Title
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Journal
IEEE Transactions on Automatic Control, 33(1)

ISSN
0018-9286

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Publication Date
1988

Peer reviewed

Stability Regions of Nonlinear Autonomous Dynamical Systems

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***Abstract*—A topological and dynamical characterization of the stability boundaries for a fairly large class of nonlinear autonomous dynamic systems is presented. The stability boundary of a stable equilibrium point is shown to consist of the stable manifolds of all the equilibrium points (and/or closed orbits) on the stability boundary. Several necessary and sufficient conditions are derived to determine whether a given equilibrium point (or closed orbit) is on the stability boundary. A method to find the stability region based on these results is proposed. The method, when feasible, will find the exact stability region, rather than a subset of it as in the Lyapunov theory approach. Several examples are given to illustrate the theoretical prediction.**

I. Introduction

THE problem of determining the stability region (region of attraction) of a stable equilibrium point for a nonlinear autonomous dynamical system is an important one in many applications, such as electric power systems [1], [2], economics [3], ecology [4], etc. The numerous methods proposed in the literature for estimating the stability region can be roughly divided into two classes [6]: those using Lyapunov functions, and all others. Most of the methods belong to the Lyapunov function approach, which is based mainly on La Salle's extension of Lyapunov theory [7]–[10]. The estimated stability region based on these methods usually is only a subset of the true stability region. Recently, methods using computer generated Lyapunov functions [11], [12] have been proposed. Another method, belonging to the Lyapunov function approach, is the Zubov method [8]. Theoretically, this method provides the true stability region via the solution of a partial differential equation. Recent advance includes the maximal Lyapunov function [30]. One of the early non-Lyapunov methods proposed for planar systems [31] requires the construction of a nontrivial integral function. The method of sinks [13], also for planar systems, utilizes the analogy between the vector field and the velocity field of an incompressible fluid. An iterative procedure using the Volterra series for estimating the stability region was proposed [14]. Another method, called the trajectory-reversing method, was recently proposed [5], [6], in which the estimation of the stability region is synthesized from a number of system trajectories obtained by integrating the system equations.

In this paper a comprehensive analysis of the stability region is conducted. It is an extension of our earlier work [32]. Several necessary and sufficient conditions for an equilibrium point (or closed orbit) to lie on the stability boundary are derived. A complete characterization of the stability boundary is presented for a fairly large class of nonlinear autonomous dynamical systems satisfying two generic conditions plus one additional condition that every trajectory on the stability boundary approaches one of the equilibrium points (or closed orbits) as the time t approaches infinity. It is shown that the stability boundary of this class of systems consists of the union of the stable manifolds of all equilibrium points (and/or closed orbits) on the stability boundary. A method to find the stability region based on these results is proposed; this method belongs to the non-Lyapunov function approach. The method is applied to several examples studied in the literature.

The organization of the paper is as follows. Some fundamental concepts in the theory of mathematical dynamical systems that are essential in the subsequent development in this paper are introduced in Section II. In Section III topological properties of the equilibrium point and closed orbit on the stability boundary are presented. In Section IV a complete characterization of the stability boundary of a class of systems is given. The class of systems is examined in Section V and is shown to be fairly large. In Section VI a new method for determining stability region is proposed. In Section VII the method is applied to several examples.

II. Concepts in Dynamical Systems

In this section we introduce some concepts that play a central role in the theory of dynamical systems. For general background on the theory of mathematical dynamical systems the reader is advised to consult the survey paper by Smale [15], or the books by Guckenheimer and Holmes [28] or Palis and De Melo [20].

Abstractly, a dynamical system (M, f) is characterized by:

1) a state-space M of the possible states for the system under consideration;

2) a vector field f, defined on M, which generates the time evolution of the states x in M.

The state-space M is assumed to be Haudsdorff; usually M is a manifold or an open subset of some topological vector space. Here the state-space M is a C^2 manifold without boundary. The time evolution is a map from $M \times I \to M$, defined by $(x, t) \to \Phi_t(x)$, where I is an interval of R and $\Phi_t(\cdot)$ is called the flow (induced by the vector field f). A vector field is said to be *complete* if $\Phi_t(x)$ is defined on $M \times R$. If M is compact, all its vector fields are complete. We may write $\Phi_t(x) = x(t)$, the map $t \to x(t)$ is the *trajectory* of $x \in M$, the image of this map is called the *orbit*. The set of all trajectories is called the *phase portrait*.

When the vector field f does not depend on time the dynamical system is said to be autonomous. A nonlinear autonomous dynamical system can be described by a set of differential equations

$$\dot{x}=f(x) \qquad x \in M. \tag{2-1}$$

We shall assume that the vector field f is C^1; this is a sufficient condition for existence and uniqueness of solution.

A zero of a vector field is referred to as an equilibrium point

Manuscript received April 15, 1986; revised December 14, 1986 and June 9, 1987. Paper recommended by Associate Editor, T. J. Tarn. This work was supported by the Department of Energy, Division of Electric Energy Systems under Contract DE-AC01-84-CE76257. M. W. Hirsch was partially supported by the National Science Foundation under Contract MCS 83-03283.

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IEEE Log Number 8717738.

0018-9286/88/0100-0016$01.00

(e.p.) or simply an equilibrium. It is a solution of the equation

$$f(x)=0. \tag{2-2}$$

We shall denote the set of equilibrium points of (2-1) by $E := \{x{:}f(x) = 0\}$.

An equilibrium point x of f is said to be *hyperbolic* if, in local coordinates, none of the eigenvalues of the Jacobian matrix $J_x f$ at x has zero real part. For a hyperbolic equilibrium point x, we can decompose the tangent space $T_x(M)$ uniquely as a direct sum of two subspaces $E^s + E^u$ such that each subspace is invariant under the linear operator $J_x f$, the eigenvalues of $J_x f$ restricted to E^s have negative real part, and the eigenvalue of $J_x f$ restricted to E^u have positive real part. Letting the dimension of E^s be n_s and the dimension of E^u be n_u, we can express such subspace as the following:

$$\text{the stable subspace } E^s = \text{span } \{v^1, v^2, \cdots, v^{n_s}\}$$

$$\text{the unstable subspace } E^u = \text{span } \{w^1, w^2, \cdots, w^{n_u}\}$$

where $v^1, v^2, \cdots, v^{n_s}$ are the n_s (generalized) eigenvectors whose eigenvalues have negative real parts, $w^1, w^2, \cdots, w^{n_u}$ are the n_u (generalized) eigenvectors whose eigenvalues have positive real parts. Obviously, $n_s + n_u = n$.

We call the value n_u the *type* of x. An equilibrium point of type 0 is called a *sink*; one of type n is called a *source*; all others are called *saddle*. Type-one equilibrium point ($n_u = 1$) will be of some importance. Note that sinks are stable equilibrium points, while sources and saddles are unstable equilibrium points.

By a *closed orbit* of a dynamical system we mean the image of a nonconstant periodic solution of (2-1), i.e., a trajectory γ is a closed orbit if γ is not an equilibrium point and $\Phi_t(x) = x$ for some $x \in \gamma$, $t \neq 0$. A closed orbit γ is said to be *hyperbolic* if for any $p \in \gamma$, $n - 1$ of the eigenvalues of the Jacobian of $\Phi_t(\gamma)$ at p have modulus not equal to 1 (one eigenvalue must always be 1). A *critical element* of the vector field f is either a closed orbit or an equilibrium point.

Let $\hat{x}$ be a hyperbolic equilibrium point. Its stable and unstable manifolds $W^s(\hat{x})$, $W^u(\hat{x})$ are defined as follows:

$$W^s(\hat{x}) = \{x \in M{:}\Phi_t(x) \rightarrow \hat{x} \text{ as } t \rightarrow \infty\} \tag{2-3a}$$

$$W^u(\hat{x}) = \{x \in M{:}\Phi_t(x) \rightarrow \hat{x} \text{ as } t \rightarrow -\infty\}. \tag{2-3b}$$

Similarly, the stable and unstable manifolds of a hyperbolic closed orbit γ are defined as the following:

$$W^s(\gamma) = \{x \in M{:}\Phi_t(x) \rightarrow \gamma \text{ as } t \rightarrow \infty\} \tag{2-4a}$$

$$W^u(\gamma) = \{x \in M{:}\Phi_t(x) \rightarrow \gamma \text{ as } t \rightarrow -\infty\}. \tag{2-4b}$$

Since the stable manifold of the critical element of the flow $\Phi_t(\cdot)$ coincides with the unstable manifold of the critical element of the flow $\Phi_{-t}(\cdot)$, this dual property enables us to translate each property of stable manifold into that of an unstable manifold. A set $S \in R^n$ is said to be an *invariant set* of (2-1) if every trajectory of (2-1) starting in S remains in S for all t. Obviously, these two sets $W^s(\cdot)$, $W^u(\cdot)$ are invariant sets. We say a map $g{:}M \rightarrow N$ is an *immersion* at x if the derivative map $df_x{:}T_x(M) \rightarrow T_{f(x)}(N)$ is injective, where $T_x(M)$ and $T_{f(x)}(N)$ denote the tangent spaces of M and N at points $x \in M$ and $f(x) \in N$, respectively. It is known that $W^s(\cdot)$ and $W^u(\cdot)$ are the images of the injective C^1 immersions of R^{n_s} and R^{n_u}, respectively, [15].

