STABILITY RESULTS FOR MULTIPLE VOLTERRA INTEGRAL EQUATIONS

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by

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GRADUATE COLLEGE

I hereby recommend that this dissertation prepared under my direction by Ronald J. DeFranco

entitled Stability Results for Multiple Volterra Integral Equations

be accepted as fulfilling the dissertation requirement of the degree of Doctor of Philosophy

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After inspection of the final copy of the dissertation, the following members of the Final Examination Committee concur in its approval and recommend its acceptance:*

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ABSTRACT

Lyapunov stability, uniform stability, and asymptotic stability for a class of systems of multiple Volterra integral equations are studied. Stability results are obtained using a representation for the solution in terms of a fundamental solution (a generalization of the fundamental matrix in the theory of ordinary differential equations).

Criteria for stability in terms of the fundamental solution are established for the general linear equation under consideration. A nonlinear perturbed equation is studied and results concerning the preservation of stabilities from the linear to the perturbed equation are given. Lipschitz and little o type nonlinearities are considered.

The general results are then applied to several special equations and conditions for various stabilities are given in terms of the kernels in these equations. The results established may also be used to study stability of the characteristic value problem for hyperbolic partial differential equations.

CHAPTER 1

INTRODUCTION

The goal of this dissertation is to define and study various types of Lyapunov stability for a class of systems of Volterra integral equations in several independent variables. The main results are concerned with preservation of stability from a linear equation to a perturbed nonlinear equation.

The initial value problem for ordinary differential equations

$$\begin{cases} u'(x) = f(x,u) \\ u(x_0) = u_0 \end{cases} \tag{1.1}$$

is equivalent to the Volterra integral equation

$$u(x) = u_0 + \int_{x_0}^{x} f(\tau,u(\tau))d\tau.$$

The general Volterra equation in one independent variable

$$u(x) = \phi(x) + \int_{x_0}^{x} h(x,\tau,u(\tau))d\tau \tag{1.2}$$

may then be thought of as a generalization of the initial value problem (1.1). This has motivated recent papers by Bownds and Cushing [5], [6], [7], [8] in which they generalized much of the stability theory for the initial value problem (1.1) to the integral equation (1.2).

In a sense, the mixed partial derivative is the natural generalization of ordinary derivative [39, pp. 147-148]. From this point of view the hyperbolic partial differential equation of the form

$$u_{xy} = g(x,y,u)$$

is the natural two dimensional generalization of the ordinary differential equation

$$u' = f(x,u).$$

Many authors (e.g., [15], [17], [39]) have been guided by this analogy, and the theory for the characteristic value problem

$$\begin{cases} u_{xy} = g(x,y,u) \\ u(x,y_0) = \phi_1(x) \quad \phi_1(x_0) = \phi_2(y_0) \\ u(x_0,y) = \phi_2(y) \end{cases} \tag{1.3}$$

parallels the theory for the initial value problem (1.1) in many respects. This suggests considering stability questions for (1.3). But in the same way that the initial value problem for ordinary differential equations is equivalent to an integral equation, the characteristic value problem (1.3) is equivalent to the integral equation

$$u(x,y) = \phi_1(x) + \phi_2(y) - \phi_1(x_0) + \int_{x_0}^{x}\int_{y_0}^{y} g(r,s,u(r,s))dsdr.$$

But this is just a special case of the general integral equation in two independent variables

$$u(x,y) = \psi(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} k(x,y,r,s,u(r,s))dsdr. \qquad (1.4)$$

Keeping in mind the relation between $u'(x)$ and $u_{xy}(x,y)$, and the progress made by Bownds and Cushing in generalizing from (1.1) to (1.2), one is then led naturally to stability considerations for the equation (1.4).

The Volterra equation to be considered here is a generalization of (1.4) in several respects, but the motivation for considering this equation stems ultimately from the initial value problem (1.1) and, as indicated in the previous paragraphs, noting the analogies and generalizations of this problem. It will be seen that the stability results obtained here are consistent in the sense that many of the results due to Bownds and Cushing and the results from ordinary differential equations follow as special cases.

One approach to stability problems for nonlinear ordinary differential equations and nonlinear Volterra integral equations in one independent variable is the following. Consider a linear equation which is stable in some sense. Suppose a nonlinear perturbation is now introduced. We study the linear equation and the perturbation to obtain results of the following general form: if the stability possessed by the linear equation is strong enough and the nonlinear perturbation is small in some sense then the nonlinear equation is also stable in some sense. This is the approach employed in this dissertation.

In this approach the concept of a fundamental solution plays a major role. This is true for two reasons:

i) the original nonlinear integral equation may be replaced by an equivalent integral equation involving the fundamental solution and

ii) the strength of the stability assumed on the linear equation may be expressed in terms of the fundamental solution.

A fundamental solution will be defined for the Volterra integral equation under consideration. It will be seen that it is a generalization of the fundamental solution used by Bownds and Cushing [5] for Volterra integral equations in one variable which is in turn a generalization of the fundamental matrix in the theory of ordinary differential equations.

Also, we will see that it is, under certain conditions, a generalization of the classical Riemann function of the adjoint operator associated with a hyperbolic operator.

We give a brief outline of the organization of the dissertation. The remainder of this chapter is dedicated to setting down a concise notation, introducing the equations to be studied, and defining the stabilities for these equations. Then the mathematical results needed to establish the stability results are considered. These consist primarily of the equivalent integral equation in terms of the fundamental solution and a Gronwall type inequality. The stability results for the general linear equation and the perturbed equation are then established. These stability criteria for the general equation are couched in terms of the fundamental solution. We then consider some special equations where it is possible to establish, via the fundamental solution, stability results directly in terms of the kernels in the equations. Finally, an application for partial differential equations is discussed.

1.1 Notation

Much of the analysis will be simplified by introducing the notation which follows. Unless otherwise specified this notation will be adopted for the remainder of the dissertation.

We shall use $x = (x_1,x_2,\ldots,x_n) \in R^n$ where R denotes the set of real numbers. If $x, y \in R^n$ then $x \leq y$ iff $x_i \leq y_i$ for $i = 1, 2, \ldots, n$. If $a, b \in R^n$, then $[a,b]$ will denote the set $[a,b] = \{x | x \in R^n, a \leq x \leq b\}$. If $\bar{a} \in R^n$, then $[\bar{a},\infty) = \{x | x \in R^n, \bar{a} \leq x < \infty\}$.

Various norms will be used and those used most often are listed as follows:

$\lvert\cdot\rvert$	an arbitrary vector norm on R^n
$\lvert\cdot\rvert_1$	the ℓ_1 norm on R^n (i.e., if $x \in R^n$ then $\lvert x\rvert_1 = \sum_{i=1}^{n} \lvert x_i\rvert$)
$\lvert\cdot\rvert_\infty$	the ℓ_∞ norm on R^n (i.e., if $x \in R^n$ then $\lvert x\rvert_\infty = \max\{\lvert x_1\rvert,\lvert x_2\rvert,\ldots,\lvert x_n\rvert\}$)
$\lVert\cdot\rVert_{0,a}$	if $a \in R^n$ and $g:[a,\infty) \to R^m$ so that g is bounded, then $\lVert g\rVert_{0,a} = \sup_{x \geq a} \lvert g(x)\rvert$
$\lVert\cdot\rVert$	the matrix norm such that if M is an $m \times n$ complex matrix then $\lVert M\rVert = \sup\{\lvert Mx\rvert \,\vert\, x \in R^n, \lvert x\rvert = 1\}$.

Let α_k denote a combination of the integers $\{1,2,\ldots,n\}$ taken k at a time. Suppose a particular combination of k elements of $\{1,2,\ldots,n\}$ has been selected. The elements in this combination may always be ordered. It will be convenient to assume that this is always done. Thus, if $\alpha_k = \{i_1,i_2,\ldots,i_k\}$ is a combination of $\{1,2,\ldots,n\}$ then $i_1 < i_2 < \ldots < i_k$. For each combination α_k, we let $\alpha_k' = \{1,2,\ldots,n\} - \alpha_k$. We note that α_k' is also a combination of the integers $\{1,2,\ldots,n\}$ and we may assume the elements in α_k' to be ordered.

Let $\alpha_k = \{i_1,i_2,\ldots,i_k\}$ be any combination. For $x \in R^n$ we define $x_{\alpha_k} = (x_{i_1},x_{i_2},\ldots,x_{i_k})$. We denote the multiple integral symbol $\int_{a_{i_1}}^{x_{i_1}}\int_{a_{i_2}}^{x_{i_2}} \cdots \int_{a_{i_k}}^{x_{i_k}}$ by $\int_{a_{\alpha_k}}^{x_{\alpha_k}}$, and the sequence of differentials $dr_{i_k}dr_{i_{k-1}} \ldots dr_{i_1}$ by dr_{α_k}. If $g:R^n \to R^m$ we define $g_{x_{i_1}x_{i_2}\cdots x_{i_k}}(x) = g_{x_{\alpha_k}}(x)$ and by a pure mixed partial of $g(x)$ we mean a partial of the form $g_{x_{\alpha_k}}(x)$ for some α_k with $1 \le k \le n$. For x, $y \in R^n$ we introduce the following: let

$$w_i(x,y;\alpha_k) = \begin{cases} x_i & i \notin \alpha_k \\ y_i & i \in \alpha_k \end{cases} \quad \text{and}$$

$w(x,y;\alpha_k) = (w_1(x,y;\alpha_k), w_2(x,y;\alpha_k), \ldots, w_n(x,y;\alpha_k))$. For example, suppose $n = 5$, $\alpha_3 = \{1,3,5\}$, $x = (x_1,x_2,x_3,x_4,x_5)$, and $y = (y_1,y_2,y_3,y_4,y_5)$ then $w(x,y;\alpha_3) = (y_1,x_2,y_3,x_4,y_5)$ and $w(y,x;\alpha_3) = (x_1,y_2,x_3,y_4,x_5)$.

1.2 The Volterra Equation and Definitions of Stability

In this section the general form of the Volterra equation will be given. The general equation will be written out without the notation introduced in Section 1.1 and then written concisely using that notation.

Let $a \in R^n$ and $u(x)$, $\phi(x)$ be functions from R^n to R^m. For each combination $\alpha_k = \{i_1, i_2, \ldots, i_k\}$ let $h_{i_1 i_2 \ldots i_k}(x_1, x_2, \ldots, x_n, r_{i_1}, r_{i_2}, \ldots, r_{i_k}, z)$ map $x \geq a$, $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k}$, and $z \in R^m$ to R^m. We shall then consider the equation of the form

$$u(x_1, x_2, \ldots, x_n) = \phi(x_1, x_2, \ldots, x_n)$$

$$+ \sum_{i_1=1}^{n} \int_{a_{i_1}}^{x_{i_1}} h_{i_1}(x_1, x_2, \ldots, x_n, r_{i_1}, \tag{1.5}$$

$$u(x_1, x_2, \ldots, x_{i_1-1}, r_{i_1}, x_{i_1+1}, \ldots, x_n))dr_{i_1}$$

$$+ \sum_{i_1=1}^{n-1} \sum_{i_2=i_1+1}^{n} \int_{a_{i_1}}^{x_{i_1}} \int_{a_{i_2}}^{x_{i_2}} h_{i_1 i_2}(x_1, x_2, \ldots, x_n, r_{i_1}, r_{i_2}, u(x_1, x_2, \ldots,$$

$$x_{i_1-1} r_{i_1}, x_{i_1+1}, \ldots, x_{i_2-1}, r_{i_2}, x_{i_2+1}, \ldots, x_n) dr_{i_2} dr_{i_1}$$

$$+$$

$$\vdots$$

$$+ \sum_{i_1=1}^{n-k+1} \sum_{i_2=i_1+1}^{n-k+2} \cdots \sum_{i_k=i_{k-1}+1}^{n} \int_{a_{i_1}}^{x_{i_1}} \int_{a_{i_2}}^{x_{i_2}}$$

$$\cdots \int_{a_{i_k}}^{x_{i_k}} h_{i_1 i_2 \ldots i_k}(x_1, x_2, \ldots, x_n, r_{i_1}, r_{i_2}, \ldots, r_{i_k}, u(x_1, \ldots,$$

$$x_{i_1-1}, r_{i_1}, x_{i_1+1}, \ldots, x_{i_k-1}, r_{i_k}, x_{i_k+1}, \ldots, x_n))$$

$$dr_{i_k} dr_{i_{k-1}} \ldots dr_{i_1}$$

$$+$$
$$\vdots$$

$$+ \int_{a_1}^{x_1} \int_{a_2}^{x_2} \cdots \int_{a_n}^{x_n} h_{12\ldots n}(x_1, x_2, \ldots, x_n, r_1, r_2, \ldots, r_n,$$

$$u(r_1, r_2, \ldots, r_n)) dr_n dr_{n-1} \ldots dr_1 .$$

This is a system of m Volterra integral equations in n independent variables. We notice that there are $2^n - 1$ integrals in the right hand side of Equation (1.5) by observing that the number of integrals in the term involving $\sum_{i_1=1}^{n-k+1} \sum_{i_2=i_1+1}^{n-k+2} \cdots \sum_{i_k=i_{k-1}+1}^{n}$ is $\binom{n}{k}$ and that $\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$.

For each combination $\alpha_k = \{i_1, i_2, \ldots, i_k\}$, let the function $h_{i_1 i_2 \ldots i_k}$ be denoted by h_{α_k}. Then using the notation introduced in Section 1.1, we have

$$\sum_{i_1=1}^{n-k+1} \sum_{i_2=i_1+1}^{n-k+2} \cdots \sum_{i_k=i_{k-1}+1}^{n} \int_{a_{i_1}}^{x_{i_1}} \int_{a_{i_2}}^{x_{i_2}} \cdots \int_{a_{i_k}}^{x_{i_k}} h_{i_1 i_2 \ldots i_k}(x_1,\ldots,$$

$$x_n, r_{i_1},\ldots,r_{i_k}, u(x_1,\ldots,x_{i_1-1}, r_{i_1}, x_{i_1+1},\ldots,$$

$$x_{i_k-1}, r_{i_k}, x_{i_k+1},\ldots,x_n))dr_{i_k} dr_{i_{k-1}} \ldots dr_{i_1}$$

$$= \sum_{\alpha_k} \int_{a_{\alpha_k}}^{x_{\alpha_k}} h_{\alpha_k}(x, r_{\alpha_k}, u(w(x,r;\alpha_k)))dr_{\alpha_k}$$

(Here the summation ranges over all combinations α_k of k of the integers $\{1,2,\ldots,n\}$). Equation (1.5) may now be written in the form

$$u(x) = \phi(x) + \sum_{\substack{\alpha_k \\ 1 \leq k \leq n}} \int_{a_{\alpha_k}}^{x_{\alpha_k}} h_{\alpha_k}(x, r_{\alpha_k}, u(w(x,r;\alpha_k)))dr_{\alpha_k}. \quad (1.6)$$

We simplify the notation further by using $\sum$ in place of the particular summation $\sum_{\substack{\alpha_k \\ 1 \leq k \leq n}}$ except in instances where use of the latter may help to clarify a discussion. Also, we will sometimes refer to the function $\phi(x)$ in Equation (1.6) as the initial function for that equation.

We will now turn to defining stability for a solution of Equation (1.6). Let $\bar{a} \in R^n$ be fixed. Let N denote a normed space of functions mapping $x \geq \bar{a}$ to R^m and denote the norm on N by $\|\cdot\|$. The normed space $(N, \|\cdot\|)$ will be referred to simply as N.

Definition 1.1. Let $\phi(x) \in N$ and let $u(x)$ be a solution of Equation (1.6) for $x \geq a \geq \bar{a}$ corresponding to $\phi(x)$. The solution $u(x)$ is stable on the space N if for each $a \geq \bar{a}$ and any $\varepsilon > 0$ there exists a $\delta(a,\varepsilon)$ such that if $\hat{\phi} \in N$ and $\|\hat{\phi} - \phi\| < \delta(a,\varepsilon)$ then any solution $\hat{u}(x)$ of Equation (1.6) corresponding to $\hat{\phi}(x)$ exists on $x \geq a$ and satisfies $\|\hat{u} - u\|_{0,a} < \varepsilon$.

Definition 1.2. Let $\phi(x) \in N$ and let $u(x)$ be a solution of Equation (1.6) for $x \geq a \geq \bar{a}$ corresponding to $\phi(x)$. The solution $u(x)$ is asymptotically stable on the space N if we have the following:

i) Equation (1.6) is stable on N

ii) for each $a \geq \bar{a}$ there exists a $\delta(a) > 0$ such that if $\hat{\phi} \in N$ and $\|\hat{\phi} - \phi\| < \delta(a)$ then any solution $\hat{u}(x)$ of Equation (1.6) corresponding to $\hat{\phi}(x)$ exists for $x \geq a$ and satisfies $\lim_{|x|\to\infty} |\hat{u}(x) - u(x)| = 0$.

We point out that part ii) of Definition 1.2 means that for each $\varepsilon > 0$ there is a $T = T(a,\varepsilon,\hat{\phi})$ so that if $\|\hat{\phi} - \phi\| < \delta(a)$ and $x \geq T(a,\varepsilon,\hat{\phi})$ then $|\hat{u}(x) - u(x)| < \varepsilon$. In particular, T may depend on $\hat{\phi}$. This dependence does not occur in asymptotic stability for ordinary differential equations and has led to consideration of a distinct type asymptotic stability for integral equations in which T does not depend on $\hat{\phi}$. For further discussion in the case $n = 1$ see [7]. The author is interested in this distinction and hopes to investigate the distinction for Equation (1.6) at some future time.

Since all vector norms on R^p are equivalent, the norm on $x \in R^n$ and the norm on $\hat{u}(x) - u(x) \in R^m$ in part ii) of Definition 1.2 need not be the same norm.

<u>Definition 1.3</u>. Let $\phi(x) \in N$ and $u(x)$ be a solution of Equation (1.6) for $x \geq a \geq \bar{a}$. corresponding to $\phi(x)$. The solution $u(x)$ is <u>uniformly stable on the space N</u> if given any $\varepsilon > 0$ and any $a \geq \bar{a}$ there exists a $\delta(\varepsilon)$ such that if $\hat{\phi}(x) \in N$ and $\|\hat{\phi} - \phi\| < \delta(\varepsilon)$ then any solution $\hat{u}(x)$ of Equation (1.6) corresponding to $\hat{\phi}(x)$ exists for $x \geq a$ and satisfies $\|\hat{u} - u\|_{0,a} < \varepsilon$.

We will occasionally use the abbreviations A.S. for "asymptotic stability" (or "asymptotically stable") and U.S. for "uniform stability" (or "uniformly stable").

The distinction between Definition 1.1 and 1.3 is that the δ in Definition 1.3 is independent of $a \geq \bar{a}$. We will see that the concept of uniform stability for Equation (1.6) on the space R^m plays a central role in preservation of stability results.

We note that in the case $n = 1$, Equation (1.6) becomes

$$u(x_1) = \phi(x_1) + \int_{a_1}^{x_1} h_1(x_1, r_1, u(r_1))dr_1$$

and the Definitions 1.1-1.3 reduce to the definitions given by Bownds and Cushing [5]. If $h_1(x_1, r_1, z)$ is independent of x_1, and if the space N in these definitions is R^m, then these definitions coincide with the stability definitions for ordinary differential equations [16].

We now list the function spaces on which the various stabilities will be studied.

$N_0 = \{\phi(x) \,|\, \phi: [\bar{a}, \infty) \to R^m,\ \phi$ is continuous and bounded$\}$. The norm to be used on N_0 is $\|\cdot\|_{0,\bar{a}}$.

$N_1 = \{\phi(x) \,|\, \phi: [\bar{a}, \infty) \to R^m,\ \phi$ is continuous and bounded, $\phi_{x_{\alpha_k}}$ is continuous and bounded for each α_k, with $1 \leq k \leq n\}$. The space N_1 will be normed by $\|\phi\| = \|\phi\|_{0,\bar{a}} + \sum_{\substack{\alpha_k \\ 1 \leq k \leq n}} \|\phi_{x_{\alpha_k}}\|_{0,\bar{a}}$

$N_2 = \{\phi(x) \,|\, \phi: [\bar{a}, \infty) \to R^m,\ \phi$ is continuous and bounded, $\phi_{x_{\alpha_k}}$ is continuous for each α_k with $1 \leq k \leq n$,

$\int_{\bar{a}_{\alpha_k}}^{\infty} \sup_{\bar{a}_{\alpha_k'} \leq x_{\alpha_k'} < \infty} |\phi_{x_{\alpha_k}}(x)| dx_{\alpha_k} < \infty\}$. N_2 will be normed by

$$\|\phi\| = \|\phi\|_{0,\bar{a}} + \sum_{\substack{\alpha_k \\ 1\le k\le n}} \int_{\bar{a}_{\alpha_k}}^{\infty} \sup_{\bar{a}_{\alpha_k'} \le x_{\alpha_k'} < \infty} |\phi_{x_{\alpha_k}}(x)|\,dx_{\alpha_k}.$$

$N_3 = \{\phi(x)\,|\,\phi:[\bar{a},\infty) \to R^m,\ \phi(x)$ is constant$\}$. N_3 will have the norm $\|\phi\| = |\phi|$.

The sup appearing in the definition of the norm on N_2 is necessary, since otherwise the integrals would be functions of $x_{\alpha_k'}$ and $\|\cdot\|$ would not be a norm. Also, we note $N_0 \supset N_2 \supset N_3$.

We will now be concerned with stability results on various spaces for equations of the form

$$u(x) = \phi(x) + \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})u(w(x,r;\alpha_k))dr_{\alpha_k} \tag{1.7}$$

and

$$u(x) = \phi(x) + \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})u(w(x,r;\alpha_k))dr_{\alpha_k}$$

$$+ \int_a^x f(x,r,u(r))dr \tag{1.8}$$

where u and ϕ map $x \ge \bar{a}$ to R^m, K_{α_k} is an $m \times m$ matrix function on $x \ge \bar{a}$ and $\bar{a}_{\alpha_k} \le r_{\alpha_k} \le x_{\alpha_k}$, and f maps $x \ge \bar{a}$, $\bar{a} \le r \le x$, R^m to R^m. It will be sufficient to assume $f(x,r,0) \equiv 0$, and to study the stability of the solution $u \equiv 0$ corresponding to $\phi \equiv 0$. This is true since a change of variables involving a particular solution under investigation will result in an equation of the form (1.8) in which the function f satisfies $f(x,r,0) \equiv 0$. We will then say Equation (1.8) is stable (A.S., or U.S.) when the solution $u \equiv 0$ is stable (A.S., or U.S.). The terminology "Equation (1.8) preserves stability (A.S., or U.S.) on N" means that if Equation (1.7) is stable (A.S., or U.S.) on N then Equation (1.8) is also stable (A.S., or U.S.) on N.

CHAPTER 2

BASIC MATHEMATICAL TOOLS

In this chapter we obtain several results that will be needed in establishing our stability results. Although developed here for the express purpose of studying stability, some of these results are interesting generalizations of known results and hold promise for application in other areas.

We include a section on existence and uniqueness not so much as a tool, but rather so we may concentrate on the other qualitative criteria set down in the stability definitions.

2.1 Existence and Uniqueness

It is not the goal of this section to give the most general conditions under which Equation (1.6) has a unique global solution, but to show there are reasonable conditions insuring global existence and uniqueness. The reader is referred to a paper by Suryanarayana [36] for other results pertaining to global existence and uniqueness for Equation (1.6).

We will be concerned in this section, and in the remainder of the dissertation, with continuous solutions for the Equation (1.6). The existence and uniqueness theorem is obtained using Banach's contraction principle and the following lemmas will be useful.

Lemma 2.1. Let Ω be a region in R^n. Let $g:\Omega \to R$ satisfy $0 < m \leq g(x) \leq M$ for all $x \in \Omega$. Let $C(\Omega) = \{\phi(x) | \phi:\Omega \to R^m,\ \phi$ is continuous and bounded on $\Omega\}$. Then the function $\|\cdot\|_g : C(\Omega) \to R$ defined by $\|\phi\|_g = \sup_{x\in\Omega} |\phi(x)|g(x)$ is a norm on $C(\Omega)$ and it is equivalent to the norm $\|\phi\|_0 = \sup_{x\in\Omega} |\phi(x)|$. Therefore $(C(\Omega), \|\cdot\|_g)$ is a Banach space.

Proof. Let $\overline{0}$ denote the function in $C(\Omega)$ mapping all $x \in \Omega$ to the zero element of R^m. Since $|\overline{0}(x)| = 0$, we have $\|\overline{0}\|_g = \sup_{x\in\Omega} |\overline{0}(x)|g(x) = 0$. Now suppose $\|\phi\|_g = 0$. Then $\sup_{x\in\Omega} |\phi(x)|g(x) = 0$, and hence $|\phi(x)|g(x) \equiv 0$. But $g(x) \geq m > 0$ implies $|\phi(x)| \equiv 0$. Therefore $\phi = \overline{0}$. Take any $c \in R$ and $\phi \in C(\Omega)$. Then $\|c\phi\|_g = \sup_{x\in\Omega} |c\phi(x)|g(x) = |c| \sup_{x\in\Omega} |\phi(x)|g(x) = |c|\|\phi\|_g$. For any ϕ_1, $\phi_2 \in C(\Omega)$ we have $\|\phi_1 + \phi_2\|_g = \sup_{x\in\Omega} |\phi_1(x) + \phi_2(x)|g(x) \leq \sup_{x\in\Omega} (|\phi_1(x)| + |\phi_2(x)|)g(x) \leq \sup_{x\in\Omega} |\phi_1(x)|g(x) + \sup_{x\in\Omega} |\phi_2(x)|g(x) = \|\phi_1\|_g + \|\phi_2\|_g$. Thus $\|\cdot\|_g$ is a norm on $C(\Omega)$. Take any $\phi \in C(\Omega)$. Using the hypothesis on the function $g(x)$ we have $m|\phi(x)| \leq |\phi(x)|g(x) \leq M|\phi(x)|$. Hence $m \sup_{x\in\Omega} |\phi(x)| \leq \sup_{x\in\Omega} |\phi(x)|g(x)$

$\leq M \sup_{x \in \Omega} |\phi(x)|$ and $m\,\|\phi\|_0 \leq \|\phi\|_g \leq M\|\phi\|_0$. Thus, the norms $\|\cdot\|_g$ and $\|\cdot\|_0$ are equivalent on $C(\Omega)$ and, since $(C(\Omega),\|\cdot\|_0)$ is a Banach space, so is $(C(\Omega),\|\cdot\|_g)$. This completes the proof.

Remark 2.1. We will use the following easily established fact. Suppose $g:[a,b] \to R$. Suppose (x_k) is any sequence in $[a,b]$ so that $\lim_{k\to\infty} x_k = y$ with $x_k \neq y$ for each k. Then $\lim_{x\to y} g(x) = L$ if and only if $\lim_{k\to\infty} g(x_k) = L$.

We will also use the following form of the dominated convergence theorem to establish the continuity of certain integrals which will be of interest to us.

Dominated Convergence Theorem [18, p. 195]. Let E be a bounded measurable subset of R^n having finite measure. Suppose $g_k:E \to R$ so that g_k is measurable for each $k = 1, 2, \ldots$. Suppose

i) $\lim_{k\to\infty} g_k(x) = g(x)$ almost everywhere in E.

ii) there is a constant M so that $|g_k(x)| \leq M$ for all $x \in E$ and $k = 1, 2, \ldots$.

Then $\lim_{k\to\infty} \int_E g_k(x)dV = \int_E g(x)dV$.

We now use the facts above to prove the following lemma.

<u>Lemma 2.2</u>. Suppose $h(x,r_{\alpha_k})$ has values in R^m and is defined for $x \in [a,b]$, $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$. Suppose for each $x \in [a,b]$, $h(x,r_{\alpha_k})$ is continuous in r_{α_k} with $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$, and for each r_{α_k} with $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$, $h(x,r_{\alpha_k})$ is continuous in x where $r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$, $a_{\alpha'_k} \leq x_{\alpha'_k} \leq b_{\alpha'_k}$. Suppose there is a constant M such that $|h(x,r_{\alpha_k})| \leq M$ for $x \in [a,b]$, $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$. Then the function $I(x) = \int_{a_{\alpha_k}}^{x_{\alpha_k}} h(x,r_{\alpha_k})dr_{\alpha_k}$ is continuous on $[a,b]$.

<u>Proof</u>. Let $\alpha_k = \{i_1,i_2,\ldots,i_k\}$ and $h(x,r_{\alpha_k}) = (h^1(x,r_{\alpha_k}),\ h^2(x,r_{\alpha_k}),\ \ldots,\ h^m(x,r_{\alpha_k}))$. Take any j so that $i \leq j \leq m$ and consider the function $I_j(x) = \int_{a_{\alpha_k}}^{x_{\alpha_k}} h^j(x,r_{\alpha_k})dr_{\alpha_k}$. The vector function $I(x)$ will be continuous if $I_j(x)$ is continuous for each j.

We define the function $\bar{h}^j(x,r_{\alpha_k})$ for $x \in [a,b]$, $a_{\alpha_k} \leq r_{\alpha_k} \leq b_{\alpha_k}$ by

$$\bar{h}^j(x,r_{\alpha_k}) = \begin{cases} h^j(x,r_{\alpha_k}) & x \in [a,b],\ a_{\alpha_k} \le r_{\alpha_k} \le x_{\alpha_k} \le b_{\alpha_k} \\ 0 & x \in [a,b],\ r_i > x_i \text{ for some } i \in \alpha_k \end{cases} .$$

For each $x \in [a,b]$, $h(x,r_{\alpha_k})$ is continuous in r_{α_k} for $a_{\alpha_k} \le r_{\alpha_k} \le x_{\alpha_k}$. Therefore, $h^j(x,r_{\alpha_k})$ is continuous in r_{α_k} for each $x \in [a,b]$ with $a_{\alpha_k} \le r_{\alpha_k} \le x_{\alpha_k}$. Then for each $x \in [a,b]$, the function $\bar{h}^j(x,r_{\alpha_k})$ is discontinuous on the set

$$E_{\alpha_k}(x) = \bigcup_{p=1}^{k} \{(r_{i_1},r_{i_2},\ldots,r_{i_p-1},x_{i_p},r_{i_p+1},\ldots r_{i_k}) \mid$$

$$a_{i_m} \le r_{i_m} \le x_{i_m}, m = 1,2,\ldots,p-1,p+1,\ldots,k\}.$$

For each $x \in [a,b]$ each set in the union forming $E_{\alpha_k}(x)$ is a subset of a $k-1$ dimensional plane. Thus, if V_k is the k-dimensional Lebesque measure on R^k, we have $V_k(E_{\alpha_k}) = 0$. Therefore, for each $x \in [a,b]$, the function $\bar{h}^j(x,r_{\alpha_k})$ is measurable in r_{α_k} for $a_{\alpha_k} \le r_{\alpha_k} \le b_{\alpha_k}$.

Now take any $\bar{x} \in [a,b]$. We will now fix r_{α_k} and let x approach $\bar{x}$. Suppose we have $r_{\alpha_k} < \bar{x}_{\alpha_k}$. Then, since the function $h^j(x,r_{\alpha_k})$ is continuous in x for each r_{α_k} with $r_{\alpha_k} \le x_{\alpha_k} \le b_{\alpha_k}$ and $a_{\alpha_k'} \le x_{\alpha_k'} \le b_{\alpha_k'}$, we see that $\lim_{x \to \bar{x}} \bar{h}^j(x,r_{\alpha_k}) = \lim_{x \to \bar{x}} h^j(x,r_{\alpha_k}) = h^j(\bar{x},r_{\alpha_k}) = \bar{h}^j(\bar{x},r_{\alpha_k})$. Suppose now r_{α_k} is such that $r_i > \bar{x}_i$ for some $i \in \alpha_k$. Then we have $\lim_{x \to \bar{x}} \bar{h}^j(x,r_{\alpha_k}) = 0 = \bar{h}^j(\bar{x},r_{\alpha_k})$. Finally, r_{α_k} may be in $E_{\alpha_k}(\bar{x})$. If $x \to \bar{x}$ so that $x_{\alpha_k} \ge r_{\alpha_k}$, we then have $\lim_{x \to \bar{x}} \bar{h}^j(x,r_{\alpha_k}) = \bar{h}^j(\bar{x},r_{\alpha_k})$. However, if $x \to \bar{x}$ so that $x_i < r_i$ for some $i \in \alpha_k$, then $\lim_{x \to \bar{x}} \bar{h}^j(x,r_{\alpha_k}) = 0$. Therefore, except on the set $E_{\alpha_k}(\bar{x})$, we have $\lim_{x \to \bar{x}} \bar{h}^j(x,r_{\alpha_k}) = \bar{h}^j(\bar{x},r_{\alpha_k})$.

Now consider any sequence $(x_n) \subset [a,b]$ so that $(x_n) \to \bar{x}$ and $x_n \ne \bar{x}$ for $n = 1, 2, \ldots$. We define the sequence of functions $g_n(r_{\alpha_k}) = \bar{h}^j(x_n,r_{\alpha_k})$ on $[a_{\alpha_k},b_{\alpha_k}]$. By the arguments above, we see that each $g_n(r_{\alpha_k})$ is measurable in r_{α_k} and that $\lim_{n \to \infty} g_n(r_{\alpha_k}) = \bar{h}^j(\bar{x},r_{\alpha_k})$ almost everywhere. Also, from the definition of $\bar{h}^j(x,r_{\alpha_k})$ and the hypothesis, it follows that $|g_n(r_{\alpha_k})| = |\bar{h}^j(x_n,r_{\alpha_k})| \le M$ for all n. Then, using the dominated convergence theorem as given above, we have

$$\lim_{n\to\infty} I_j(x_n) = \lim_{n\to\infty} \int_{a_{\alpha_k}}^{b_{\alpha_k}} \bar{h}^j(x_n, r_{\alpha_k})\,dr_{\alpha_k} = \lim_{n\to\infty} \int_{a_{\alpha_k}}^{b_{\alpha_k}} g_n(r_{\alpha_k})\,dr_{\alpha_k}$$

$$= \int_{a_{\alpha_k}}^{b_{\alpha_k}} \bar{h}^j(\bar{x}, r_{\alpha_k})\,dr_{\alpha_k} = \int_{a_{\alpha_k}}^{\bar{x}_{\alpha_k}} \bar{h}^j(\bar{x}, r_{\alpha_k})\,dr_{\alpha_k} = I_j(\bar{x}).$$

Thus, by Remark 2.1, we see that $\lim_{x\to\bar{x}} I_j(x) = I_j(\bar{x})$ and $I_j(x)$ is continuous on $[a,b]$. Hence, $I(x)$ is continuous on $[a,b]$ and the proof is complete.

