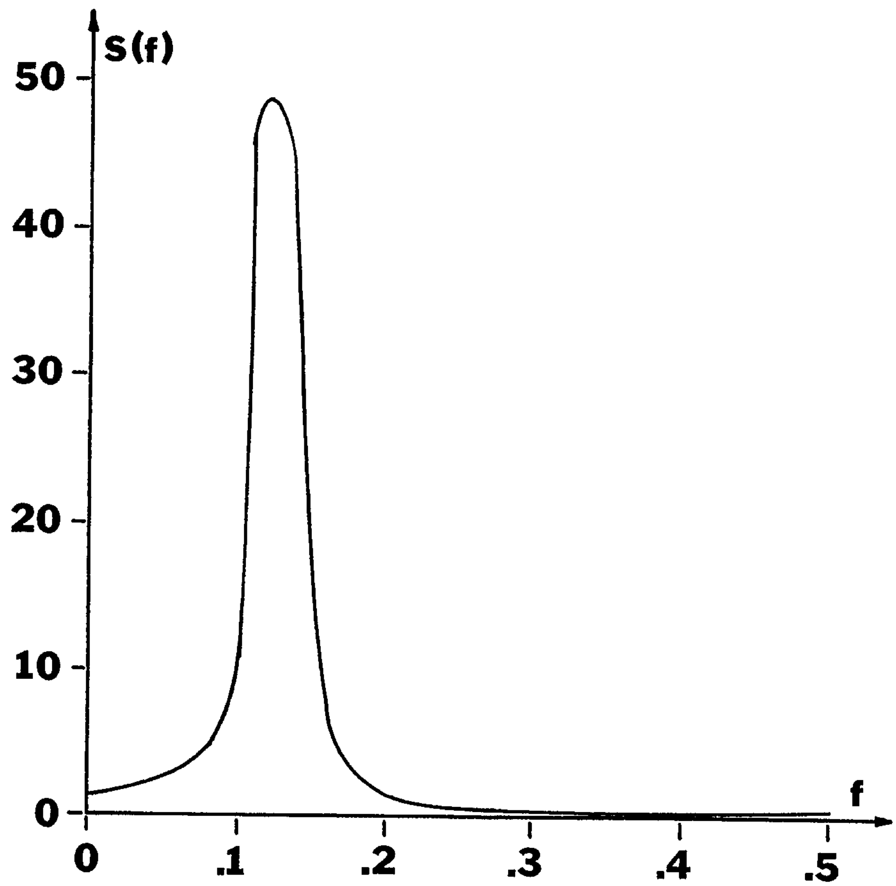

Fig. 11 Peak Spectra for $a_1 = 1.00$, $a_2 = -.50$, $a_3 = -.25$

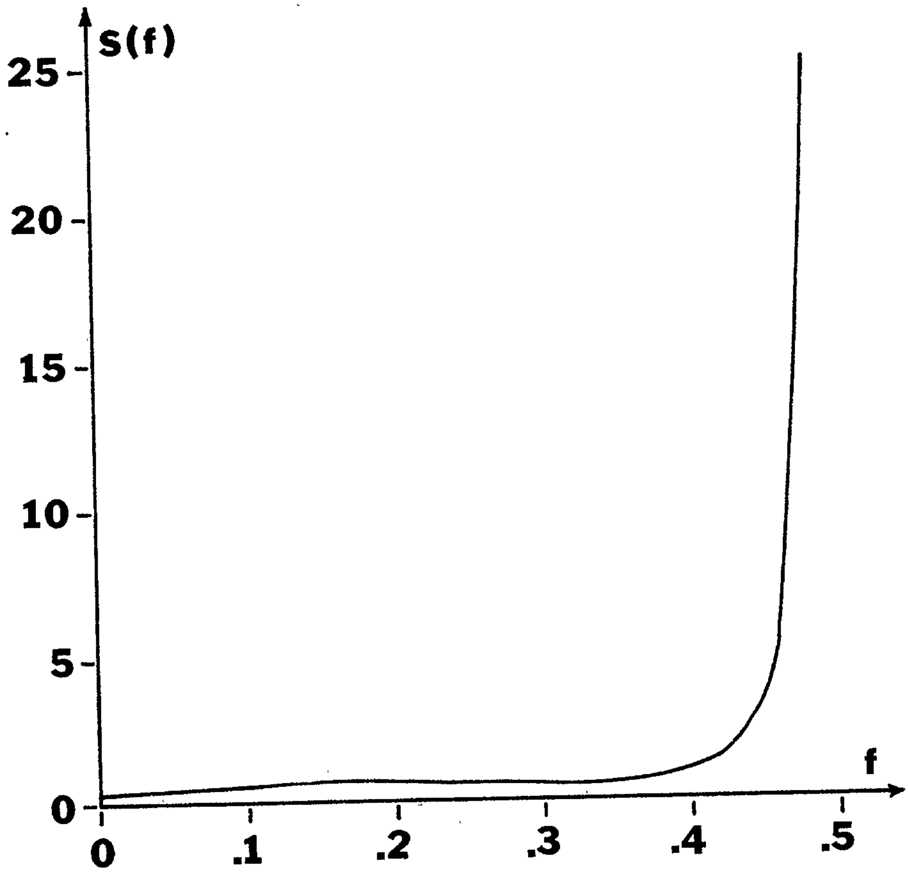

Fig. 12 Trough-Peak Spectra for $a_1 = -.50$, $a_2 = .25$, $a_3 = -.25$

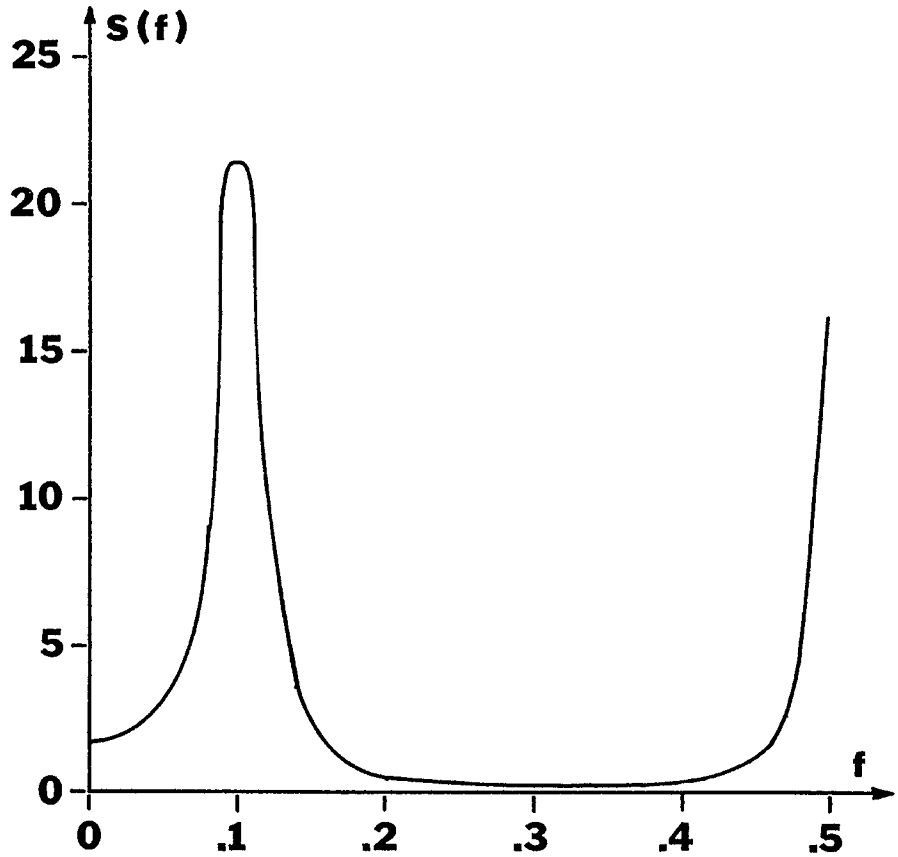

Fig. 13 Peak-Trough Spectra for $a_1 = .50$, $a_2 = .50$, $a_3 = -.75$

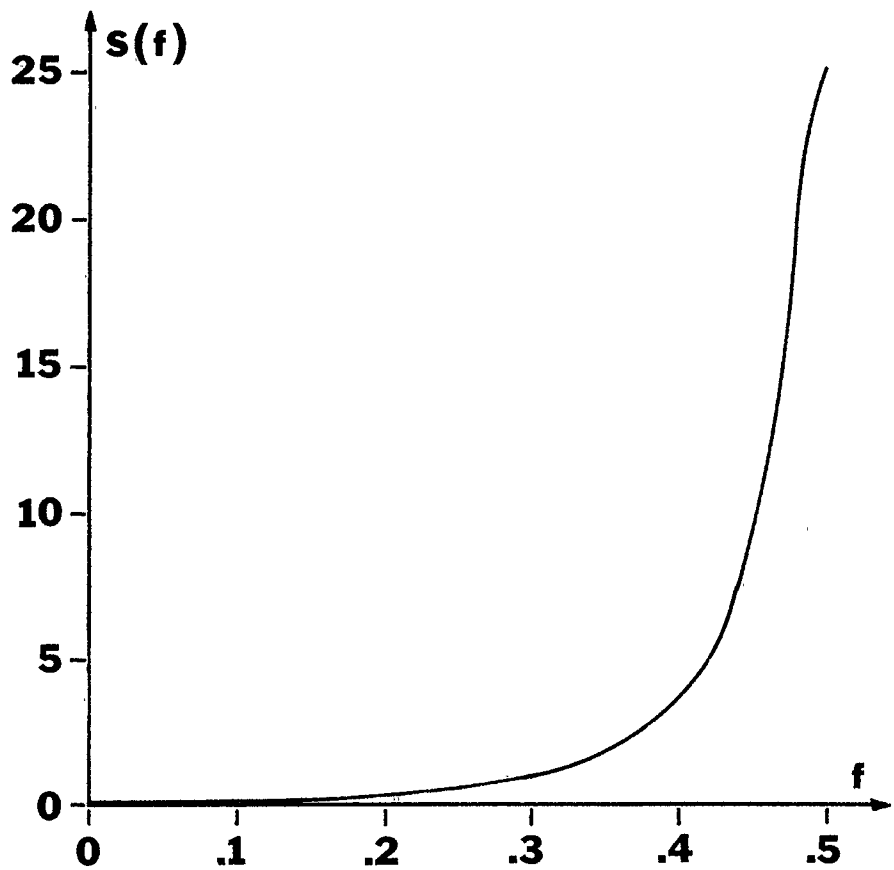