The long-term behavior of the trajectory can be studied in terms of its *ω-limit set.* We say y is in the ω-limit set of x, denoted as $\omega(x)$, if there is a sequence $\{t_i\}$ in R, $t_i \rightarrow \infty$, such that

$$y = \lim_{i \rightarrow \infty} \Phi_{t_i}(x).$$

The α-limit set $\alpha(x)$ is defined similarly by letting $t_i \rightarrow -\infty$. It can be shown that these limit sets are closed invariant subsets of M [27, p. 198]. For example, an equilibrium point is its own ω-limit set; it is also the ω-limit set of trajectories in its stable manifold and the α-limit set of trajectories in its unstable manifold. A closed orbit γ is the ω-limit set and the α-limit set of every point on γ.

The idea of transversality is basic in the study of dynamical systems. If A, B are injectively immersed manifolds in M, we say they satisfy the *transversality condition* if either i) at every point of intersection $x \in A \cap B$, the tangent spaces of A and B span the tangent space of M at x,

$$\text{i.e., } T_x(A) + T_x(B) = T_x(M) \qquad \text{for } x \in A \cap B$$

or ii) they do not intersect at all.

One of the most important features of a hyperbolic equilibrium point $\hat{x}$ is that its stable and unstable manifolds intersect transversely at $\hat{x}$. This transversal intersection is important because it persists under perturbation of the vector field.

III. Equilibrium Points on the Stability Boundary

We will show in Section IV that under fairly general conditions, the stability boundary of a stable equilibrium point is the union of the stable manifolds of the equilibrium points (and/or closed orbits) on the stability boundary. Therefore, in this section we shall derive conditions to characterize the equilibrium points and closed orbits on the stability boundary. The necessary and sufficient conditions for an equilibrium point (or closed orbit) to be on the stability boundary are derived in terms of both the stable manifold and the unstable manifold of the equilibrium point (or closed orbit). We also study the number of equilibrium points on the stability boundary.

Consider a nonlinear autonomous dynamical system described by the differential equation

$$\dot{x} = f(x) \tag{3-1}$$

where x is an n-dimensional vector and the vector field f is C^1.

Suppose x_s is a stable equilibrium point of the vector field f. The *stability region* (or *region of attraction*) of x_s is defined to be $W^s(x_s)$, that is, the set of all points x such that

$$\lim_{t \rightarrow \infty} \Phi_t(x) \rightarrow x_s. \tag{3-2}$$

We will also denote the stability region of x_s by $A(x_s)$, its boundary and its closure by $\partial A(x_s)$ and $\bar{A}(x_s)$, respectively. When it is clear from the context, we write A for $A(x_s)$, etc. Alternatively, the stability region can be expressed as

$$A(x_s) = \{x \in R^n{:}\omega(x) = x_s\}. \tag{3-3}$$

Based on the properties of the stable manifold of x_s, we have the following proposition [15].

Proposition 3-1: $A(x_s)$ is an open, invariant set which is diffeomorphic to R^n.

Since the boundary of an invariant set is also invariant and the boundary of any set is closed, therefore, we have the following.

Proposition 3-2: $\partial A(x_s)$ is a closed invariant set of dimension $< n$. $A(x_s)$ is not dense in R^n, then $\partial A(x_s)$ is of dimension $n - 1$.

Proof: A general result [29, p. 46] states that, if U is an open set in R^n, then ∂U is of dimension $< n$; moreover if U is not dense in R^n, then ∂U is of dimension $n - 1$.

Remark: If there are at least two stable equilibrium points, then the dimension of stability boundary of each of them is $n - 1$; in particular, stability boundaries are nonempty in this case.

Next, we give conditions for an equilibrium point to be on the stability boundary, which is a key step in the characterization of the stability region $A(x_s)$. We do this in two steps. First we impose only one assumption on the dynamical system (3-1), namely, that equilibrium points are hyperbolic, and derive conditions for an equilibrium point to be on the stability boundary in terms of both its stable and unstable manifolds (Theorem 3-3). Additional conditions are then imposed on the dynamical system and the results are further sharpened (Theorem 3-7). We also derive the characterizations of closed orbits on the stability boundary. We use the notation $A - B$ to denote those elements which belong to A but not to B.

Let x be a hyperbolic critical element. Let U be a neighborhood of x in $W^s(x)$ whose boundary ∂U is transversal to the vector field f. We call ∂u a *fundamental domain* of $W^s(x)$. A cross section $V \subset R^n$ of a vector field f is a manifold V of dimension $n - 1$, which need not be a hyperplane but must be in a manner such that the flow of f is everywhere transversal to it. Any cross section of f containing ∂U and transversal to $W^s(x)$ is the so-called *fundamental neighborhood* $G(x)$ associated with $W^s(x)$. It follows that $W^s(x) = \cup_{t\in R} \Phi_t(\partial U) \cup \{x\}$ and $\cup_{t\geqslant 0} \Phi_t(G(x)) \cup W^u(x)$ contains a neighborhood of x.

Theorem 3-3 (Characterization of the Equilibrium Point on the Stability Boundary): Let A be the stability region of a stable equilibrium point x_s. Let $\hat{x} \neq x_s$ be a hyperbolic equilibrium point. Then:

i) if $\{W^u(\hat{x}) - \hat{x}\} \cap \bar{A} \neq \phi$, then $\hat{x} \in \partial A$. Conversely, if $\hat{x} \in \partial A$, the $\{W^u(\hat{x}) - \hat{x}\} \cap \bar{A} \neq \phi$;

ii) suppose $\hat{x}$ is not a source (i.e., $\{W^s(\hat{x}) - \hat{x}\} \neq \phi$). Then $\hat{x} \in \partial A$ if and only if $\{W^s(\hat{x}) - \hat{x}\} \cap \partial A \neq \phi$.

Proof: i) If $y \in W^u(x) \cap \bar{A}$, then

$$\lim_{t\to\infty} \Phi_{-t}(y) = \hat{x}.$$

But since $\bar{A}$ is invariant, we have

$$\Phi_{-t}(y) \in \bar{A}.$$

It follows that

$$\hat{x} \in \bar{A}.$$

Since $\hat{x}$ cannot be in the stability region, $\hat{x}$ is on the stability boundary.

Suppose conversely that $\hat{x} \in \partial A$. Let $G \subset \{W^u(\hat{x}) - \hat{x}\}$ be a fundamental domain of $W^u(\hat{x})$; this means that G is a compact set such that

$$\bigcup_{t\in R} \Phi_t(G) = \{W^u(\hat{x}) - \hat{x}\}. \tag{3-4}$$

Let G_ϵ be the ϵ-neighborhood of G in R^n. Then $\cup_{t<0} \Phi_t(G_\epsilon)$ contains a set of the form $\{U - W^s(\hat{x})\}$, where U is a neighborhood of $\hat{x}$. Since $\hat{x} \in \partial A$, it follows that $U \cap A \neq \phi$. But, by assumption, $\hat{x} \in \partial A$, so $W^s(\hat{x}) \cap A = \phi$. Therefore, we have

$$\{U - W^s(\hat{x})\} \cap A \neq \phi \tag{3-5}$$

or

$$\bigcup_{t<0} \Phi_t(G_\epsilon) \cap A \neq \phi. \tag{3-6}$$

This implies that $G_\epsilon \cap \Phi_t(A) \neq 0$ for some t. Since A is invariant under the flow it follows that

$$G_\epsilon \cap A \neq \phi.$$

Since $\epsilon > 0$ is arbitrary and G is a compact set, we conclude that G contains at least a point of $\bar{A}$.

The proof of ii) is similar to the proof of i). ##

Similarly, we can characterize the closed orbit on the stability boundary as follows.