The following theorem gives sufficient conditions for existence and uniqueness of a continuous global solution of Equation (1.6).

<u>Theorem 2.1</u>. Suppose for each α_k with $1 \le k \le n$, the functions $h_{\alpha_k}(x, r_{\alpha_k}, z)$ with $x \in [a,b]$, $a_{\alpha_k} \le r_{\alpha_k} \le x_{\alpha_k} \le b_{\alpha_k}$, and $z \in R^m$ map into R^m and satisfy:

i) $h_{\alpha_k}(x, r_{\alpha_k}, z)$ is continuous on its domain

ii) there is a constant $M \ge 0$ such that $|h_{\alpha_k}(x, r_{\alpha_k}, z_1) - h_{\alpha_k}(x, r_{\alpha_k}, z_2)| \le M|z_1 - z_2|$ for all z_1, $z_2 \in R^m$.

Suppose $\phi: [a,b] \to R^m$ continuously. Then Equation (1.6) has a unique continuous solution on $[a,b]$.

<u>Proof</u>. Let $C[a,b] = \{g(x) | g:[a,b] \to R^m$ and g continuous$\}$. Let λ be any positive number such that $\lambda > 1$ and $\frac{M(2^n-1)}{\lambda} < 1$. Consider the norm, denoted by $\|\cdot\|_\lambda$ on $C[a,b]$ such that if $g \in C[a,b]$ then $\|g\|_\lambda = \sup_{x \in [a,b]} |g(x)| \exp[-\lambda(\sum_{i=1}^{n} x_i)]$. The space $(C[a,b], \|\cdot\|_\lambda)$ will be denoted by $C_\lambda[a,b]$. By Lemma 2.1, $C_\lambda[a,b]$ is a Banach space.

Define the map T on $C_\lambda[a,b]$ such that if $g \in C_\lambda[a,b]$ then

$$(Tg)(x) = \phi(x) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} h_{\alpha_k}(x, r_{\alpha_k}, g(w(x,r;\alpha_k)))dr_{\alpha_k}.$$

Since each h_{α_k} is continuous in x, r_{α_k}, and z and g is continuous, we see that $h_{\alpha_k}(x, r_{\alpha_k}, g(w(x,r;\alpha_k)))$ is continuous on the compact set $x \in [a,b]$, $a_{\alpha_k} \le r_{\alpha_k} \le x_{\alpha_k} \le b_{\alpha_k}$ for each α_k. Thus from Lemma 2.2 it follows that the function $\int_{a_{\alpha_k}}^{x_{\alpha_k}} h_{\alpha_k}(x, r_{\alpha_k}, g(w(x,r;\alpha_k)))dr_{\alpha_k}$ is continuous on $[a,b]$ for each α_k. Then since $\phi(x)$ is continuous $T:C_\lambda[a,b] \to C_\lambda[a,b]$.

Take any $g_1, \; g_2 \in C_\lambda[a,b]$.

$$|(Tg_1)(x) - (Tg_2)(x)| \le \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} |h_{\alpha_k}(x,r_{\alpha_k},g_1(w(x,r;\alpha_k)))$$

$$- h_{\alpha_k}(x,r_{\alpha_k},g_2(w(x,r;\alpha_k)))|dr_{\alpha_k}.$$

Using hypothesis ii), we have

$$|(Tg_1)(x) - (Tg_2)(x)| \le \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} M|g_1(w(x,r;\alpha_k))$$

$$- g_2(w(x,r;\alpha_k))|dr_{\alpha_k}$$

$$= M\sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} |g_1(w(x,r;\alpha_k)) - g_2(w(x,r;\alpha_k))|$$

$$\cdot \exp[-\lambda(\sum_{i=1}^{n} w_i(x,r;\alpha_k))]\exp[\lambda(\sum_{i=1}^{n} w_i(x,r;\alpha_k))]dr_{\alpha_k}$$

$$\le M\|g_1 - g_2\|_\lambda \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} \exp[\lambda(\sum_{i=1}^{n} w_i(x,r;\alpha_k))]dr_{\alpha_k}$$

$$\le M\|g_1 - g_2\|_\lambda \sum\frac{1}{\lambda^k}\exp[\lambda(\sum_{i=1}^{n} x_i)].$$

Recalling that $\lambda > 1$, we have

$$|(Tg_1)(x) - (Tg_2)(x)| \le \frac{M\|g_1-g_2\|_\lambda}{\lambda}\sum \exp[\lambda(\sum_{i=1}^{n} x_i)]$$

$$= \frac{M\|g_1-g_2\|_\lambda}{\lambda}\exp[\lambda(\sum_{i=1}^{n} x_i)]\sum 1 = \frac{M(2^n-1)}{\lambda}\|g_1 - g_2\|_\lambda \exp[\lambda(\sum_{i=1}^{n} x_i)].$$

Therefore $|(Tg_1)(x) - (Tg_2)(x)|\exp[-\lambda(\sum_{i=1}^{n} x_i)]$ $\le \frac{M(2^n-1)}{\lambda}\|g_1 - g_2\|_\lambda$ and thus $\|Tg_1 - Tg_2\|_\lambda$ $\le \frac{M(2^n-1)}{\lambda}\|g_1 - g_2\|_\lambda$. But we have taken λ so that $\frac{M(2^n-1)}{\lambda}$ < 1. Therefore T is a contraction on $C_\lambda[a,b]$ and has a unique fixed point in $C_\lambda[a,b]$. Thus Equation (1.6) has a unique continuous solution on $[a,b]$. This completes the proof.

The contraction principle of course has been used by many authors to establish existence and uniqueness for both differential and integral equations. In the usual treatment, a map is defined from the continuous functions normed by the sup norm to itself. Then, even though the hypotheses are sufficient to insure global existence, it is necessary to choose a smaller interval so that the map will be a contraction. In Theorem 2.1 we have avoided this problem by observing that there are many ways in which $C[a,b]$ may be normed to obtain a Banach space, and that T will have a fixed point if it is a contraction on just one of these spaces. We see that $\{C_\lambda(a,b] \mid -\infty < \lambda < \infty\}$ is a family of Banach spaces and that in Theorem 2.1, we have selected one on which T will be a contraction. This approach to global existence and uniqueness for the initial value problem of ordinary differential equations and for a special case of Equation (1.6) has been considered by Bielecki [3], [4]. These ideas have also been given in a different form by Chu and Diaz [12]. Their approach is to show that the map under consideration has a fixed point if and only if a related composition map has a fixed point. Using their approach one has no need to change from sup norm on $C[a,b]$.

We will use Theorem 2.1 in the form of the following corollaries.

<u>Corollary 2.1</u>. Suppose for each combination α_k with $1 \leq k \leq n$ the functions $h_{\alpha_k}(x,r_{\alpha_k},z)$ with $x \geq a$, $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$, and $z \in R^m$ map to R^m and satisfy:

i) $h_{\alpha_k}(x,r_{\alpha_k},z)$ is continuous on its domain

ii) there are continuous scalar functions $\gamma_{\alpha_k}(x,r_{\alpha_k}) \geq 0$ defined for $x \geq a$, $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$ such that $|h_{\alpha_k}(x,r_{\alpha_k},z_1) - h_{\alpha_k}(x,r_{\alpha_k},z_2)| \leq \gamma_{\alpha_k}(x,r_{\alpha_k})|z_1 - z_2|$ for all z_1, $z_2 \in R^m$.

Suppose $\phi(x)$ is continuous for $x \geq a$. Then Equation (1.6) has a unique continuous solution on $x \geq a$.

Proof. Suppose this is not true. Then there is a point $b \in R^n$ with $b \geq a$ so that on $[a,b]$ Equation (1.6) fails to have a solution or has more than one solution. But the hypotheses of Corollary 2.1 insure the hypotheses of Theorem 2.1 so there must be a unique continuous solution on $[a,b]$. This contradiction completes the proof.

For the linear equation we obtain the following corollary directly.

Corollary 2.2. Suppose for each α_k with $1 \leq k \leq n$ the $m \times m$ matrix functions $K_{\alpha_k}(x, r_{\alpha_k})$ are continuous for $x \geq a$ and $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$. Suppose $\phi(x)$ is continuous for $x \geq a$. Then Equation (1.7) has a unique continuous solution for $x \geq a$.

2.2 The Fundamental Solution and Representation Theorems

We now turn our attention specifically to the Equations (1.7) and (1.8). As mentioned previously, the concept of a fundamental solution is important for our stability analysis. Here we will introduce the fundamental solution and use it to establish an integral representation for the solution of Equation (1.7). We will also use the fundamental solution to obtain an integral equation which is equivalent to Equation (1.8).

The following lemma will be useful.

Lemma 2.3. Suppose $K(x)$ is an $m \times m$ matrix function on $x \geq a$ and $g(x)$ maps $x \geq a$ to R^m. Suppose both functions have continuous pure mixed partials of all orders less than or equal to n on $x \geq a$. Then

$$\int_a^x K(r)g_r(r)dr = K(x)g(x) - K(a)g(a)$$

$$- \sum_{\substack{\alpha_k \\ 0 \leq k \leq n-1}} \int_{a_{\alpha_k}}^{x_{\alpha_k}} K(w(a,r;\alpha_k))g_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k}$$

$$+ \sum_{\substack{\alpha_k \\ 1 \leq k \leq n}} (-1)^k \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{r_{\alpha_k}}(w(x,r;\alpha_k))g(w(x,r;\alpha_k)dr_{\alpha_k}. \qquad (2.1)$$

Proof. The proof is by induction. In the case $n = 1$, Equation (2.1) is just the ordinary integration by parts formula,

$$\int_{a_1}^{x_1} K(r_1)g_{r_1}(r_1)dr_1 = K(x)g(x) - K(x_1)g(x_1) - \int_{a_1}^{x_1} K_{r_1}(r_1)g(r_1)dr_1.$$

Now assume the result is true for any n. We now consider

x, a, $r \in R^{n+1}$ such that $a \leq r \leq x$. We also introduce the following: if $r = (r_1, r_2, \ldots, r_n, r_{n+1})$ then $\bar{r} = (r_2, r_3, \ldots, r_n, r_{n+1})$, if $x = (x_1, x_2, \ldots, x_{n+1})$ then $\bar{x} = (x_2, x_3, \ldots, x_n, x_{n+1})$, and if $a = (a_1, a_2, \ldots, a_{n+1})$ then $\bar{a} = (a_2, a_3, \ldots, a_n, a_{n+1})$. We notice then that $r = (r_1, \bar{r})$, $x = (x_1, \bar{x})$, and $a = (a_1, \bar{a})$. With this notation we consider

$$\int_a^x K(r) g_r(r)\,dr = \int_{a_1}^{x_1} [\int_{\bar{a}}^{\bar{x}} K(r_1, \bar{r}) g_{r_1\bar{r}}(r_1, \bar{r})\,d\bar{r}]\,dr_1. \qquad (2.2)$$

Let β_k denote any combination of the n integers $\{2, 3, \ldots, n, n+1\}$ taken k at a time. Once agin if $\beta_k = \{i_1', i_2', \ldots, i_k'\}$, we will assume $i_1' < i_2' < \ldots < i_k'$. Now using the induction hypothesis in Equation (2.2) we have

$$\int_a^x K(r) g_r(r)\,dr = \int_{a_1}^{x_1} [K(r_1, \bar{x}) g_{r_1}(r_1, \bar{x}) - K(r_1, \bar{a}) g_{r_1}(r_1, \bar{a})$$

$$- \sum_{\substack{\beta_k \\ 0 \leq k \leq n-1}} \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K(r_1, w(\bar{a}, \bar{r}; \beta_k)) g_{r_1\bar{r}_{\beta_k}}(r_1, w(\bar{a}, \bar{r}; \beta_k))\,d\bar{r}_{\beta_k}$$

$$+ \sum_{\substack{\beta_k \\ 1 \leq k \leq n}} (-1)^k \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(r_1, w(\bar{x}, \bar{r}; \beta_k)) g_{r_1}(r_1, w(\bar{x}, \bar{r}; \beta_k))\,d\bar{r}_{\beta_k}]\,dr_1$$

$$= \int_{a_1}^{x_1} K(r_1, \bar{x}) g_{r_1}(r_1, \bar{x})\,dr_1 - \int_{a_1}^{x_1} K(r_1, \bar{a}) g_{r_1}(r_1, \bar{a})\,dr_1$$

$$- \sum_{\substack{\beta_k \\ 0\le k\le n-1}} \int_{a_1}^{x_1}\int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K(r_1,w(\bar{a},\bar{r};\beta_k))\, g_{r_1\bar{r}_{\beta_k}}(r_1,w(\bar{a},\bar{r};\beta_k))\, d\bar{r}_{\beta_k}\, dr_1$$

$$+ \sum_{\substack{\beta_k \\ 1\le r\le n}} (-1)^k \int_{a_1}^{x_1}\int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(r_1,w(\bar{x},\bar{r};\beta_k))\, g_{r_1}(r_1,w(\bar{x},\bar{r};\beta_k))\, d\bar{r}_{\beta_k}\, dr_1 \tag{2.3}$$

For each β_k, using Fubini's theorem (see Appendix A, or [32]) and the integration by parts formula, we obtain

$$\int_{a_1}^{x_1}\int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(r_1,w(\bar{x},\bar{r};\beta_k))\, g_{r_1}(r_1,w(\bar{x},\bar{r};\beta_k))\, d\bar{r}_{\beta_k}\, dr_1$$

$$= \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(x_1,w(\bar{x},\bar{r};\beta_k))\, g(x_1,w(\bar{x},\bar{r};\beta_k))\, d\bar{r}_{\beta_k}$$

$$- \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(a_1,w(\bar{x},\bar{r};\beta_k))\, g(a_1,w(\bar{x},\bar{r};\beta_k))\, d\bar{r}_{\beta_k}$$

$$- \int_{a_1}^{x_1}\int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{r_1\bar{r}_{\beta_k}}(r_1,w(\bar{x},\bar{r};\beta_k))\, g(r_1,w(\bar{x},\bar{r};\beta_k))\, d\bar{r}_{\beta_k}\, dr_1 \tag{2.4}$$

Using Equation (2.4) and

$$\int_{a_1}^{x_1} K(r_1,\bar{x})g_{r_1}(r_1,\bar{x})dr_1 = K(x)g(x) - K(a_1,\bar{x})g(a_1,\bar{x}) - \int_{a_1}^{x_1} K_{r_1}(r_1,\bar{x})g(r_1,\bar{x})dr_1$$

in Equation (2.3) we see that

$$\int_a^x K(r)g_r(r)dr = K(x)g(x) - K(a_1,\bar{x})g(a_1,\bar{x})$$

$$- \int_{a_1}^{x_1} K_{r_1}(r_1,\bar{x})g(r_1,\bar{x})dr_1 - \int_{a_1}^{x_1} K(r_1,\bar{a})g_{r_1}(r_1,\bar{a})dr_1$$

$$- \sum_{\substack{\beta_k \\ 0\leq k\leq n-1}} \int_{a_1}^{x_1}\int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K(r_1,w(\bar{a},\bar{r};\beta_k))g_{r_1\bar{r}_{\beta_k}}(r_1,w(\bar{a},\bar{r};\beta_k))d\bar{r}_{\beta_k}dr_1$$

$$+ \sum_{\substack{\beta_k \\ 1\leq k\leq n}} (-1)^k [\int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(x_1,w(\bar{x},\bar{r};\beta_k)g(x_1,w(\bar{x},\bar{r};\beta_k))d\bar{r}_{\beta_k}$$

$$- \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(a_1,w(\bar{x},\bar{r};\beta_k))g(a_1,w(\bar{x},\bar{r};\beta_k))d\bar{r}_{\beta_k}$$

$$- \int_{a_1}^{x_1}\int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{r_1\bar{r}_{\beta_k}}(r_1,w(\bar{x},\bar{r};\beta_k)g(r_1,w(\bar{x},\bar{r};\beta_k))d\bar{r}_{\beta_k}dr_1] \qquad (2.5)$$

Using the induction hypothesis again we have

$$\int_{\bar{a}}^{\bar{x}} K(a_1\bar{r})\, g_{\bar{r}}(a_1,\bar{r})\, d\bar{r} = K(a_1,\bar{x})\, g(a_1,\bar{x}) - K(a)\, g(a)$$

$$- \sum_{\substack{\beta_k \\ 0 \le k \le n-1}} \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K(a_1, w(\bar{a},\bar{r};\beta_k))\, g_{\bar{r}_{\beta_k}}(a_1, w(\bar{a},\bar{r};\beta_k))\, d\bar{r}_{\beta_k}$$

$$+ \sum_{\substack{\beta_k \\ 1 \le k \le n}} (-1)^k \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(a_1, w(\bar{x},\bar{r};\beta_k))\, g(a_1, w(\bar{x},\bar{r};\beta_k))\, d\bar{r}_{\beta_k}.$$

Thus

$$K(a_1,\bar{x})\, g(a_1,\bar{x})$$

$$+ \sum_{\substack{\beta_k \\ 1 \le k \le n}} (-1)^k \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(a_1, w(\bar{x},\bar{r};\beta_k))\, g(a_1, w(\bar{x},\bar{r};\beta_k))\, d\bar{r}_{\beta_k}$$

$$= K(a)\, g(a) + \sum_{\substack{\beta_k \\ 0 \le k \le n-1}} \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K(a_1, w(\bar{a},\bar{r};\beta_k))\, g_{\bar{r}_{\beta_k}}(a_1, w(\bar{a},\bar{r};\beta_k))\, d\bar{r}_{\beta_k}$$

$$+ \int_{\bar{a}}^{\bar{x}} K(a_1,\bar{r})\, g_{\bar{r}}(a_1,\bar{r})\, d\bar{r}.$$

Using this in Equation (2.5) we obtain

$$\int_a^x K(r)g_r(r)dr = K(x)g(x) - \int_{a_1}^{x_1} K_{r_1}(r_1,\bar{x})g(r_1,\bar{x})dr_1$$

$$- \int_{a_1}^{x_1} K(r_1,\bar{a})g_{r_1}(r_1,\bar{a})dr_1$$

$$- \sum_{\substack{\beta_k \\ 0\leq k\leq n-1}} \int_{a_1}^{x_1}\int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K(r_1,w(\bar{a},\bar{r};\beta_k)g_{r_1\bar{r}_{\beta_k}}(r_1,w(\bar{a},\bar{r};\beta_k))d\bar{r}_{\beta_k}dr_1$$

$$+ \sum_{\substack{\beta_k \\ 1\leq k\leq n}} (-1)^k \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(x_1,w(\bar{x},\bar{r};\beta_k))g(x_1,w(\bar{x},\bar{r};\beta_k))d\bar{r}_{\beta_k}$$

$$- \sum_{\substack{\beta_k \\ 1\leq k\leq n}} (-1)^k \int_{a_1}^{x_1}\int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{r_1\bar{r}_{\beta_k}}(r_1,w(\bar{x},\bar{r};\beta_k))g(r_1,w(\bar{x},\bar{r};\beta_k))d\bar{r}_{\beta_k}dr_1$$

$$- K(a)g(a) - \sum_{\substack{\beta_k \\ 0\leq k\leq n-1}} \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K(a_1,w(\bar{a},\bar{r};\beta_k))g_{\bar{r}_{\beta_k}}(a_1,w(\bar{a},\bar{r};\beta_k))d\bar{r}_{\beta_k}$$

$$- \int_{\bar{a}}^{\bar{x}} K(a_1,\bar{r})g_{\bar{r}}(a_1,\bar{r})d\bar{r}.$$

Therefore

$$\int_a^x K(r)g_r(r)dr = K(x)g(x) - K(a)g(a)$$

$$- [\int_{a_1}^{x_1} K(r_1,\bar{a})g_{r_1}(r_1,\bar{a})dr_1$$

$$+ \sum_{\beta_1} \int_{\bar{a}_{\beta_1}}^{\bar{x}_{\beta_1}} K(a_1,w(\bar{a},\bar{r};\beta_1))g_{\bar{r}_{\beta_1}}(a_1,w(\bar{a},\bar{r};\beta_1))d\bar{r}_{\beta_1}]$$

$$- [\sum_{\beta_1} \int_{a_1}^{x_1} \int_{\bar{a}_{\beta_1}}^{\bar{x}_{\beta_1}} K(r_1,w(\bar{a},\bar{r};\beta_1))g_{r_1\bar{r}_{\beta_1}}(r_1,w(\bar{a},\bar{r};\beta_1))d\bar{r}_{\beta_1}dr_1$$

$$+ \sum_{\beta_2} \int_{\bar{a}_{\beta_2}}^{\bar{x}_{\beta_2}} K(a_1,w(\bar{a},\bar{r};\beta_2))g_{\bar{r}_{\beta_2}}(a_1,w(\bar{a},\bar{r};\beta_2))d\bar{r}_{\beta_2}]$$

$$-$$

$$\vdots$$

$$- [\sum_{\beta_k} \int_{a_1}^{x_1} \int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K(r_1,w(\bar{a},\bar{r};\beta_k))g_{r_1\bar{r}_{\beta_k}}(r_1,w(\bar{a},\bar{r};\beta_k))d\bar{r}_{\beta_k}dr_1$$

$$+ \sum_{\beta_{k+1}} \int_{\bar{a}_{\beta_{k+1}}}^{\bar{x}_{\beta_{k+1}}} K(a_1,w(\bar{a},\bar{r};\beta_{k+1})g_{\bar{r}_{\beta_{k+1}}}(a_1,w(\bar{a},\bar{r};\beta_{k+1})d\bar{r}_{\beta_{k+1}}]$$

$$-\ \cdot\ \cdot\ \cdot\ -\ [\sum_{\beta_{n-1}}\int_{a_1}^{x_1}\int_{\bar{a}_{\beta_{n-1}}}^{\bar{x}_{\beta_{n-1}}} K(r_1, w(\bar{a},\bar{r};\beta_{n-1}))$$

$$g_{r_1\bar{r}_{\beta_{n-1}}}(r_1, w(\bar{a},\bar{r};\beta_{n-1}))\,d\bar{r}_{\beta_{n-1}}\,dr_1 + \int_{\bar{a}}^{\bar{x}} K(a_1,\bar{r})\,g_{\bar{r}}(a_1,\bar{r})\,d\bar{r}]$$

$$-\ [\int_{a_1}^{x_1} K_{r_1}(r_1,\bar{x})\,g(r_1,\bar{x})\,dr_1$$

$$+\ \sum_{\beta_1}\int_{\bar{a}_{\beta_1}}^{\bar{x}_{\beta_1}} K_{\bar{r}_{\beta_1}}(x_1, w(\bar{x},\bar{r};\beta_1))\,g(x_1, w(\bar{x},\bar{r};\beta_1))\,d\bar{r}_{\beta_1}]$$

$$(-1)^2[\sum_{\beta_2}\int_{\bar{a}_{\beta_2}}^{\bar{x}_{\beta_2}} K_{\bar{r}_{\beta_2}}(x_1, w(\bar{x},\bar{r};\beta_2))\,g(x_1, w(\bar{x},\bar{r};\beta_2))\,d\bar{r}_{\beta_2}$$

$$+\ \sum_{\beta_1}\int_{a_1}^{x_1}\int_{\bar{a}_{\beta_1}}^{\bar{x}_{\beta_1}} K_{r_1\bar{r}_{\beta_1}}(r_1, w(\bar{x},\bar{r};\beta_1))\,g(r_1, w(\bar{x},\bar{r};\beta_1))\,d\bar{r}_{\beta_1}\,dr_1]$$

$$-\ \cdot\ \cdot\ \cdot$$

$$(-1)^k[\sum_{\beta_k}\int_{\bar{a}_{\beta_k}}^{\bar{x}_{\beta_k}} K_{\bar{r}_{\beta_k}}(x_1,w(\bar{x},\bar{r};\beta_k))g(x_1,w(\bar{x},\bar{r};\beta_k))d\bar{r}_{\beta_k}$$

$$+\sum_{\beta_{k-1}}\int_{a_1}^{x_1}\int_{\bar{a}_{\beta_{k-1}}}^{\bar{x}_{\beta_{k-1}}} K_{r_1\bar{r}_{\beta_{k-1}}}(r_1,w(\bar{x},\bar{r};\beta_{k-1}))$$

$$g(r_1,w(\bar{x},\bar{r};\beta_{k-1}))d\bar{r}_{\beta_{k-1}}dr_1]$$

$$\vdots$$

$$(-1)^{n+1}\int_a^x K_r(r)g(r)dr.$$

Hence

$$\int_a^x K(r)g_r(r)dr = K(x)g(x) - K(a)g(a)$$

$$-\sum_{\substack{\alpha_k \\ 0\le k\le n}}\int_{a_{\alpha_k}}^{x_{\alpha_k}} K(w(a,r;\alpha_k))g_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k}$$

$$+\sum_{\substack{\alpha_k \\ 1\le k\le n+1}}(-1)^k\int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{r_{\alpha_k}}(w(x,r;\alpha_k))g(w(x,r;\alpha_k))dr_{\alpha_k}.$$

This completes the proof of Lemma 2.3

We now give the definition of a fundamental solution for Equation (1.7),

$$u(x) = \phi(x) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) u(w(x,r;\alpha_k)) dr_{\alpha_k} \quad \text{for} \quad x \geq a \geq \bar{a}.$$

Definition 2.1. Suppose the matrix function $A(x;\xi)$ satisfies the matrix equation

$$A(x;\xi) = I + \sum \int_{\xi_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) A(w(x,r;\alpha_k);\xi) dr_{\alpha_k} \tag{2.6}$$

for $\bar{a} \leq a \leq \xi \leq x < \infty$. Then $A(x;\xi)$ will be called a fundamental solution for Equation (1.7).

If the matrix functions $K_{\alpha_k}(x, r_{\alpha_k})$ are continuous for $x \geq \bar{a}$, $\bar{a}_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$ then the fundamental solution exists, is unique, and is continuous in x for each fixed ξ with $x \geq \xi$. This may be seen by applying corollary 2.2 to the vector equation

$$u_j(x) = e_j + \sum \int_{\xi_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) u_j(w(x,r;\alpha_k)) dr_{\alpha_k}$$

where e_j is the j^{th} column of I. The matrix with $u_j(x)$ as its j^{th} column is then the unique fundamental solution.

The following theorem and its corollaries are basic to our analysis.

<u>Theorem 2.2</u>. Suppose for each α_k with $1 \leq k \leq n$ the matrix function $K_{\alpha_k}(x,r_{\alpha_k})$ is continuous for $a \leq x < \infty$ and $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$. Suppose $\phi(x)$ is continuous and has continuous pure mixed partials of all orders less than or equal to n for $a \leq x < \infty$. Let $A(x;\xi)$ be the fundamental solution for Equation (1.7). Then the unique continuous solution of Equation (1.7) for $x \geq a$ is

$$u(x) = A(x;a)\phi(a)$$

$$+ \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k}. \tag{2.7}$$

<u>Remark 2.2.</u> We will prove Theorem 2.2 under the following additional hypotheses on the fundamental solution:

i) suppose for each $x \geq a$ the fundamental solution $A(x;\xi)$ is continuous in ξ for $a \leq \xi \leq x$, and

ii) suppose for each $b \geq a$ there is a constant $M(b)$ so that $\|A(x;\xi)\| \leq M(b)$ for $a \leq \xi \leq x \leq b$.

We will see in Section 2.3, with the aid of the inequality developed there, that continuity of the matrix functions $K_{\alpha_k}(x, r_{\alpha_k})$ is sufficient to insure the additional assumptions made here. Thus, Theorem 2.2 will be complete once these facts are established in Section 2.3.

Proof of Theorem 2.2. (Under the additional assumptions given in Remark 2.2). Take any $b \geq a$. From the discussion preceding this theorem it follows that for each fixed $\xi \geq a$ the fundamental solution $A(x;\xi)$ is continuous in x for $x \geq \xi$. Thus, for each α_k and each r_{α_k} with $a_{\alpha_k} \leq r_{\alpha_k} \leq b_{\alpha_k}$, the function $A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))$ is continuous in x for $x \in [a,b]$ and $r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$. By Assumption i) of Remark 2.2 and the continuity of $\phi_{x_{\alpha_k}}(x)$ it follows that for each $x \in [a,b]$ the function $A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))$ is continuous in r_{α_k} with $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k}$. Using Assumption ii) of Remark 2.2 and the continuity of $\phi_{x_{\alpha_k}}(x)$ we also see that $A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))$ is bounded for $x \in [a,b]$ and $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$. Thus, by Lemma 2.2, the integrals in Equation (2.7) are continuous on $[a,b]$ and $u(x)$ given by Equation (2.7) is also continuous on $[a,b]$.

We now show by direct substitution that $u(x)$ given by Equation (2.7) satisfies Equation (1.7) on $[a,b]$. Let γ_k also denote a combination of the integers $\{1,2,\ldots,n\}$ taken k at a time and let $s = (s_1, s_2, \ldots, s_n) \in R^n$. Then putting $u(x)$ as given by Equation (2.7) into the right hand side of Equation (1.7) we obtain

$$\phi(x) + \sum_{\substack{\alpha_k \\ 1\le k\le n}} \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) [A(w(x,r;\alpha_k);a)\phi(a)$$

$$+ \sum_{\substack{\gamma_i \\ 1\le i\le n}} \int_{a_{\gamma_i}}^{w_{\gamma_i}(x,r;\alpha_k)} A(w(x,r;\alpha_k);w(a,s;\gamma_i))$$

$$\phi_{s_{\gamma_i}}(w(a,s;\gamma_i))ds_{\gamma_i}]dr_{\alpha_k} \tag{2.8}$$

$$= \phi(x) + [\sum_{\substack{\alpha_k \\ 1\le k\le n}} \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) A(w(x,r;\alpha_k);a)dr_{\alpha_k}]\phi(a)$$

$$+ \sum_{\substack{\alpha_k \\ i\le k\le n}} \sum_{\substack{\gamma_i \\ i\le i\le n}} \int_{a_{\alpha_k}}^{x_{\alpha_k}} \int_{a_{\gamma_i}}^{w_{\gamma_i}(x,r;\alpha_k)} K_{\alpha_k}(x, r_{\alpha_k})$$

$$A(w(x,r;\alpha_k);w(a,s;\gamma_i))\phi_{s_{\gamma_i}}(w(a,s;\gamma_i))ds_{\gamma_i}dr_{\alpha_k}.$$

Now consider any integers p and q such that $1 \leq p \leq n$ and $1 \leq q \leq n$. We verify the following in Appendix A.

$$\int_{a_{\alpha_p}}^{x_{\alpha_p}} \int_{a_{\gamma_q}}^{w_{\gamma_q}(x,r;\alpha_p)} K_{\alpha_p}(x,r_{\alpha_p})$$

$$A(w(x,r;\alpha_p);w(a,s;\gamma_q))\phi_{s_{\gamma_q}}(w(a,s;\gamma_q))ds_{\gamma_q}dr_{\alpha_p}$$

$$= \int_{a_{\gamma_q}}^{x_{\gamma_q}} \int_{w_{\alpha_p}(a,s;\gamma_q)}^{x_{\alpha_p}} K_{\alpha_p}(x,r_{\alpha_p})$$

$$A(w(x,r;\alpha_p);w(a,s;\gamma_q))\phi_{s_{\gamma_q}}(w(a,s;\gamma_q))dr_{\alpha_p}ds_{\gamma_q}.$$

Using this in Equation (2.8) we have the right side of Equation (2.8) equal to

$$\phi(x) + \sum_{\substack{\alpha_k \\ 1\leq k\leq n}} \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})A(w(x,r;\alpha_k);a)dr_{\alpha_k}]\phi(a)$$

$$+ \sum_{\substack{\gamma_i \\ 1\leq i\leq n}} \int_{a_{\gamma_i}}^{x_{\gamma_i}} (\sum_{\substack{\alpha_k \\ 1\leq k\leq n}} \int_{w_{\alpha_k}(a,s;\gamma_i)}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})$$

$$A(w(x,r;\alpha_k);w(a,s;\gamma_i))dr_{\alpha_k})\phi_{s_{\gamma_i}}(w(a,s;\gamma_i))ds_{\gamma_i}$$

$$= \phi(x) + [A(x;a) - I]\phi(a)$$

$$+ \sum_{\substack{\gamma_i \\ 1\le i\le n}} \int_{a_{\gamma_i}}^{x_{\gamma_i}} [A(x;w(a,s;\gamma_i)) - I]\phi_{s_{\gamma_i}}(w(a,s;\gamma_i))ds_{\gamma_i}$$

$$= A(x;a)\phi(a) + \sum_{\substack{\gamma_i \\ 1\le i\le n}} \int_{a_{\gamma_i}}^{x_{\gamma_i}} A(x;w(a,s;\gamma_i))\phi_{s_{\gamma_i}}(w(a,s;\gamma_i))ds_{\gamma_i}$$

$$+ [\phi(x) - \phi(a) - \sum_{\substack{\gamma_i \\ 1\le i\le n}} \int_{a_{\gamma_i}}^{x_{\gamma_i}} \phi_{s_{\gamma_i}}(w(a,s;\gamma_i))ds_{\gamma_i}].$$

Using Lemma 2.3 with $K(x) \equiv I$ and $g(x) \equiv \phi(x)$ we see that

$$\phi(x) - \phi(a) - \sum_{\substack{\gamma_i \\ 1\le i\le n}} \int_{a_{\gamma_i}}^{x_{\gamma_i}} \phi_{s_{\gamma_i}}(w(a,s;\gamma_i))ds_{\gamma_i} \equiv 0.$$

Therefore

$$\phi(x) + \sum_{\substack{\alpha_k \\ 1\le k\le n}} \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k}) [A(w(x,r;\alpha_k);a)\phi(a)$$

$$+ \sum_{\substack{\gamma_i \\ 1\le i\le n}} \int_{a_{\gamma_i}}^{w_{\gamma_i}(x,r;\alpha_k)} A(w(x,r;\alpha_k);w(a,s;\gamma_i))$$

$$\phi_{s_{\gamma_i}}(w(a,s;\gamma_i))ds_{\gamma_i}]dr_{\alpha_k} = A(x;a)\phi(a)$$

$$+ \sum_{\substack{\gamma_i \\ 1\le i\le n}} \int_{a_{\gamma_i}}^{x_{\gamma_i}} A(x;w(a,s;\gamma_i))\phi_{s_{\gamma_i}}(w(a,s;\gamma_i))ds_{\gamma_i} = u(x).$$

But since b is arbitrary, $u(x)$ given by Equation (2.7) is the unique continuous solution for $x \geq a$. This completes the proof.