Fig. 14 High Frequency Spectra for $a_1 = -1.70$, $a_2 = -.75$, $a_3 = -.25$

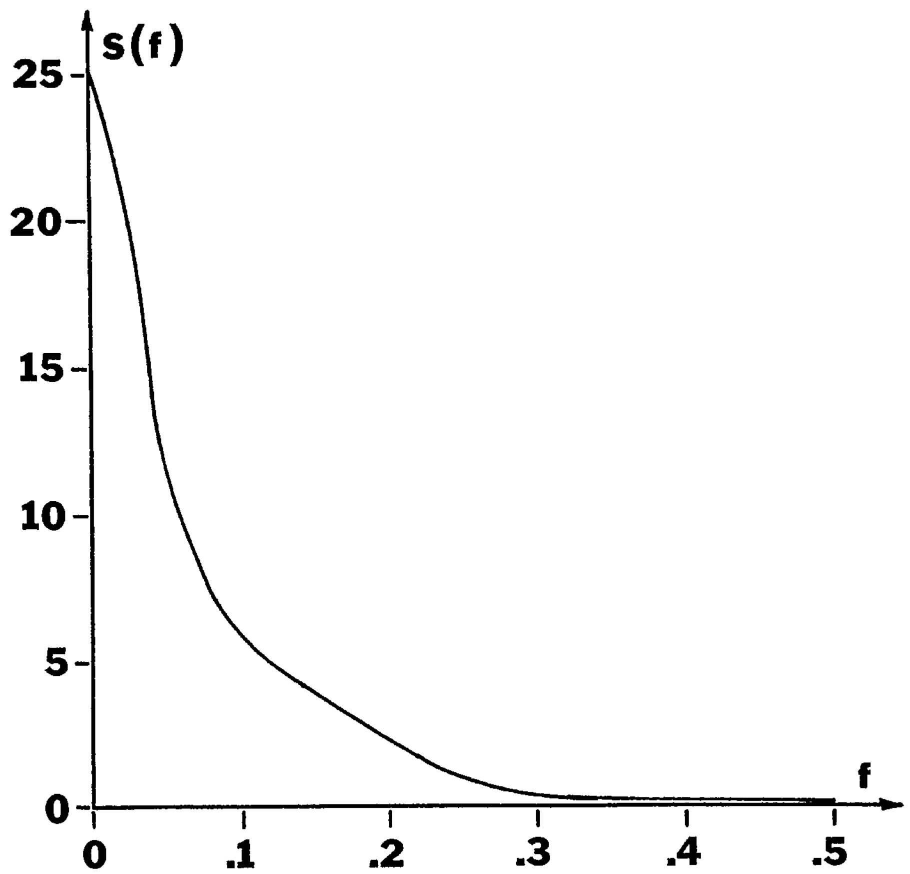

Fig. 15 Low Frequency Spectra for $a_1 = 1.25$, $a_2 = -.70$, $a_3 = .25$