Corollary 3-4: (Characterization of the Closed Orbit on the Stability Boundary): Let A be the stability region of a stable equilibrium point. Let γ be a hyperbolic closed orbit. Then

i) $\gamma \subseteq \partial A$ if and only if $\{W^u(\gamma) - \gamma\} \cap \bar{A} \neq \phi$;

ii) Suppose $\{W^s(\gamma) - \gamma\} \neq \phi$. Then $\gamma \subseteq \partial A$ if and only if $\{W^s(\gamma) - \gamma\} \cap \partial A \neq \phi$.

As a corollary to Theorem 3-3, if $\{W^u(\hat{x}) - \hat{x}\} \cap A \neq \phi$, then $\hat{x}$ must be on the stability boundary. Since any trajectory in $A(x_s)$ approaches x_s, we see that a sufficient condition for $\hat{x}$ to be on the stability boundary is the existence of a trajectory in $W^u(\hat{x})$ which approaches x_s. The nice thing about this condition is that it can be checked numerically. From a practical point of view, therefore, we would like to see when this condition is also necessary. We are going to show that this condition becomes necessary under two additional assumptions which are reasonable.

So far we have assumed only that the critical elements are hyperbolic. This is a generic property for dynamical systems. Roughly speaking, we say a property is generic for a class of systems if that property is true for *almost all* systems in this class. A formal definition is given in [15]. It has been shown [16] that among $C^r(r \geqslant 1)$ vector fields, the following properties are generic: i) all equilibrium points and closed orbits are hyperbolic and ii) the intersections of the stable and unstable manifolds of critical elements satisfy the transversality condition. Theorem 3-3 can be sharpened under two conditions, one of which is generic for the dynamical system (3-1). That is the transversality condition. The other condition requires that every trajectory on the stability boundary approach one of the critical elements.

The following lemma [15] is used in the proofs of the next two theorems.

Lemma 3-5: Let x_i and x_j be hyperbolic critical elements of (3-1). Suppose that the intersection of stable and unstable manifolds of x_i, x_j satisfy the transversality condition and $\{W^u(x_i) - x_i\} \cap \{W^s(x_j) - x_j\} \neq \phi$. Then $\dim W^u(x_i) \geqslant \dim W^u(x_j)$, where the equality sign is true only when x_i is an equilibrium point and x_j is a closed orbit.

The following lemma, which is a weak version of the λ-lemma [23], is useful in the proof of the next theorem. Recall that the type of an equilibrium point is the dimension of its unstable manifold. An m-disk is a disk of dimension m.

Lemma 3-6: Let $\hat{v}$ be a hyperbolic critical element of (3-1) with $\dim W^u(\hat{v}) = m$. If $\hat{v}$ is an equilibrium point, let D be an m-disk in $W^u(\hat{v})$. If $\hat{v}$ is a closed orbit, let D be an $(m - 1)$-disk in $W^u(\hat{v}) \cap S$, where S is a cross section at $p \in \hat{v}$. Let N be an m-disk (if $\hat{v}$ is an equilibrium point) or $(m - 1)$-disk (if $\hat{v}$ is a closed orbit) having a point of transversal intersection with $W^s(\hat{v})$. Then D is contained in the closure of the set $\cap_{t\geqslant 0} \Phi_t(N)$.

Now, we present the key theorem of this section which characterizes an equilibrium point being on the stability boundary, in terms of both its stable and unstable manifolds. From the practical point of view, this result is more useful than Theorem 3-3.

Theorem 3-7 (Further Characterization of the Equilibrium Point on the Stability Boundary): Let A be the stability region of a stable equilibrium point. Let $\hat{x}$ be an equilibrium point. Assume the following.

i) All the equilibrium points on ∂A are hyperbolic.

ii) The stable and unstable manifolds of equilibrium points on ∂A satisfy the transversality condition.

iii) Every trajectory on ∂A approaches one of the equilibrium points as $t \to \infty$.

Then

1) $\hat{x} \in \partial A$ if and only if $W^u(\hat{x}) \cap A \neq \phi$.

2) $\hat{x} \in \partial A$ if and only if $W^s(\hat{x}) \subseteq \partial A$.

Proof: 1) Because of Theorem 3-3 we only need to prove that, under these assumptions $W^u(\hat{x}) \cap \bar{A} \neq \phi$ implies $W^u(\hat{x}) \cap A \neq \phi$. Let $n_u(x)$ denote the type of an equilibrium point x, i.e., the dimension of its unstable manifold. It follows from assumption i) that $n_u(x) \geqslant 1$ for all equilibrium points $x \in \partial A$. Let $\hat{x} \in \partial A$ and $n_u(\hat{x}) = h$. By Theorem 3-3 there exists a point $y \in \{W^u(\hat{x}) - \hat{x}\} \cap \bar{A}$. If $y \in A$, the proof is complete. If $y \in \partial A$, by assumption iii) there exists an equilibrium point $\hat{z} \in \partial A$ and $y \in$

$\{W^s(\hat{z}) - \hat{z}\}$. Let $n_u(\hat{z}) = m$. By assumption ii) $W^u(\hat{x})$ and $W^s(\hat{z})$ meet transversely at y, thus by Lemma 3-5, $h > m$. Now, consider two cases.

a) $h = 1$: Then m must be zero (i.e., $\hat{z}$ must be a stable equilibrium point), which is a contradiction to the fact that no stable equilibrium point exists on the stability boundary. Consequently, $W^u(\hat{x}) \cap A \neq \phi$.

b) $h > 1$: Without loss of generality, we assume inductively that $W^u(\hat{z}) \cap A \neq \phi$. Since $W^u(\hat{x})$ and $W^s(\hat{z})$ intersect transversely at y, $W^u(\hat{x})$ contains an m-disk N centered at y, transverse to $W^s(\hat{z})$. Applying Lemma 3-6 with $\hat{v} = \hat{z}$, we have $\Phi_t(N) \cap A \neq \phi$ for some $t > 0$. Since A is invariant, this implies $N \cap A \neq \phi$, hence, $W^u(\hat{x}) \cap A \neq \phi$. This completes part a).

2) If $W^s(\hat{x}) \subset \partial A$, then $\hat{x} \in \partial A$ since $\hat{x} \in W^s(\hat{x})$. Conversely, suppose $\hat{x} \in \partial A$. By part a), $W^u(\hat{x}) \cap A \neq \phi$. Let $D \subset W^u(\hat{x}) \cap A$ be an m-disk, $m = \dim W^u(\hat{x})$. Let $y \in W^s(\hat{x})$ be arbitrary. For any $\epsilon > 0$, let N be an m-disk transversal to $W^s(\hat{x})$ at y, contained in the ϵ-neighborhood of y. By Lemma 3-6 with $\hat{v} = \hat{x}$, there exists $t > 0$ such that $\Phi_t(N)$ is so close to D that $\Phi_t(N)$ contains a point $p \in A$. Thus, $\Phi_{-t}(p) \in N$. Since A is invariant, this shows that $N \cap A \neq \phi$. Letting $\epsilon \to 0$ proves $y \in \bar{A}$. Thus, $W^s(\hat{x}) \subset \bar{A}$. Since $W^s(\hat{x})$ is disjoint from A, it follows that $W^s(\hat{x}) \subset \in A$. ##

Remarks:

1) Fig. 1 shows an example for which the assumption that every trajectory on the stability boundary approaches one of the equilibrium points does not hold. For this system, the unstable manifold of x_1 does not intersect with the stability region (see Theorem 3-7) and a part of the stable manifold of x_1 is not on the stability boundary (see Theorem 3-7).

2) To show that the transversality condition is needed in Theorem 3-7, let us consider the example taken from [17]. In Fig. 2 the transversality condition is not satisfied because the intersection of the unstable manifold of x_1 and the stable manifold of x_2 is a portion of the manifold whose tangent space has dimension 1. Note that the unstable manifold of x_1 intersects with the stability boundary (see Theorem 3-3), but not the stability region (see Theorem 3-7). A part of the stable manifold of x_1 (upper part in Fig. 2) is not in the stability boundary (see Theorem 3-7).

Theorem 3-8 below extends the result of Theorem 3-7 to accommodate closed orbits on the stability boundary.

Theorem 3-8 (Characterization of the Critical Element on the Stability Boundary): Let A be the stability region of a stable equilibrium point. Let r be a critical element. Assume the following.

i) All the critical elements on ∂A are hyperbolic.

ii) The stable and unstable manifolds of critical elements on ∂A satisfy the transversality condition.

iii) Every trajectory on ∂A approaches one of the critical elements as $t \to \infty$.