Remark 2.3. We notice that in the special case $\phi(x) \equiv \phi$ (a constant), the solution of Equation (1.7) is just $u(x) = A(x;a)\phi$ and there is no need for the assumption on $A(x;\xi)$ given in Remark 2.2. (In this case the continuity of $u(x)$ follows directly.) We will later use this fact to prove that the continuity of the functions $K_{\alpha_k}(x,r_{\alpha_k})$ implies the continuity of $A(x;\xi)$ in the ξ variable.

We have the following important corollary for the nonlinear Equation (1.8). The importance of this result, as mentioned earlier, is that we have an equivalent integral equation for the solution of the nonlinear Equation (1.8) in terms of the fundamental solution.

Corollary 2.3. Suppose for each α_k with $1 \leq k \leq n$, the function $K_{\alpha_k}(x,r_{\alpha_k})$ is continuous for $a \leq x < \infty$ and

$a_{\alpha_k} \le r_{\alpha_k} \le x_{\alpha_k} < \infty$. Suppose $\phi(x)$ is continuous and has continuous pure mixed partials of all orders less than or equal to n for $x \ge a$. Suppose $f(x,r,z)$ is continuous for $a \le r \le x < \infty$, $z \in R^m$ and for each α_k with $1 \le k \le n$ the function $\frac{\partial^k f}{\partial x_{\alpha_k}}(x,w(x,r;\alpha_k),z)$ is continuous for $a \le x < \infty$, $a_{\alpha_k} \le r_{\alpha_k} \le x_{\alpha_k} < \infty$, and $z \in R^m$. Let $A(x;\xi)$ be the fundamental solution for Equation (1.7). Then $u(x)$ is a continuous solution of Equation (1.8) if and only if $u(x)$ is a continuous solution of

$$u(x) = A(x;a)\phi(a) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k}$$

$$+ \int_a^x A(x;r)\frac{\partial}{\partial r}\left[\int_a^r f(r,s,u(s))ds\right]dr \tag{2.9}$$

<u>Proof</u>. We first prove necessity. Suppose $\bar{u}(x)$ is a continuous solution of Equation (1.8) for $x \ge a$ and consider the equation

$$u(x) = \phi(x) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})u(w(x,r;\alpha_k))dr_{\alpha_k}$$

$$+ \int_a^x f(x,r,\bar{u}(r))dr. \tag{2.10}$$

This is a linear equation since the last term depends only on x. The unique continuous solution of Equation (2.10) is $\bar{u}(x)$, since $\bar{u}(x)$ satisfies Equation (1.8). Letting $\bar{\phi}(x) = \phi(x) + \int_a^x f(x,r,\bar{u}(r))dr$, Equation (2.10) becomes

$$u(x) = \bar{\phi}(x) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})u(w(x,r;\alpha_k))dr_{\alpha_k}. \quad (2.11)$$

But $\bar{\phi}_{r_{\alpha_k}}(w(a,r;\alpha_k)) = \phi_{r_{\alpha_k}}(w(a,r;\alpha_k)) +$

$+ \frac{\partial}{\partial r_{\alpha_k}} \int_a^{w(a,r;\alpha_k)} f(w(a,r;\alpha_k),s,\bar{u}(s))ds = \phi_{r_{\alpha_k}}(w(a,r;\alpha_k))$ for $1 \leq k \leq n-1$ and $\bar{\phi}_r(r) = \phi_r(r) + \frac{\partial}{\partial r}\int_a^r f(r,s,\bar{u}(s))ds$. The continuity assumptions on f and its partials imply that $\bar{\phi}_r(r)$ is continuous. Thus using Theorem 2.2 the solution of Equation (2.11) is

$$u(x) = \bar{u}(x) = A(x;a)\phi(a) +$$

$$+ \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k}$$

$$+ \int_a^x A(x;r)\frac{\partial}{\partial r}[\int_a^r f(r,s,\bar{u}(s))ds]dr$$

Therefore $\bar{u}$ satisfies Equation (2.9).

To prove sufficiency, assume $u^*(x)$ is continuous and satisfies Equation (2.9). Consider the linear equation

$$u(x) = \phi(x) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) u(w(x, r; \alpha_k)) dr_{\alpha_k}$$

$$+ \int_a^x f(x, r, u^*(r)) dr.$$

But the solution of this is given by

$$u(x) = A(x;a)\phi(a)$$

$$+ \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x; w(a, r; \alpha_k)) \phi_{r_{\alpha_k}} (w(a, r; \alpha_k)) dr_{\alpha_k}$$

$$+ \int_a^x A(x;r) \frac{\partial}{\partial r} [\int_a^r f(r, s, u^*(s)) ds dr = u^*(x).$$

Therefore $u^*(x)$ satisfies Equation (1.8) and the proof is complete.

The following corollary will be useful in establishing various stability criteria.

<u>Corollory 2.4</u>. Suppose all of the hypotheses of Theorem 2.2 hold. Suppose in addition the fundamental solution $A(x;\xi)$ for Equation (1.7) has continuous pure mixed partials of all orders less than or equal to n in the ξ

variable for $a \leq \xi \leq x$. Then the solution of Equation (1.7) for $x \geq a$ is

$$u(x) = \phi(x)$$

$$+ \sum(-1)^k \int_{a_{\alpha_k}}^{x_{\alpha_k}} A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\phi(w(x,r;\alpha_k))dr_{\alpha_k}. \qquad (2.12)$$

<u>Proof</u>. Fix $x \geq a$. Let $K(\xi) = A(x;\xi)$ and $g(\xi) = \phi(\xi)$ in Lemma 2.3. Then noting that $K(x) = A(x;x) = I$ we obtain

$$\int_a^x A(x;r)\phi_r(r)dr = \phi(x) - A(x;a)\phi(a)$$

$$- \sum_{\substack{\alpha_k \\ 0 \leq k \leq n-1}} \int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k}$$

$$+ \sum_{\substack{\alpha_k \\ 1 \leq k \leq n}} (-1)^k \int_{a_{\alpha_k}}^{x_{\alpha_k}} A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\phi(w(x,r;\alpha_k))dr_{\alpha_k}.$$

Therefore

$$u(x) = A(x;a)\phi(a)$$

$$+ \sum_{\substack{\alpha_k \\ 1\le k\le n}} \int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k}$$

$$= \phi(x) + \sum_{\substack{\alpha_k \\ 1\le k\le n}} (-1)^k \int_{a_{\alpha_k}}^{x_{\alpha_k}} A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\phi(w(x,r;\alpha_k)dr_{\alpha_k}.$$

This completes the proof.

In the case $n = 1$ Equation (1.8) becomes

$$u(x_1) = \phi(x_1) + \int_{a_1}^{x_1} [K(x_1,r_1)u(r_1)$$

$$+ f(x_1,r_1,u(r_1))]dr_1 \tag{2.13}$$

with a_1, x_1, $r_1 \in R$, $x_1 \ge a_1$ and $a_1 \le r_1 \le x_1$. In this case the fundamental solution for the linear equation satisfies

$$A(x_1;\xi_1) = I + \int_{\xi_1}^{x_1} K(x_1,r_1)A(r_1;\xi_1)dr_1, \quad a_1 \le \xi_1 \le x_1,$$

and the integral equation equivalent to Equation (2.13) is

$$u(x_1) = A(x_1;a_1)\phi(a_1) + \int_{a_1}^{x_1} [A(x_1;r_1)\phi_{r_1}(r_1)$$

$$+ \frac{d}{dr_1}\{\int_{a_1}^{r_1} f(r_1,s_1,u(s_1))ds_1]dr_1. \qquad (2.14)$$

Equation (2.14) is used by Bownds and Cushing in their investigations of stability for Equation (2.13). When Equation (2.13) is equivalent to an initial value problem the fundamental solution is the fundamental matrix $Y(x_1)Y^{-1}(\xi_1)$ from the theory of ordinary differential equations, and Equation (2.14) is the well known variation of constants formula.

When $n = 2$, Equation (1.8) is

$$u(x_1,x_2) = \phi(x_1,x_2) + \int_{a_1}^{x_1} K_1(x_1,x_2,r_1)u(r_1,x_2)dr_1$$

$$+ \int_{a_2}^{x_2} K_2(x_1,x_2,r_2)u(x_1,r_2)dr_2$$

$$+ \int_{a_1}^{x_1}\int_{a_2}^{x_2} K_2(x_1,x_2,r_1,r_2)u(r_1,r_2)dr_2dr_1$$

$$+ \int_{a_1}^{x_1}\int_{a_2}^{x_2} f(x_1,x_2,r_1,r_2,u(r_1,r_2))dr_2dr_1. \qquad (2.15)$$

The fundamental solution $A(x_1,x_2;\xi_1,\xi_2)$ for the linear equation satisfies

$$A(x_1,x_2;\xi_1,\xi_2) = I + \int_{a_1}^{x_1} K_1(x_1,x_2,r_1)A(r_1,x_1;\xi_1,\xi_2)dr_1$$

$$+ \int_{a_2}^{x_2} K_2(x_1,x_2,r_2)A(x_1,r_2;\xi_1,\xi_2)dr_2$$

$$+ \int_{a_1}^{x_1}\int_{a_2}^{x_2} K_{12}(x_1,x_2,r_1,r_2)A(r_1,r_2;\xi_1,\xi_2)dr_2dr_1.$$

The equation equivalent to Equation (2.15) is

$$u(x_1,x_2) = A(x_1,x_2;a_1,a_2)\phi(a_1,a_2)$$

$$+ \int_{a_1}^{x_1} A(x_1,x_2;r_1,a_2)\phi_{r_1}(r_1,a_2)dr_1$$

$$+ \int_{a_2}^{x_2} A(x_1,x_2;a_1,r_2)\phi_{r_2}(a_1,r_2)dr_2$$

$$+ \int_{a_1}^{x_1}\int_{a_2}^{x_2} [A(x_1,x_2;r_1,r_2)\phi_{r_1r_2}(r_1,r_2)$$

$$+ \frac{\partial^2}{\partial r_1 \partial r_2}\{\int_{a_1}^{r_1}\int_{a_2}^{r_2} f(r_1,r_2,s_1,s_2,u(s_1,s_2))ds_2ds_1\}]dr_2dr_1.$$

We will return to stability considerations for special cases of Equation (2.15) later in the dissertation.

2.3 A Gronwall Type Inequality

In 1919, T. H. Gronwall [21] made use of a lemma which, in a generalized form, is a basic tool in the theory of ordinary differential equations. The following generalized version of that original lemma is now known as Gronwall's lemma or Bellman's lemma (sometimes the Gronwall-Bellman lemma) [2].

Gronwall's Lemma. Suppose $u(t)$ and $g(t)$ are real valued, nonnegative, and continuous functions of the real variable t for $t_0 \leq t \leq \tau$. Suppose

$$u(t) \leq c + \int_{t_0}^{t} g(s)u(s)ds \quad (t_0 \leq t \leq \tau) \tag{2.16}$$

holds where c is a nonnegative constant. Then

$$u(t) \leq c \exp[\int_{t_0}^{t} g(s)ds] \quad (t_0 \leq t \leq \tau). \tag{2.17}$$

This result is useful in the theory of ordinary differential equations for such topics as, uniqueness, continuous dependence, comparison results, and stability considerations.

The lemma has been generalized in several ways and for a variety of motives [1], [11], [14], [15], [33], [34], [39], [41]. In this section we will establish a generalization of this lemma which is particularly suited to our stability study. We will return to a discussion of the connection between this generalization and those of other authors later in this section.

Our result is based on the following theorem which can be found in [24, p. 18].

Theorem 2.3. Suppose F is a complete metric space and is partially ordered (the symbol of the partial order will be $<$) in such a way, that if an increasing sequence $(y_n) \subset F$ has the limit y_0, then $y_n < y_0$ for all n. Let T be an order preserving ($f_1 < f_2 \Rightarrow Tf_1 < Tf_2$) contraction on F. Let f_0 be the unique fixed point of T. Then $f \in F$ and $f < T(f)$ implies $f < f_0$.

Proof. Suppose $f \in F$ and $f < T(f)$. Since T is order preserving $f < T(f) < T^2(f) < T^3(f) < \ldots < T^n(f)$. Since T is a contraction we have $\lim_{n\to\infty} T^n(f) = f_0$. But $(T^n(f))$ is then an increasing sequence in F and therefore $T^n(f) < f_0$ for each n. In particular, $f < T(f) < f_0$ and the proof is complete.

As in Section 2.1, let $C[a,b] = \{g(x) \mid g:[a,b] \to R^m$, g continuous$\}$ and let $C_\lambda[a,b]$ be the Banach space of functions $C[a,b]$ normed by

$\|g\|_\lambda = \sup_{x\in[a,b]} |g(x)| \exp[-\lambda\{\sum_{i=1}^{n} x_i\}]$ for any real λ. Let $K \subset C_\lambda[a,b]$ be the positive cone of functions such that $\phi \in K$ if and only if each of its component functions is non-negative. We may use K to establish a partial order on $C_\lambda[a,b]$. The partial order is defined such that if g_1,

$g_2 \in C_\lambda[a,b]$, $g_1 < g_2$ iff $g_2 - g_1 \in K$. We notice that this partial order has the property that if (g_n) is an increasing sequence in $C_\lambda[a,b]$ converging to g_0, then $g_n < g_0$ for all n.

We have the following generalization of Gronwall's lemma.

Theorem 2.4. Suppose $\phi(x)$ is continuous on $[a,b]$. Suppose for each α_k with $1 \leq k \leq n$, the $m \times m$ matrix function $K_{\alpha_k}(x,r_{\alpha_k})$, defined for $x \in [a,b]$ and $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$, has nonnegative elements on this domain. Then if $u(x) \in C[a,b]$ and

$$u(x) < \phi(x) + \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})u(w(x,r;\alpha_k))dr_{\alpha_k} \tag{2.18}$$

then $u(x) < v(x)$ where $v(x)$ is the unique continuous solution of the equation

$$v(x) = \phi(x) + \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})v(w(x,r;\alpha_k))dr_{\alpha_k}$$

$$\text{for } x \in [a,b]. \tag{2.19}$$

Proof. Since each $K_{\alpha_k}(x,r_{\alpha_k})$ is continuous on the compact set $a \leq x \leq b$ and $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$, there is an

M so that $\|K_{\alpha_k}(x,r_{\alpha_k})\| \leq M$ on this domain. Choose λ so that $\lambda > 1$ and $\frac{M(2^n-1)}{\lambda} < 1$. Define T on $C_\lambda[a,b]$ such that for $g \in C_\lambda[a,b]$,

$$(Tg)(x) = \phi(x) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k}) g(w(x,r;\alpha_k))dr_{\alpha_k}.$$

The argument used in Theorem 2.1 shows that T is a contraction on $C_\lambda[a,b]$.

Suppose g_1, $g_2 \in C_\lambda[a,b]$ such that $g_1 < g_2$; then $g_1(w(x,r;\alpha_k)) < g_2(w(x,r;\alpha_k))$. Since the elements in the matrix $K_{\alpha_k}(x,r_{\alpha_k})$ are nonnegative we have

$$K_{\alpha_k}(x,r_{\alpha_k})g_1(w(x,r;\alpha_k)) < K_{\alpha_k}(x,r_{\alpha_k})g_2(w(x,r;\alpha_k)),$$

hence

$$\int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})g_1(w(x,r;\alpha_k))dr_{\alpha_k}$$

$$< \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})g_2(w(x,r;\alpha_k))dr_{\alpha_k}$$

and

$$(Tg_1)(x) = \phi(x) + \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})g_1(w(x,r;\alpha_k))dr_{\alpha_k}$$

$$< \phi(x) + \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})g_2(w(x,r;\alpha_k))dr_{\alpha_k} = (Tg_2)(x).$$

Therefore T is an order preserving contraction on $C_\lambda[a,b]$. The hypothesis on $u(x)$ implies that $u < Tu$. Then by Theorem 2.3, $u < v$ where v is the unique solution of $v = Tv$. This completes the proof.

The following corollary gives a useful property of the fundamental solution.

Corollary 2.5. Suppose $\phi \in K \subset C_\lambda[a,b]$. Suppose the $m \times m$ matrix functions $K_{\alpha_k}(x,r_{\alpha_k})$ are continuous and have nonnegative entries for $x \in [a,b]$ and $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$. Then the unique continuous solution $v(x)$ of

$$v(x) = \phi(x) + \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})v(w(x,r;\alpha_k))dr_{\alpha_k}$$

is in K. (i.e., $v > 0$).

Proof. Since $\phi \in K$ we have $\phi > 0$ and thus $u \equiv 0$ satisfies inequality (2.18). Therefore the solution v of Equation (2.19) satisfies $0 = u < v$. This completes the proof.

Corollary 2.5 may also be established by noting that, in the case $\phi > 0$, the restriction of the map T, defined in Theorem 2.4, to K is a contraction on K. Since K is a closed subset of the complete space $C_\lambda[a,b]$, K is complete. Thus, the unique fixed point of the restriction of T to K is in K.

Remark 2.4. If we consider the partial order as given above on the continuous functions defined for $x \geq a$ then the results of Theorem 2.4 and Corollary 2.5 may easily be extended to hold for the unbounded region $x \geq a$.

Remark 2.5. Suppose for each α_k, $K_{\alpha_k}(x,r_{\alpha_k})$ is continuous and has nonnegative elements for $x \geq a$ and $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$. Let $A_j(x;\xi)$ be the j^{th} column of the fundamental solution, and let e_j be the j^{th} column of the identity matrix I. Then, since $A(x;\xi)$ satisfies Equation (2.7) we see that

$$A_j(x;\xi) = e_j + \sum\int_{\xi_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x,r_{\alpha_k})A_j(w(x,r;\alpha_k);\xi)dr_{\alpha_k}.$$

Thus from Corollary 2.5 and Remark 2.4 it follows that $A_j(x;\xi) > 0$ for $a \leq \xi \leq x < \infty$ and each j. Therefore all the elements in the matrix $A(x;\xi)$ are nonnegative on $a \leq \xi \leq x < \infty$.

We will now use the Gronwall inequality to show that continuity of the kernels $K_{\alpha_k}(x,r_{\alpha_k})$ implies the additional assumptions made on $A(x;\xi)$ in Remark 2.2. We begin with a preliminary lemma.

Lemma 2.4. Let ξ and ξ^0 be such that $\xi \geq \xi^0 \geq a$. Suppose each of the matrix functions K_{α_k} is constant. Then the fundamental solution for Equation (1.7) satisfies $A(x;\xi) = A(x - (\xi - \xi^0);\xi^0)$ for $x \geq \xi \geq \xi^0 \geq a$.

Proof. Take $\xi \geq \xi^0$ and $x \geq \xi$. We recall that $A(x;\xi^0)$ satisfies $A(x;\xi^0) = I + \sum\int_{\xi^0_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k} A(w(x,r;\alpha_k);\xi^0)dr_{\alpha_k}$.

For each $\alpha_k = \{i_1,i_2,\ldots,i_k\}$ let $r'_{i_p} = r_{i_p} - (\xi_{i_p} - \xi^0_{i_p})$ for $1 \leq p \leq k$. When $r_{i_p} = \xi_{i_p}$ we have $r'_{i_p} = \xi^0_{i_p}$, and when $r_{i_p} = x_{i_p}$ we have $r'_{i_p} = x_{i_p} - (\xi_{i_p} - \xi^0_{i_p})$. Thus

$$I + \sum\int_{\xi_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k} A(w(x - (\xi - \xi^0);r - (\xi - \xi^0);\alpha_k);\xi^0)dr_{\alpha_k}$$

$$= I + \sum\int_{\xi^0_{\alpha_k}}^{x_{\alpha_k}-(\xi_{\alpha_k}-\xi^0_{\alpha_k})} K_{\alpha_k} A(w(x - (\xi - \xi^0);r';\alpha_k);\xi^0)dr'_{\alpha_k}$$

$$= A(x - (\xi - \xi^0);\xi^0).$$

Therefore by uniqueness it follows that $A(x;\xi) \equiv A(x - (\xi - \xi^0);\xi^0)$ for $a \leq \xi \leq \xi^0 \leq x$. This completes the proof.

Lemma 2.5. Suppose each of the matrix functions $K_{\alpha_k}(x,r_{\alpha_k})$ is continuous for $x \geq a$, $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$. Let $A(x;\xi)$ be the fundamental solution for Equation (1.7). Then

i) for each $b \geq a$ there is a constant $M(b)$ such that $\|A(x;\xi)\| \leq M(b)$ for $a \leq \xi \leq x \leq b$;

ii) for each fixed $x \geq a$, $A(x;\xi)$ is continuous in ξ for $a \leq \xi \leq x$.

Proof. i) Take any $b \geq a$. From the continuity of the functions $K_{\alpha_k}(x,r_{\alpha_k})$ it follows that there exists a constant $\bar{M}(b)$ so that $\|K_{\alpha_k}(x,r_{\alpha_k})\| \leq \bar{M}(b)$ for each α_k and $x \in [a,b]$, $a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$. Since $A(x;\xi)$ satisfies Equation (2.7) we have

$$\|A(x;\xi)\| \leq 1 + \bar{M}\sum\int_{\xi_{\alpha_k}}^{x_{\alpha_k}} \|A(w(x,r;\alpha_k);\xi)\| dr_{\alpha_k}.$$

Let $A^*(x;\xi)$ be the fundamental solution for the scalar equation

$$u(x) = 1 + \bar{M}\sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} u(w(x,r;\alpha_k))dr_{\alpha_k}. \tag{2.20}$$

Using Theorem 2.4 we see that $\|A(x;\xi)\| \leq A^*(x;\xi)$ for $a \leq \xi \leq x \leq b$.

The function $A^*(x;a)$ is continuous on $[a,b]$ so there is a constant $M(b)$ such that $A^*(x;a) \leq M(b)$ for all $x \in [a,b]$. Take any ξ and x such that $a \leq \xi \leq x \leq b$. Using Lemma 2.4 we have $A^*(x;\xi) = A^*(x - (\xi - a);a) \leq M(b)$. Thus $\|A(x;\xi)\| \leq M(b)$ for $a \leq \xi \leq x \leq b$.

ii) Take any $x \geq a$, $\varepsilon > 0$, and ξ, $\xi^0 \in [a,x]$. Choose any $b \geq a$. As in part i) there are constants $\bar{M}(b)$, $M(b)$ so that for each α_k, $\|K_{\alpha_k}(x,r_{\alpha_k})\| \leq \bar{M}$ and $\|A(x;\xi)\| \leq A^*(x;\xi) \leq M$ for $a \leq \xi \leq x \leq b$. ($A^*(x;\xi)$ is the fundamental solution for Equation (2.20).)

Let $R_{\alpha_k}(\xi) = \{r_{\alpha_k} | \xi_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k}\}$ and let V_k denote the k-dimensional Lebesque measure on R^k. Then, if $\alpha_k = \{i_1,i_2,\ldots,i_k\}$, we have $V_k(R_{\alpha_k}(\xi)) = \prod_{p=1}^{k}(x_{i_p} - \xi_{i_p})$. For each α_k, $V_k(R_{\alpha_k}(\xi))$ is a uniformly continuous function of ξ on $[a,x]$. Thus there is a δ_{α_k} such that if ξ_1, $\xi_2 \in [a,x]$ with $|\xi_1 - \xi_2|_\infty < \delta_{\alpha_k}$ then $|V_k(R_{\alpha_k}(\xi_1)) - V_k(R_{\alpha_k}(\xi_2))| < \frac{\varepsilon}{2(2^n-1)\bar{M}M^2}$.

Let $\overline{\xi}_i = \max\{\xi_i, \xi_i^0\}$ and $\overline{\xi} = (\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_n)$. Let $\delta = \min_{\substack{\alpha_k \\ 1 \leq k \leq n}} \{\delta_{\alpha_k}\}$ and take ξ, ξ^0 so that $|\xi - \xi^0|_\infty < \delta$. Thus $|\xi - \overline{\xi}|_\infty = \max_{1 \leq i \leq n} \{|\xi_i - \overline{\xi}_i|\} < \delta$. Likewise $|\xi^0 - \overline{\xi}|_\infty < \delta$. Then observing that

$$\left.\begin{aligned}
&R_{\alpha_k}(\xi) = R_{\alpha_k}(\overline{\xi}) \cup (R_{\alpha_k}(\xi) - R_{\alpha_k}(\overline{\xi})) \quad \text{where} \\
&R_{\alpha_k}(\overline{\xi}) \cap (R_{\alpha_k}(\xi) - R_{\alpha_k}(\overline{\xi})) = \phi \qquad \text{and} \\
&R_{\alpha_k}(\xi^0) = R_{\alpha_k}(\overline{\xi}) \cup (R_{\alpha_k}(\xi^0) - R_{\alpha_k}(\overline{\xi})) \quad \text{where} \\
&R_{\alpha_k}(\xi^0) \cap (R_{\alpha_k}(\xi^0) - R_{\alpha_k}(\overline{\xi})) \overset{!}{=} \phi
\end{aligned}\right\} \tag{2.21}$$

we see that

$$\left.\begin{aligned}
&\int_{R_{\alpha_k}(\xi) - R_{\alpha_k}(\overline{\xi})} dr_{\alpha_k} = V_k(R_{\alpha_k}(\xi) - R_{\alpha_k}(\overline{\xi})) = V_k(R_{\alpha_k}(\xi)) \\
&- V_k(R_{\alpha_k}(\overline{\xi})) < \frac{\varepsilon}{2(2^n-1)\overline{M}M^2} \qquad \text{and} \\
&\int_{R_{\alpha_k}(\xi^0) - R_{\alpha_k}(\overline{\xi})} dr_{\alpha_k} = V_k(R_{\alpha_k}(\xi^0) - R_{\alpha_k}(\overline{\xi})) \\
&= V_k(R_{\alpha_k}(\xi^0)) - V_k(R_{\alpha_k}(\overline{\xi})) < \frac{\varepsilon}{2(2^n-1)\overline{M}M^2}
\end{aligned}\right\} \tag{2.22}$$

.

Thus for $|\xi - \xi^0|_\infty < \delta$ we have

$$\|A(x;\xi) - A(x;\xi^0)\|$$

$$\leq \sum \Big\| \int_{\xi_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) A(w(x,r;\alpha_k);\xi)\, dr_{\alpha_k}$$

$$- \int_{\xi^0_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) A(w(x,r;\alpha_k);\xi^0)\, dr_{\alpha_k} \Big\|.$$

From the expressions in (4.21), (4.22) and the estimates for $K_{\alpha_k}(x, r_{\alpha_k})$ and $A(x;\xi)$ it follows that

$$\|A(x;\xi) - A(x;\xi^0)\|$$

$$\leq \sum \Big[\int_{R_{\alpha_k}(\xi) - R_{\alpha_k}(\bar{\xi})} \|K_{\alpha_k}(x, r_{\alpha_k}) A(w(x,r;\alpha_k);\xi)\| dr_{\alpha_k}$$

$$+ \int_{R_{\alpha_k}(\xi^0) - R_{\alpha_k}(\bar{\xi})} \|K_{\alpha_k}(x, r_{\alpha_k}) A(w(x,r;\alpha_k);\xi^0)\| dr_{\alpha_k} \Big]$$

$$+ \sum \Big\| \int_{\bar{\xi}_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) [A(w(x,r;\alpha_k);\xi) - A(w(x,r;\alpha_k);\xi^0)]\, dr_{\alpha_k} \Big\|$$

$$\leq \bar{M}M \sum \Big[\int_{R_{\alpha_k}(\xi) - R_{\alpha_k}(\bar{\xi})} dr_{\alpha_k} + \int_{R_{\alpha_k}(\xi^0) - R_{\alpha_k}(\bar{\xi})} dr_{\alpha_k} \Big]$$

$$+ \bar{M}\sum\int_{\xi_{\alpha_k}}^{x_{\alpha_k}} \|A(w(x,r;\alpha_k);\xi) - A(w(x,r;\alpha_k);\xi^0)\| dr_{\alpha_k}$$

$$< \frac{\varepsilon}{M} + \bar{M} \int_{\xi_{\alpha_k}}^{x_{\alpha_k}} \|A(w(x,r;\alpha_k);\xi) - A(w(x,r;\alpha_k);\xi^0)\| dr_{\alpha_k}.$$

Using Theorem 2.4, Remark 2.3, and the fact that $A^*(x;\xi) \leq M$ we have $\|A(x;\xi) - A(x;\xi^0)\| < A^*(x;\xi)(\frac{\varepsilon}{M}) \leq \varepsilon$. This completes the proof of Lemma 2.5.

<u>Remark 2.6</u>. The conclusions of Lemma 2.5 show that we need not have made the additional assumptions on $A(x;\xi)$ in Remark 2.2. Thus Lemma 2.5 completes the proof of Theorem 2.2 as stated in Section 2.2.

<u>Remark 2.7</u>. If we assume the hypotheses of Theorem 2.4 for $x \geq a$ and in addition assume that $\phi(x)$ has continuous pure mixed partials of all orders less than or equal to n on $x \geq a$ then the conclusion of Theorem 2.4 in terms of the fundamental solution $A(x;\xi)$ for Equation (1.7) is that $u(x)$ satisfies

$$u(x) < A(x;a)\phi(a) + \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k}.$$

If we assume further that $A(x;\xi)$ has continuous pure mixed partials of all orders in ξ for $a \leq \xi \leq x$ then, by Corollary 2.4, the conclusion of Theorem 2.4 becomes

$$u(x) < \phi(x) + \sum(-1)^k \int_{a_{\alpha_k}}^{x_{\alpha_k}} A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\phi(w(x,r;\alpha_k))dr_{\alpha_k}.$$

We will now discuss some other generalizations of Gronwall's lemma and show how some of these follow from Theorem 2.4 (and Remark 2.7). We first note that Gronwall's lemma as stated at the beginning of this section is a particular case of Theorem 2.4. In that lemma $u(t)$ satisfies the scalar inequality

$$u(t) \leq c + \int_{t_0}^{t} g(s)u(s)ds \quad t,\ s \ \text{real}, \quad t_0 \leq t \leq \tau.$$

The fundamental solution for integral equation

$$u(t) = c + \int_{t_0}^{t} g(s)u(s)ds$$

is $A(t;\xi) = \exp[\int_{\xi}^{t} g(s)ds]$. Then using Theorem 2.4 and Remark 2.7, we have

$$u(t) \leq A(t;t_0)c = c\ \exp[\int_{t_0}^{t} g(s)ds] \quad (t_0 \leq t \leq \tau).$$

Conlan and Diaz [15] have used the following generalization of Gronwall's lemma to study existence for an n^{th}

order hyperbolic partial differential equation. The statement of their result is as follows: if γ, M, L are nonnegative constants, and if in the region $0 \leq x \leq b (b \in R^n, b > 0)$ the real valued function $u(x)$ is continuous and nonnegative, and if

$$u(x) \leq \gamma + L \sum_{\substack{\alpha_k \\ 1 \leq k \leq n-1}} \int_{0_{\alpha_k}}^{x_{\alpha_k}} u(w(x,r;\alpha_k))\,dr_{\alpha_k} + M\int_0^x u(r)\,dr \qquad (2.23)$$

for $x \in [0,b]$ then $u(x) \leq \gamma K$ for all $x \in [0,b]$, where K is a constant depending on M, L, and b.

We see that under their hypotheses we may apply Theorem 2.4 and Remark 2.7.to infer that $u(x) \leq A(x;0)\gamma$. However, from Lemma 2.5, Part i) it follows that there is a constant $K = K(b)$ depending on b, L and M so that $|A(x;0)| \leq K(b)$. Thus using our approach we obtain the result given by Conlan and Diaz.

Another generalization for a scalar equation in two independent variables has been given by Snow [33]. Throughout this discussion x, y, r, s, a, b, ξ, η will be real and u, ϕ and g will be scalar functions. We state here a modified version of the result given in [33]. Let D be the rectangle given by $0 \leq r \leq a$, $0 \leq s \leq b$. Suppose $u(r,s)$, $\phi(r,s)$, and $g(r,s)$ are continuous on D and $g(r,s) \geq 0$ on D. Let $P(x,y)$ be a point in D and

and let G be the rectangle with opposite corners $(0,0)$ and $P(x,y)$. Suppose $R(r,s;x,y)$ is the solution of the problem

$$R_{rs}(r,s;x,y) = g(r,s)R(r,s;x,y) \tag{2.24}$$

$$R(x,s;x,y) = 1, \quad R(r,y;x,y) = 1.$$

Let D^+ be connected subdomain of D which contains $P(x,y)$ and on which $R(r,s;x,y) \geq 0$. Then if $G \subset D^+$ and if $u(x,y)$ satisfies

$$u(x,y) \leq \phi(x,y) + \int_0^x\int_0^y g(r,s)u(r,s)dsdr \tag{2.25}$$

then $u(x,y)$ also satisfies

$$u(x,y) \leq \phi(x,y) + \int_0^x\int_0^y \phi(r,s)g(r,s)R(r,s;x,y)dsdr. \tag{2.26}$$

The function $R(r,s;x,y)$ introduced here is the Riemann function for a special hyperbolic partial differential equation. We will discuss this function R in more detail in Section 4.5 (see also [19], [35], [37]) and use h re some facts which will be given in that section.

We will now show that Snow's result also follows from Theorem 2.4. Let $A(x,y;\xi,\eta)$ be the fundamental solution for the equation

$$u(x,y) = \phi(x,y) + \int_0^x\int_0^y g(r,s)u(r,s)\,ds\,dr. \tag{2.27}$$

If ϕ, g, and u are as above and if u satisfies inequality (2.25) then by Remark 2.7, u satisfies

$$u(x,y) \leq \phi(x,y) - \int_0^x A_r(x,y;r,y)\phi(r,y)\,dr \tag{2.28}$$

$$- \int_0^y A_s(x,y;x,s)\phi(x,s)\,ds + \int_0^x\int_0^y A_{rs}(x,y;r,s)\phi(r,s)\,ds\,dr$$

provided $A_r(x,y;r,y)$, $A_s(x,y;x,s)$, and $A_{rs}(x,y;r,s)$ are continuous. We will use the following relation between $R(r,s;x,y)$ as given by Equation (2.24) and the fundamental solution $A(x,y;\xi,\eta)$ for Equation (2.27):

$$A(x,y;\xi,\eta) = A(\xi,\eta;x,y) = R(x,y;\xi,\eta) = R(\xi,\eta;x,y) \tag{2.29}$$

(Here we are assuming $A(\xi,\eta;x,y)$ is the solution of the usual equation even though $\xi \leq x$, $\eta \leq y$.) We will establish the identity (2.29) in Section 4.5.