Then

1) $\hat{r}$ is on the stability boundary ∂A if and only if $W^u(\hat{r}) \cap A \neq \phi$,

2) $\hat{r}$ is on the stability boundary ∂A if and only if $W^s(\hat{r}) \subseteq \partial A$.

Proof: 1) Because of Theorem 3-3 and Corollary 3-4 we only need to prove that, under these assumptions, $\hat{r}$ is on the stability boundary ∂A implies $W^u(\hat{r}) \cap A \neq \phi$. We also use the notation $n_u(r)$ to denote the dimension of $W^u(\hat{r})$. From assumption i) we have that $n_u(r) \geqslant 1$ if $\hat{r}$ is an equilibrium point and $n_u(r) \geqslant 2$ if $\hat{r}$ is a closed orbit. Let $\hat{x} \in \partial A$ and $n_u(x) = h$. By Theorem 3-3 or Corollary 3-4, there exists a point $y \in \{W^u(\hat{r}) - \hat{r}\} \cap \bar{A}$. If $y \in A$, the proof is complete. Suppose that $y \in \partial A$; then by assumption iii) there exists a critical element $\hat{z} \in \partial A$ and $y \in \{W^s(\hat{z}) - \hat{z}\}$.

Let $n_u(\hat{z}) = m$; then by Lemma 3-5 we have $h \geqslant m$. Consider the following cases: a) $h = 1$: then $m = 0$ (i.e., $\hat{z}$ is a stable equilibrium point) or $m = 1$ (i.e., $\hat{z}$ is a stable closed orbit), which is a contradiction. Thus, $W^u(\hat{r}) \cap A(x_s) \neq \phi$. b) $h = 2$: then two subcases are possible; b′) $m = 1$ (i.e., $\hat{z}$ is a type-one

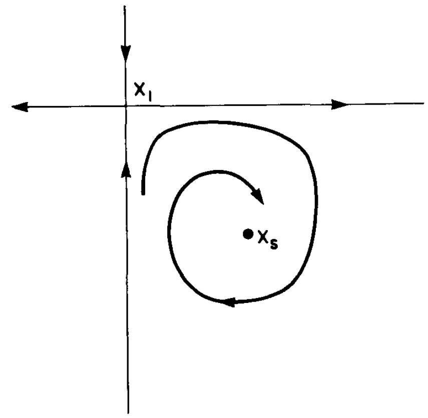

Fig. 1. An example of a dynamical system whose trajectories on the stability boundary do not all approach its critical elements.

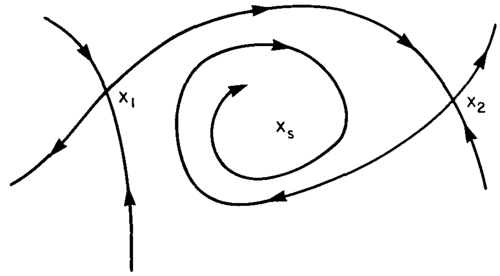

Fig. 2. The intersection between the unstable manifold of x_1 and the stable manifold of x_2 does not satisfy the transversality condition.

equilibrium point), then $W^u(\hat{r})$ contains an m-disk N centered at y and transversal to $W^s(\hat{z})$; applying Lemma 3-6 with $\hat{v} = \hat{z}$, we have $N \cap A \neq \phi$. b″) $m = 2$ (i.e., $\hat{z}$ is a closed orbit with $\dim W^u(\hat{z}) = 2$), then $W^u(\hat{r})$ contains an $(m - 1)$-disk N centered at y and transversal to $W^s(\hat{z})$. After applying Lemma 3-6 with $\hat{v} = \hat{z}$, we have $N \cap A \neq \phi$. Consequently, $W^u(\hat{r}) \cap A \neq \phi$. The proof is completed by an induction similar to that in the proof of Theorem 3-7 part i).

2) This part is similar to the proof of part 2) of Theorem 3-7, using Lemma 3-6 with $\hat{z} = \hat{r}$. ##

The next result concerns the number of equilibrium points on the stability boundary. We say that $S \subset R^n$ is a *smooth* manifold of dimension s if, for each point $p \in S$, there exist a neighborhood $U \subset S$ of p and a homeomorphism $h: U \to V$, where V is an open subset of R^s, such that the inverse homeomorphism $h^{-1}: V \to U \subset R^n$ is an immersion of class C^1.

Theorem 3-9 (Number of Equilibrium Points on the Stability Boundary): If the stability boundary ∂A of a stable equilibrium point is a smooth compact manifold and all the equilibrium points on ∂A are hyperbolic, then the number of equilibrium points on ∂A is even.

Proof: The proof is based on the following fact [24, Exercise 7, p. 139]: the Euler characteristic of the boundary of a compact manifold is even. From the Poincare–Hopf index Theorem [25, p. 134], it follows that the sum of the indexes of equilibrium points of f on the smooth, compact stability boundary ∂A are even. But the index of f at a hyperbolic equilibrium point is either $+1$ or -1 [26, p. 37]. Consequently, Theorem 3-9 follows. ##

Remarks:

1) Genesio and Vicino [22] have shown that Theorem 3-9 is true for a special case, namely; an odd order system ($n \neq 5$) without *degenerate* equilibrium point.

2) Fig. 3 shows an example that the assumption of smoothness

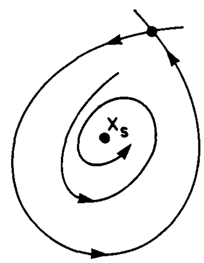

Fig. 3. The stability boundary of x_s is not smooth.

of stability boundary does not hold. For this system, the stability boundary contains only one equilibrium point (see Theorem 3-9).

3) Theorem 3-9 is also true if instead of hyperbolicity, we assume only that every equilibrium point is *nondegenerate* in the sense that the corresponding Jacobian matrix of the vector field is invertible. The proof is the same.

IV. The Stability Boundary

In this section we characterize the stability boundary for a fairly large class of nonlinear autonomous dynamical systems (3-1) whose stability boundary is nonempty. We make the following assumptions concerning the vector field.

A1) All the equilibrium points on the stability boundary are hyperbolic.

A2) The stable and unstable manifolds of equilibrium points on the stability boundary satisfy the transversality condition.

A3) Every trajectory on the stability boundary approaches one of the equilibrium points as $t \to \infty$.

Remark: Assumption A1) is a generic property of C^1 dynamical systems, and can be checked for a particular system by direct computation of the eigenvalues of the corresponding Jacobian matrix of the vector field. Assumption A2) is also a generic property, however, it is not easy to check. Assumption A3) is not a generic property, but in many systems it can be verified by means of a V-function or by direct analysis (see Section V and Example 7.1).

Theorem 4-1 asserts that if assumptions A1) to A3) are satisfied, then the stability boundary *is* the union of the stable manifolds of the equilibrium points on the stability boundary.

Theorem 4-1 (Characterization of Stability Boundary): For a nonlinear autonomous dynamical system (3-1) which satisfies assumptions A1) to A3), let x_i, $i = 1, 2, \cdots$ be the equilibrium points on the stability boundary ∂A of the stable equilibrium point. Then

$$\partial A = \bigcup_i W^s(x_i). \tag{4-1}$$

Proof: Let x_i, $i = 1, 2, \cdots$ be the equilibrium points on the stability boundary. Theorem 3-7 implies

$$\bigcup_i W^s(x_i) \subseteq \partial A. \tag{4-2}$$

The assumption A3) implies

$$\partial A \subseteq \bigcup_i W^s(x_i). \tag{4-3}$$

Combining (4-2) and (4-3) we have the required result. ##

Remarks:

1) Earlier attempts to characterize the stability boundary was presented by Tsolas, Arapostathis, and Varaiya in [17].

2) Results similar to Theorem 3-7 and Theorem 4-1 under a stronger condition than A3) have been derived previously [32].

3) Zaborszky *et al.* [33] have developed independent proofs of Theorems 3.3, 3.7, and 4.1 for power system models.

Theorem 4-1 can be generalized to allow closed orbits to exist on the stability boundary.

Theorem 4-2 (Characterization of Stability Boundary): Consider a dynamical system (3-1) whose vector field satisfies the following assumptions.

B1) All the critical elements on the stability boundary are hyperbolic.

B2) The stable and unstable manifolds of critical elements on the stability boundary satisfy the transversality condition.

B3) Every trajectory on the stability boundary approaches one of the critical elements as $t \to \infty$.

Let x_i, $i = 1, 2, \cdots$, be the equilibrium points and γ_j, $j = 1, 2, \cdots$, be the closed orbits on the stability boundary ∂A of a stable equilibrium point. Then

$$\partial A = \bigcup_i W^s(x_i) \bigcup_j W^s(\gamma_j).$$

Proof: By Theorem 3-8, the stable manifolds of critical elements which are on ∂A lie in ∂A. By assumption B3), every point of ∂A is on the stable manifold of one of the critical elements on ∂A. Combining these two we have the required result. ##

Remark: Assumption B1) is a generic property of C^1 dynamical systems, however, because closed orbits are hard to determine, it is difficult to check assumption B1) for a given system except perhaps for planar systems. Assumption B2) is also a generic property; but it is even harder to check. Assumption B3) is a generic property only for planar systems. For higher dimensional systems no general methods of verifying it are known. Nevertheless we believe Theorem 4-2 has considerable theoretical interest, since the class of systems satisfying assumptions B1) to B3) may be considered the simplest class of systems having closed orbits.

The following theorem gives an interesting result on the structure of the equilibrium points on the stability boundary. Moreover, it presents a necessary condition for the existence of certain types of equilibrium points on a *bounded* stability boundary.

Theorem 4-3 (Structure of Equilibrium Points on the Stability Boundary): For the nonlinear autonomous dynamical system (3-1) containing two or more stable equilibrium points and satisfying the assumptions A1) to A3), the stability boundary must contain at least one type-one equilibrium point. If, furthermore, the stability region is bounded, then ∂A must contain at least one type-one equilibrium point and one source.

Proof: Since there are at least two stable equilibrium points including, say x_s, it follows that the dimension of $\partial A(x_s)$ is $(n - 1)$ (see the proof of Proposition 3-2). Since $\partial A(x_s) = \cup\ W^s(x_j)$, where $x_j \in \partial A(x_s)$, at least one of the x_j must be a type-one equilibrium point, say x_1, so that the dimension of $\cup\ W^s(x_j)$ is $(n - 1)$. Repeating the same argument, if $\partial W^s(x_1)$ is nonempty, then the dimension of $\partial W^s(x_1)$ is $\leqslant (n - 2)$, say $(n - k)$. The application of Theorem 4-1 yields $\partial W^s(x_1) = \cup\ W^s(x_j)$, $x_j \in \partial W^s(x_1)$. In order for $\cup\ W^s(x_j)$ to have dimension $(n - k)$, at least one of the x_j must be a type-k equilibrium point. If the stability region is bounded, the same argument can be repeated until we reach a type-n equilibrium point (a source).

The contrapositive of Theorem 4-3 leads to the following corollary, which is useful in predicting unboundedness of the stability region.

Corollary 4-4 (Sufficient Condition for the Stability Region to be Unbounded): For the nonlinear autonomous dynamical systems (3-1), if assumptions A1) to A3) are satisfied and if $\partial A(x_s)$ contains no source, then the stability region $A(x_s)$ is unbounded.

V. Sufficient Condition for Assumption A3)

The characterization of stability boundaries in the previous section is valid for dynamical systems satisfying assumptions A1) to A3). Since assumptions A1) and A2) are generic properties, assumption A3) is the crucial one in the application of Theorem 4-1. In this section, we will show that many dynamical systems arising from physical system models satisfy assumption A3). We first present two theorems that give sufficient conditions for this assumption.

It should be stressed that the main results in this paper are independent of the existence of Lyapunov functions. For a convenient sufficient condition for guaranteeing assumption A3), however, we will introduce a function in the following theorems which bears some resemblance to a Lyapunov function. Recall that E denotes the set of equilibrium points of (3-1). If V is a scalar function on R^n, then $\dot V(x) := d/dt|_{t=0} V(\Phi_t(x)) = \nabla V(x) \cdot f(x)$.

Theorem 5-1: Suppose that there exists a C^1 function $V:R^n \to R$ for the system (3-1) such that

1) $\dot V(x) < 0 \quad$ if $x \notin E$.

Suppose also that there exists $\delta > 0$ such that for any $\hat x \in E$, the open ball $B_\delta(\hat x) := \{x : |x - \hat x| < \delta\}$ contains no other point in E, and that the distance between any two such balls is at least δ. Furthermore, suppose that there exist a positive continuous function $\alpha : R^n \to R^+$ and two constants, $c_1 > 0$ and $c_2 > 0$, such that

2) $\alpha(x)|f(x)| < c_1 \quad$ for all $x \in \partial A$;

and

3) $\alpha(x)\dot V(x) < -c_2 \quad$ unless $x \in B_\delta(\hat x)$ for some $\hat x \in E \cap \partial A$.

Then the assumption A3) is true: every trajectory on ∂A converges to an equilibrium point as $t \to \infty$.

Proof: Let $x(t) := \Phi_t(x)$ be a trajectory on the stability boundary. Suppose $x(t)$ does not approach one of the equilibrium points, we show this leads to a contradiction. We consider two cases.

Case 1: There exists a $T > 0$ such that for all $t > T$, $\Phi_t(x)$ is not in any $B_\delta(\hat x)$.

Therefore, by condition (3) we have

$$\dot V(x(t)) < -\frac{c_2}{\alpha(x(t))} \qquad \text{for all } t > T.$$

We estimate for $t > T$

$$\begin{aligned} V(x(t)) - V(x(T)) &= \int_t^T \dot V(x(\tau))\, d\tau \\ &< -c_2 \int_T^t \frac{1}{\alpha(x(\tau))}\, d\tau \\ &< -\frac{c_2}{c_1} \int_T^t |f(x(\tau))|\, d\tau \\ &= -\frac{c_2}{c_1} \int_T^t |\dot x(\tau)|\, d\tau \\ &\leqslant -\frac{c_2}{c_1} \left| \int_T^t \dot x(\tau)\, d\tau \right| \\ &= -\frac{c_2}{c_1} |x(t) - x(T)|. \end{aligned} \tag{5-1}$$

This shows that $\lim_{t\to\infty} V(x(t)) = -\infty$. But this contradicts the fact that $V(\cdot)$ is bounded below [by $V(x_s)$] along any trajectory on the stability boundary, which follows from condition 1) and the continuity property of the function $V(\cdot)$.

Case 2: There is an infinite sequence $\{\hat p_i\}$ of equilibrium points and increasing sequence $r_i \to \infty$ such that $x(r_j) \in B_\delta(\hat p_i)$.

Let us define two increasing sequences $\{t_i\}$ and $\{s_i\}$: t_i is the first time $x(t)$ enters the δ-ball $B_\delta(\hat p_i)$ and s_i is the first time $> t_i$ that $x(t)$ leaves the 2δ-ball $B_{2\delta}(\hat p_i)$.

Fix an integer $m > 0$; then for $t \geqslant t_{m+1}$ we have

$$\begin{aligned} V(x(t)) - V(x(0)) &= \int_0^t \dot V(x(\tau))\, d\tau \\ &< \sum_{i=1}^m \int_{t_i}^{s_i} \dot V(x(\tau))\, d\tau \\ &< -\frac{c_2}{c_1} \sum_{i=1}^m \left| \int_{t_i}^{s_i} \dot x(\tau)\, d\tau \right| \\ &< -\frac{c_2}{c_1} m\delta. \end{aligned}$$

Letting $m \to \infty$, we contradict the fact that $V(\cdot)$ is bounded below on the stability boundary. Therefore, every trajectory on the stability boundary must approach one of the equilibrium points. ##

Corollary 5-2: Suppose that the system (3-1) has a finite number of equilibrium points on its stability boundary and there exists a C^1 function $V:R^n \to R$, and two positive numbers ϵ, δ for the system (3-1) such that

$$\dot V(x) < 0 \qquad \text{if } x \notin E; \tag{5-2}$$

$$\dot V(x) < -\delta \qquad \text{if } x \notin B_\epsilon(\hat x),\ \hat x \in E$$

and

$$|f(x)| \text{ is bounded for } x \in R^n. \tag{5-3}$$

Then assumption A3) is true.

Theorem 5-3: Suppose there exists C^1 function $V:R^n \to R$ for the system (3-1) such that

1) $\dot V(x) \leqslant 0$ at every point $x \notin E$;

2) if $x \notin E$, then the set $\{t \in R : \dot V(\Phi_t(x)) = 0\}$ has measure 0 in R;

and either

3) the map $V:R^n \to R$ is proper;

or

3') : for each $x \in R^n$, if $\{V(\Phi_t(x))\}_{t \geqslant 0}$ is bounded, then $\{\Phi_t(x)\}_{t\geqslant 0}$ is bounded.