Using identity (2.29) we have

$$A(x,y;\xi,\eta) = A(\xi,\eta;x,y) = 1 + \int_x^\xi\int_y^\eta g(r,s)A(r,s;x,y)\,ds\,dr.$$

Thus $A_\xi(x,y;\xi,\eta) = \int_y^\eta g(\xi,s)A(\xi,s;x,y)\,ds$ and $A_\xi(x,y;\xi,y) = 0$. Similarly $A_\eta(x,y;x,\eta) = 0$. Also, $A_{\xi\eta}(x,y;\xi,\eta)$

$= g(\xi,\eta)A(\xi,\eta;x,y) = g(\xi,\eta)A(x,y;\xi,\eta)$. The functions A_ξ, A_η, and $A_{\xi\eta}$ are continuous and inequality (2.28) becomes

$$u(x,y) \leq \phi(x,y) + \int_a^x\int_b^y g(r,s)\phi(r,s)A(x,y;r,s)dsdr.$$

Then using identity (2.29) we obtain (2.26), which is Snow's conclusion.

Snow has given a similar result for systems in two independent variables [34] and Young [41] has generalized Snow's result for the scalar case in two independent variables to the case of n independent variables. The author feels that these inequalities may be obtained using Theorem 2.4 in a manner similar to that given above and plans to study these in more detail at some future time.

Although it does not follow from Theorem 2.4, we mention another inequality of interest given in [1]. It is referred to there as Wendroff's inequality. Suppose c is a real nonnegative constant and $u(x,y)$, $v(x,y)$ are nonnegative continuous scalar functions. Suppose $u(x,y)$ satisfies

$$u(x,y) \leq c + \int_\xi^x\int_\eta^y v(r,s)u(r,s)dsdr \qquad (2.30)$$

for $x \geq \xi$, $y \geq \eta$. Then

$$u(x,y) \leq c\ \exp[\int_\xi^x\int_\eta^y v(r,s)dsdr].$$

This result gives an exponential estimate for the fundamental solution $A(x,y;\xi,\eta)$ for the equation

$$u(x,y) = c + \int_{\xi}^{x}\int_{\eta}^{y} v(r,s)u(r,s)\,ds\,dr$$

where $c \geq 0$, $v(x,y)$ is continuous and nonnegative. We see that $A(x,y;\xi,\eta)$ must satisfy

$$0 \leq A(x,y;\xi,\eta) \leq \exp\left[\int_{\xi}^{x}\int_{\eta}^{y} v(r,s)\,ds\,dr\right].$$

We point out that although Wendroff's inequality gives an explicit bound for u satisfying (2.30) it is not the best estimate which may be obtained for u.

CHAPTER 3

GENERAL STABILITY RESULTS

In this chapter we will establish our main stability results. As mentioned earlier our approach is one of seeking conditions for preservation of stability from the linear equation to the nonlinear perturbed equation. It is then natural that we begin by considering different kinds of stability for the linear equation (1.7) before pursuing questions of preservation.

We will assume throughout this chapter that $\bar{a} \in R^n$ is fixed, and that the matrix functions $K_{\alpha_k}(x, r_{\alpha_k})$ are continuous for $x \geq \bar{a}$ and $\bar{a}_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$.

3.1 Stability of the General Linear Equation

The following theorem gives important characterizations of stability, uniform stability, and asymptotic stability on the space N_3. We point out that if $\phi(x)$ is any of the spaces N_0, N_1, N_2, or N_3 then it is continuous for $x \geq \bar{a}$, so we always have existence (and uniqueness) for Equation (1.7) for $\bar{a} \leq a \leq x < \infty$.

Theorem 3.1. Let $A(x;\xi)$ be the fundamental solution for Equation (1.7). Then

i) Equation (1.7) is stable on N_3 if and only if there exists a constant $M(a)$ (M depends on $a \geq \bar{a}$) such that $\|A(x;a)\| \leq M(a)$ for $x \geq a$.

ii) Equation (1.7) is uniformly stable on N_3 if and only if there is a constant M, independent of a, so that $\|A(x;a)\| \leq M$ for $\bar{a} \leq a \leq x < \infty$.

iii) Equation (1.7) is asymptotically stable on N_3 if and only if $\lim_{|x|\to\infty} \|A(x;a)\| = 0$ for each $a \geq \bar{a}$.

Proof. i) We first proof sufficiency. By Theorem 2.2, the solution of Equation (1.7) for any $\phi \in N_3$ and for $x \geq a \geq \bar{a}$ is $u(x) = A(x;a)\phi$. Then $|u(x)| \leq \|A(x;a)\| |\phi| \leq M(a)\|\phi\|$. For any $\varepsilon > 0$, take $\phi \in N_3$ such that $\|\phi\| < \frac{\varepsilon}{2M(a)} = \delta(a)$. Thus $|u(x)| < \frac{\varepsilon}{2}$ for $x \geq a$ and therefore $\|u\|_{0,a} \leq \frac{\varepsilon}{2} < \varepsilon$. This means Equation (1.7) is stable on N_3.

We now prove necessity. Take any $\varepsilon > 0$ and any $a \geq \bar{a}$. There is a $\delta(a) > 0$ such that if $\phi \in N_3$ with $\|\phi\| < \delta(a)$ and if u is the solution of Equation (1.7) corresponding to ϕ then $\|u\|_{0,a} < \varepsilon$. Now take $\bar{\phi} \in N_3$ so that $\|\bar{\phi}\| = 1$. Let $\bar{u}(x)$ be the solution of Equation (1.7) corresponding to $\frac{\delta}{2}\bar{\phi}$. Since $\|\bar{u}\|_{0,a} = \|A(x;a)\frac{\delta}{2}\bar{\phi}\|_{0,a} < \varepsilon$ it follows that $|A(x;a)\bar{\phi}| < \frac{2\varepsilon}{\delta}$ for $x \geq a$. Thus $\|A(x;a)\| = \sup_{|\bar{\phi}|=1} |A(x;a)\bar{\phi}| \leq \frac{2\varepsilon}{\delta}$ for $x \geq a$.

ii) The argument here is the same as that given in part i) except that δ and M are now independent of $a \geq \bar{a}$.

iii) We first prove sufficiency. Take any $a \geq \bar{a}$. Since $\lim_{|x|\to\infty} \|A(x;a)\| = 0$ and $A(x;a)$ is continuous in x there is an $M(a)$ so that $\|A(x;a)\| \leq M(a)$ for $x \geq a$. Thus by part i) of this theorem Equation (1.7) is stable on N_3. Take any $\varepsilon > 0$ and any $\phi \in N_3$. The solution of Equation (1.7) for $x \geq a$ corresponding to ϕ is $u(x) = A(x;a)\phi$. Since $\lim_{|x|\to\infty} \|A(x;a)\| = 0$ there is a $T = T(a,\varepsilon,\phi)$ such that if $|x| \geq T$ then $\|A(x;a)\| < \frac{\varepsilon}{\|\phi\|}$. Thus $|u(x)| \leq \|A(x;a)\|\|\phi\| < \varepsilon$ for $|x| \geq T$ and $\lim_{|x|\to\infty} |u(x)| = 0$.

We now prove necessity. In this proof we specialize the vector norm and use $|\cdot|_1$. If K is any constant $m \times m$ matrix with elements k_{ij} and $\phi \in N_3$, then we have $\|K\| = \sup_{|\phi|_1=1} |K\phi|_1 = \max_{1\leq j\leq m} [\sum_{i=1}^{m} |k_{ij}|]$ (see [16], p. 41). Let $e_j = (0,0,\ldots,0_{j-1},1,0_{j+1},\ldots,0_m)$ for $j = 1, 2, \ldots, m$. Then for each j, $\|e_j\| = |e_j|_1 = 1$. There is a $\delta(a)$ such that $\phi \in N_3$ and $\|\phi\| = |\phi|_1 < \delta$ then the solution $u(x)$ of Equation (1.7) corresponding to ϕ satisfies $\lim_{|x|_1\to\infty} |u(x)|_1 = 0$. Take any $\varepsilon > 0$. For each j, let $u_j(x)$ be the solution of Equation (1.7) corresponding to $\phi_j = \frac{\delta}{2}e_j$. Thus for each j there exists a T_j such that if $|x|_1 \geq T_j$ then $|u_j(x)|_1 < \frac{\delta\varepsilon}{2}$. Let $T = \max_{1\leq j\leq m} \{T_j\}$, and let $A_{ij}(x;a)$ denote the elements in $A(x;a)$. Then for x so that $|x|_1 \geq T$ we have $|u_j(x)|_1 = |A(x;a)\frac{\delta}{2}e_j|_1 = \sum_{i=1}^{m} |\sum_{k=1}^{m} A_{ik}(x;a)\frac{\delta}{2}\delta_{kj}|$

$< \frac{\delta\varepsilon}{2}$. Therefore $\sum_{i=1}^{m} |A_{ij}(x;a)| < \varepsilon$ for $j = 1, 2, \ldots, m$ and $\|A(x;a)\| = \max_{1 \le j \le m} [\sum_{i=1}^{m} |A_{ij}(x;a)|] < \varepsilon$. Thus $\lim_{|x|_1 \to \infty} \|A(x;a)\| = 0$ and since all vector norms are equivalent on R^n, $\lim_{|x| \to \infty} \|A(x;a)\| = 0$. This completes the proof of Theorem 3.1.

Remark 3.1. If in Equation (1.7) the matrix functions $K_{\alpha_k}(x, r_{\alpha_k}) \equiv 0$ for $1 \le k \le n - 1$, then Equation (1.7) cannot be asymptotically stable on N_3. In this case Equation (1.7) becomes

$$u(x) = \phi + \int_a^x K_{\alpha_n}(x,r)u(r)dr. \tag{3.1}$$

Thus $u(a_1, x_2, x_3, \ldots, x_n) = \phi$ for all $(x_2, \ldots, x_n) \in R^{n-1}$ and $u(x)$ cannot go to zero as $|x| \to \infty$. We also point out that this means the fundamental solution $A(x;\xi)$ for Equation (3.1) cannot approach zero as $|x| \to \infty$.

The following theorems give sufficient conditons for stability, U.S., and A.S., for Equation (1.7) on the space N_2, N_1 and N_0 respectively.

Theorem 3.2. Let $A(x;\xi)$ be the fundamental solution for Equation (1.7). Then we have the following:

i) Suppose there is a constant $M(a)$ such that $\|A(x;\xi)\| \le M(a)$ for $\bar{a} \le a \le \xi \le x < \infty$. Then Equation (1.7) is stable on N_2.

ii) If the constant M in part i) is independent of a, then Equation (1.7) is uniformly stable on N_2.

iii) Suppose Equation (1.7) is uniformly stable on N_3 and $\lim_{|x|\to\infty} \|A(x;\xi)\| = 0$ for each $\xi \ge \bar{a}$. Then Equation (1.7) is asymptotically stable on N_2.

Proof. i) Take any $a \ge \bar{a}$ and any $\varepsilon > 0$. From Theorem 2.2, the solution of Equation (1.7) for any $\phi \in N_2$ is given by Equation (2.7). Therefore

$$|u(x)| \le \|A(x;a)\|\,|\phi(a)|$$

$$+ \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A(x;w(a,r;\alpha_k))\|\,|\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))|\,dr_{\alpha_k}$$

$$\le M(a)\,[\|\phi\|_{0,\bar{a}} + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} |\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))|\,dr_{\alpha_k}]$$

$$\le M(a)\,[\|\phi\|_{0,\bar{a}} + \sum \int_{\bar{a}_{\alpha_k}}^{\infty} \sup_{\bar{a}_{\alpha'_k} \le r_{\alpha'_k} < \infty} |\phi_{r_{\alpha_k}}(r)|\,dr_{\alpha_k}] = M(a)\|\phi\|.$$

If $\phi \in N_2$ and $\|\phi\| < \frac{\varepsilon}{2M(a)} = \delta(a)$ then $|u(x)| < \frac{\varepsilon}{2}$ for

$x \geq a$. Thus $\|u\|_{0,a} \leq \frac{\varepsilon}{2} < \varepsilon$ and Equation (1.7) is stable on N_2,

ii) The proof here is the same as part i) except the choice of δ is independent of a and hence we have uniform stability for Equation (1.7).

iii) Since Equation (1.7) is uniformly stable on N_3 there is a constant M such that $\|A(x;\xi)\| \leq M$ for $\bar{a} \leq a \leq \xi \leq x < \infty$. Therefore by part i) Equation (1.7) is stable on N_2.

Take any $a \geq \bar{a}$. The fundamental solution $A(x;\xi)$ for Equation (1.7) is defined only for $a \leq \xi \leq x < \infty$. We therefore introduce the matrix function $\bar{A}(x;\xi)$ defined for $x, \ \xi \geq a$ given by

$$\bar{A}(x;\xi) = \begin{cases} A(x;\xi) & a \leq \xi \leq x < \infty \\ 0 & \xi \geq a, \ \ \xi_i > x_i \text{ for some } i \text{ with } 1 \leq i \leq n. \end{cases} \tag{3.2}$$

From the assumption that $\lim_{|x|\to\infty} \|A(x;\xi)\| = 0$ for each $\xi \geq \bar{a}$, it follows that $\lim_{|x|\to\infty} \|\bar{A}(x;\xi)\| = 0$ for each $\xi \geq a$. We see that $\|\bar{A}(x;\xi)\| = \|A(x;\xi)\| \leq M$ for $a \leq \xi \leq x < \infty$ and $\|\bar{A}(x;\xi)\| = 0$ if $\xi_i > x_i$ for some i.

For each fixed x and $\alpha_k = \{i_1, i_2, \ldots, i_k\}$ the matrix function $\bar{A}(x;w(a,\xi;\alpha_k))$ is discontinuous on the $k - 1$ dimensional set

$$E_{\alpha_k} = \bigcup_{p=1}^{k} \{(a_1, a_2, \ldots, a_{i_1}-1, r_{i_1}, a_{i_1+1}, \ldots a_{i_p-1}, x_{i_p}, a_{i_p+1},$$

$$\ldots, a_{i_k-1}, r_{i_k}, a_{i_k+1}, \ldots, a_n) \mid a_{i_q} \le r_{i_q} \le x_{i_q},\ i_q \in \alpha_k, i_q \ne i_p\}.$$

Recalling that V_k is the k-dimensional Lebesque measure on R^k, we see that $V_k(E_{\alpha_k}) = 0$. Thus for each x the function $\|\bar{A}(x;w(a,\xi;\alpha_k))\|$ is integrable on sets of the form $[a_{\alpha_k}, b_{\alpha_k}]$.

Take any $\varepsilon > 0$. For any $x \ge a$ and any α_k let $R_{\alpha_k}(x) = \{r_{\alpha_k} \mid a_{\alpha_k} \le r_{\alpha_k} \le x_{\alpha_k}\}$ and $\bar{R}_{\alpha_k}(x) = [a,\infty) - R_{\alpha_k}(x)$. Take any $\phi \in N_2$. There is an $\bar{x} \ge a$ such that

$$\int_{\bar{R}_{\alpha_k}(\bar{x})} \sup_{\bar{a}_{\alpha'_k} \le r_{\alpha'_k} < \infty} |\phi_{r_{\alpha_k}}(r)| dr_{\alpha_k} < \frac{\varepsilon}{3M(2^n-1)} .$$

Take any x so that $|x|_\infty \ge |\bar{x}|_\infty$ and let $p \in R^n$ be such that $p = (|x|_\infty, |x|_\infty, \ldots, |x|_\infty)$. The solution of Equation (1.7) corresponding to ϕ is given by Equation (2.7). Therefore

$$|u(x)| \le \|A(x;a)\| |\phi(a)|$$

$$+ \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A(x;w(a,r;\alpha_k))\| |\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))| dr_{\alpha_k}$$

and thus

$$|u(x)| \leq \|A(x;a)\| |\phi(a)|$$

$$+ \sum \int_{a_{\alpha_k}}^{p_{\alpha_k}} \|\bar{A}(x;w(a,r;\alpha_k))\| |\phi_{r_{\alpha_k}}(w(a,r;\alpha_k)| dr_{\alpha_k}.$$

Thus

$$|u(x)| \leq \|A(x;a)\| |\phi(a)|$$

$$+ \sum \int_{R_{\alpha_k}(\bar{x})} \|\bar{A}(x;w(a,r;\alpha_k))\| |\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))| dr_{\alpha_k}$$

$$+ M \sum \int_{R_{\alpha_k}(p) - R_{\alpha_k}(\bar{x})} \sup_{\bar{a}_{\alpha'_k} \leq r_{\alpha'_k} < \infty} |\phi_{r_{\alpha_k}}(r)| dr_{\alpha_k}$$

and since $R_{\alpha_k}(p) - R_{\alpha_k}(\bar{x}) \subset \bar{R}_{\alpha_k}(\bar{x})$ we obtain

$$|u(x)| \leq \|A(x;a)\| |\phi(a)|$$

$$+ \sum \int_{R_{\alpha_k}(\bar{x})} \|\bar{A}(x;w(a,r;\alpha_k))\| |\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))| dr_{\alpha_k} + \frac{\varepsilon}{3}.$$

For each x the function $\|\bar{A}(x;w(a,r;\alpha_k))\|$ is discontinuous in r_{α_k} on $E_{\alpha_k} \cap [a_{\alpha_k}, \bar{x}_{\alpha_k}]$. Then since $\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))$ is continuous on the compact set $R_{\alpha_k}(\bar{x})$ and $\|\bar{A}(x;\xi)\| \leq M$, we see that $\|\bar{A}(x;w(a,r;\alpha_k))\| |\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))|$ is measurable and uniformly bounded for r_{α_k} in $R_{\alpha_k}(\bar{x})$. Since $\lim_{|x|\to\infty} \|\bar{A}(x;\xi)\| = 0$ for each $\xi \geq a$ it follows that $\lim_{|x|\to\infty} \|\bar{A}(x;w(a,r;\alpha_k)\| |\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))| = 0$ for $r_{\alpha_k} \in R_{\alpha_k}(\bar{x})$. Using the dominated convergence theorem (see Section 2.1) we have

$$\lim_{|x|\to\infty}\int_{R_{\alpha_k}(\bar{x})} \|\bar{A}(x;w(a,r;\alpha_k))\| |\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))| dr_{\alpha_k} = 0$$

for each α_k. Thus there is a T_1 so that for each α_k,

$$\int_{R_{\alpha_k}(\bar{x})} \|\bar{A}(x;w(a,r;\alpha_k))\| |\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))| dr_{\alpha_k} < \frac{\varepsilon}{3(2^n-1)} \text{ when}$$

$|x|_\infty \geq T_1$. Also, there is a T_2 such that $\|A(x;a)\| < \frac{\varepsilon}{3|\phi(a)|}$ when $|x|_\infty \geq T_2$. Let $T = \max\{|\bar{x}|_\infty, T_1, T_2\}$. Therefore for x with $|x|_\infty \geq T$ we see that $|u(x)| < \frac{\varepsilon}{3|\phi(a)|} \cdot |\phi(a)| + \sum \frac{\varepsilon}{3(2^n-1)} + \frac{\varepsilon}{3} = \varepsilon$. Thus $\lim_{|x|\to\infty} |u(x)| = 0$ and the proof is complete.

<u>Theorem 3.3</u>. Suppose $A(x;\xi)$ is the fundamental solution for Equation (1.7). Then we have the following.

i) If there exists a constant $M(a)$ such that

$$\|A(x,a)\| + \sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A(x;w(a,r;\alpha_k))\| dr_{\alpha_k} \leq M(a) \quad \text{for}$$

$x \geq a \geq \bar{a}$, then Equation (1.7) is stable on N_1.

ii) If the constant M in part i) is independent of a, then Equation (1.7) is uniformly stable on N_1.

iii) If for each $a \geq \bar{a}$,

$$\|A(x;a)\| + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A(x;w(a,r;\alpha_k))\| dr_{\alpha_k} \to 0 \quad \text{as} \quad |x| \to \infty,$$

then Equation (1.7) is asymptotically stable on N_1.

<u>Proof</u>. i) Take any $a \geq \bar{a}$, $\varepsilon > 0$, and any $\phi \in N_1$. From Theorem 2.2, the solution of Equation (1.7) for $x \geq a$ is given by Equation (2.7). Thus

$$|u(x)| \leq \|A(x;a)\| |\phi(a)|$$

$$+ \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A(x;w(a,r;\alpha_k))\| |\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))| dr_{\alpha_k}$$

$$\leq \|\phi\| [\|A(x;a)\| + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A(x;w(a,r;\alpha_k))\| dr_{\alpha_k}] \leq M(a)\|\phi\|.$$

For $\phi \in N_1$ such that $\|\phi\| < \frac{\varepsilon}{2M(a)} = \delta(a)$ then $|u(x)| < \frac{\varepsilon}{2}$ and thus $\|u\|_{0,a} \leq \frac{\varepsilon}{2} < \varepsilon$.

ii) This follows from the argument used in part i) except δ will now be independent of a.

iii) From Lemma 2.5, Lemma 2.2, and the hypothesis of part iii) it follows that the expression given in part i) is bounded for $x \geq a$. Thus by part i) we see that Equation (1.7) is stable on N_1. In the proof of part i) we obtained the estimate

$$|u(x)| \le \|\phi\| [\|A(x;a)\| + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A(x;w(a,r;\alpha_k))\| dr_{\alpha_k}]$$

for the solution of Equation (1.7) for any $\phi \in N_1$. It follows directly from this and the hypothesis of part iii) that $\lim_{|x| \to \infty} |u(x)| = 0$. This completes the proof.

Theorem 3.4. Let $A(x;\xi)$ be the fundamental solution for Equation (1.7). Suppose for each $x \ge a$, and each α_k, $A_{\xi_{\alpha_k}}(x;\xi)$ is continuous in ξ such that $a \le \xi \le x$. Let $\overline{N}_0 = N_0 \cap \{\phi | \phi : R^n \to R^m,\ \phi(x)$ continuous and $\phi_{x_{\alpha_k}}(x)$ continuous on $x \ge \overline{a}\}$ ($\overline{N}_0$ is normed by $\|\cdot\|_{0,\overline{a}}$). Then we have the following.

i) If there is a constant $M(a)$ so that

$$\sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\| dr_{\alpha_k} \le M(a),$$ then Equation (1.7) is stable on $\overline{N}_0$.

ii) If the constant M in part i) is independent of $a \ge \overline{a}$ then Equation (1.7) is uniformly stable on $\overline{N}_0$.

iii) If $\sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\| dr_{\alpha_k} \to 0$ as $|x| \to \infty$, Then Equation (1.7) is asymptotically stable on $\overline{N}_0 \cap \{\phi | \phi : R^n \to R^m,\ |\phi(x)| \to 0$ as $|x| \to \infty\}$.

<u>Proof</u>. i) Take any $a \geq \bar{a}$, $\varepsilon > 0$, and $\phi \in \bar{N}_0$. By Corollary 2.4, the solution of Equation (1.7) corresponding to ϕ is given by Equation (2.12). Thus $|u(x)| \leq \|\phi\|_{0,\bar{a}} + \|\phi\|_{0,\bar{a}}[\sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\|dr_{\alpha_k}] \leq \|\phi\|_{0,\bar{a}}[1 + M(a)]$. Hence for $\phi \in \bar{N}_0$ with $\|\phi\|_{0,\bar{a}} < \frac{\varepsilon}{2[1+M(a)]} = \delta(a)$ we have $\|u\|_{0,\bar{a}} \leq \frac{\varepsilon}{2} < \varepsilon$.

ii) This proof is like that of part i) except now δ is independent of a.

iii) Take any $a \geq \bar{a}$. Since for each α_k, $\sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\|dr_{\alpha_k}$ is continuous on $x \geq a$ and since by hypothesis $\sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\|dr_{\alpha_k} \to 0$ as $|x| \to \infty$ it follows that $\sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\|dr_{\alpha_k}$ is bounded for $x \geq a$. Then part i) implies Equation (1.7) is stable on $\bar{N}_0 \cap \{\phi(x)|\phi:R^n \to R^m, \ |\phi(x)| \to 0 \text{ as } |x| \to \infty\}$. For $\phi \in \bar{N}_0$ the solution u of Equation (1.7) satisfies

$$|u(x)| \leq |\phi(x)| + \|\phi\|_{0,\bar{a}}[\sum\int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\|dr_{\alpha_k}].$$

Asymptotic stability of Equation (1.7) on $\bar{N}_0 \cap \{\phi(x) | \phi: R^n \to R^m, |\phi(x)| \to 0 \text{ as } |x| \to \infty\}$ now follows from this estimate, hypothesis iii), and the fact that $|\phi(x)| \to 0$ as $|x| \to \infty$. This completes the proof.

These theorems generalize results obtained by Bownds and Cushing [5] for the case $n = 1$. Also, in the case $n = 1$, when Equation (1.7) is equivalent to an initial value problem, we see that Theorem 3.1 specializes to give well known results for the linear ordinary differential equations [16].

3.2 Preservation of Stability for Lipschitz Type Nonlinear Perturbations

We now turn our attention to the nonlinear Equation (1.8). We will prove that, under conditions to be specified on the function $f(x,r,z)$, the nonlinear equation will remain stable when the linear equation is sufficiently stable.

The following lemmas will be useful.

Lemma 3.1. Let $g:[\bar{a},\infty) \to R$ such that $g \in L_1[\bar{a},\infty) \cap C[\bar{a},\infty)$ and such that $g(x) \geq 0$ for $x \geq \bar{a}$. Then the unique continuous solution of the scalar integral equation

$$u(x) = 1 + \int_a^x g(r)u(r)dr \qquad (3.3)$$

is uniformly bounded in a and x with $\bar{a} \leq a \leq x < \infty$.

Proof. For each $a \geq \bar{a}$ the successive approximations for Equation (3.3) converge uniformly to the unique continuous solution. Consider the successive approximations

$$u_0(x) \equiv 0 \qquad u_k(x) = 1 + \int_a^x g(r)u_{k-1}(r)dr \qquad k = 1, 2, \ldots .$$

Let $z_k(x) = u_{k+1}(x) - u_k(x)$ $k = 0, 1, 2, \ldots$. We see that $u_k(x) = \sum_{j=0}^{k-1} z_j(x)$. If $u(x)$ is the unique continuous solution of Equation (3.3) for $x \geq a$, then $u(x) = \lim_{k\to\infty} u_k(x)$ uniformly on $x \geq a$.

Since $z_0(x) = 1$ and $z_k(x) = \int_a^x g(r)z_{k-1}(r)dr$ for $k = 1, 2, \ldots$ we have

$$|z_1(x)| = \int_a^x g(r)z_0(r)dr \leq \int_a^x g(r)\exp[2\int_a^r g(s)ds]dr.$$

Since

$$\frac{\partial}{\partial r}\{\exp[2\int_a^r g(s)ds]\} \geq 2g(r)\exp[2\int_a^r g(s)ds] \qquad (3.4)$$

we have

$$|z_1(x)| \leq \frac{1}{2}\int_a^x \frac{\partial}{\partial r}\{\exp[2\int_a^r g(s)ds]\}dr$$

$$\leq \frac{1}{2}\exp[2\int_a^x g(r)dr] \qquad (3.5)$$

for $x \geq a$ (see Walter [39], p. 142 and p. 148). Assume $|z_k(x)| \leq \frac{1}{2^k}\exp[2\int_a^x g(r)dr]$ for $x \geq a$. We then have

$$|z_{k+1}(x)| \leq \int_a^x g(r)|z_k(r)|dr \leq \frac{1}{2^k}\int_a^x g(r)\exp[2\int_a^r g(s)ds]dr.$$

Using (3.4) and (3.5) again, we see that

$$|z_{k+1}(x)| \leq \frac{1}{2^{k+1}}\int_a^x \frac{\partial}{\partial r}\{\exp[2\int_a^r g(s)ds]\}dr \leq \frac{1}{2^{k+1}}\exp[2\int_a^x g(r)dr]$$

and hence

$$|z_j(x)| \leq \frac{1}{2^j}\exp[2\int_a^x g(r)dr] \quad \text{for} \quad x \geq a \quad \text{and} \quad j = 0, 1, 2, \ldots .$$

Thus

$$|u(x)| \leq \sum_{j=0}^{\infty}|z_j(x)| \leq \exp[2\int_a^x g(s)ds]\sum_{j=0}^{\infty}(\tfrac{1}{2})^j = 2\exp[2\int_a^x g(s)ds].$$

But since $g \in L_1[\bar{a},\infty)$ and $g(x) \geq 0$ it follows that $|u(x)| \leq 2\exp[\int_{\bar{a}}^{\infty} g(s)ds]$ and this bound holds for any x and a such that $\bar{a} \leq a \leq x < \infty$. This completes the proof.

Consider the following hypotheses on the m-dimensional vector valued function $f(x,r,z)$ defined for $\bar{a} \leq a \leq r \leq x < \infty$, $z \in R^m$, $|z| \leq b$ where b is a positive constant.

(H1) The function $f(x,r,z)$ is continuous and there exists a continuous function $\beta(x,r) \geq 0$ so that

$$|f(x,r,z_1) - f(x,r,z_2)| \leq \beta(x,r)|z_1 - z_2| \quad \text{for} \quad z_1,$$

$$z_2 \in R^m, \quad |z_1|, \quad |z_2| \leq b, \quad \bar{a} \leq a \leq r \leq x < \infty.$$

(H2) There exists a continuous scalar function $\gamma_0(x) \geq 0$ such that $|f(x,x,z)| \leq \gamma_0(x)|z|$ for $|z| \leq b$, $x \geq \bar{a}$ and such that $\int_{\bar{a}}^{\infty}\gamma_0(r)dr < \infty$. For each $\alpha_k (1 \leq k \leq n)$, the function $\frac{\partial^k f}{\partial x_{\alpha_k}}(x,w(x,r;\alpha_k),z)$ is continuous for $x \geq a \geq \bar{a}$, $\bar{a}_{\alpha_k} \leq a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$, and $|z| \leq b$. Suppose for each α_k there exists a continuous scalar function $\gamma_{\alpha_k}(x,r_{\alpha_k}) \geq 0$ such that $|\frac{\partial^k f}{\partial x_{\alpha_k}}(x,w(x,r;\alpha_k),z)| \leq \gamma_{\alpha_k}(x,r_{\alpha_k})|z|$ for $x \geq a \geq \bar{a}$. $\bar{a}_{\alpha_k} \leq a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$, $|z| \leq b$ and such that

$$\int_{\bar{a}}^{\infty}\int_{\bar{a}_{\alpha_k}}^{r_{\alpha_k}} \gamma_{\alpha_k}(r,s_{\alpha_k})ds_{\alpha_k}dr < \infty.$$

Let $\Phi(x;a)$ be the solution of the linear Equation (1.7) associated with the nonlinear Equation (1.8) for

$\bar{a} \leq a \leq x < \infty$ and corresponding to $\phi(x)$. If $\phi_{x_{\alpha_k}}(x)$ is continuous on $x \geq a$ for each α_k we have, by Theorem 2.2,

$$\Phi(x;a) = A(x;a)\phi(a)$$
$$+ \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x;w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k} \qquad (3.6)$$

where $A(x;\xi)$ is the fundamental solution for Equation (1.7).

We now give an important lemma.

Lemma 3.2. Suppose $\phi(x)$ and $\phi_{x_{\alpha_k}}(x)$ are continuous on $x \geq a \geq \bar{a}$. Let $A(x;\xi)$ be the fundamental solution for Equation (1.7) and suppose $f(x,r,z)$ satisfies (H1) and (H2). Suppose Equation (1.7) is uniformly stable on N_3. Then there is a constant $L > 0$ independent of a such that if the solution $\Phi(x;a)$ of Equation (1.7) corresponding to ϕ satisfies $\|\Phi(x;a)\|_{0,a} \leq bL^{-1}$ then the solution $u(x)$ of Equation (1.8) exists and satisfies $|u(x)| \leq L\|\Phi(x;a)\|_{0,a}$ for $x \geq a$.

Proof. The function $f(x,r,z)$ may be extended in such a way that the extension $\bar{f}(x,r,z)$ is defined and satisfies (H1), (H2) for $\bar{a} \leq a \leq r \leq x < \infty$, $z \in R^m$. From Corollary 2.1 it follows that Equation (1.8) with the perturbation taken to be $\bar{f}(x,r,z)$ has a unique continuous solution $u(x)$ on $x \geq a$. Using Equation (3.6) and Corollary 2.3 we see that

$$u(x) = \Phi(x;a) + \int_a^x A(x;r)\frac{\partial}{\partial r}\left[\int_a^r \overline{f}(r,s,u(s))ds\right]dr$$

$$= \Phi(x;a) + \int_a^x A(x;r)\left[\overline{f}(r,r,u(r))\right.$$

$$\left. + \sum\int_{a_{\alpha_k}}^{r_{\alpha_k}} \frac{\partial^k \overline{f}}{\partial r_{\alpha_k}}(r,w(r,s;\alpha_k),u(w(r,s;\alpha_k)))ds_{\alpha_k}\right]dr.$$

Since Equation (1.7) is uniformly stable on N_3 there exists an M such that $\|A(x;r)\| \leq M$ for $\overline{a} \leq a \leq r \leq x < \infty$. Using this and (H2) we obtain

$$|u(x)| \leq \|\Phi(x;a)\|_{0,a} + M\int_a^x [\gamma_0(r)|u(r)|$$

$$+ \sum\int_{a_{\alpha_k}}^{r_{\alpha_k}} \gamma_{\alpha_k}(r,s_{\alpha_k})|u(w(r,s;\alpha_k))|ds_{\alpha_k}]dr.$$

Let $v(x) = \sup\limits_{a \leq r \leq x} |u(r)|$. We then have

$$v(x) \leq \|\Phi(x;a)\|_{0,a} + M\int_a^x \overline{\gamma}(r)v(r)dr$$

where $\overline{\gamma}(r) = \gamma_0(r) + \sum\int_{a_{\alpha_k}}^{r_{\alpha_k}} \gamma_{\alpha_k}(r,s_{\alpha_k})ds_{\alpha_k}$. Let $A^*(x;\xi)$ be the fundamental solution for the scalar equation

$$u(x) = \phi(x) + M\int_a^x \bar{\gamma}(r)u(r)dr.$$

It follows from (H2) that $\bar{\gamma}(r) \geq 0$ and $\bar{\gamma}(r) \in L_1[\bar{a},\infty) \cap C[\bar{a},\infty)$. Thus by Lemma 3.1 there exists a constant $L > 0$ such that $0 \leq A^*(x;\xi) \leq L$ for $\bar{a} \leq \xi \leq x < \infty$. Continuity of $v(x)$ for $x \geq a$ follows from the continuity of $u(x)$ and we may therefore apply Theorem 2.4 in the form of Remark 2.7 to obtain

$$v(x) \leq A^*(x;a)\|\Phi(x;a)\|_{0,a} \leq L\|\Phi(x;a)\|_{0,a}.$$

Thus $\|u\|_{0,a} \leq L\|\Phi(x;a)\|_{0,a}$. If $\|\Phi(x;a)\|_{0,a} \leq bL^{-1}$ we see that $|u(x)| \leq b$ for all $x \geq a$ and, since $\bar{f}(x,r,z) = f(x,r,z)$ for z with $|z| \leq b$, $u(x)$ must satisfy Equation (1.8) on $x \geq a$. This completes the proof of Lemma 3.2.