Then the assumption A3) is true.

Proof: From the well-known Lyapunov-type argument, conditions 1) and 2) imply that all the limit sets of trajectories consist of equilibrium points [27, p. 203]. Since the stability boundary is a closed invariant set, by the continuity property of the function $V(\cdot)$ and conditions 1) and 2) we have the value of $V(\cdot)$ along every trajectory on the stability boundary is bounded below by $V(x_s)$. Hence, condition 3) or condition 3') implies $\{x(t)\}_{t\geqslant 0}$ is bounded. Since the limit set of any compact trajectory is nonempty, thus A3) follows. ##

Remarks:

1) It can be shown, by applying Corollary 5-2, that the following dynamical systems satisfy assumption A3).

$$\dot x = Df(x)$$

where $f:R^n \rightarrow R^n$ is a bounded gradient vector field with only finitely many equilibrium points on $\partial A(x_s)$ and the matrix D is a positive definite matrix.

2) It has been shown [19] that many second-order dynamical systems frequently encountered in physical system models satisfy assumption A3). These are systems of the form

$$M\ddot{x}+D\dot{x}+f(x)=0$$

whose state-space representation is

$$\dot{x}=y \quad (5\text{-}4a)$$

$$M\dot{y}=-Dy-f(x) \quad (5\text{-}4b)$$

where M is a diagonal matrix with positive elements, D is a symmetric, diagonally dominant matrix with positive diagonal elements, $f:R^n \rightarrow R^n$ is a bounded gradient vector field with bounded Jacobian. In addition, $f(\cdot)$ is assumed to satisfy the following condition. There exists $\epsilon > 0$ and $\delta > 0$ such that the distance between the two balls $B_\epsilon(x_i)$ and $B_\epsilon(x_j)$ is greater than ϵ, for all $x_i, x_j \in E$ and $|f(x)| > \delta$, for $x \notin \cup_{\bar{x}\in E} B_\epsilon(\bar{x})$, where $E := \{x: f(x) = 0\}$, $B_r(\hat{x}) := \{x: |x - \hat{x}| \leqslant r\}$.

VI. An Algorithm to Determine the Stability Region

Theorems 3-7 and 4-1 lead to the following conceptual algorithm to determine the stability boundary of a stable equilibrium point of (3-1), *assuming* that assumptions A1) to A3) of Section IV hold.

Algorithm: To determine the stability boundary $\partial A(x_s)$

Step 1: Find all the equilibrium points.

Step 2: Identify those equilibrium points whose unstable manifolds contain trajectories approaching the stable equilibrium point x_s.

Step 3: The stability boundary of x_s is the union of the stable manifolds of the equilibrium points identified in Step 2.

Step 1 in the algorithm involves finding all the solutions of $f(x) = 0$. Step 2 can be accomplished numerically. The following procedure is suggested.

i) Find the Jacobian at the equilibrium point (say, $\hat{x}$).

ii) Find many of the generalized unstable eigenvectors of the Jacobian having unit length.

iii) Find the intersection of each of these normalized, generalized unstable eigenvectors (say, y_i) with the boundary of an ϵ-ball of the equilibrium point. (The intersection points are $\hat{x} + \epsilon y_i$ and $\hat{x} - \epsilon y_i$.)

iv) Integrate the vector field backward (reverse time) from each of these intersection points up to some specified time. If the trajectory remains inside this ϵ-ball, then go to the next step. Otherwise, we replace the value ϵ by $\alpha\epsilon$ and also the intersection points $\hat{x} \pm \epsilon y_i$ by $\hat{x} \pm \alpha\epsilon y_i$, where $0 < \alpha < 1$. Repeat this step.

v) Numerically integrate the vector field starting from these intersection points.

vi) Repeat the steps iii) through v). If any of these trajectories approaches x_s, then the equilibrium point is on the stability boundary.

For a planar system, the type of the equilibrium point on the stability boundary is either one (saddle) or two (source). The stable manifold of a type-one equilibrium point in this case has dimension one, which can easily be determined numerically as follows.

a) Find a normalized stable eigenvector y of the Jacobian at the equilibrium point $\hat{x}$.

b) Find the intersection of this stable eigenvector with the boundary of an ϵ-ball of the equilibrium point $\hat{x}$. (The intersection points are $\hat{x} + \epsilon y$ and $\hat{x} - \epsilon y$.)

c) Integrate the vector field from each of these intersection points after some specified time. If the trajectory remains inside this ϵ-ball, then go to the next step. Otherwise, we replace the value ϵ by $\alpha\epsilon$ and also the intersection points $\hat{x} \pm \epsilon y_i$ by $\hat{x} \pm \alpha\epsilon y_i$, where $0 < \alpha < 1$. Repeat this step.

d) Numerically integrate the vector field backward (reverse time) starting from these intersection points.

e) The resulting trajectories are the stable manifold of the equilibrium point.

For higher dimensional systems, the numerical procedure similar to the one above can only provide a set of trajectories on the stable manifold. Finding the stable and unstable manifold of an equilibrium point is a nontrivial problem. A power series expansion of the stable manifold of an equilibrium point is derived in [18].

VII. Examples

The method for the determination of stability region proposed in Section VI has been applied to some examples we have found in the literature; almost all of them are planar systems. In this section we present these examples to illustrate the results of this paper. In each example we give two figures; one compares the estimated stability region by previous methods and the present one, the other gives the phase portrait of the system to verify the results of this paper. Throughout these examples we *assume* the transversality condition A2) is satisfied.

Example 1: This is an example studied in [22], [10]

$$\dot{x}_1 = -2x_1 + x_1x_2$$
$$\dot{x}_2 = -x_2 + x_1x_2. \quad (7\text{-}1)$$

There are two equilibrium points: (0.0, 0.0) is a stable equilibrium point and (1, 2) is a type-one equilibrium point. The assumption A1) is satisfied. The trajectory on the unstable manifold of (1, 2) converges to the stable equilibrium point (0.0, 0.0), hence (1, 2) is on the stability boundary (Theorem 3-7). Next, we check assumption A3). Consider the following function:

$$V(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2.$$

The derivative of $V(x_1, x_2)$ along the trajectory of (7-1) is

$$\dot{V}(x_1, x_2) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = -2(2x_1 - x_2)(x_1 - x_2).$$

Hence,

$$\dot{V}(x_1, x_2) < 0 \qquad \text{for } (x_1, x_2) \in B^c := R^2 - B$$

where $B := \{(x_1, x_2): 2x_1 - x_2 \geqslant 0 \text{ and } x_1 - x_2 \leqslant 0\}$.

Define the following sets:

$$\tilde{B} := B_1 \cup B_2 \cup B_3$$

where $B_1 = \{(x_1, x_2): x_1 < 1, x_2 < 2\}$, $B_2 = \{(x_1, x_2): x_1 \geqslant 1, x_2 \leqslant 2\} \cap B$, and $B_3 = \{(x_1, x_2): x_1 > 1, x_2 > 2\}$.

Since, in the set B_1, both $|x_1(t_i)|$ and $|x_2(t_i)|$ are strictly decreasing sequences, we conclude that B_1 is inside the stability region of (0, 0). In other words, the stability boundary $\partial A(0, 0)$ cannot lie in B_1. On the other hand, every trajectory of (7-1) in the set B_3 is unbounded as $t \rightarrow \infty$, therefore, the stability boundary $\partial A(0, 0)$ cannot lie in B_3 either. However, by checking the vector field of (7-1) in B_2 we find that every trajectory in B_2 will either enter into B_1 or B_3, or converge to the point (1, 2). Hence, we have shown that the stability boundary $\partial A(0, 0)$ cannot be in B_1 nor in B_3, the part of the stability boundary $\partial A(0, 0)$ in B_2 must converge to (1, 2). Next, we will show that the part of the stability boundary $\partial A(0, 0)$ in $R^2 - \tilde{B}$ also converges to (1, 2). Then, we may claim that assumption A3) is satisfied. Note that $\dot{V}(x_1, x_2) < 0$ for $(x_1, x_2) \in R^2 - \tilde{B} \subseteq B^c$ and that the function $V(x_1, x_2)$ is a proper map in $R^2 - \tilde{B}$. Thus, following the same argument as in