In the remainder of the dissertation we will refer to an arbitrary space of functions on $x \geq \bar{a}$ as N. We will assume that functions in N are continuous and have continuous pure mixed partials of all orders less than or equal to n.

We now give preservation of stability results.

<u>Theorem 3.5</u>. Suppose f satisfies (H1) and (H2). Suppose Equation (1.7) is uniformly stable on N_3. Then

Equation (1.8) preserves stability and uniform stability on any space N and preserves asymptotic stability on any space N such that $N_3 \subset N$.

Proof. We see that preservation of stability and uniform stability for Equation (1.8) are immediate consequences of Lemma 3.2.

We now turn to preservation of asymptotic stability. Take $a \geq \bar{a}$ and $\varepsilon > 0$. Since Equation (1.7) is asymptotically stable on N it is stable on N and since stability is preserved we see that Equation (1.8) is stable on N. Thus for any $c > 0$ there exists a $\delta_0(a)$ such that if $\|\phi\|_N < \delta_0(a)$ then the solution $u(x)$ of Equation (1.8) satisfies $\|u\|_{0,a} < c$. Asymptotic stability of Equation (1.7) on N means there exists a $\delta_1(a)$ such that if $\|\phi\|_N < \delta_1(a)$ then the solution $\Phi(x;a)$ of Equation (1.7) corresponding to ϕ satisfies $\lim_{|x|\to\infty} |\Phi(x;a)| = 0$. Let $\delta(a) = \min\{\delta_0(a), \delta_1(a)\}$. Take $\phi \in N$ such that $\|\phi\|_N < \delta(a)$ and let $u(x)$ be the solution of Equation (1.8) corresponding to ϕ. With $v(x)$ and $\bar{\gamma}(r)$ as in Lemma 3.2 and using Corollary 2.3 we see that

$$|u(x)| \leq |\Phi(x;a)| + \int_a^x \|A(x;r)\| \bar{\gamma}(r) v(r)\,dr$$

$$\leq |\Phi(x;a)| + c\int_a^x \|A(x;r)\| \bar{\gamma}(r)\,dr.$$

By Theorem 3.1 there exists an M such that $\|A(x;\xi)\| \leq M$ for $\bar{a} \leq a \leq \xi \leq x < \infty$. Let $R(x) = \{r | r \in R^n, a \leq r \leq x\}$ and $\bar{R}(x) = [a,\infty) - R(x)$. Since $\int_{\bar{a}}^{\infty} \bar{\gamma}(r)dr < \infty$ there exists an $\bar{x}$ such that $\int_{\bar{R}(\bar{x})} \bar{\gamma}(r)dr < \frac{\varepsilon}{3Mc}$. Then for x such that $|x|_\infty \geq |\bar{x}|_\infty$ we have

$$|u(x)| \leq |\Phi(x;a)| + c\int_{R(x)\cap R(\bar{x})} \|A(x;r)\|\bar{\gamma}(r)dr$$

$$+ c\int_{R(x)-[R(x)\cap R(\bar{x})]} \|A(x;r)\|\bar{\gamma}(r)dr$$

and

$$|u(x)| \leq |\Phi(x;a)| + c\int_{R(\bar{x})} \|\bar{A}(x;r)\|\bar{\gamma}(r)dr$$

$$+ c\int_{\bar{R}(\bar{x})} \|\bar{A}(x;r)\|\bar{\gamma}(r)dr \tag{3.7}$$

where $\bar{A}(x;\xi)$ is defined by Equation (3.2).

Since Equation (1.7) is asymptotically stable on N and $N_3 \subset N$ it follows that Equation (1.7) is asymptotically stable on N_3. Thus from Theorem 3.1 we see that $\lim_{|x|_\infty \to \infty} \|A(x;\xi)\| = 0$ for each $\xi \geq \bar{a}$, $x \geq \xi \geq \bar{a}$ and hence $\lim_{|x|_\infty \to \infty} \|\bar{A}(x;\xi)\| = 0$ for each $\xi \geq \bar{a}$. Recalling that $\|A(x;\xi)\| \leq M$ we see $\|\bar{A}(x;\xi)\|\bar{\gamma}(r)$ is uniformly bounded and

$\lim_{|x|_\infty \to \infty} \|\bar{A}(x;r)\| \bar{\gamma}(r) = 0$ for $r \in R(\bar{x})$. Thus using the dominated convergence theorem we obtain $\lim_{|x|_\infty \to \infty} \int_{R(\bar{x})} \|\bar{A}(x;r)\| \bar{\gamma}(r)\,dr = 0$. Then there is an M_1 such that for x with $|x|_\infty \geq M_1$ we have $\int_{R(\bar{x})} \|\bar{A}(x;r)\| \bar{\gamma}(r)\,dr < \frac{\varepsilon}{3c}$. Also there exists an M_2 so that $|\phi(x;a)| < \frac{\varepsilon}{3}$ when $|x|_\infty \geq M_2$. Let $\bar{M} = \max\{|\bar{x}|_\infty, M_1, M_2\}$. Then for x with $|x|_\infty \geq \bar{M}$ we have from Equation (3.7)

$$|u(x)| \leq \frac{\varepsilon}{3} + c\left(\frac{\varepsilon}{3c}\right) + cM\int_{\bar{R}(\bar{x})} \bar{\gamma}(r)\,dr < \varepsilon.$$

Thus $\lim_{|x| \to \infty} |u(x)| = 0$ and Equation (1.8) is asymptotically stable on N. This completes the proof.

For $n = 1$, Theorem 3.5 reduces to a result obtained by Bownds and Cushing [5]. Further specialization to the case when the Volterra equation is equivalent to an initial value problem in ordinary differential equations $(K_1(x_1,r_1) \equiv \bar{K}_1(r_1), f(x_1,r_1,z) \equiv \hat{f}(r_1,z), \phi \in N_3)$, yields results given in Coppel [16].

Examples given in [16] show that the hypothesis of uniform stability on N_3 in Theorem 3.5 cannot be weakened to stability on N_3, and also that the integrability condition on the function $\gamma_0(x)$ in (H2) cannot be weakened to $\gamma_0(x) \to 0$ as $|x| \to \infty$. The scalar equation in one independent variable with $a = 0$, $K_1 \equiv 0$, and $f(x,r,z) = [e^{(x-r)} - 1]z$ shows that the hypotheses on the derivatives in (H2) cannot be dropped. In this example $f(x,x,z) \equiv 0$. However $\frac{\partial f}{\partial x}(x,r,z) = e^{(x-r)}z$ does not satisfy the derivative assumption in (H2). Although the unperturbed equation $(u(x) = \phi)$ is uniformly stable on N_3 it may be shown that the perturbed equation is not stable on N_3. Other aspects of this example are discussed in [7].

Corollary 3.1. Suppose Equation (1.7) is uniformly (and asymptotically) stable on N_3 and $f(x,r,z)$ satisfies (H1) and (H2). Then Equation (1.8) is uniformly (and asymptotically) stable on N_2.

Proof. From Theorems 3.1 and 3.2 we see that Equation (1.7) is uniformly stable on N_2 iff Equation (1.7) is uniformly stable on N_3. Thus it follows from Theorem 3.5 that Equation (1.8) is uniformly stable on N_2. If in addition Equation (1.7) is asymptotically stable on N_3, part iii) of Theorem 3.2 implies Equation (1.7) is also asymptotically stable on N_2. Hence, by Theorem 3.5, Equation (1.8) is also asymptotically stable on N_2. This completes the proof.

We also have a corollary pertaining to stability of Equation (1.8) on the space N_1. It follows directly from Theorem 3.3 and Theorem 3.5.

Corollary 3.2. Suppose Equation (1.7) is uniformly stable on N_3 and f satisfies the Hypotheses (H1) and (H2). Then

i) if $A(x;\xi)$ satisfies the condition in part i) of Theorem 3.3, Equation (1.8) is stable on N_1.

ii) If $A(x;\xi)$ satisfies the condition in part ii) of Theorem 3.3, Equation (1.8) is uniformly stable on N_1.

iii) If $A(x;\xi)$ satisfies the hypothesis of part iii) of Theorem 3.3, Equation (1.8) is asymptotically stable on N_1.

3.3 Preservation of Stability for Little o Type Nonlinear Perturbations

As in the case of ordinary differential equations linearization of the multiple Volterra equation gives rise to stability questions for Equation (1.8) where $f(x,r,z)$ is higher order in z. For some perturbations of this type the results of the previous section may be used to establish various stabilities for Equation (1.8). However if the perturbation $f(x,r,z)$ is independent of x and r then the results of Section 3.2 cannot be applied. We establish results in this section which allow us to consider perturbations which are little o in z and are independent of x and r.

Let $A(x;\xi)$ be the fundamental solution for Equation (1.7). Consider the following hypothesis on $f(x,r,z)$ and $A(x;\xi)$.

(H3) Suppose $f(x,r,z)$ is continuous for $\bar{a} \leq r \leq x < \infty$, $z \in R^m$ and for each α_k, $\frac{\partial^k f}{\partial x_{\alpha_k}}(x,w(x,r;\alpha_k),z)$ is continuous for $\bar{a} \leq a \leq x < \infty$, $\bar{a}_{\alpha_k} \leq a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} < \infty$, $z \in R^m$. Suppose there exists continuous functions $\psi(x) \geq 0$, $\psi_{\alpha_k}(x,r_{\alpha_k}) \geq 0$ and a constant $M(a)$ having the following properties:

a) for each $\varepsilon > 0$ there exists an $\eta > 0$ such that $|z_1|$, $|z_2| \leq \eta$ implies $|f(x,x,z_1) - f(x,x,z_2)| \leq \varepsilon\psi(x)|z_1 - z_2|$ for $x \geq a \geq \bar{a}$ and

$$\left|\frac{\partial^k f}{\partial x_{\alpha_k}}(x,w(x,r;\alpha_k),z_1) - \frac{\partial^k f}{\partial x_{\alpha_k}}(x,w(x,r;\alpha_k),z_2)\right| \leq \varepsilon\psi_{\alpha_k}(x,r_{\alpha_k})|z_1 - z_2|$$

for $\bar{a} \leq a \leq x$, $\bar{a}_{\alpha_k} \leq a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k}$,

b) $\int_a^x \|A(x,r)\|\psi(r)dr \leq M(a)$ and

$$\int_a^x \int_{a_{\alpha_k}}^{r_{\alpha_k}} \|A(x;r)\|\psi_{\alpha_k}(r,s_{\alpha_k})ds_{\alpha_k}dr \leq M(a)$$

for $x \geq a \geq \bar{a}$.

Although (H3) is not just an assumption on the perturbation $f(x,r,z)$, in cases when b) of hypothesis (H3) is satisfied with $\psi(r) = \psi_{\alpha_k}(x, r_{\alpha_k}) \equiv 1$ hypothesis (H3) does define a class of perturbations. Perturbations of this type have been studied for Volterra equations in one independent variable in [27], [28], [29], [30], [31]. For our study of preservation of stability the hypothesis as stated above yields more general results.

To illustrate the connection between hypothesis (H3) and the usual little o assumption we suppose the fundamental solution satisfies $\int_a^x \|A(x,;r)\| dr \le M(a)$ and consider perturbations $f(x,r,z) = \bar{f}(r,z)$ which are independent of x. Then $\bar{f}(r,z)$ will satisfy hypothesis (H3) if for each $\varepsilon > 0$ there is an $\eta > 0$ such that $|\bar{f}(r,z_1) - \bar{f}(r,z_2)| \le \varepsilon|z_1 - z_2|$ uniformly in r when $|z_1|, |z_2| \le \eta$. We point out that this condition is slightly stronger than the little o assumption $\bar{f}(r,z) = o(|z|)$ uniformly in r, which is often assumed in ordinary differential equations. However, it may be shown that if the matrix $\frac{\partial \bar{f}}{\partial z}(r,z)$ is continuous in z for some neighborhood of $z = 0$ uniformly in r then the two conditions are equivalent.

The following theorem will be used to obtain preservation of stability results.

<u>Theorem 3.6</u>. Suppose $f(x,r,z)$ and the fundamental solution $A(x;\xi)$ satisfy (H3). Suppose $\phi(x)$, $\phi_{x_{\alpha_k}}(x)$

are continuous for $x \geq a$ and $\Phi(x;a)$ is given by Equation (3.6). For each $\lambda \in (0,1)$ there exists an $\varepsilon_0 = \varepsilon_0(\lambda) > 0$ such that if ε satisfies $0 < \varepsilon \leq \varepsilon_0$ and if $\|\Phi(x;a)\|_{0,a} \leq \lambda\varepsilon$ then there exists a unique continuous solution $u(x)$ of Equation (1.8) such that $\|u\|_{0,a} \leq \varepsilon$.

Proof. Take any $\lambda \in (0,1)$. Take $\rho > 0$ so that $\rho M(a) < 1$. By hypothesis (H3) there exists an η such that $|z_1|$, $|z_2| \leq \eta$ implies $|f(x,x,z_1) - f(x,x,z_2)| \leq \frac{\rho}{2^n}\psi(x)|z_1 - z_2|$ and

$$|\frac{\partial^k f}{\partial x_{\alpha_k}}(x,w(x,r;\alpha_k),z_1) - \frac{\partial^k f}{\partial x_{\alpha_k}}(x,w(x,r;\alpha_k),z_2)|$$

$$\leq \frac{\rho}{2^n}\psi_{\alpha_k}(x,r_{\alpha_k})|z_1 - z_2|$$

for each α_k. Take $\mu > 0$ such that $\mu M(a) < (1 - \lambda)$. From hypothesis (H3) and the fact that $\frac{\partial^k f}{\partial x_{\alpha_k}}(x,w(x,r;\alpha_k),0) \equiv 0$, there exists a δ such that $|z| \leq \delta$ implies $|f(x,x,z)| \leq \frac{\mu}{2^n}\psi(x)|z|$ and $|\frac{\partial^k f}{\partial x_{\alpha_k}}(x,w(x,r;\alpha_k),z)| \leq \frac{\mu}{2^n}\psi_{\alpha_k}(x,r_{\alpha_k})|z|$ for each α_k. Let $\varepsilon_0 = \min\{\delta,\eta\}$ and take any ε such that $0 < \varepsilon \leq \varepsilon_0$. Consider the set $S(\varepsilon) = \{g(x)|g:R^n \to R^m$, $g \in C[a,\infty), \|g\|_{0,a} \leq \varepsilon\}$. The set $S(\varepsilon)$ is a closed subset of a Banach space and is therefore complete.

We define the map T on $S(\varepsilon)$ so that for each $g \in S(\varepsilon)$ then

$$(Tg)(x) = \Phi(x;a) + \int_a^x A(x;r)[f(r,r,g(r))$$

$$+ \sum \int_{a_{\alpha_k}}^{r_{\alpha_k}} \frac{\partial^k f}{\partial r_{\alpha_k}}(r,w(r,s;\alpha_k),g(w(r,s;\alpha_k)))ds_{\alpha_k}]dr.$$

It follows from Lemmas 2.2 and 2.5 that $(Tg)(x)$ is continuous for $x \geq a$. Using the estimates above we have

$$|(Tg)(x)| \leq \|\Phi(x;a)\|_{0,a} + \frac{\mu\varepsilon}{2^n}[\int_a^x \|A(x;r)\|\psi(r)dr$$

$$+ \sum \int_a^x \int_{a_{\alpha_k}}^{r_{\alpha_k}} \|A(x;r)\|\psi_{\alpha_k}(r,s_{\alpha_k})ds_{\alpha_k}dr]$$

$$\leq \lambda\varepsilon + \frac{\mu\varepsilon}{2^n} \cdot 2^n \cdot M(a) = \lambda\varepsilon + \varepsilon\mu M(a) \leq \lambda\varepsilon + \varepsilon(1 - \lambda) = \varepsilon.$$

Therefore $\|Tg\|_{0,a} \leq \varepsilon$ and $T:S(\varepsilon) \to S(\varepsilon)$.

Take g_1, $g_2 \in S(\varepsilon)$. Then using the facts that $\|g_1\|_{0,a} \leq \varepsilon$, $\|g_2\|_{0,a} \leq \varepsilon$, and the estimates above we see that

$$|(Tg_1)(x) - (Tg_2)(x)| \leq \frac{\rho}{2^n}[\int_a^x \|A(x;r)\|\psi(r)|g_1(r) - g_2(r)|dr$$

$$+ \sum \int_{a_{\alpha_k}}^{r_{\alpha_k}} \| A(x;r) \| \, \psi_{\alpha_k}(r,s_{\alpha_k}) \, | g_1(w(r,s;\alpha_k))$$

$$- g_2(w(r,s;\alpha_k)) | ds_{\alpha_k} dr$$

$$\leq \frac{\rho}{2^n} \| g_1 - g_2 \|_{0,a} \cdot 2^n \cdot M(a) = \rho M(a) \| g_1 - g_2 \|_{0,a}.$$

Thus $\| Tg_1 - Tg_2 \|_{0,a} \leq \rho M(a) \| g_1 - g_2 \|_{0,a}$. Since $\rho M(a) < 1$, T is a contraction on $S(\varepsilon)$ and has a unique fixed point $u(x) \in S(\varepsilon)$. By Corollary 2.3, $u(x)$ is a solution of Equation (1.8) and the proof is complete.

Arguments similar to the one employed in Theorem 3.6 have been used in [27], [28], [31] to study perturbed nonlinear Volterra equations in one independent variable. Their approach is based on a representation for the nonlinear equation in terms of the resolvent for the associated linear equation.

We will now introduce further conditions on the perturbation which will be needed for preservation of asymptotic stability. Let $a \geq \bar{a}$ and let $X \in R$ so that $X \geq \max_{1 \leq i \leq n} \{a_i\}$. Let $X^* \in R^n$ and be given by $X^* = (X, X, \ldots, X)$. Certain problems arise in attempting to prove Theorem 3.7 below when the perturbation $f(x,r,z)$ is not independent of x. We therefore consider the following hypothesis on the perturbation $f(x,r,z)$ and $A(x;\xi)$.

(H4) Suppose $f(x,r,z) \equiv \bar{f}(r,z)$. Suppose there exists a continuous scalar function $\bar{\psi}(r) \geq 0$ such that for each $\varepsilon > 0$ there exists an η such that $|z_1|$, $|z_2| \leq \eta$ implies $|\bar{f}(r,z_1) - \bar{f}(r,z_2)| \leq \varepsilon\bar{\psi}(r)|z_1 - z_2|$ for $\bar{a} \leq a < r < \infty$, and there exists a constant $\bar{M}(a)$ such that $\int_a^x \|A(x;r)\|\bar{\psi}(r)dr \leq \bar{M}(a)$ for all $x \geq a$. Suppose for each $X^* \geq a$ and each α_k we have

$$\lim_{|x_{\alpha_k}| \to \infty} \int_a^{w(x,X^*;\alpha_k)} \|A(x;r)\|\bar{\psi}(r)dr = 0 \quad \text{for} \quad x_{\alpha_k} \geq X^*_{\alpha_k},$$

uniformly in x_j such that $a_j \leq x_j \leq X$ with $j \neq \alpha_k$.

If we assume f satisfies (H4) and demand more of the solution of the linear equation we then have the following theorem concerning the solution $u(x)$ of Equation (1.8).

<u>Theorem 3.7</u>. Suppose $f(x,r,z)$ satisfies (H4), $\phi(x)$, $\phi_{x_{\alpha_k}}(x)$ are continuous for $x \geq a$, and $\Phi(x;a)$ is given by Equation (3.6). For each $\lambda \in (0,1)$ there exists an $\varepsilon_0 = \varepsilon_0(\lambda) > 0$ such that if ε satisfies $0 < \varepsilon \leq \varepsilon_0$ and if $\|\Phi(x;a)\|_{0,a} \leq \lambda\varepsilon$ then there exists a unique continuous solution $u(x)$ of Equation (1.8) such that $\|u\|_{0,a} \leq \varepsilon$. If, in addition, $\lim_{|x| \to \infty} |\Phi(x;a)| = 0$ then $\lim_{|x| \to \infty} |u(x)| = 0$.

Proof. Take $\lambda \in (0,1)$. Take any $\rho > 0$ such that $\rho\overline{M}(a) < 1$. By hypothesis (H4) there exists an η such that $|z_1|$, $|z_2| \leq \eta$ implies $|\overline{f}(r,z_1) - \overline{f}(r,z_2)| \leq \rho\overline{\psi}(r)|z_1 - z_2|$. Take $\mu > 0$ such that $\mu\overline{M}(a) < (1 - \lambda)$. From Hypothesis (H4) it follows that there exists a δ such that $|z| \leq \delta$ implies $|\overline{f}(r,z)| \leq \mu\overline{\psi}(r)|z|$. Let $\varepsilon_0 = \min\{\delta,\eta\}$ and take any ε such that $0 < \varepsilon \leq \varepsilon_0$. Let $S(\varepsilon) = \{g(x)|g:R^n \to R^m; g \in C[a,\infty), \|g\|_{0,a} \leq \varepsilon\}$ and define T on $g \in S(\varepsilon)$ so that for $g \in S(\varepsilon)$ we have

$$(Tg)(x) = \Phi(x;a) + \int_a^x A(x;r)\overline{f}(r,g(r))dr.$$

Proceding as in Theorem 3.6 we see that T is a contraction on the complete set $S(\varepsilon)$.

Now let $S_0(\varepsilon) = \{g(x)|g \in S(\varepsilon), \lim_{|x|\to\infty} |g(x)| = 0\}$. The set $S_0(\varepsilon)$ is a closed subset of the complete set $S(\varepsilon)$ and is therefore complete. Take any $\sigma > 0$ and any $g \in S_0(\varepsilon)$. Take $X_0^* \geq a$ such that $|x|_\infty \geq X_0$ implies $|\Phi(x;a)| \leq \frac{\sigma}{3}$ and $|g(x)| \leq \frac{\sigma}{3\overline{M}\mu}$. By hypothesis (H4) there exists a $\tau > X_0$ such that $|x|_\infty \geq \tau$ implies $\int_a^{w(x,X_0^*;\alpha_k)} \|A(x;r)\| \overline{\psi}(r)dr < \frac{\sigma}{3\mu\varepsilon}$ for any α_k, uniformly in x_j such that $a_j \leq x_j \leq X_0$, $j \notin \alpha_k$. Take $x = (x_1,x_2,\ldots,x_n)$ such that $|x|_\infty \geq \tau$ and let α_k^* be the special combination such that if $|x_i| \geq X_0$ then $i \in \alpha_k^*$ and if $|x_i| < X_0$ then $i \notin \alpha_k^*$ (α_k^* will be fixed in regions of the set $\{x \in R^n \mid |x|_\infty \geq X_0\}$). Let $R(x) = \{r \mid |r|_\infty > X_0, r \leq x\}$. We observe that $[a,x] = R(x) \cup [a,w(x,X_0^*;\alpha_k^*)]$ and $R(x) \cap [a,w(x,X_0^*;\alpha_k^*)] = \phi$.

Thus we have

$$|(Tg)(x)| \leq |\Phi(x;a)| + \mu\int_a^x \|A(x;r)\|\overline{\psi}(r)|g(r)|dr$$

$$\leq \frac{\sigma}{3} + \mu\int_a^{w(x,X_0^*;\alpha_k^*)} \|A(x;r)\|\overline{\psi}(r)|g(r)|dr$$

$$+ \mu\int_{R(x)} \|A(x;r)\|\overline{\psi}(r)|g(r)|dr$$

$$\leq \frac{\sigma}{3} + \mu\varepsilon\int_a^{w(x,X_0^*;\alpha_k^*)} \|A(x;r)\|\overline{\psi}(r)dr + \frac{\sigma}{3\overline{M}}\int_{R(x)} \|A(x;r)\|\overline{\psi}(r)dr$$

$$\leq \frac{\sigma}{3} + \mu\varepsilon\left(\frac{\sigma}{3\mu\varepsilon}\right) + \frac{\sigma}{3\overline{M}}\int_a^x \|A(x;r)\|\overline{\psi}(r)dr \leq \sigma$$

for x such that $|x|_\infty \geq \tau$. Therefore $\lim_{|x|_\infty \to \infty} |(Tg)(x)| = 0$ and $T:S_0(\varepsilon) \to S_0(\varepsilon)$. But T is a contraction on $S(\varepsilon)$ and thus is also a contradiction on $S_0(\varepsilon)$. Hence the unique fixed point $u(x)$ of T in $S(\varepsilon)$ is also in $S_0(\varepsilon)$. Since $u(x)$ is a solution of the Equation (1.8) the proof is complete.

We now have the following preservation results.

Theorem 3.8. Suppose $f(x,r,z)$ and $A(x;\xi)$ satisfy (H3).

i) If Equation (1.7) is stable on any space N, then Equation (1.8) is stable on N.

ii) If the constant $M(a)$ in (H3) is independent of a for $\bar{a} \le a$ and if Equation (1.7) is uniformly stable on any space N, then Equation (1.8) is uniformly stable on N.

Suppose $f(x,r,z) \equiv \bar{f}(r,z)$ and $A(x;\xi)$ satisfy (H4).

iii) If Equation (1.7) is asymptotically stable on any space N, then Equation (1.8) is asymptotically stable on N.

Proof. i) Take $\varepsilon > 0$, $a \ge \bar{a}$, and $\lambda \in (0,1)$. By Theorem 3.6 there exists an ε_0 such that if $0 < \gamma \le \varepsilon_0$ and if $\|\Phi(x;a)\|_{0,a} \le \lambda\gamma$ then there is a unique solution $u(x)$ of Equation (1.8) such that $\|u\|_{0,a} \le \gamma$. Let $\bar{\varepsilon} = \min\{\varepsilon_0, \frac{\varepsilon}{2}\} \le \varepsilon_0$. Since Equation (1.7) is stable on N there exists a $\delta = \delta(a)$ such that if $\|\phi\|_N < \delta(a)$ then $\|\Phi(x;a)\|_{0,a} < \lambda\bar{\varepsilon}$. Thus there is a unique solution $u(x)$ of Equation (1.8) corresponding to ϕ such that $\|u\|_{0,a} \le \bar{\varepsilon} \le \frac{\varepsilon}{2} < \varepsilon$. Therefore Equation (1.8) is stable on N.

ii) In the proof of Theorem 3.6, ε_0 was chosen so that $\varepsilon_0 = \min\{\delta,\eta\}$. In the proof δ and η depend on the point a since ρ and μ as defined there depend on the constant $M(a)$. But if M is independent of the point a, then ε_0 is also independent of a. Then for any $a \geq \bar{a}$ there exists a fixed ε_0 with the property insured by Theorem 3.6. Now take any $\varepsilon > 0$, $a \geq \bar{a}$, and $\lambda \in (0,1)$. Let $\bar{\varepsilon} = \min\{\varepsilon_0, \frac{\varepsilon}{2}\}$. From the uniform stability of Equation (1.7) on N it follows that there exists a $\delta = \delta(\bar{\varepsilon})$ such that if $\|\phi\|_N < \delta$ then $\|\Phi(x;a)\| < \lambda\bar{\varepsilon}$. Then the solution u of Equation (1.8) satisfies $\|u\|_{0,a} \leq \bar{\varepsilon} < \varepsilon$ and Equation (1.8) is uniformly stable on N.

iii) Choose any $a \geq \bar{a}$. Stability of Equation (1.8) follows directly from part i) of this theorem. Take any $\lambda \in (0,1)$. Then there exists an ε_0 having the properties given in Theorem 3.7. Since Equation (1.7) is asymptotically stable on N there exists a δ such that if $\|\phi\|_N < \delta$ then $\|\Phi(x;a)\|_{0,a} < \lambda\varepsilon_0$, and there exists a $\delta_0 < \delta$ such that if $\|\phi\|_N < \delta_0$ then $\lim_{|x|\to\infty} |\Phi(x;a)| = 0$. Thus if $\|\phi\| \in N$ and $\|\phi\|_N < \delta_0$, it follows from Theorem 3.7 that the solution u of Equation (1.8) corresponding to ϕ satisfies $\lim_{|x|\to\infty} |u(x)| = 0$. This completes the proof.

A rather detailed example discussed in [7] shows that Theorem 3.8 is false if the conditions on the derivatives in (H3) are dropped. We also mention that the hypotheses made here are related to those assumed in the study of little o perturbations for ordinary differential equations. It may be shown for ordinary differential equations that the assumption that the linear equation is uniformly asymptotically stable is equivalent to the assumption that the fundamental solution satisfy the condition $\int_a^x \|A(x;r)\| dr \leq M$ for $x \geq a \geq \bar{a}$ [22, p. 290]. Using the equivalence of these conditions we may obtain results in the theory of ordinary differential equations from Theorem 3.8 [22], [23].

We now see that Theorem 3.8 may be used for perturbations which are independent of x and r and are higher order in z. As mentioned previously, perturbations of this form cannot satisfy hypothesis (H2) of Section 3.2. This important class of perturbations is not the only instance in which Theorem 3.8 may yield results while Theorem 3.5 does not. Examples illustrating this are not difficult to construct.

We point out another aspect of Theorem 3.8. Part i) of Theorem 3.8 shows that ordinary stability of Equation (1.7) may be preserved to Equation (1.8). For example, suppose Equation (1.7) is stable on a space N such that $N_3 \subset N$. Suppose the perturbation $f(x,r,z) \equiv \bar{f}(r,z)$ satisfies the following: there is a continuous function

$\psi(r) \geq 0$ so that for each $\varepsilon > 0$ there is an η such that $|z_1|$, $|z_2| \leq \eta$ implies $|\bar{f}(r,z_1) - \bar{f}(r,z_2)| \leq \varepsilon\psi(r)|z_1 - z_2|$ and $\int_a^\infty \psi(r)dr \leq M(a)$. It then follows from Theorem 3.8 part i) that Equation (1.8) is stable on N. This is interesting in light of examples from the theory of ordinary differential equations [16] where stability is not preserved for either little o type perturbations or perturbations of the type considered in Section 3.2.

CHAPTER 4

SPECIAL EQUATIONS AND APPLICATIONS

In general the fundamental solution $A(x:\xi)$ is difficult to obtain. This is the case even for the differential system $\frac{du}{dt} = K(t)u$ $(t \in R)$, when K is not constant. In this chapter we will study the fundamental solutions for some special linear equations. The information obtained concerning these fundamental solutions will then be used, along with the results from previous chapters, to establish stability results for these special equations.

We will also study the relationship between the fundamental solution and the Riemann function for hyperbolic partial differential equations and give stability results for the characteristic value problem.

Most of the results in this chapter will be for integral equations in two independent variables. We therefore drop the notation used in the previous chapters and, unless stated otherwise, the variables x, y, r, s ξ, η, etc. will be real. We will assume that $\bar{a}$ and $\bar{b}$ are fixed throughout the discussion.

Since we will be concerned here with results for equations in two independent variables we give the hypotheses on the perturbation introduced in the previous chapter for this special case. Let d be a positive constant. Then hypotheses (H1) and (H2) specialize to the following.

(H1)' Suppose $f(x,y,r,s,z)$ is continuous for $\bar{a} \le a \le r \le x < \infty$, $\bar{b} \le b \le s \le y < \infty$, $z \in R^m$, $|z| \le d$ and there exists a continuous function $\beta(x,y,r,s)$ so that $|f(x,y,r,s,z_1) - f(x,y,r,s,z_2)| \le \beta(x,y,r,s)|z_1 - z_2|$ for $|z_1|$, $|z_2| \le d$.

(H2)' Suppose there exists a continuous function $\gamma_0(x,y) \ge 0$ such that $|f(x,y,x,y,z)| \le \gamma_0(x,y)|z|$ for all $x \ge a \ge \bar{a}$, $y \ge b \ge \bar{b}$, $z \in R^m$, $|z| \le d$ and such that $\int_{\bar{a}}^{\infty}\int_{\bar{b}}^{\infty}\gamma_0(r,s)dsdr < \infty$. Suppose the functions $\frac{\partial f}{\partial x}(x,y,r,y,z)$, $\frac{\partial f}{\partial y}(x,y,x,s,z)$, $\frac{\partial^2 f}{\partial x \partial y}(x,y,r,s,z)$ are continuous for $\bar{a} \le a \le r \le x < \infty$, $\bar{b} \le b \le s \le y < \infty$, $z \in R^m$, $|z| \le d$. Suppose there are continuous non-negative functions $\gamma_1(x,y,r)$, $\gamma_2(x,y,s)$, $\gamma_3(x,y,r,s)$ such that $|\frac{\partial f}{\partial x}(x,y,r,y,z)| \le \gamma_1(x,y,r)|z|$, $|\frac{\partial f}{\partial y}(x,y,x,s,z)| \le \gamma_2(x,y,s)|z|$, $|\frac{\partial^2 f}{\partial x \partial y}(x,y,r,s,z)| \le \gamma_3(x,y,r,s)|z|$ for $\bar{a} \le a \le r \le x < \infty$, $\bar{b} \le b \le s \le y < \infty$, $|z| \le d$ and $\int_{\bar{a}}^{\infty}\int_{\bar{b}}^{\infty}\int_{\bar{a}}^{r}\gamma_1(r,s,p)dpdsdr < \infty$, $\int_{\bar{a}}^{\infty}\int_{\bar{b}}^{\infty}\int_{\bar{b}}^{s}\gamma_2(r,s,q)dqdsdr < \infty$, $\int_{\bar{a}}^{\infty}\int_{\bar{b}}^{\infty}\int_{\bar{a}}^{r}\int_{\bar{b}}^{s}\gamma_3(r,s,p,q)dqdpdsdr < \infty$.

As mentioned in Section 3.3, hypothesis (H3) is not just an assumption on the perturbation. However in cases where the fundamental solution satisfies the integral conditions in (H3) with $\psi \equiv \psi_{\alpha_k} \equiv 1$, then (H3) describes a class of perturbations. The fundamental solutions studied here unfortunately do not satisfy all of these conditions. We therefore assume the perturbation f is independent of x and y and consider the following hypothesis which will allow us to obtain preservation results based on the results of Section 3.3.

(H3)' Let $f(x,y,r,s,z) = h(r,s,z)$. Suppose $h(r,s,z)$ is continuous for $\bar{a} \leq a \leq r$, $\bar{b} \leq b \leq s$, $z \in R^m$ and suppose for each $\varepsilon > 0$ there exists an η such that $|z_1|$, $|z_2| \leq \eta$ implies $|h(r,s,z_1) - h(r,s,z_2)| \leq \varepsilon |z_1 - z_2|$.

4.1 The Equation $u(x,y) = \phi(x,y) + \int_a^x k_1(r)u(r,y)dr + \int_b^y k_2(s)u(x,s)ds + \int_a^x\int_b^y [k_3(r,s)u(r,s) + f(x,y,r,s,u(r,s))]dsdr$

In this and the next section we will be concerned with special cases of the scalar equations $(m = 1)$ in two independent variables of the form

$$u(x,y) = \phi(x,y) + \int_a^x k_1(x,y,r)u(r,y)dr$$

$$+ \int_b^y k_2(x,y,s)u(x,s)ds + \int_a^x\int_b^y k_3(x,y,r,s)u(r,s)dsdr \qquad (4,1)$$

and

$$u(x,y) = \phi(x,y) + \int_a^x k_1(x,y,r)u(r,y)dr + \int_b^y k_2(x,y,s)u(x,s)ds$$

$$+ \int_a^x\int_b^y [k_3(x,y,r,s)u(r,s) + f(x,y,r,s,u(r,s))]dsdr. \qquad (4.2)$$

This section will be devoted primarily to the study of results for the linear equation

$$u(x,y) = \phi(x,y) + \int_a^x k_1(r)u(r,y)dr + \int_b^y k_2(s)u(x,s)ds$$

$$+ \int_a^x\int_b^y k_3(r,s)u(r,s)dsdr \qquad (4.3)$$

and the perturbed equation

$$u(x,y) = \phi(x,y) + \int_a^x k_1(r)u(r,y)dr + \int_b^y k_2(s)u(x,s)ds$$

$$+ \int_a^x\int_b^y [k_3(r,s)u(r,s) + f(x,y,r,s,u(r,s))]dsdr \qquad (4.4)$$

where $k_1(r)$, $k_2(s)$, and $k_3(r,s)$ will be assumed continuous.

The following lemma concerning the fundamental solution for Equation (4.3) together with the Gronwall inequality will yield a stability result for the more general Equation (4.2).

<u>Lemma 4.1</u>. Suppose $k_1(r) \in L_1[\bar{a},\infty)$, $k_2(s) \in L_1[\bar{b},\infty)$, $k_3(r,s) \in L_1([\bar{a},\infty) \times [\bar{b},\infty))$ and each of these functions is nonnegative on its domain. Then there exists a constant M (independent of (ξ,η)) such that the fundamental solution $A(x,y;\xi,\eta)$ for Equation (4.3) satisfies, $0 \leq A(x,y;\xi,\eta) \leq M$ for $\bar{a} \leq \xi \leq x < \infty$ and $\bar{b} \leq \eta \leq y < \infty$.

<u>Proof</u>. Take any $\xi \geq \bar{a}$, $\eta \geq \bar{b}$ and let $v(x,y) = A(x,y;\xi,\eta)$. Then we have

$$v(x,y) = 1 + \int_{\xi}^{x} k_1(r)v(r,y)dr + \int_{\eta}^{y} k_2(s)v(x,s)ds$$

$$+ \int_{\xi}^{x}\int_{\eta}^{y} k_3(r,s)v(r,s)dsdr \quad \text{for} \quad x \geq \xi, \quad y \geq \eta. \tag{4.5}$$

The successive approximations for this equation converge uniformly for $x \geq \xi$, $y \geq \eta$. These approximations are defined by

$$v_0(x,y) \equiv 0$$

and

$$v_k(x,y) = 1 + \int_{\xi}^{x} k_1(r)v_{k-1}(r,y)dr + \int_{\eta}^{y} k_2(s)v_{k-1}(x,s)ds$$

$$+ \int_{\xi}^{x}\int_{\eta}^{y} k_3(r,s)v_{k-1}(r,s)dsdr \qquad k = 1, 2, \ldots .$$

Let $z_k = v_{k+1} - v_k$, $k = 0, 1, 2 \ldots$. Then $v_n(x,y) = \sum_{j=0}^{n-1} z_j(x,y)$ and the solution $v(x,y)$ of Equation (4.5) is $v(x,y) = \sum_{j=0}^{\infty} z_j(x,y)$.

Now $z_0(x,y) \equiv 1$ and $z_k(x,y) = \int_\xi^x k_1(r) z_{k-1}(r,y)dr + \int_\eta^y k_2(s) z_{k-1}(x,s)ds + \int_\xi^x \int_\eta^y k_3(r,s) z_{k-1}(r,s)dsdr$. Let $H(x,y) = 6[\int_\xi^x k_1(p)dp + \int_\eta^y k_2(q)dq + \int_\xi^x \int_\eta^y k_3(p,q)dqdp]$ for $x \geq \xi$, $y \geq \eta$. Then since $H(x,y) \geq 0$ for $x \geq \xi$, $y \geq \eta$ we see that

$$|z_1(x,y)| \leq \int_\xi^x k_1(r)\exp[H(r,y)]dr + \int_\eta^y k_2(s)\exp[H(x,s]ds$$

$$+ \int_\xi^x \int_\eta^y k_3(r,s)\exp[H(r,s)]dsdr \tag{4.6}$$

But $\frac{\partial}{\partial r}\{\exp[H(r,y)]\} = 6[k_1(r) + \int_\eta^y k_3(r,q)dq]\exp[H(r,y)] \geq 6k_1(r)\exp[H(r,y)]$ and hence

$$k_1(r)\exp[H(r,y] \leq \frac{1}{6}\,\frac{\partial}{\partial r}\{\exp[H(r,y)]\}. \tag{4.7}$$

In a similar way we have

$$k_2(s)\exp[H(x,s)] \leq \frac{1}{6}\,\frac{\partial}{\partial s}\{\exp[H(x,s)]\}. \tag{4.8}$$

Also $\frac{\partial^2}{\partial s \partial r}\{\exp[H(r,s)]\} = 6k_3(r,s)\exp[H(r,s)] + (6)^2[k_1(r) + \int_\eta^s k_3(r,q)dq][k_2(s) + \int_\xi^r k_3(p,s)dp]\exp[H(r,s)]$, thus $\frac{\partial^2}{\partial s \partial r}\{\exp[H(r,s)]\} \geq 6k_3(r,s)\exp[H(r,s)]$, and therefore

$$k_3(r,s)\exp[H(r,s)] \leq \frac{1}{6}\frac{\partial^2}{\partial s \partial r}\{\exp[H(r,s)]\}. \qquad (4.9)$$

Using Equations (4.7)-(4.9) in Equation (4.6) we see that

$$|z_1(x,y)| \leq \frac{1}{6}\Big[\int_\xi^x \frac{\partial}{\partial r}\{\exp[H(r,y)]\}dr + \int_\eta^y \frac{\partial}{\partial s}\{\exp[H(x,s)]\}ds$$

$$+ \int_\xi^x\int_\eta^y \frac{\partial^2}{\partial s \partial r}\{\exp[H(r,s)]\}dsdr\Big] \leq \frac{1}{6}\{\exp[H(x,y)] - \exp[H(\xi,y)]$$

$$+ \exp[H(x,y)] - \exp[H(x,\eta)] + \exp[H(x,y)] - \exp[H(x,\eta)]$$

$$- \exp[H(\xi,y)] + \exp[H(\xi,\eta)]\}.$$

Therefore $|z_1(x,y)| \leq \frac{1}{2}\exp[H(x,y)]$ for $x \geq \xi$, $y \geq \eta$.

Now assume $|z_n(x,y)| \leq \frac{1}{2^n}\exp[H(x,y)]$ for any n. We then have

$$|z_{n+1}(x,y)| \leq \frac{1}{2^n}\Big[\int_\xi^x k_1(r)\exp[H(r,y)]dr + \int_\eta^y k_2(s)\exp[H(x,s)]ds$$

$$+ \int_\xi^x\int_\eta^y k_3(r,s)\exp[H(r,s)]dsdr\Big].$$

Using Equations (4.7)-(4.9) and the calculations above we obtain

$$|z_{n+1}(x,y)| \leq \frac{1}{2^{n+1}}\exp[H(x,y)] \quad \text{for} \quad x \geq \xi, \quad y \geq \eta.$$

Therefore $|z_n(x,y)| \leq \frac{1}{2^n}\exp[H(x,y)]$ for $n = 0, 1, 2, \ldots,$ and $x \geq \xi$, $y \geq \eta$. Thus we have

$$|v(x,y)| \leq \sum_{i=0}^{\infty}|z_j(x,y)| \leq \exp[H(x,y)]\sum_{j=0}^{\infty}\frac{1}{2^j} = 2\exp[H(x,y)]$$

$$\leq 2\exp[6\{\int_{\bar{a}}^{\infty}k_1(r)dr + \int_{\bar{b}}^{\infty}k_2(s)ds + \int_{\bar{a}}^{\infty}\int_{\bar{b}}^{\infty}k_3(r,s)dsdr\}]$$

for any ξ, η, and $x \geq \xi$, $y \geq \eta$. Using Corollary 2.5 we see that

$$0 \leq A(x,y;\xi,\eta) \leq 2\exp[6\{\int_{\bar{a}}^{\infty}k_1(r)dr + \int_{\bar{b}}^{\infty}k_2(s)ds]$$

$$+ \int_{\bar{a}}^{\infty}\int_{\bar{b}}^{\infty}k_3(r,s)dsdr\}]$$

for any $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$. This completes the proof of Lemma 4.1.

We may now prove a stability theorem for Equation (4.2).

Theorem 4.1. Suppose there exists continuous functions $\gamma_1(r)$, $\gamma_2(s)$, and $\gamma_3(r,s)$ defined respectively on $\bar{a} \leq r$, $\bar{b} \leq s$, and $\bar{a} \leq r$, $\bar{b} \leq s$ such that $|k_1(x,y,r)| \leq \gamma_1(r)$ for $\bar{a} \leq a \leq r \leq x < \infty$, $y \geq \bar{b}$, $|k_2(x,y,s)| \leq \gamma_2(s)$ for $\bar{b} \leq b \leq s \leq y < \infty$, $x \geq \bar{a}$, and $|k_3(x,y,r,s)| \leq \gamma_3(r,s)$ for $\bar{a} \leq a \leq r \leq x < \infty$, $\bar{b} \leq b \leq s \leq y < \infty$. Suppose $\gamma_1(r)$, $\gamma_2(s)$ and $\gamma_3(r,s)$ are in L_1 on their respective domains, and $f(x,y,r,s,z)$ satisfies (H1)' and (H2)'. Then Equation (4.2) preserves stability and uniform stability on any space N. In particular, Equation (4.2) is uniformly stable on N_2.

Proof. Let $A(x,y;\xi,\eta)$ be the fundamental solution for Equation (4.1). We then have

$$|A(x,y;\xi,\eta)| \leq 1 + \int_\xi^x \gamma_1(r)|A(r,y;\xi,\eta)|dr$$

$$+ \int_\eta^y \gamma_2(s)|A(x,s;\xi,\eta)|ds + \int_\xi^x\int_\eta^y \gamma_3(r,s)|A(r,s;\xi,\eta)|dsdr$$

for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$. Let $A^*(x,y;\xi,\eta)$ be the fundamental solution for the equation

$$v(x,y) = 1 + \int_a^x \gamma_1(r)v(r,y)dr + \int_b^y \gamma_2(s)v(x,s)ds$$

$$+ \int_a^x\int_b^y \gamma_3(r,s)v(r,s)dsdr.$$

By Lemma 4.1 there exists constant M such that $0 \le A^*(x,y;\xi,\eta) \le M$ for $\bar{a} \le \xi \le x < \infty$, $\bar{b} \le \eta \le y < \infty$. Using the Gronwall inequality (Theorem 2.4) we have $|A(x,y;\xi,\eta)| \le A^*(x,y;\xi,\eta) \le M$ for $\bar{a} \le \xi \le x < \infty$, $\bar{b} \le \eta \le y < \infty$. It then follows from Theorem 3.2 that Equation (4.1) is uniformly stable on N_2. The result now follows directly from Theorem 3.5. This completes the proof.

We will now consider several lemmas leading to explicit solutions for the fundamental solution of Equation (4.3) when $k_3(r,s)$ is of certain special forms. The following lemma is well known. We include the proof for completeness.

Lemma 4.2. Suppose $f(r)$ is continuous for $a \le x$. Then for $x \ge a$ we have the following identity.

$$\int_a^x f(r)\left(\int_a^r f(r_1)dr_1\right)\left(\int_a^{r_1} f(r_2)dr_2\right) \cdots \left(\int_a^{r_{n-2}} f(r_{n-1})dr_{n-1}\right)dr$$

$$= \frac{1}{n!}\left(\int_a^x f(r)dr\right)^n \quad \text{for} \quad n \ge 2. \tag{4.10}$$

Proof. Take $n = 2$. Let $u(r) = \int_a^r f(r_1)dr_1$. Then

$$\int_a^x f(r)\left(\int_a^r f(r_1)dr_1\right)dr = \int_a^x u(r)du = \frac{1}{2}\left[\left(\int_a^r f(r_1)dr_1\right)^2\right|_a^x$$

$$= \frac{1}{2}\left(\int_a^x f(r_1)dr_1\right)^2.$$

Assume Equation (4.10) holds for any $n > 2$. Then using this assumption we have

$$\int_a^x f(r)\left(\int_a^r f(r_1)dr_1\right)\left(\int_a^{r_1} f(r_2)dr_2\right) \cdots \left(\int_a^{r_{n-1}} f(r_n)dr_n\right)dr$$

$$= \int_a^x f(r)\left[\frac{1}{n!}\left(\int_a^r f(r_1)dr_1\right)^n\right]dr = \frac{1}{n!}\int_a^x u^n du$$

$$= \frac{1}{(n+1)!}\left[\left(\int_a^r f(r_1)dr_1\right)^{n+1}\right]_a^x = \frac{1}{(n+1)!}\left(\int_a^x f(r)dr\right)^{n+1}.$$

Thus, by induction, the proof is complete.

Consider the integral equation

$$u(x,y) = 1 + \int_\xi^x\int_\eta^y f(r)g(s)u(r,s)dsdr \qquad (4.11)$$

with $x \geq \xi$, $y \geq \eta$ and $f(r)$, $g(s)$ continuous. The next lemma establishes the solution for Equation (4.11). Results for the scalar equation in several variables ($x \in R^n$, $r = (r_1, r_2, \ldots, r_n)$) of the form

$$u(x) = 1 + \int_a^x f_1(r_1)f_2(r_2) \cdots f_n(r_n)u(r)dr.$$

may be found in Walter [39, pp. 142-143].

<u>Lemma 4.3</u>. The solution of Equation (4.11) for $x \geq \xi$, $y \geq \eta$ is

$$u(x,y) = I_0\left(2\sqrt{\left(\int_\xi^x f(r)dr\right)\left(\int_\eta^y g(s)ds\right)}\right)$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero [25], [40].

<u>Proof</u>. Consider the successive approximations for Equation (4.11). They are $u_0(x,y) \equiv 1$,

$$u_k(x,y) = 1 + \int_\xi^x\int_\eta^y f(r)g(s)u_{k-1}(r,s)dsdr.$$

Then,

$$u_1(x,y) = 1 + \int_\xi^x\int_\eta^y f(r)g(s)dsdr = 1 + \left(\int_\xi^x f(r)dr\right)\left(\int_\eta^y g(s)ds\right)$$

and an easy induction shows that

$$u_n(x,y) = 1 + \left(\int_\xi^x f(r)dr\right)\left(\int_\eta^y g(s)ds\right) +$$

$$+ \left[\int_\xi^x f(r)\left(\int_\xi^r f(r_1)dr_1\right)dr\right]\left[\int_\eta^y g(s)\left(\int_\eta^s g(s_1)ds_1\right)ds\right]$$

$$+ \dots + \left[\int_\xi^x f(r)\left(\int_\xi^r f(r_1)dr_1\right)\left(\int_\xi^{r_1} f(r_2)dr_2\right)\right.$$

$$\left.\dots\left(\int_\xi^{r_{n-2}} f(r_{n-1})dr_{n-1}\right)dr\right]\left[\int_\eta^y g(s)\left(\int_\eta^s g(s_1)ds_1\right)\left(\int_\eta^{s_1} g(s_2)ds_2\right)\right.$$

$$\left.\dots\left(\int_\eta^{s_{n-2}} g(s_{n-1})ds_{n-1}\right)ds\right].$$

Using Lemma 4.2, we have

$$u_n(x,y) = 1 + (\int_\xi^x f(r)dr)(\int_\eta^y g(s)ds) + \frac{1}{2^2}(\int_\xi^x f(r)dr)^2(\int_\eta^y g(s)ds)^2$$

$$+ \ldots + \frac{1}{(n!)^2}(\int_\xi^x f(r)dr)^n(\int_\eta^y g(s)ds)^n$$

and thus

$$u_n(x,y) = \sum_{k=0}^{n} \frac{\{2[(\int_\xi^x f(r)dr)(\int_\eta^y g(s)ds)]^{1/2}\}^{2k}}{2^{2k}(k!)^2}.$$

These successive approximations converge to the solution $u(x,y)$ of Equation (4.11) for $x \geq \xi$, $y \geq \eta$. Hence

$$u(x,y) = \lim_{n\to\infty} u_n(x,y) = \sum_{k=0}^{\infty} \frac{\{2[(\int_\xi^x f(r)dr)(\int_\eta^y g(s)ds)]^{1/2}\}^{2k}}{2^{2k}(k!)^2}.$$

Since $I_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2}$ ([25], p. 108), we have

$u(x,y) = I_0(2\sqrt{(\int_\xi^x f(r)dr)(\int_\eta^y g(s)ds)})$ and the proof is complete.

Lemma 4.3 cannot in general be extended to systems. This is not surprising since in the case of the differential system $y' = A(x)y$ with variable coefficients, the solution can no longer be expressed as $\exp[\int_a^x A(s)ds]$ as in the scalar case. We will consider systems with constant kernels in Section 4.3.

Remark 4.1. Using Lemma 4.3 we see that the solution of the scalar equation

$$u(x,y) = 1 + \int_{\xi}^{x}\int_{\eta}^{y} cu(r,s)dsdr \quad \text{for} \quad x \geq \xi, \quad y \geq \eta \qquad (4.12)$$

where c is a constant is $u(x,y) = I_0(2\sqrt{c(x-\xi)(y-\eta)})$ [19], [37].

Lemma 4.4. Suppose $v(x,y)$ is the solution of the scalar integral equation

$$v(x,y) = 1 \qquad (4.13)$$

$$+ \int_{a}^{x}\int_{b}^{y} [k_1(r)k_2(s) + k_3(r,s)]v(r,s)dsdr \quad \text{for} \quad x \geq a, \quad y \geq b.$$

Then the solution $u(x,y)$ of Equation (4.3) for $x \geq a$, $y \geq b$ with $\phi \equiv 1$ is

$$u(x,y) = v(x,y)\exp[\int_{a}^{x} k_1(r)dr + \int_{b}^{y} k_2(s)ds]. \qquad (4.14)$$

Proof. The function $u(x,y)$ is the solution of Equation (4.3) with $\phi \equiv 1$ iff $u(x,y)$ is the solution of the characteristic value problem

$$\left.\begin{aligned} u_{xy}(x,y) &= k_1(x)u_y(x,y) + k_2(y)u_x(x,y) + k_3(x,y)u(x,y) \\ u(x,b) &= \exp[\int_a^x k_1(r)dr], \quad u(a,y) = \exp[\int_b^y k_2(s)ds] \end{aligned}\right\}. \quad (4.15)$$

Now let $u(x,y) = \exp[\int_a^x k_1(r)dr + \int_b^y k_2(s)ds]v(x,y)$.

Then

$$u_x(x,y) = [k_1(x)v(x,y) + v_x(x,y)]\exp[\int_a^x k_1(r)dr + \int_b^y k_2(s)ds],$$

$$u_y(x,y) = [k_2(y)v(x,y) + v_y(x,y)]\exp[\int_a^x k_1(r)dr + \int_b^y k_2(s)ds]$$

and

$$u_{xy}(x,y) = [k_1(x)v_y + v_{xy} + k_1(x)k_2(y)v(x,y) + k_2(y)v(x,y)]$$

$$\cdot \exp[\int_\xi^x k_1(r)dr + \int_\eta^y k_2(s)ds].$$

The differential equation in terms of v then becomes

$$v_{xy}(x,y) = [k_1(x)k_2(y) + k_3(x,y)]v(x,y).$$

The characteristic data for $v(x,y)$ is determined from

$$u(a,y) = \exp[\int_b^y k_2(s)ds] = \exp[\int_b^y k_2(s)ds]v(a,y)$$

and

$$u(x,b) = \exp[\int_a^x k_1(r)dr] = \exp[\int_a^x k_1(r)dr]v(x,b).$$

Therefore $v(x,y)$ satisfies the characteristic value problem

$$\left.\begin{aligned} v_{xy}(x,y) &= [k_1(x)k_2(y) + k_3(x,y)]v(x,y) \\ v(a,y) &= 1 \quad v(x,b) = 1 \end{aligned}\right\}. \tag{4.16}$$

If $v(x,y)$ is the solution of the problem (4.16) then $u(x,y) = \exp[\int_a^x k_1(r)dr + \int_b^y k_2(s)ds]v(x,y)$ is the solution of problem (4.15) and is therefore the solution of the integral Equation (4.3) with $\phi \equiv 1$. But $v(x,y)$ is the solution of problem (4.16) iff $v(x,y)$ satisfies the integral equation

$$v(x,y) = 1 + \int_a^x\int_b^y [k_1(r)k_2(s) + k_3(r,s)]v(r,s)dsdr.$$

This completes the proof.

We see that if $k_3(r,s)$ is such that $k_1(r)k_2(s) + k_3(r,s)$ can be written as the product of a function of r and a function of s, then Lemma 4.3 will give the solution of Equation (4.13). Then, by Lemma 4.4, Equation (4.14) yields the fundamental solution for Equation (4.3) for this particular $k_3(r,s)$. This discussion allows us to give the following lemma in which we list the fundamental solution for kernels $k_3(r,s)$ which may be expressed as the product of a function of r and a function of s.

Lemma 4.5. Assume that the functions $g_1(r)$ and $g_2(s)$ are continuous on $\bar{a} \le r$ and $\bar{b} \le s$ respectively. Let $A(x,y;\xi,\eta)$ be the fundamental solution for Equation (4.3).

i) If k_1, k_2, and k_3 are constants then

$$A(x,y;\xi,\eta) \tag{4.17}$$

$$= I_0(2\sqrt{(k_1k_2+k_3)(x-\xi)(y-\eta)})\exp[k_1(x - \xi) + k_2(y - \eta)].$$

ii) If $k_3(r,s) \equiv 0$ then

$$A(x,y;\xi,\eta) \tag{4.18}$$

$$= I_0(2\sqrt{(\int_\xi^x k_1(r)dr)(\int_\eta^y k_2(s)ds)})\exp[\int_\xi^x k_1(r)dr + \int_\eta^y k_2(s)ds]$$

iii) If $k_3(r,s) = g_1(r)k_1(r)k_2(s)$, then

$$A(x,y;\xi,\eta)$$

$$= I_0\left(2\sqrt{\left(\int_\xi^x k_1(r)\{g_1(r)+1\}dr\right)\left(\int_\eta^y k_2(s)ds\right)}\right)\exp\left[\int_\xi^x k_1(r)dr\right.$$

$$\left. + \int_\eta^y k_2(s)ds\right] \tag{4.19}$$

and if $k_3(r,s) = g_2(s)k_1(r)k_2(s)$ then

$A(x,y;\xi,\eta)$

$$= I_0\left(2\sqrt{\left(\int_\xi^x k_1(r)dr\right)\left(\int_\eta^y k_2(s)\{g_2(s)+1\}ds\right)}\right)\exp\left[\int_\xi^x k_1(r)dr\right.$$

$$\left. + \int_\eta^y k_2(s)ds\right]. \tag{4.20}$$

iv) Suppose k_1 and k_2 are constant. If $k_3(r,s)$ $= g_1(r)$ then

$A(x,y;\xi,\eta)$ (4.21)

$$= I_0\left(2\sqrt{\left(\int_\xi^x [k_1k_2+g_1(r)]dr\right)(y-\eta)}\right)\exp[k_1(x - \xi) + k_2(y - \eta)]$$

and if $k_3(r,s) = g_2(s)$

$A(x,y;\xi,\eta)$ (4.22)

$$= I_0\left(2\sqrt{(x-\xi)\left(\int_\eta^y [k_1k_2+g_2(s)]ds\right)}\right)\exp[k_1(x - \xi) + k_2(y - \eta)].$$

v) If $k_3(r,s) = k_1(r)g_2(s) + k_2(s)g_1(r) + g_1(r)g_2(s)$ then

$$A(x,y;\xi,\eta)$$

$$= I_0\left(2\sqrt{\left(\int_\xi^x [k_1(r)+g_1(r)]dr\right)\left(\int_\eta^y [k_2(s)+g_2(s)]ds\right)}\right)\exp\left[\int_\xi^x k_1(r)dr + \int_\eta^y k_2(s)ds\right]. \quad (4.23)$$

Results along these lines have been given in [35]. The solutions given in iii)-v) appear to be new.

We notice in Lemma 4.5 that the fundamental solution in each case is a product of exponentials and the Bessel function. The following lemmas give results concerning such products and will be useful in establishing stability results; they follow from a basic representation for $I_0(t)$.

<u>Lemma 4.6</u>. Let t be a real variable. Then $\lim_{t\to\infty} I_0(t)e^{-t} = 0$ and $0 < I_0(t)e^{-t} \le 1$ for all $t \ge 0$.

<u>Proof</u>. The Bessel function $I_\nu(t)$ is given by

$$I_\nu(t) = \frac{(1/2t)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)}\int_0^\pi e^{t\cos\theta}\sin^{2\nu}\theta\, d\theta \quad ([40], \text{ p. } 79).$$

Since $\Gamma(1/2) = \sqrt{\pi}$ we have

$$I_0(t)e^{-t} = \frac{1}{\pi}\int_0^{\pi} e^{t\cos\theta} e^{-t} d\theta = \frac{1}{\pi}\int_0^{\pi} e^{-t(1-\cos\theta)} d\theta.$$

Consider the function $h(t,\theta) = e^{-t(1-\cos\theta)}$ on $[0,\infty) \times [0,\pi]$. Then $\lim_{t\to\infty} h(t,0) = 1$, and for $\theta \in (0,\pi]$ we have $\lim_{t\to\infty} h(t,\theta) = 0$. We also see that $0 < h(t,\theta) \leq 1$. By the dominated convergence theorem (see Section 2.1)

$$\lim_{t\to\infty} I_0(t)e^{-t} = \frac{1}{\pi} \lim_{t\to\infty}\int_0^{\pi} e^{-t(1-\cos\theta)} d\theta$$

$$= \frac{1}{\pi} \lim_{t\to\infty}\int_0^{\pi} h(t,\theta) d\theta = \frac{1}{\pi}\int_0^{\pi} (\lim_{t\to\infty} h(t,\theta)) d\theta = 0.$$

Also, $0 < \frac{1}{\pi}\int_0^{\pi} e^{-t(1-\cos\theta)} d\theta \leq \frac{1}{\pi}\int_0^{\pi} d\theta = 1$. Hence $0 < I_0(t)e^{-t} \leq 1$ and the proof is complete.

<u>Lemma 4.7</u>. Let c_1 and c_2 be real positive constants. Then

$$\lim_{|(t_1,t_2)|\to\infty} I_0(2\sqrt{c_1c_2t_1t_2})e^{-c_1t_1}e^{-c_2t_2} = 0 \quad \text{and}$$

$$0 < I_0(2\sqrt{c_1c_2t_1t_2})e^{-c_1t_1}e^{-c_2t_2} \leq 1 \quad \text{for all } t_1,\ t_2 \geq 0.$$

<u>Proof</u>. $I_0(t)$ is a positive increasing function of t with $I_0(0) = 1$. Thus $I_0(2\sqrt{c_1c_2t_1t_2})e^{-c_1t_1}e^{-c_2t_2} > 0$. Since

$[(c_1t_1)^{1/2} - (c_2t_2)^{1/2}]^2 \geq 0$ we have $2\sqrt{c_1c_2t_1t_2} \leq c_1t_1 + c_2t_2$ for all t_1, $t_2 \geq 0$, and thus $I_0(2\sqrt{c_1c_2t_1t_2}) \leq I_0(c_1t_1 + c_2t_2)$. Using Lemma 4.6 we see that

$$I_0(2\sqrt{c_1c_2t_1t_2})e^{-(c_1t_1+c_2t_2)}$$

$$\leq I_0(c_1t_1 + c_2t_2)e^{-(c_1t_1+c_2t_2)} \leq 1.$$

Now take any $\varepsilon > 0$. By Lemma 4.6 there exists an M such that if $t \geq M$ then $I_0(t)e^{-t} < \varepsilon$. Take $|(t_1,t_2)|_1 \geq \frac{M}{\min\{c_1,c_2\}}$. Then $c_1t_1 + c_2t_2 \geq \min\{c_1,c_2\}(|t_1| + |t_2|) \geq \min\{c_1,c_2\} \cdot \frac{M}{\min\{c_1,c_2\}} = M$. Hence $I_0(2\sqrt{c_1c_2t_1t_2})e^{-(c_1t_1+c_2t_2)} \leq I_0(c_1t_1+c_2t_2)e^{-(c_1t_1+c_2t_2)} < \varepsilon$ and this completes the proof.

<u>Remark 4.2</u>. We introduce the Bessel function $J_0(x)$ and note some of the properties of this function [25]. We have the following relation between I_0 and J_0:

$$I_0(z) = J_0(-iz) \text{ for } z \text{ complex, } -\frac{\pi}{2} < \arg z < \pi.$$

By taking $z = ix$, x real and $x \geq 0$, it then follows that

$$J_0(x) = I_0(ix). \tag{4.24}$$

We also have $|J_0(x)| \le 1$ for x real and $x \ge 0$. Then since $I_0(0) = 1$ and I_0 is increasing it follows that

$$|J_0(x)| = |I_0(ix)| \le 1 \le I_0(y) \tag{4.25}$$

for x, y real, x, $y \ge 0$.

Remark 4.3. Using Remarks 4.1 and 4.2 we see that, in contrast to the behavior of the fundamental solution for the scalar equation in one independent variable, the fundamental solution in two independent variables can be negative for some values of the independent variables.

Due to the fact that several different assumptions lead to the same stability conclusions we will now give a list of hypotheses and then a theorem based on these hypotheses.

(H5) Suppose there are positive constants M_1, M_2 such that $\int_\xi^x k_1(r)dr \le M_1$ and $\int_\eta^y k_2(s)ds \le M_2$ for $\bar{a} \le \xi \le x < \infty$, $\bar{b} \le \eta \le y < \infty$. Suppose $k_3(r,s) \in L_1([\bar{a},\infty) \times [\bar{b},\infty))$.

(H5.a) Suppose (H5) holds and $\int_{\bar{a}}^x k_1(r)dr \to -\infty$ as $x \to \infty$, $\int_{\bar{b}}^y k_2(s)ds \to -\infty$ as $y \to \infty$.

(H6) Suppose $k_3(r,s) = g_1(r)k_1(r)k_2(s)$ (or $k_3(r,s) = g_2(s)k_1(r)k_2(s)$). Suppose $k_1(r) \in L_1([\bar{a},\infty))$,

$k_2(s) \in L_1([\bar{b},\infty))$ and $k_1(r)g_1(r) \in L_1([\bar{a},\infty))$ (or $k_2(s)g_2(s) \in L_1([\bar{b},\infty)))$.

(H7) Suppose $k_3(r,s) = g_1(r)k_1(r)k_2(s)$ (or $k_3(r,s) = g_2(s)k_1(r)k_2(s))$. Suppose $k_1(r)$, $k_2(s) \geq 0$ and $k_1(r) \in L_1([\bar{a},\infty))$ and $k_2(s) \in L_1([\bar{b},\infty))$. Suppose $g_1(r) \leq 0$ for $r \geq a \geq \bar{a}$ (or $g_2(s) \geq 0$, $s \geq b \geq \bar{b}$).

(H8) Suppose $k_3(r,s) = g_1(r)k_1(r)k_2(s)$ (or $k_3(r,s) = g_2(s)k_1(r)k_2(s))$. Suppose $k_1(r) \leq 0$ for $r \geq \bar{a}$ and $k_2(s) \leq 0$ for $s \geq \bar{b}$. Suppose $g_1(r) \leq 0$ for $r \geq \bar{a}$ (or $g_2(s) \leq 0$ for $s \geq \bar{b}$).

(H8.a) Suppose (H8) holds. Suppose $\int_{\bar{a}}^{x} k_1(r)dr \to -\infty$, as $x \to \infty$. $\int_{\bar{b}}^{y} k_2(s)ds \to -\infty$ as $y \to \infty$.