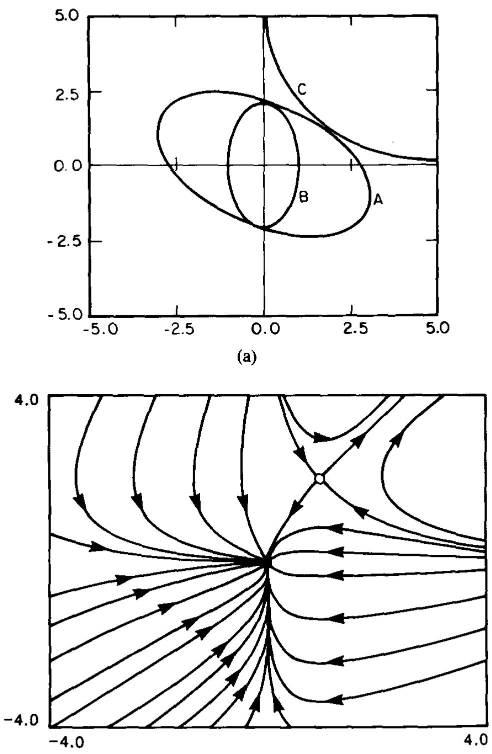

Fig. 4. (a) Predictions of the stability region of Example 1 by different methods. Curves A and B are obtained by the methods in [10] and [22]. Curve C is obtained by the present method. (b) The phase portrait of this system. Note that all the points inside the curve C converge to the stable equilibrium point which verified that curve C is the true stability region.

the proof of Theorem 5-3 we conclude that every trajectory of $\partial A(0, 0)$ in $R^2 - \bar{B}$ is bounded and, if it converges in $R^2 - \bar{B}$, it must converge to an equilibrium point in $R^2 - \bar{B}$. However, there is no equilibrium point in $R^2 - \bar{B}$. So, the part of the stability boundary $\partial A(0, 0)$ in $R^2 - \bar{B}$ must enter the set $\bar{B}$. But, we have shown that the stability boundary $\partial A(0, 0)$ in $\bar{B}$ converges to (1, 2). Therefore, the trajectories on the stability boundary $\partial A(0, 0)$ converge to (1, 2), and assumption A3) is shown to be satisfied.

Consequently, the stability boundary is the stable manifold of (1, 2) (Theorem 4-1), which is the curve C in Fig. 4(a). Because there is no source, the stabililty region is unbounded (Corollary 4-4). Curves A and B in Fig. 4(a) are obtained by the methods in [10] and [22], respectively. The approximately true stability boundary mentioned in [22] seems to agree with curve C. Fig. 4(b) is the phase portrait of this system.

Example 2: The following system is considered in [10]:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 0.301 - \sin(x_1 + 0.4136) + 0.138 \sin 2(x_1 + 0.4136) - 0.279\, x_2. \quad (7\text{-}2)$$

Since this system is of the same form as (5-4), it follows that the assumption A3) is satisfied. The equilibrium points of (7-2) are periodic in the subspace $\{(x_1, x_2) | x_2 = 0\}$; the Jacobian matrix of (7-2) at (x_1, x_2) is

$$J(x) = \begin{bmatrix} 0 & 1 \\ a & -0.279 \end{bmatrix} \quad (7\text{-}5)$$

where $a = -\cos(x_1 + 0.4136) + 0.276 \cos 2(x_1 + 0.4136)$.

Let λ_1, λ_2 be the eigenvalue of $J(x)$:

$$\lambda_1 + \lambda_2 = -0.279 \quad (7\text{-}6a)$$

$$\lambda_1 \times \lambda_2 = -a. \quad (7\text{-}6b)$$

The following observations are immediate.

1) Assumption A1) is satisfied.

2) At least one of the eigenvalues must be negative, which implies there is no source in the system (7-2). By Corollary 4-4 we conclude that the stability region (with respect to any stable equilibrium point) is unbounded.

3) The stable equilibrium points and the type-one equilibrium points are located alternately on the x_1-axis.

It can be shown that (6.284098, 0.0) is a stable equilibrium point of (7-2). Let us consider its stability region. The application of Theorem 3-7 shows that the type-one equilibrium points (2.488345, 0.0) and (8.772443, 0.0) are on the stability boundary. The stability region is again unbounded owing to the absence of a source. The stability boundary obtained by the present method is the curve B shown in Fig. 5(a) which is the union of stable manifolds of the equilibrium points (2.488345, 0.0) and (8.772443, 0.0). Curve A is the stability boundary obtained in [10] (after a shift in coordinates). It is clear from the phase portrait in Fig. 5(b) that the trajectories of the points inside curve B converge to the stable equilibrium point which verifies that the curve B is the exact stability boundary.

In the following examples, assumptions A1) and A3) have been checked; the details are omitted.

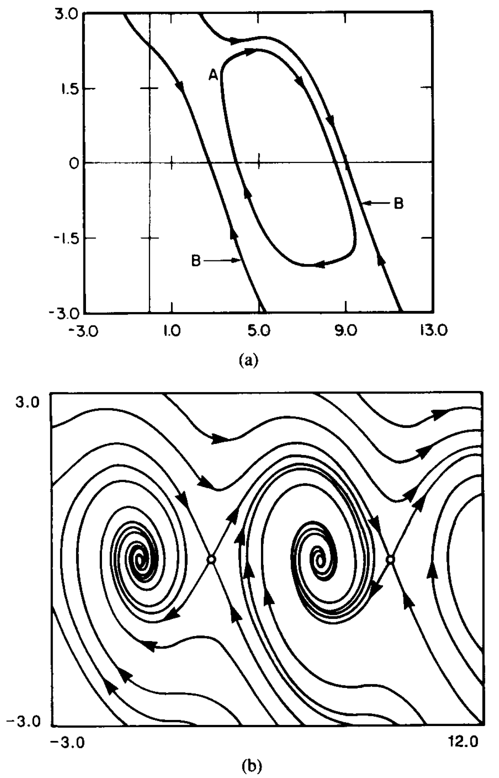

Fig. 5. (a) Predictions of the stability region of Example 2 by different methods. Curve A is obtained by the methods in [10] (after a shift in coordinates). Curve B is obtained by the present method. (b) The phase portrait of this system.

Example 3: The following system was also considered in [6]:

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = 0.1\, x_1 - 2.0\, x_2 - x_1^2 - 0.1\, x_1^3. \tag{7-7}$$

There are three equilibrium points: (0.0, 0.0) is stable, (−2.55, −2.55) is type-one, and (−7.45, −7.45) is also stable. We are interested in the stability region of (0.0, 0.0). Again the stability region in this case is unbounded. Fig. 6(a) shows the stability region obtained by our method. Fig. 6(b) represents the phase portrait of this system.

Example 4: A simple nonlinear speed-control system studied by Fallside, etc. [21] and Jocic [6] shown in Fig. 7(a) can be described by the following equation:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -K_d x_2 - x_1 - g x_1^2 \left(\frac{x_2}{K_d} + x_1 + 1 \right). \tag{7-8}$$

For $K_d = 1$ and $g = 6$, there are three equilibrium points: (−0.78865, 0.0) is stable (the corresponding Jacobian has two real negative eigenvalues), (−0.21135, 0.0) is type-one, and (0.0, 0.0) is also stable (the corresponding Jacobian has two complex eigenvalues with negative real parts). The type-one equilibrium point is on the stability boundary of (0.0, 0.0) and also on the stability boundary of (−0.78865, 0.0) because the two branches of its unstable manifold approaches them. Thus, by Theorem 4-1 we conclude that the stability region of (0.0, 0.0) is the open set containing (0.0, 0.0); its boundary is characterized by the stable manifold of (−0.211325, 0.0); the stability region of (−0.78865, 0.0) is the open set containing (−0.78865, 0.0) with the same boundary as that of (0.0, 0.0). The region in Fig. 7(b) is the stability region predicted by this method. The region denoted by A_J in Fig. 7(c) shows the stability region predicted by method of sinks [13], and the region A_w is predicted by [21]. The phase portrait of this control system is in Fig. 7(d).