(H8.b) Suppose (H8) holds. Suppose there exists an $r_0 \geq \bar{a}$ and an $s_0 \geq \bar{b}$ and positive constants α, β such that $k_1(r) \leq -\alpha$, $k_2(s) \leq -\beta$ for $r \geq r_0$, $s \geq s_0$. Suppose $g_1(r) \leq -1$ for $r \geq \bar{a}$ (or $g_2(s) \leq -1$ for $s \geq \bar{b}$).

(H9) Suppose $k_3(r,s) = g_1(r)$ (or $k_3(r,s) = g_2(s))$. Suppose k_1, k_2 are nonpositive constants and $g_1(r) \leq 0$ for $r \geq \bar{a}$ (or $g_2(s) \leq 0$ for $s \geq \bar{b}$).

(H9.a) Suppose (H9) holds and in addition $k_1 < 0$, $k_2 < 0$.

(H9.b) Suppose (H9) and (H9.a) hold and $g_1(r) \leq -k_1k_2$ for $r \geq \bar{a}$ (or $g_2(s) \leq -k_1k_2$ for $s \geq \bar{b}$).

(H10) Suppose $k_3(r,s) = k_1(r)g_2(s) + k_2(s)g_1(r) + g_1(r)g_2(s)$. Suppose $k_1(r)$, $g_1(r) \in L_1([\bar{a},\infty))$ and $k_2(s)$, $g_2(s) \in L_1([\bar{b},\infty))$.

(H11) Suppose $k_3(r,s) = k_1(r)g_2(s) + k_2(s)g_1(r) + g_1(r)g_2(s)$ where $k_1(r) \leq 0$, $k_2(s) \leq 0$. Suppose any one of the following hold for $g_1(r)$, $g_2(s)$:

(A1) $g_1(r) \geq |k_1(r)|$ and $g_2(s) \leq 0$.

(A2) $g_2(s) \geq |k_2(s)|$ and $g_1(r) \leq 0$.

(A3) $g_1(r) \geq 0$, $g_2(s) \geq 0$ and $g_1(r) \leq |k_1(r)|$, $g_2(s) \leq |k_2(s)|$.

(H11.a) Suppose (H11) holds and $\int_{\bar{a}}^{x} k_1(r)dr \to -\infty$ as $x \to \infty$, $\int_{\bar{b}}^{y} k_2(s)ds \to -\infty$ as $y \to \infty$.

(H11.b) Suppose (H11) with (A1) or (A2) holds. Suppose there exists an $r_0 \geq \bar{a}$ and an $s_0 \geq \bar{b}$ and positive constants α, β such that $k_1(r) \leq -\alpha$, $k_2(s) \leq -\beta$ for $r \geq r_0$, $s \geq s_0$.

(12) Suppose k_1, k_2 and k_3 are constants such that k_1, k_2, $k_3 \leq 0$.

(H12.a) Suppose (H12) holds with $k_1 < 0$ and $k_2 < 0$.

(H12.b) Suppose (H12) and (H12.a) hold and $k_1k_2 + k_3 \leq 0$.

We now return to stability results for Equation (4.4).

Theorem 4.2. Suppose $f(x,y,r,s,z)$ satisfies (H1)' and (H2)'.

i) Suppose any one of the hypotheses (H5)-(H12) is satisfied. Then Equation (4.4) preserves stability and uniform stability on any space N and preserves asymptotic stability on any space N such that $N_3 \subset N$. In particular, Equation (4.4) is uniformly stable on N_2.

ii) Suppose any one of the hypotheses (H5.a), (H8.a), (H9.a), (H11.a), or (H12.a) is satisfied. Then Equation (4.4) is asymptotically stable on N_2.

iii) Suppose any one of the hypotheses (H8.b), (H9.b), (H11.b), or (H12.b) is satisfied. Then Equation (4.4) is uniformly stable on N_1.

Proof. We notice that some of the hypotheses above contain alternate assumptions. Since in each case the method of proof is the same for these alternate assumptions we give the proof for only one set of assumptions. Throughout this proof $A(x,y;\xi,\eta)$ will be the fundamental solution for Equation (4.3).

i) Assume (H5) holds. Consider the equation

$$u(x,y) = \phi(x,y) + \int_a^x k_1(r)u(r,y)dr + \int_b^y k_2(s)u(x,s)ds \qquad (4.26)$$

for $\bar{a} \leq a \leq x < \infty$, $\bar{b} \leq b \leq y < \infty$ and let $\hat{A}(x,y;\xi,\eta)$ be the fundamental solution for this equation. Suppose for some $x \geq \xi \geq \bar{a}$ and $y \geq \eta \geq \bar{b}$ we have $\int_\xi^x k_1(r)dr \geq 0$ and $\int_\eta^y k_2(s)ds \geq 0$. Then since I_0 is an increasing function we see from Equation (4.18) that $|\hat{A}(x,y;\xi,\eta)| \leq I_0(2\sqrt{M_1M_2})\exp[M_1 + M_2]$. If for some $x \geq \xi \geq \bar{a}$, $y \geq \eta \geq \bar{b}$ we have $\int_\xi^x k_1(r)dr \leq 0$ and $\int_\eta^y k_2(s)ds \geq 0$ or $\int_\xi^x k_1(r)dr \geq 0$ and $\int_\eta^y k_2(s)ds \leq 0$ then from Equation (4.18) and inequality (4.25) we see that $|\hat{A}(x,y;\xi,\eta)| \leq \exp[M_1 + M_2]$. If we have $\int_\xi^x k_1(r)dr < 0$ and $\int_\eta^y k_2(s)ds < 0$ it follows from Lemma 4.7 and Equation (4.18) that $|\hat{A}(x,y;\xi,\eta)| \leq 1$. Thus in each case we have $|\hat{A}(x,y;\xi,\eta)| \leq I_0(2\sqrt{M_1M_2})\exp[M_1 + M_2]$ for $x \geq \xi \geq \bar{a}$, $y \geq \eta \geq \bar{b}$. Thus from Theorem 3.2 it follows that Equation (4.26) is uniformly stable on N_2. We then have by Theorem 3.5 and the assumption on the function $k_1(r,s)$ that the Equation (4.3) is uniformly stable on N_2. It follows from Theorem 3.5 that Equation (4.4) preserves stability, and uniform stability on any space N and preserves asymptotic stability on any space N such that $N_3 \subset N$. Thus Equation (4.4) is uniformly stable on N_2.

In each of the remaining cases (H6)-(H12) the assumption will lead to a uniform bound on the fundamental solution $A(x,y;\xi,\eta)$. The result then follows from Theorems 3.2 and 3.5. We indicate how the uniform bound is obtained for the hypotheses (H6)-(H12).

Assume (H6) holds. Using Equation (4.19) and Remark (4.2) we see that

$$|A(x,y;\xi,\eta)|$$

$$\leq I_0\left(2\sqrt{\left(\int_{\bar{a}}^{\infty}|k_1(r)g_1(r)|dr\right)\left(\int_{\bar{b}}^{\infty}|k_2(s)|ds\right) + \left(\int_{\bar{a}}^{\infty}|k_1(r)|dr\right)\left(\int_{\bar{b}}^{\infty}k_2(s)ds\right)}\right)\exp\left[\int_{\bar{a}}^{\infty}|k_1(r)|dr + \int_{\bar{b}}^{\infty}|k_2(s)|ds\right]$$

for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$.

Suppose hypothesis (H7) holds. It follows that

$$\left(\int_{\xi}^{x}k_1(r)g_1(r)dr\right)\left(\int_{\eta}^{y}k_2(s)ds\right) + \left(\int_{\xi}^{x}k_1(r)dr\right)\left(\int_{\eta}^{y}k_2(s)ds\right)$$

$$\leq \left(\int_{\xi}^{x}k_1(r)dr\right)\left(\int_{\eta}^{y}k_2(s)ds\right) \tag{4.27}$$

for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$. It then follows from Equation (4.19) and Remark (4.2) that

$$|A(x,y;\xi,\eta)|$$

$$\leq I_0\left(2\sqrt{\left(\int_{\bar{a}}^{\infty}|k_1(r)|dr\right)\left(\int_{\bar{b}}^{\infty}|k_2(s)|ds\right)}\right)\exp\left[\int_{\bar{a}}^{\infty}|k_1(r)|dr + \int_{\bar{b}}^{\infty}|k_2(s)|ds\right] \quad \text{for } \bar{a} \leq \xi \leq x < \infty,\ \bar{b} \leq \eta \leq y < \infty.$$

Assume (H8) holds. It follows that inequality (4.27) holds. Thus from Equation (4.19), Remark (4.2), and Lemma 4.7 we have $|A(x,y;\xi,\eta)| \leq 1$ for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$.

Suppose hypothesis (H9) holds. It follows that $k_1k_2(x-\xi)(y-\eta) + (y-\eta)\left(\int_{\xi}^{x} g_1(r)dr\right) \leq k_1k_2(x-\xi)(y-\eta)$ for $x \geq \xi$, $y \geq \eta$. Thus by Remark 4.2 and Lemma 4.7 we have $|A(x,y;\xi,\eta)| \leq 1$.

Using hypothesis (H10), Equation (4.22), and Remark 4.2 we obtain

$$|A(x,y;\xi,\eta)|$$

$$\leq I_0\left(2\sqrt{\left[\int_{\bar{a}}^{\infty}|k_1(r) + g_1(r)|dr\right]\left[\int_{\bar{b}}^{\infty}|k_2(s) + g_2(s)|ds\right]}\right)$$

$$\exp\left[\int_{\bar{a}}^{\infty}|k_1(r)|dr + \int_{\bar{b}}^{\infty}|k_2(s)|ds\right]$$

for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$.

Suppose (H11) with the assumption (A1) holds. It follows that $(\int_{\xi}^{x}[k_1(r) + g_1(r)]dr)(\int_{\eta}^{y}[k_2(s) + g_2(s)]ds) \leq 0$ and hence from Remark 4.2 we see that $|A(x,y;\xi,\eta)| \leq \exp[\int_{\xi}^{x}k_1(r)dr + \int_{\eta}^{y}k_2(s)ds] \leq 1$. The proof under the assumption (A2) is similar. Now assume (H11) with (A3) holds. It follows that $(\int_{\xi}^{x}[k_1(r) + g_1(r)]dr)(\int_{\eta}^{y}[k_2(s) + g_2(s)]ds) \leq (\int_{\xi}^{x}k_1(r)dr)(\int_{\eta}^{y}k_2(s)ds)$ and thus by Lemma 4.7 we have $|A(x,y;\xi,\eta)| \leq 1$ for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$.

Assume (H12) holds. Then $k_1k_2 + k_3 \leq k_1k_2$ and we have by Lemma 4.7 that $|A(x,y;\xi,\eta)| \leq 1$ for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$.

ii) Assume (H5.a) holds. Take any $\varepsilon > 0$ and $\xi \geq \bar{a}$, $\eta \geq \bar{b}$, Since $\int_{\bar{a}}^{x}k_1(r)dr \to -\infty$ as $x \to \infty$ it follows that $\int_{\xi}^{x}k_1(r)dr \to -\infty$ as $x \to \infty$. Likewise $\int_{\eta}^{y}k_2(s)ds \to -\infty$ as $y \to \infty$. From Lemma 4.7 there exists an $M > 0$ such that if $\int_{\xi}^{x}k_1(r)dr + \int_{\eta}^{y}k_2(s)ds \leq -M$ then

$$I_0(2\sqrt{(\int_{\xi}^{x}k_1(r)dr)(\int_{\eta}^{y}k_2(s)ds)})\exp[\int_{\xi}^{x}k_1(r)dr + \int_{\eta}^{y}k_2(s)ds] < \varepsilon.$$

There exists constants $\bar{M}_1$, $\bar{M}_2 > 0$ such that $\int_{\xi}^{x}k_1(r)dr \leq -M - M_2$ for $x \geq \bar{M}_1$ and $\int_{\eta}^{y}k_2(s)ds \leq -M - M_1$ for $y \geq \bar{M}_2$. Take (x,y) such that $|(x,y)|_\infty \geq \max\{\bar{M}_1,\bar{M}_2\}$.

Suppose $|(x,y)|_\infty = x \geq \max\{\bar{M}_1,\bar{M}_2\}$. Then $\int_\xi^x k_1(r)dr + \int_\eta^y k_2(s)ds \leq -M - M_1 + M_1 = -M$. In a similar way $\int_\xi^x k_1(r)dr + \int_\eta^y k_2(s)ds \leq -M$ when $|(x,y)|_\infty = y \geq \max\{\bar{M}_1,\bar{M}_2\}$. Thus using Equation (4.18), we see that the fundamental solution $\hat{A}(x,y;\xi,\eta)$ for Equation (4.26) satisfies $\lim_{|(x,y)|\to\infty} |\hat{A}(x,y;\xi,\eta)| = 0$. Thus Equation (4.26) is asymptotically stable on N_2. Therefore by part i) of this theorem Equation (4.3) is asymptotically stable on N_2. Using part i) again we see that Equation (4.4) is asymptotically stable on N_2.

For the remaining hypotheses we show that $\lim_{|(x,y)|\to\infty} |A(x,y;\xi,\eta)| = 0$ for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$. It then follows from Theorem 3.2 and part i) of this theorem that Equation (4.4) is asymptotically stable on N_2.

Assume (H8.a) holds. From inequality (4.27) and Equation (4.19) we see that $|A(x,y;\xi,\eta)|$
$\leq I_0(2\sqrt{(\int_\xi^x k_1(r)dr)(\int_\eta^y k_2(s)ds)})\exp[\int_\xi^x k_1(r)dr + \int_\eta^y k_2(s)ds]$ for $x \geq \xi$, $y \geq \eta$. An argument similar to that used above to show $A(x,y;\xi,\eta) \to 0$ as $|(x,y)| \to \infty$ establishes here that $\lim_{|(x,y)|\to\infty} |A(x,y;\xi,\eta)| = 0$.

Assume (H9.a) holds. Since $k_1k_2(x - \xi)(y - \eta) + (y - \eta)(\int_\xi^x g_1(r)dr) \leq k_1k_2(x - \xi)(y - \eta)$ it follows from

Equation (4.21) that $|A(x,y;\xi,\eta)| \leq I_0(2\sqrt{k_1 k_2 (x-\xi)(y-\eta)})$ $\exp[k_1(x - \xi) + k_2(y - \eta)]$. Then, as above we have

$$\lim_{|(x,y)|\to\infty} |A(x,y;\xi,\eta)| = 0.$$

Suppose (H11.a) is satisfied. Using the discussion in part i) and Equation (4.23) we see that the assumptions (A1) and (A2) both lead to the estimate $|A(x,y;\xi,\eta)|$ $\leq \exp[\int_\xi^x k_1(r)dr + \int_\eta^y k_2(s)ds]$. The assumption (A3) implies $|A(x,y;\xi,\eta)| \leq I_0(2\sqrt{(\int_\xi^x k_1(r)dr)(\int_\eta^y k_2(s)ds)})\exp[\int_\xi^x k_1(r)dr$ $+ \int_\eta^y k_2(s)ds]$. In either case we see that

$$\lim_{|(x,y)|\to\infty} |A(x,y;\xi,\eta)| = 0.$$

It follows from (H12.a) and Equation (4.17) that $|A(x,y;\xi,\eta)| \leq I_0(2\sqrt{k_1 k_2 (x-\xi)(y-\eta)})\exp[k_1(x - \xi) + k_2(y - \eta)]$. Hence $\lim_{|(x,y)|\to\infty} |A(x,y;\xi,\eta)| = 0$.

iii) We may treat (H9.b) and (H12.b) together. These assumptions both lead, via Equations (4.21) and (4.17) respectively, to the estimate $|A(x,y;\xi,\eta)| \leq \exp[k_1(x - \xi)$ $+ k_2(y - \eta)] \leq 1$ for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$. Thus

$$\int_a^x |A(x,y;r,b)|dr \leq \int_a^x e^{k_1(x-r)} e^{k_2(y-b)} dr$$

$$= \frac{-e^{k_2(y-b)}}{k_1}[1 - e^{k_1(x-a)}] \leq \frac{1}{|k_1|} \quad \text{for } x \geq a, \; y \geq b.$$

Similarly, $\int_b^y |A(x,y;a,s)|ds \leq \frac{1}{|k_2|}$ and $\int_a^x\int_b^y |A(x,y;r,s)|dsdr \leq \frac{1}{k_1k_2}$ for $x \geq a$, $y \geq b$. It then follows from Theorem 3.3 that Equation (4.3) is uniformly stable on N_1. Thus part i) of this theorem implies Equation (4.4) is uniformly stable on N_1.

We may also treat hypotheses (H8.b) and (H11.b) together. These hypotheses imply that $|A(x,y;\xi,\eta)| \leq \exp[\int_\xi^x k_1(r)dr + \int_\eta^y k_2(s)ds] \leq 1$ for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$. Thus

$$\int_a^x |A(x,y;r,b)|dr \leq \int_a^x \exp[\int_r^x k_1(p)dp]dr \quad \text{for} \quad x \geq a, \quad y \geq b.$$

Suppose $a \leq r_0$. Take any x such that $a \leq x \leq r_0$. Then $\int_a^x \exp[\int_r^x k_1(p)dp]dr \leq \int_a^x dr \leq (r_0 - \bar{a})$. Take any x such such that $x > r_0$. Since $k_1(r) \leq -\alpha$ for $r \geq r_0$ we see that $\int_r^x k_1(p)dp \leq -\alpha(x - r)$ and hence $\exp[\int_r^x k_1(p)dp] \leq \exp[-\alpha(x - r)]$ for $r \geq r_0$. Thus for $x \geq r_0$ we have

$$\int_a^x \exp[\int_r^x k_1(p)dp]dr = \int_a^{r_0} \exp[\int_r^x k_1(p)dp]dr$$

$$+ \int_{r_0}^x \exp[\int_r^x k_1(p)dp]dr \leq (r_0 - \bar{a})$$

$$+ \int_{r_0}^{x} \exp[-\alpha(x - r)]dr \leq (r_0 - \bar{a}) + \frac{1}{\alpha}.$$

Thus for $a \geq r_0$ and $x \geq a$ we see that $\int_a^x |A(x,y;r,b)|$ $\leq (r_0 - \bar{a}) + \frac{1}{\alpha}$. For $a \geq r_0$ and any $x \geq a$ a calculation similar to that given above shows that $\int_a^x |A(x,y;r,b)|dr$ $\leq \frac{1}{\alpha}$. Hence for $\bar{a} \leq a \leq x < \infty$, $\bar{b} \leq b \leq y < \infty$, we have $\int_a^x |A(x,y;r,b)|dr \leq (r_0 - \bar{a}) + \frac{1}{\alpha}$. Similar arguments show that $\int_b^y |A(x,y;a,s)|ds \leq (s_0 - \bar{b}) + \frac{1}{\beta}$ and $\int_a^x\int_b^y |A(x,y;r,s)|dsdr \leq [(r_0 - \bar{a}) + \frac{1}{\alpha}][(s_0 - \bar{b}) + \frac{1}{\beta}]$ for $\bar{a} \leq a \leq x < \infty$, $\bar{b} \leq b \leq y < \infty$. Thus by Theorem 3.3, Equation (4.3) is uniformly stable on N_1. It then follows from part i) of this theorem that Equation (4.4) is uniformly stable on N_1. This completes the proof of Theorem (4.2).

Remark 4.4. We notice that the conditions in (H3) of Section 3.3 will be satisfied (with $\psi \equiv 1$ and $\psi_{\alpha_k} \equiv 0$) if (H3)' holds and $A(x,y;\xi,\eta)$ satisfies

$$\int_a^x\int_b^y |A(x,y;r,s)|dsdr \leq M(a,b) \quad \text{for } x \geq a, \quad y \geq b. \qquad (4.28)$$

Also, the conditions in (H4) will hold if in addition we have for each fixed $t \geq \max\{a,b\}$

$$\left.\begin{array}{l}
\lim\limits_{|(x,y)|_\infty \to \infty} \int_a^t \int_b^t |A(x,y;r,s)|dsdr = 0 \\[2ex]
\text{for} \quad x \geq t, \quad y \geq t, \\[2ex]
\lim\limits_{x \to \infty} \int_a^t \int_b^y |A(x,y;r,s)|dsdr = 0 \\[2ex]
\text{uniformly in} \quad y \quad \text{for} \quad b \leq y \leq t \\[2ex]
\lim\limits_{y \to \infty} \int_a^x \int_b^t |A(x,y;r,s)|dsdr = 0 \\[2ex]
\text{uniformly in} \quad x \quad \text{for} \quad a \leq x \leq t
\end{array}\right\} \tag{4.29}$$

We now turn our attention to a stability theorem for the class of perturbations satisfying hypothesis (H3)'.

Theorem 4.3. Suppose $f(x,y,r,s,z)$ satisfies (H3)'. Suppose any one of the hypotheses (H8.b), (H9.b), (H11.b), or (H12.b) hold.

i) Then Equation (4.4) preserves stability and uniform stability on any space N. In particular, Equation (4.4) is uniformly stable on N_1 and N_2.

ii) If either (H8.b) or (H11.b) hold, assume in addition that the functions $k_1(r)$, $k_2(s)$ are negative constants. Then Equation (4.4) preserves asymptotic stability on any space N. In particular, Equation (4.4) is asymptotically stable on N_2.

Proof. Throughout this proof $A(x,y;\xi,\eta)$ will denote the fundamental solution for Equation (4.3).

i) Both hypothesis (H9.b) and (H12.b) lead to the estimate $|A(x,y;\xi,\eta)| \le e^{k_1(x-\xi)} e^{k_2(y-\eta)}$. Thus $\int_a^x\int_b^y |A(x,y;r,s)|dsdr \le \frac{1}{k_1k_2}[1 - e^{k_1(x-a)}][1 - e^{k_2(y-b)}]$ $\le \frac{1}{k_1k_2}$ for $a \ge \bar{a}$, $b \ge \bar{b}$. The hypotheses (H8.b) and (H11.b) each yield the inequality $|A(x,y;\xi,\eta)| \le \exp[\int_\xi^x k_1(r)dr + \int_\eta^y k_2(s)ds]$. Thus, as in the proof of Theorem 4.2, we have $\int_a^x\int_b^y |A(x,y;r,s)|dsdr \le [(r_0 - \bar{a}) + \frac{1}{\alpha}][(s_0 - \bar{b}) + \frac{1}{\beta}]$ for $\bar{a} \le a$, $\bar{b} \le b$. We see that these bounds are independent of a and b. Therefore by Remark 4.4 and Theorem 3.8 it follows that Equation (4.4) preserves stability and uniform stability on any space N. Hence by Theorem 4.2 Equation (4.4) is uniformly stable on N_1 and N_2.

ii) Take any (a,b) and fix $t \ge \max\{a,b\}$. Under each of the hypotheses we have $|A(x,y;\xi,\eta)| \le \exp[k_1(x - \xi) + k_2(y - \eta)]$. Thus

$$\int_a^t\int_b^t |A(x,y;r,s)|dsdr$$

$$\leq \frac{1}{k_1k_2}\{\exp[k_1(x - t)] - \exp[k_1(x - a)]\}\{\exp[k_2(y - t)]$$

$$- \exp[k_2(y - b)]\} \leq \frac{1}{k_1k_2}\exp[k_1(x - t) + k_2(y - t)]$$

for $x \geq t$, $y \geq t$. Take any $\varepsilon > 0$ and let $k = \max\{k_1,k_2\}$. There exists an R such that if $(x - t) + (y - t) \geq R$ then $\exp\{k[(x - t) + (y - t)]\} < k_1k_2\varepsilon$. Now take (x,y) such that $x \geq t$, $y \geq t$ and $|(x,y)|_\infty \geq R + |t|$. Then $\max\{|x - t|,|y - t|\} = |(x,y) - (t,t)|_\infty \geq |(x,y)|_\infty - |(t,t)| \geq R$. Thus for $x \geq t$, $y \geq t$ such that $|(x,y)|_\infty \geq R + |t|$ we have $(x - t) + (y - t) \geq R$ and hence

$$\int_a^t\int_b^t|A(x,y;r,s)|dsdr \leq \frac{1}{k_1k_2}\exp[k_1(x - t) + k_2(y - t)]$$

$$\leq \frac{1}{k_1k_2}\exp\{k[(x - t) + (y - t)]\} < \varepsilon.$$

Therefore $\lim\limits_{|(x,y)|_\infty\to\infty} \int_a^t\int_b^t|A(x,y;r,s)|dsdr = 0$. We also see that

$$\int_a^t\int_b^y|A(x,y;r,s)|dsdr \leq \frac{1}{k_1k_2}\{\exp[k_1(x - t)] - \exp[k_1(x - a)]\}$$

$$\{1 - \exp[k_2(y - b)]\} \leq \frac{1}{k_1k_2}\{\exp[k_1(x - t)] - \exp[k_1(x - a)]\}$$

for all y such that $b \leq y \leq t$. From this estimate it follows easily that $\lim_{x\to\infty} \int_a^t\int_b^y |A(x,y;r,s)|dsdr = 0$ uniformly in y such that $b \le y \le t$. Similarly we have $\lim_{y\to\infty} \int_a^x\int_b^t |A(x,y;r,s)|dsdr = 0$ uniformly in x such that $a \le x \le t$. The result now follows directly from Remark 4.4 Theorem 3.8, and Theorem 4.2. This completes the proof.

4.2 The Equation $u(x,y) = \phi(x,y) + \int_a^x k_1(r,y)u(r,y)dr + \int_b^y k_2(x,s)u(x,s)ds - \int_a^x\int_b^y [k_1(r,s)k_2(r_1 s) + k_{1s}(r,s)]u(r,s)dsdr + \int_a^x\int_b^y f(x,y,r,s,u(r,s))dsdr$

In this section we will give some results for the equations

$$u(x,y) = \phi(x,y) + \int_a^x k_1(r,y)u(r,y)dr + \int_b^y k_2(x,s)u(x,s)ds - \int_a^x\int_b^y k_3(r,s)u(r,s)dsdr, \tag{4.30}$$

$$u(x,y) = \phi(x,y) + \int_a^x k_1(r,y)u(r,y)dr + \int_b^y k_2(x,s)u(x,s)ds - \int_a^x\int_b^y k_3(r,s)u(r,s)dsdr + \int_a^x\int_b^y f(x,y,r,s,u(r,s))dsdr \tag{4.31}$$

where $k_3(r,s) = k_1(r,s)k_2(r,s) + k_{1s}(r,s)$ or $k_3(r,s) = k_1(r,s)k_2(r,s) + k_{2r}(r,s)$ (here we use $k_{1s} = \frac{\partial k_1}{\partial s}$ and $k_{2r} = \frac{\partial k_2}{\partial r}$). We assume that the functions $k_1(r,s)$, $k_2(r,s)$, $k_{1s}(r,s)$, and $k_{2r}(r,s)$ are continuous.

We consider Equation (4.30) because this equation, with $\phi(x,y) \equiv 1$, is equivalent to a characteristic value problem which may be solved explicitly: hence we may obtain the fundamental solution for Equation (4.30). The method used to obtain the following lemma is called Laplace's cascade method and is discussed in [26], [35]. We use the following lemma for stability purposes and include the proof for completeness.

Lemma 4.8. The fundamental solution for Equation (4.30) with $k_3(r,s) = k_1(r,s)k_2(r,s) + k_{1s}(r,s)$ is

$$A(x,y;\xi,\eta) = \exp\left[\int_{\xi}^{x} k_1(r,\eta)\,dr + \int_{\eta}^{y} k_2(x,s)\,ds\right] \qquad (4.32)$$

for $\bar{a} \le \xi \le x$, $\bar{b} \le \eta \le y$. For $k_3(r,s) = k_1(r,s)k_2(r,s) + k_{2r}(r,s)$ we have

$$A(x,y;\xi,\eta) = \exp\left[\int_{\xi}^{x} k_1(r,y)\,dr + \int_{\eta}^{y} k_2(\xi,s)\,ds\right]. \qquad (4.33)$$

Proof. We will give the proof that A given by (4.32) is the fundamental solution for Equation (4.30) with

$k_3(r,s) = k_1(r,s)k_2(r,s) + k_{1s}(r,s)$. The proof for $k_3(r,s) = k_1(r,s)k_2(r,s) + k_{2r}(r,s)$ is similar. Take $\xi \geq \bar{a}$, $\eta \geq \bar{b}$. If $u(x,y)$ satisfies the characteristic value problem

$$\left.\begin{aligned} &u_{xy}(x,y) = k_1(x,y)u_y + k_2(x,y)u_x(x,y) \\ &+ [k_{2x}(x,y) - k_1(x,y)k_2(x,y)]u(x,y) \\ &u(x,\eta) = \exp[\int_\xi^x k_1(r,\eta)dr], \quad u(\xi,y) = \exp[\int_\eta^y k_2(\xi,s)ds] \end{aligned}\right\} \quad (4.34)$$

for $x \geq \xi$, $y \geq \eta$, then u satisfies Equation (4.30) with $\phi \equiv 1$, $k_3 = k_1k_2 + k_{1s}$ and hence $u(x,y) = A(x,y;\xi,\eta)$. We have the following identity:

$$(u_y - k_2u)_x - k_1(u_y - k_2u) = u_{xy} - k_1u_y - k_2u_x + (k_1k_2 - k_{2x})u.$$

Thus u satisfies (4.34) if and only if u satisfies

$$\left.\begin{aligned} &(u_y - k_2u)_x - k_1(u_y - k_2u) = 0 \\ &u(x,\eta) = \exp[\int_\xi^x k_1(r,\eta)dr], \quad u(\xi,y) = \exp[\int_\eta^y k_2(\xi,s)ds] \end{aligned}\right\}. \quad (4.35)$$

Let $h(x,y) = u_y(x,y) - k_2(x,y)u(x,y)$. From problem (4.35) it follows that the function $h(x,y)$ satisfies $h_x(x,y) - k_1(x,y)h(x,y) = 0$. Then integrating with respect to x we see that $h(x,y)\exp[-\int_{\xi}^{x} k_1(r,y)dr] = g_1(y)$. To determine $g_1(y)$ we set $x = \xi$ and use (4.35) to obtain $g_1(y) = h(\xi,y) = u_y(\xi,y) - k_2(\xi,y)u(\xi,y) \equiv 0$. Therefore we must have $h(x,y) = u_y(x,y) - k_2(x,y)u(x,y) = 0$. Integrating with respect to y we obtain $u(x,y)\exp[-\int_{\eta}^{y} k_2(x,s)ds] = g_2(x)$. Setting $y = \eta$ we have $g_2(x) = u(x,\eta) = \exp[\int_{\xi}^{x} k_1(r,\eta)dr]$. Thus $u(x,y) = \exp[\int_{\xi}^{x} k_1(r,\eta)dr + \int_{\eta}^{y} k_2(x,s)ds]$ is the solution of (4.35) and the proof is complete.

We point out that when $k_1(r,s)$ is independent of s and $k_2(r,s)$ is independent of r then Equation (4.19) with $g_1(r) \equiv -1$ and Equation (4.32) are identical.

We consider the following hypotheses on the functions $k_1(x,y)$ and $k_2(x,y)$.

(H13) Suppose there exists a constant M such that $\int_{\xi}^{x} k_1(r,\eta)dr + \int_{\eta}^{y} k_2(x,s)ds \leq M$ for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$ (or $\int_{\xi}^{x} k_1(r,y)dr + \int_{\eta}^{y} k_2(\xi,s)ds \leq M$ for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$).

(H13.a) Suppose (H13) holds and in addition for $\xi \geq \bar{a}$, $\eta \geq \bar{b}$ we have $\int_{\xi}^{x} k_1(r,\eta)dr + \int_{\eta}^{y} k_2(x,s)ds \to -\infty$ as $|(x,y)| \to \infty$ (or $\int_{\xi}^{x} k_1(r,y)dr + \int_{\eta}^{y} k_2(\xi,s)ds \to -\infty$ as $|(x,y)| \to \infty$).

(H14) Suppose there exists constants α_1, $\alpha_2 > 0$ such that $k_1(x,y) \leq -\alpha_1$, $k_2(x,y) \leq -\alpha_2$ for $x \geq \bar{a}$, $y \geq \bar{b}$.

We have the following stability theorem for Equation (4.31).

Theorem 4.4. a) Suppose $f(x,y,r,s,z)$ satisfies (H1)' and (H2)'.

i) Suppose we have (H13). Then Equation (4.31) with $k_3 = k_1k_2 + k_{1s}$ (or $k_3 = k_1k_2 + k_{2r}$) preserves stability and uniform stability on any space N; it preserves asymptotic stability on any space N such that $N \subset N_3$. Equation (4.31) is uniformly stable on N_2.

ii) Suppose (H13.a) holds. Equation (4.31) with $k_3 = k_1k_2 + k_{1s}$ (or $k_3 = k_1k_2 + k_{2r}$) is asymptotically stable on N_2.

iii) Suppose (H14). Then Equation (4.31) with $k_3 = k_1k_2 + k_{1s}$ (or $k_3 = k_1k_2 + k_{2r}$) is uniformly stable on N_1.

b) Suppose $f(x,y,r,s,z)$ satisfies (H3)'. Suppose (H14) is satisfied. Then Equation (4.31) with $k_3 = k_1k_2 + k_{1s}$ (or $k_3 = k_1k_2 + k_{2r}$) preserves stability, uniform stability and asymptotic stability on any space N. Equation (4.31) is uniformly stable on N_1 and N_2 and asymptotically stable on N_2.