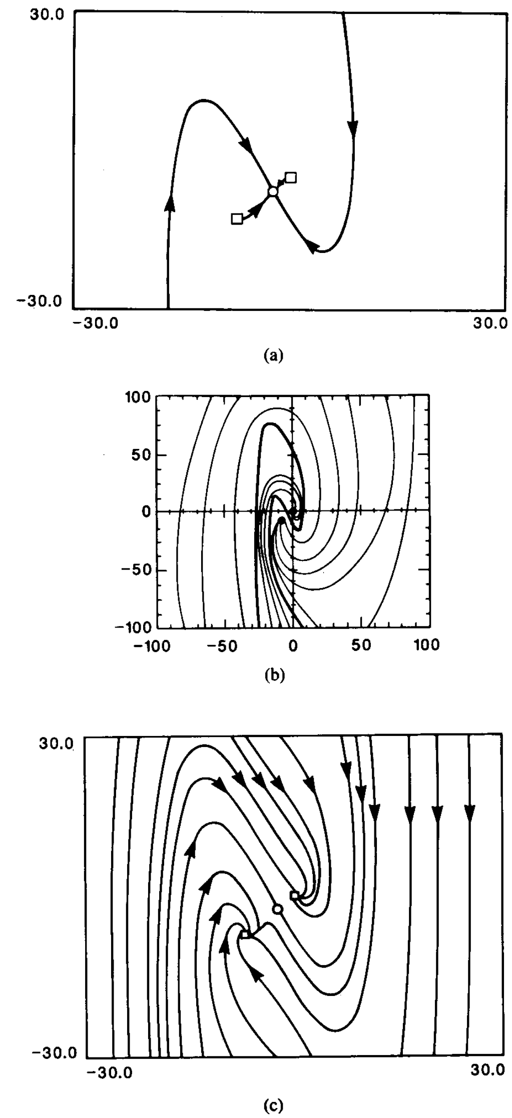

Fig. 6. (a) Predictions of the stability region of Example 3 by the present method. The curves in this figure are the stable manifold of the type-one equilibrium point (−2.55, −2.55). The stability region of (0.0, 0.0) is the region inside these curves which contains (−2.55, −2.55). (b) The phase portrait of this system. (c) The phase portrait of this system with the coordinate system rescaled.

Example 5: Consider the following system which is similar to (7-8) except the term $-K_d x_2$ is replaced by $K_d x_2$.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = K_d x_2 - x_1 - g x_1^2 \left(\frac{x_2}{K_d} + x_1 + 1 \right). \tag{7-9}$$

For $K_d = 1$ and $g = 6$, there are three equilibrium points:

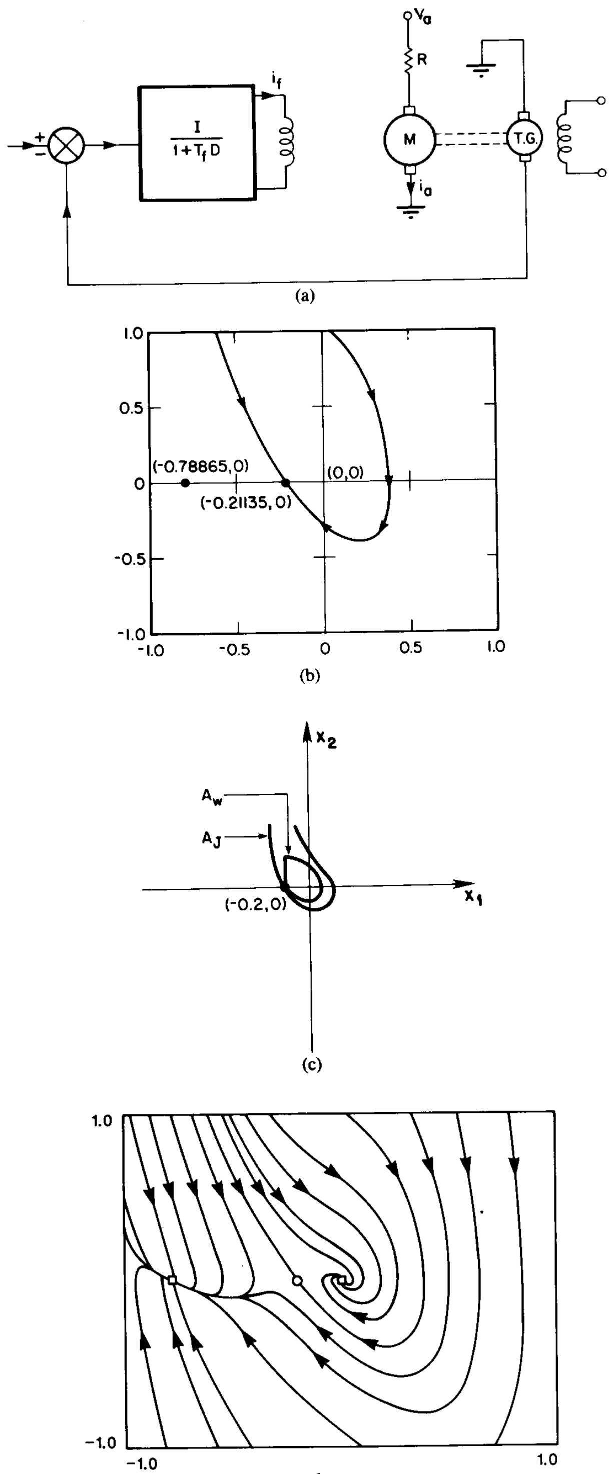

Fig. 7. (a) A simple nonlinear speed-control system. (b) The stability region of Example 4 predicted by the present method. (c) Predictions of the stability region of Example 4 by different methods. The regions denoted by A_J and A_W are obtained by the methods in [13] and [21]. (d) The phase portrait of this system.

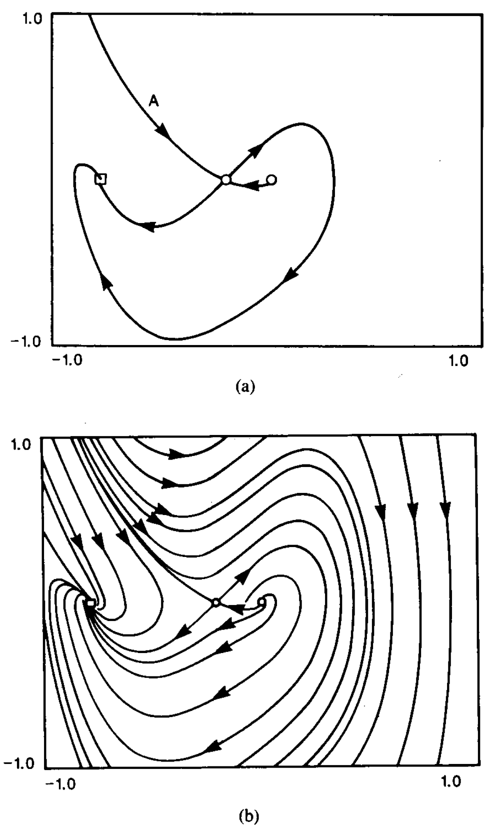

Fig. 8. (a) The stability region of Example 5 predicted by the present method is the whole state space except for the stable manifold of $(-0.21135, 0.0)$, denoted by the curve A and the source $(0.0, 0.0)$. (b) The phase portrait of this system. Note that all the points except for the curve A converge to the stable equilibrium point.

$(-0.78865, 0.0)$ is stable, $(-0.21135, 0.0)$ is type-one, and $(0.0, 0.0)$ is a source. It can be shown that both the type-one equilibrium points and the source are on the stability boundary. Both parts of the unstable manifold of the type-one equilibrium point $(-0.21135, 0.0)$ approach the stable equilibrium point. We conclude that they both belong to the stability region; consequently, the stability region is the whole state-space except for the stable manifold of $(-0.21135, 0.0)$ and the source $(0.0, 0.0)$. Fig. 8(a) shows the stable manifold and unstable manifold of $(-0.21135, 0.0)$. The phase portrait of this system is in Fig. 8(b). Comparing system (7-9) to system (7-8) we found that the stability region of $(0.0, 0.0)$ for (7-8) is shrunk to a point for (7-9) while the stability region of the stable equilibrium point of (7-9) is expanded to fill almost all of the state space.

VIII. Conclusion

A comprehensive theory of stability regions of stable equilibrium points for nonlinear autonomous dynamical systems is presented. A complete dynamical characterization of the stability boundary of a fairly large class of nonlinear autonomous dynamical systems is derived. A method for finding the stability region based on its topological properties is proposed.

The proposed method requires the determination of the stable manifold of an equilibrium point. For lower dimensional systems this may be done by numerical methods. For higher dimensional systems efficient computational methods to derive the stable manifolds are needed.

Acknowledgment

The authors wish to thank P. P. Varaiya and L. O. Chua for their helpful discussions.

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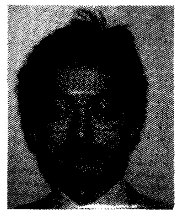

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