Proof. We give the proof only for the case $k_3 = k_1k_2 + k_{1s}$. Throughout the proof $A(x,y;\xi,\eta)$ is the fundamental solution for Equation (4.30).

a) i) From Equation (4.32) and hypothesis (H13) it follows that $|A(x,y;\xi,\eta)| \leq e^M$ for $\bar{a} \leq \xi \leq x < \infty$, $\bar{b} \leq \eta \leq y < \infty$. Thus, by Theorem 3.2, Equation (4.31) is uniformly stable on N_2. The result now follows from Theorem 3.5.

ii) It follows immediately from (H13.a) and Equation (4.32) that $\lim_{|(x,y)|\to\infty} |A(x,y;\xi,\eta)| = 0$. Hence by part i) and Theorem 3.2 we see that Equation (4.31) is asymptotically stable on N_2.

iii) From (H14) and Equation (4.32) we see that $|A(x,y;\xi,\eta)| \leq \exp[-\alpha_1(x - \xi) - \alpha_2(y - \eta)]$. An easy calculation similar to the one made in Theorem 4.2 shows that

$$|A(x,y;a,b)| + \int_a^x |A(x,y;r,b)|dr + \int_b^y |A(x,y;a,s)|ds$$

$$+ \int_a^x\int_b^y |A(x,y;r,s)|dsdr \leq 1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_1\alpha_2}$$

for $\bar{a} \leq a \leq x < \infty$, $\bar{b} \leq b \leq y < \infty$. Thus by Theorem 3.3 and part i) it follows that Equation (4.31) is uniformly stable on N_1.

b) As in part iii) above we have $|A(x,y;\xi,\eta)| \leq \exp[-\alpha_1(x - \xi) - \alpha_2(y - \eta)]$. This result now follows from calculations obtained in the proof of Theorem 4.3, Theorem 3.8, and part a) of this theorem. This completes the proof.

4.3 Results for Systems with Constant Kernels

In this section we will consider the linear equation

$$u(x,y) = \phi(x,y) + \int_a^x K_1 u(r,y)dr + \int_b^y K_2 u(x,s)ds + \int_a^x\int_b^y K_3 u(r,s)dsdr \qquad (4.36)$$

and the nonlinear equation

$$u(x,y) = \phi(x,y) + \int_a^x K_1 u(r,y)dr + \int_b^y K_2 u(x,s)ds + \int_a^x\int_b^y [K_3 u(r,s) + f(x,y,r,s,u(r,s))]dsdr \qquad (4.37)$$

where u, ϕ map R^2 to R^m, K_1, K_2, K_3 are constant real $m \times m$ matrices, and f has values in R^m with domain $\bar{a} \leq r \leq x < \infty$, $\bar{b} \leq s \leq y < \infty$, $z \in R^m$.

In constructing the solution of the differential system $y' = By$ (where B is an $m \times m$ constant matrix), we define the matrix e^R for each constant matrix R and then establish e^{tB} as a fundamental matrix for $y' = By$. This is a generalization of the fundamental solution for the scalar version of $y' = By$. We will see that it is possible, under certain commutativity conditions on the matrices K_1, K_2, K_3, to carry out a similar generalization here.

<u>Remark 4.5</u>. Let B be any $m \times m$ complex matrix It follows easily that the series $I + \sum_{k=1}^{\infty} \frac{B^k}{(k!)^2}$ converges. Our motive for considering this series is that the fundamental solution for Equation (4.36) when $m = 1$ is given by Equation (4.17) and that $I_0(2\sqrt{x}) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}$ [25].

We may therefore make the following definition.

<u>Definition 4.1</u>. For each complex matrix B let $I_0(2\sqrt{B}) = I + \sum_{k=1}^{\infty} \frac{B^k}{(k!)^2}$.

The following lemma gives a useful property of the matrix $I_0(2\sqrt{B})$.

<u>Lemma 4.9</u>. Suppose B is an $m \times m$ complex matrix which is similar to a diagonal matrix Λ and T is such that $\Lambda = T^{-1}BT$. Then

$$I_0(2\sqrt{B}) = TI_0(2\sqrt{\Lambda})T^{-1}.$$

Also, if $\Lambda = \operatorname{diag}_{1\le i\le m} (\lambda_i)$ then $I_0(2\sqrt{\Lambda}) = \operatorname{diag}_{1\le i\le m} (I_0(2\sqrt{\lambda_i}))$.

Proof. We have

$$I_0(2\sqrt{B}) = I + \sum_{k=1}^{\infty} \frac{B^k}{(k!)^2} = TT^{-1} + \sum_{k=1}^{\infty} \frac{(T\Lambda T^{-1})^k}{(k!)^2}$$

$$= TT^{-1} + T\left(\sum_{k=1}^{\infty} \frac{\Lambda^k}{(k!)^2}\right)T^{-1} = T\left(I + \sum_{k=1}^{\infty} \frac{\Lambda^k}{(k!)^2}\right)T^{-1}$$

$$= TI_0(2\sqrt{\Lambda})T^{-1}.$$

If $\Lambda = \operatorname{diag}_{1\le i\le m} (\lambda_i)$ it follows that $\Lambda^k = \operatorname{diag}_{1\le i\le m} (\lambda_i^k)$. Therefore, we have

$$I_0(2\sqrt{\Lambda}) = I + \sum_{k=1}^{\infty} \frac{\Lambda^k}{(k!)^2} = \operatorname{diag}_{1\le i\le m} (I_0(2\sqrt{\lambda_i})).$$

This completes the proof.

We now use our definition of $I_0(2\sqrt{B})$ to obtain the following lemma.

Lemma 4.10. Let B be a real $m \times m$ matrix. Then the solution of the equation

$$u(x,y) = I + \int_{\xi}^{x}\int_{\eta}^{y} Bu(r,s)\,ds\,dr \quad \text{for} \quad x \ge \xi, \quad y \ge \eta \tag{4.38}$$

is

$$u(x,y) = I_0(2\sqrt{B(x-\xi)(y-\eta)}).$$

Proof. The successive approximations for this equation converge to the unique continuous solution. We have

$$u_1(x,y) = I$$

and

$$u_2(x,y) = I + \int_\xi^x\int_\eta^y BI\,ds\,dr = I + B(x - \xi)(y - \eta).$$

Assume that

$$u_n(x,y) = I + \sum_{k=1}^{n-1} \frac{[B(x-\xi)(y-\eta)]^k}{(k!)^2}$$

Then

$$u_{n+1}(x,y) = I + \int_\xi^x\int_\eta^y B[I + \sum_{k=1}^{n-1} \frac{[B(r-\xi)(s-\eta)]^k}{(k!)^2}]dsdr$$

$$= I + B(x - \xi)(y - \eta) + \sum_{k=1}^{n-1} [\int_\xi^x\int_\eta^y \frac{B^{k+1}(r-\xi)^k(s-\eta)^k}{(k!)^2}]dsdr$$

$$= I + B(x - \xi)(y - \eta) + \sum_{k=1}^{n-1} \frac{B^{k+1}(x-\xi)^{k+1}(y-\eta)^{k+1}}{[(k+1)!]^2}$$

$$= I + \sum_{k=1}^{n} \frac{[B(x-\xi)(y-\eta)]^k}{(k!)^2}.$$

Thus $u_n(x,y) = I + \sum_{k=1}^{n-1} \frac{[B(x-\xi)(y-\eta)]^k}{(k!)^2}$ for each $n = 1, 2,$ $\ldots$. Therefore the solution $u(x,y)$ of Equation (4.38) is given by

$$u(x,y) = \lim_{n\to\infty} u_n(x,y) = I_0(2\sqrt{B(x-\xi)(y-\eta)}).$$

This completes the proof.

The following lemma gives the fundamental solution for Equation (4.36) under the specified conditions on the kernels. This result is a generalization of the fundamental solution given by Equation (4.17) for the scalar equation with constant kernels.

Lemma 4.11. Suppose the matrices K_1, K_2, K_3 satisfy $K_1K_2 = K_2K_1$, $K_1K_3 = K_3K_1$, and $K_2K_3 = K_3K_2$. Then the fundamental solution for Equation (4.36) is

$$A(x,y;\xi,\eta) = I_0(2\sqrt{(K_1K_2+K_3)(x-\xi)(y-\eta)})e^{K_1(x-\xi)}e^{K_2(y-\eta)}. \quad (4.39)$$

Proof. For any ξ, η let $u(x,y) = A(x,y;\xi,\eta)$. Thus $u(x,y)$ must satisfy Equation (4.36) with $\phi \equiv I$, $a = \xi$, $b = \eta$. But $u(x,y)$ is the solution of Equation (4.36) with $\phi \equiv I$, $a = \xi$, $b = \eta$ if and only if $u(x,y)$ satisfies the characteristic value problem

$$\left.\begin{aligned} u_{xy}(x,y) &= K_1 u_y(x,y) + k_2 u_x(x,y) + K_3 u(x,y) \\ u(x,\xi) &= \exp[K_1(x-\xi)], \quad u(\xi,y) = \exp[K_2(y-\eta)] \end{aligned}\right\}. \tag{4.40}$$

Let

$$u(x,y) = e^{K_1(x-\xi)} e^{K_2(y-\eta)} v(x,y). \tag{4.41}$$

Since K_1 and K_2 commute we have,

$$u(x,y) = e^{K_1(x-\xi)+K_2(y-\eta)} v(x,y) = e^{K_2(y-\xi)+K_1(x-\eta)} v(x,y)$$

$$= e^{K_2(y-\xi)} e^{K_1(x-\eta)} v(x,y).$$

We then have,

$$u_x(x,y) = K_1 e^{K_1(x-\xi)} e^{K_2(y-\eta)} v + e^{K_1(x-\xi)} e^{K_2(y-\eta)} v_x$$

and

$$u_y(x,y) = K_2 e^{K_1(x-\xi)} e^{K_2(y-\eta)} v + e^{K_1(x-\xi)} e^{K_2(y-\eta)} v_y.$$

Therefore

$$u_{xy}(x,y) = K_1K_2e^{K_1(x-\xi)}e^{K_2(y-\eta)}v + K_1e^{K_1(x-\xi)}e^{K_2(y-\eta)}v_y$$

$$+ K_2e^{K_1(x-\xi)}e^{K_2(y-\eta)}v_x + e^{K_1(x-\xi)}e^{K_2(y-\eta)}v_{xy}.$$

Putting u_x, u_y, and u_{xy} in terms of v into the equation in (4.40), we obtain

$$e^{K_1(x-\xi)}e^{K_2(y-\eta)}v_{xy} = (K_1K_2 + K_3)e^{K_1(x-\xi)}e^{K_2(y-\eta)}v. \quad (4.42)$$

Using the fact that if B and C are $m \times m$ matrices such that $BC = CB$ then $Be^C = e^CB$, we see that Equation (4.42) becomes

$$e^{K_1(x-\xi)}e^{K_2(y-\eta)}v_{xy} = e^{K_1(x-\xi)}e^{K_2(y-\eta)}(K_1K_2 + K_3)v. \quad (4.43)$$

Since e^B is nonsingular for each $m \times m$ matrix B, Equation (4.43) takes the form

$$v_{xy}(x,y) = (K_1K_2 + K_3)v. \quad (4.44)$$

Using the characteristic data for $u(x,y)$ in problem (4.40), we obtain the following conditions on $v(x,y)$:

$$v(x,\eta) = I, \quad v(\xi,y) = I. \quad (4.45)$$

Now if $v(x,y)$ satisfies (4.44) and (4.45) and if $u(x,y)$ is given by (4.41), then $u(x,y)$ is the solution of Equation (4.36) with $\phi \equiv I$, $a = \xi$, $b = \eta$. The integral equation for $v(x,y)$ is

$$v(x,y) = I + \int_{\xi}^{x}\int_{\eta}^{y}(K_1K_2 + K_3)v(r,s)dsdr.$$

Thus, by Lemma 4.10, $v(x,y)$ is given by

$$v(x,y) = I_0(2\sqrt{(K_1K_2+K_3)(x-\xi)(y-\eta)})$$

and hence

$$A(x,y;\xi,\eta) = u(x,y) = e^{K_1(x-\xi)} e^{K_2(y-\eta)} I_0(2\sqrt{(K_1K_2+K_3)(x-\xi)(y-\eta)})$$

$$= I_0(2\sqrt{(K_1K_2+K_3)(x-\xi)(y-\eta)})e^{K_1(x-\xi)} e^{K_2(y-\eta)}.$$

This completes the proof.

We will make use of the following hypotheses.

(H15) Suppose the matrix $K_1K_2 + K_3$ is similar to the diagonal matrix Λ, and the eigenvalues $K_1K_2 + K_3$ are real and nonpositive. Suppose the eigenvalues of K_1 and K_2 have nonpositive real parts and those eigenvalues with zero real parts are simple.

(H16) Suppose $K_1K_2 + K_3$ is similar to the diagonal matrix Λ, and the eigenvalues of $K_1K_2 + K_3$ are real and nonpositive. Suppose the eigenvalues of K_1 and K_2 have negative real parts.

We now give a stability theorem for Equation (4.37).

Theorem 4.5. Suppose $K_1K_2 = K_2K_1$, $K_1K_3 = K_3K_1$, and $K_2K_3 = K_3K_2$.

a) Suppose $f(x,y,r,s,z)$ satisfies (H1)' and (H2)'.

i) Suppose (H15) holds. Then Equation (4.37) preserves stability and uniform stability on any space N; it preserves asymptotic stability on any space N such that $N_3 \subset N$. In particular, Equation (4.37) is uniformly stable on N_2.

ii) Suppose (H16) holds. Then Equation (4.37) is uniformly stable on N_1 and N_2; it is asymptotically stable on N_2.

b) Suppose $f(x,y,r,s,z)$ satisfies (H3)' and (H16) holds. Then Equation (4.37) preserves stability, uniform stability, and asymptotic stability on any space N. In particular, Equation (4.37) is uniformly stable on N_1 and N_2; it is asymptotically stable on N_2.

<u>Proof</u>. a) i) Take any $\xi \geq \bar{a}$, $\eta \geq \bar{b}$. By Lemma 4.11 the fundamental solution $A(x,y;\xi,\eta)$ for Equation (4.36) is given by (4.39). In this proof we will use a special matrix norm; if B is an $m \times m$ matrix with elements b_{ij} we will use $\|B\| = \max_{1 \leq i \leq m} [\sum_{j=1}^{m} |b_{ij}|]$. Since $K_1K_2 + K_3$ is similar to Λ, there exists a nonsingular matrix T such that $\Lambda = T^{-1}(K_1K_2 + K_3)T$. Then using Lemma 4.9, we have

$$I_0(2\sqrt{(K_1K_2+K_3)(x-\xi)(y-\eta)}) = TI_0(2\sqrt{\Lambda(x-\xi)(y-\eta)})T^{-1}.$$

Suppose $\Lambda = \operatorname{diag}_{1 \leq i \leq m} (\lambda_i)$. By assumption we have $\lambda_i \leq 0$ for $i = 1, 2, \ldots, m$. Using this, Remark 4.2, and Lemma 4.9 again we see that

$$I_0(2\sqrt{\Lambda(x-\xi)(y-\eta)}) = \operatorname{diag}_{1 \leq i \leq m} (I_0(2\sqrt{\lambda_i(x-\xi)(y-\eta)})$$

$$= \operatorname{diag}_{1 \leq i \leq m} (J_0(2\sqrt{|\lambda_i|(x-\xi)(y-\eta)})).$$

Therefore $\|I_0(2\sqrt{\Lambda(x-\xi)(y-\eta)})\| \leq 1$ for $x \geq \xi$, $y \geq \eta$. Since all the eigenvalues of K_1 and K_2 have negative or zero real parts and those with a zero real part are simple, there are constants M_1 and M_2 independent of ξ, η so that

$\|e^{K_1(x-\xi)}\| \le M_1$ and $\|e^{K_2(y-\eta)}\| \le M_2$ for $x \ge \xi$, $y \ge \eta$

(see [9], p. 81). Therefore

$$\|A(x,y;\xi,\eta)\|$$

$$\le \|T\|\|I_0(2\sqrt{\Lambda(x-\xi)(y-\eta)})\|\|T^{-1}\|\|e^{K_1(x-\xi)}\|\|e^{K_2(y-\eta)}\|$$

$$\le \|T\|\|T^{-1}\|M_1M_2.$$

Thus it follows from Theorem 3.2 that Equation (4.36) is uniformly stable on N_2. Theorem 3.5 then implies the preservation results.

ii) Since (H16) implies (H15), it follows by part i) of this theorem that Equation (4.37) is uniformly stable on N_2. Also, as in the proof of part i), we have

$$\|A(x,y;\xi,\eta)\| \le \|T\|\|T^{-1}\|\|e^{K_1(x-\xi)}\|\|e^{K_2(y-\eta)}\|.$$

However, the additional assumptions now insure existence of positive constants M_1, M_2, ρ_1, ρ_2 such that

$$\|e^{K_1(x-\xi)}\| \le M_1e^{-\rho_1(x-\eta)} \quad \text{and} \quad \|e^{K_2(y-\eta)}\| \le M_2e^{-\rho_2(y-\eta)}.$$

Therefore, we have

$$\|A(x,y;\xi,\eta)\| \leq \|T\|\|T^{-1}\|M_1M_2e^{-\rho_1(x-\xi)}e^{-\rho_2(y-\eta)}.$$

From this it follows that $\lim_{|(x,y)|\to\infty} \|A(x,y;\xi,\eta)\| = 0$ for each $\xi \geq \bar{a}$, $\eta \geq \bar{b}$. Thus Equation (4.36) is asymptotically stable on N_2; hence, by part i), Equation (4.37) is also asymptotically stable on N_2. Using the inequality above and proceeding as in Theorem 4.2, part iii), we see that Equation (4.37) is uniformly stable on N_1.

b) Using the estimate in part ii) above, we obtain this result by the same argument that was employed in the proof of Theorem 4.3.

4.4 Results for Equations With a Pincherle-Goursat Kernel

In this section we will return to the notation used in the earlier chapters. We shall consider the equations

$$u(x) = \phi(x) + \int_a^x [\sum_{i=1}^{p} B_i(x)C_i(r)]u(r)dr \qquad (4.46)$$

and

$$u(x) = \phi(x) + \int_a^x [\sum_{i=1}^{p} B_i(x)C_i(r)]u(r)dr$$
$$+ \int_a^x f(x,r,u(r))dr, \qquad (4.47)$$

where a, r, $x \in R^n$, u, ϕ map $x \geq \bar{a}$ to R^m, $B_i(x)$, $C_i(r)$ are continuous $m \times m$ matrices, and f maps $\bar{a} \leq r \leq x$, $z \in R^m$ to R^m. Kernels of this type, namely $K_{\alpha_n} = \sum_{i=1}^{p} B_i(x)C_i(r)$, are known as Pincherle-Goursat kernels, or PG-kernels [38]. Equations with kernels of this form appear in the literature for a variety of reasons. We will obtain stability results for Equations (4.46) and (4.47).

The fundamental solution $A(x;\xi)$ for Equation (4.46) satisfies

$$A(x;\xi) = I + \int_{\xi}^{x} [\sum_{i=1}^{p} B_i(x)C_i(r)]A(r;\xi)dr \qquad x \geq \xi \geq \bar{a}. \tag{4.48}$$

We may obtain $A(x;\xi)$ in this case in terms of the solution of a related problem. Consider the $pm \times pm$ matrix $M(x)$ given by

$$M(x) = \begin{bmatrix} C_1(x)B_1(x) & C_1(x)B_2(x) & \dots & C_1(x)B_p(x) \\ C_2(x)B_1(x) & C_2(x)B_2(x) & \dots & C_2(x)B_p(x) \\ \vdots & & & \\ C_p(x)B_1(x) & C_p(x)B_2(x) & \dots & C_p(x)B_p(x) \end{bmatrix}. \tag{4.49}$$

We now suppose that $A^*(x;\xi)$ is the solution of the integral equation

$$A^*(x;\xi) = I + \int_\xi^x M(r)A^*(r;\xi)dr. \tag{4.50}$$

Suppose $A^*(x;\xi)$ is blocked off into $m \times m$ submatrices. Let $A^*_{ij}(x;\xi)$ be the $m \times m$ submatrix in the i^{th} row and j^{th} column of the resulting $p \times p$ matrix of submatrices.

The following lemma gives $A(x;\xi)$ in terms of $A^*(x;\xi)$. Results related to this lemma have been discussed in [6], [13], [20]. The author has obtained a similar result for a special case of the more general Equation (1.7) with PG-kernels and hopes to obtain further results along these lines.

Lemma 4.12. Let $A^*(x;\xi)$ be the solution of Equation (4.50) for $x \geq \xi \geq \bar{a}$. Then the fundamental solution for Equation (4.46) is given by

$$A(x;\xi) = I + \sum_{i,j=1}^{p} \int_\xi^x B_i(x)A^*_{ij}(x;r)C_j(r)dr \tag{4.51}$$

for $a \leq \xi \leq x < \infty$.

Proof. Putting Equation (4.51) into Equation (4.48) we obtain

$$I + \int_\xi^x [\sum_{i=1}^{p} B_i(x)C_i(r)][I + \sum_{\ell,q=1}^{p} \int_\xi^r B_\ell(r)A^*_{\ell q}(r;s)C_q(s)ds]dr$$

$$= I + \int_{\xi}^{x} [\sum_{i=1}^{p} B_i(x)C_i(r)]dr$$

$$+ \sum_{i,\ell,q=1}^{p} \int_{\xi}^{x}\int_{\xi}^{r} B_i(x)C_i(r)B_\ell(r)A^*_{\ell q}(r;s)C_q(s)dsdr.$$

From Fubini's theorem, (see Appendix A), it follows that the above becomes

$$I + \int_{\xi}^{x} [\sum_{i=1}^{p} B_i(x)C_i(r)]dr$$

$$+ \sum_{i,\ell,q=1}^{p} \int_{\xi}^{x}\int_{s}^{x} B_i(x)C_i(r)B_\ell(r)A^*_{\ell q}(r;s)C_q(s)drds$$

$$= I + \int_{\xi}^{x} [\sum_{i=1}^{p} B_i(x)C_i(r)]dr$$

$$+ \sum_{i,q=1}^{p} \int_{\xi}^{x} B_i(x)[\int_{s}^{x} \sum_{\ell=1}^{p} C_i(r)B_\ell(r)A^*_{\ell q}(r;s)dr]C_q(s)ds. \qquad (4.52)$$

Since $A^*(x;\xi)$ is the solution of Equation (4.50) it follows that

$$A^*_{ij}(x;\xi) = \delta_{ij}I + \int_{\xi}^{x} [\sum_{k=1}^{p} C_i(r)B_k(r)A^*_{kj}(r;\xi)]dr.$$

Now using this in Equation (4.52) we see that the right hand fo that equation becomes

$$I + \int_{\xi}^{x} [\sum_{i=1}^{p} B_i(x)C_i(r)]dr$$

$$+ \sum_{i,q=1}^{p} \int_{\xi}^{x} B_i(x)[A_{iq}^*(x;s) - \delta_{iq}I]C_q(s)ds$$

$$= I + \int_{\xi}^{x} [\sum_{i=1}^{p} B_i(x)C_i(r)]dr + \sum_{i,q=1}^{p} \int_{\xi}^{x} B_i(x)A_{iq}^*(x;s)C_q(s)ds$$

$$- \sum_{i=1}^{p} \int_{\xi}^{x} B_i(x)C_i(s)ds = I + \sum_{i,q=1}^{p} \int_{\xi}^{x} B_i(x)A_{iq}^*(x;s)C_q(s)ds.$$

This completes the proof.

We point out that Equation (4.50) is equivalent to a characteristic value problem for a system of hyperbolic partial differential equations. Thus Lemma 4.12 gives the fundamental solution for Equation (4.46) in terms of an associated characteristic value problem. This generalizes a result obtained in [6].

We may now establish the following stability theorem which also generalizes a result given in [6]. The space $\overline{N}_0$, introduced in Theorem 3.4 and used in the next theorem is given by $\overline{N}_0 = N_0 \cap \{\phi | \phi : R^n \to R^m, \phi(x)$ is continuous for $x \geq \overline{a}$ and for each α_k, $\phi_{x_{\alpha_k}}(x)$ is continuous on $x \geq \overline{a}\}$.

<u>Theorem 4.6</u>. Suppose $A(x;\xi)$ has continuous pure mixed partials of all orders less than or equal to n in

ξ for $a \leq \xi \leq x < \infty$. Suppose the equation

$$u(x) = \phi(x) + \int_a^x M(r)u(r)dr \tag{4.53}$$

is uniformly stable on N_3 and there exists a constant $L(a)$ such that $\sum_{i,j=1}^{p} \|B_i(x)\| \int_a^x \|C_j(r)\| dr \leq L(a)$ for $x \geq a \geq \bar{a}$.

a) Then Equation (4.46) is stable on $\bar{N}_0$; if L is independent of a, Equation (4.46) is uniformly stable on $\bar{N}_0$. If for each $a \geq \bar{a}$

$$\sum_{i,j=1}^{p} \|B_i(x)\| \int_a^x \|C_j(r)\| dr \to 0 \quad \text{as} \quad |x| \to \infty$$

then Equation (4.46) is asymptotically stable on $\bar{N}_0 \cap \{\phi(x) \mid \phi: R^n \to R^m, |\phi(x)| \to 0 \text{ as } |x| \to \infty\}$.

b) If f satisfies (H1) and (H2) of Chapter 3 and L in the hypotheses above is independent of a, then Equation (4.47) preserves stability and uniform stability on any space N; it preserves asymptotic stability on any space N such that $N \supset N_3$. In particular, Equation (4.47) is uniformly stable on $\bar{N}_0$.

Proof. a) From Lemma 4.12, the fundamental solution $A(x;\xi)$ for Equation (4.46) is given by Equation (4.51) where $A^*(x;\xi)$ is the solution of Equation (4.50). Since

Equation (4.53) is uniformly stable on N_3, there is a positive constant $\overline{M}$ such that $\|A^*(x;\xi)\| \leq \overline{M}$ for $x \geq \xi \geq \overline{a}$. For $k < n$, we have

$$A_{\xi_{\alpha_k}}(x;\xi)$$

$$= (-1)^k \sum_{i,j=1}^{p} \int_{\xi_{\alpha'_k}}^{x_{\alpha'_k}} B_i(x) A^*_{ij}(x;w(r,\xi;\alpha_k)) C_j(w(r,\xi;\alpha_k)) dr_{\alpha'_k},$$

and for $k = n$

$$A_{\xi}(x;\xi) = (-1)^n \sum_{i,j=1}^{p} B_i(x) A^*_{ij}(x;\xi) C_j(\xi).$$

Therefore, for $k < n$ we obtain $A_{r_{\alpha_k}}(x;w(x,r;\alpha_k)) = 0$. Thus

$$\sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\| dr_{\alpha_k} = \int_a^x \|A_r(x;r)\| dr$$

$$= \int_a^x \| (-1)^n \sum_{i,j=1}^{p} B_i(x) A^*_{ij}(x;r) C_j(r) \| dr$$

$$\leq \sum_{i,j=1}^{p} \int_a^x \|B_i(x)\| \|A^*_{ij}(x;r)\| \|C_j(r)\| dr$$

$$\leq \overline{M} \sum_{i,j=1}^{p} \|B_i(x)\| \int_a^x \|C_j(r)\| dr \leq ML(a).$$

Therefore, by Theorem 3.4, Equation (4.46) is stable on $\overline{N}_0$, and uniformly stable on $\overline{N}_0$ when L is independent of a. If $\sum_{i,j=1}^{p} \|B_i(x)\| \int_a^x \|C_j(r)\| dr \to 0$ as $|x| \to \infty$, it follows from the estimate above that

$$\int_{a_{\alpha_k}}^{x_{\alpha_k}} \|A_{r_{\alpha_k}}(x;w(x,r;\alpha_k))\| dr_{\alpha_k} \to 0 \quad \text{as} \quad |x| \to \infty.$$

Thus by Theorem 3.4 Equation (4.46) is asymptotically stable on $\overline{N}_0 \cap \{\phi(x) | \phi:R^n \to R^m, |\phi(x)| \to 0$ as $|x| \to \infty\}$.

b) By part a) above, Equation (4.46) is uniformly stable on $\overline{N}_0$ when L is independent of a. Since $N_3 \subset \overline{N}_0$, Equation (4.46) is uniformly stable on N_3. The result now follows directly from Theorem 3.5. This completes the proof.

4.5 Connection Between the Fundamental Solution and the Classical Riemann Function

In the theory of hyperbolic partial differential equations, the Riemann function may be used to give integral representations for the Cauchy problem and the characteristic value problem. In Chapter 2, we introduced the idea of a fundamental solution for an integral equation and gave an integral representation for the solution of this integral equation. Special cases of the integral equation are equivalent to a characteristic value problem and thus a natural question arises as to the connection between the fundamental solution and the Riemann function. The purpose of this section is to give this connection. We begin by briefly describing Riemann's method of solution for the characteristic value problem for a hyperbolic scalar equation in two independent variables. We will follow closely the presentation given by Garabedian [19].

Consider the characteristic value problem

$$\left.\begin{aligned} Lu = u_{xy} - k_2(x,y)u_x - k_1(x,y)u_y - k_3(x,y)u = g(x,y) \\ u(x,b) = \phi_1(x), \quad u(a,y) = \phi_2(y), \quad \phi_1(a) = \phi_2(b) \end{aligned}\right\} \quad (4.54)$$

where (a,b) is fixed, $\phi_1(x)$, $\phi_2(y)$ are continuously differentiable, k_{1y}, k_{2x} are continuous, and $g(x,y)$ is continuous. The adjoint operator M associated with the operator L is given by $Mv = v_{xy} + (k_2v)_x + (k_1v)_y - k_3v$. We have the following identity for any two functions u and v:

$$vLu - uMv = (-k_2uv - vu_y)_x + (-k_1uv + vu_x)y. \qquad (4.55)$$

Let (x_0,y_0) be any point such that $a \le x_0$, $b \le y_0$ and let $D = \{(x,y)\,|\,a \le x \le x_0, b \le y \le y_0\}$. Using identity (4.55), Green's theorem, and integration by parts we have

$$\iint_D [vLu - uMv]dsdr = \int_{\partial D} (-k_2uv - vu_s)ds - (-k_1uv + vu_r)dr$$

$$= u(x_0,y_0)v(x_0,y_0) - v(a,b)u(a,b) - \int_a^{x_0} [-k_1(r,b)u(r,b)$$

$$+ u_r(r,b)]v(r,b)dr - \int_a^{x_0} [k_1(r,y_0)v(r,y_0) + v_r(r,y_0)]u(r,y_0)dr$$

$$- \int_b^{y_0} [-k_2(a,s)u(a,s) + u_s(a,s)]v(a,s)ds$$

$$- \int_a^{y_0} [k_2(x_0,s)v(x_0,s) + v_s(x_0,s)]u(x_0,s)ds. \qquad (4.56)$$

Let $R(x,y;x_0,y_0)$ be the solution of the following special characteristic value problem.

$$\left.\begin{aligned} &MR(x,y;x_0,y_0) = 0 \\ &R(x,y_0;x_0,y_0) = \exp[-\int_{x_0}^{x} k_1(r,y_0)dr], \\ &R(x_0,y;x_0,y_0) = \exp[-\int_{y_0}^{y} k_2(x_0,s)ds] \end{aligned}\right\}. \qquad (4.57)$$

The function $R(x,y;x_0,y_0)$ is known as the Riemann function associated with the operator L. We now suppose u is the solution of problem (4.54) and take $v(r,s) = R(r,s;x_0,y_0)$. Then using the characteristic data and replacing x_0, y_0 by x, y we see that Equation (4.56) becomes

$$u(x,y) = R(a,b;x,y)\phi_1(a)$$

$$+ \int_a^x [-k_1(r,b)\phi_1(r) + \phi_1'(r)]R(r,b;x,y)dr$$

$$+ \int_b^y [-k_2(a,s)\phi_2(s) + \phi_2'(s)]R(a,s;x,y)ds$$

$$+ \int_a^x \int_b^y R(r,s;x,y)g(r,s)dsdr. \qquad (4.58)$$

Thus $u(x,y)$ given by Equation (4.58) is the solution of problem (4.54).

Suppose we introduce the Riemann function $\overline{R}(x,y;r,s)$ for the adjoint operator M. It is easy to see that the adjoint of M is L and therefore $\overline{R}(x,y;r,s)$ satisfies

$$\left.\begin{aligned} &L\overline{R}(x,y;r,s) = 0 \\ &\overline{R}(x,s;r,s) = \exp[\int_r^x k_1(\sigma,s)d\sigma], \\ &\overline{R}(r,y;r,s) = \exp[\int_s^y k_2(r,\sigma)d\sigma] \end{aligned}\right\}. \qquad (4.59)$$

If we assume $g(x,y) \equiv 0$, then $\overline{R}(x,y;r,s)$ must satisfy Equation (4.58) with $a = r$, $b = s$. Therefore, since the integrals vanish, we obtain

$$\overline{R}(x,y;r,s) = R(r,s;x,y). \qquad (4.60)$$

Equation (4.60) may be used to write the Equation (4.58) in terms of the Riemann function $\bar{R}(x,y;r,s)$. Equation (4.58) becomes

$$u(x,y) = \bar{R}(x,y;a,b)\phi_1(a)$$
$$+ \int_a^x [-k_1(r,b)\phi_1(r) + \phi_1'(r)]\bar{R}(x,y;r,b)dr$$
$$+ \int_b^y [-k_2(a,s)\phi_2(s) + \phi_2'(s)]\bar{R}(x,y;a,s)ds$$
$$+ \iint_D g(r,s)\bar{R}(x,y;r,s)dsdr. \quad (4.61)$$

When the operator L is self-adjoint, that is $k_1 = k_2 \equiv 0$, then the operators L and M are identical. It then follows from problems (4.57) and (4.59) that $R(x,y;r,s) = \bar{R}(x,y;r,s)$. Thus using Equation (4.60) we have

$$R(x,y;r,s) = R(r,s;x,y) = \bar{R}(x,y;r,s) = \bar{R}(r,s;x,y). \quad (4.62)$$

Remark 4.6. If $k_{2x}(x,y)$ and $k_{1y}(x,y)$ are continuous it follows easily that $u(x,y)$ is a solution of problem (4.54) if and only if $u(x,y)$ satisfies the integral equation

$$u(x,y) = \phi_1(x) + \phi_2(y) - \phi_1(a) - \int_a^x k_1(r,b)\phi_1(r)dr$$

$$- \int_b^y k_2(a,s)\phi_2(s)ds + \int_a^x k_1(r,y)u(r,y)dr + \int_b^y k_2(x,s)u(x,s)ds$$

$$+ \int_a^x\int_b^y [k_3(r,s) - k_{2r}(r,s) - k_{1s}(r,s)]u(r,s)dsdr$$

$$+ \int_a^x\int_b^y g(r,s)dsdr \tag{4.63}$$

We see by Equations (4.58) and (4.61) that the solution of problem (4.54) may be expressed in terms of the function $R(x,y;r,s)$ or the function $\overline{R}(x,y;r,s)$. By Remark 4.6 we see that the characteristic value problem (4.54) is equivalent to an integral equation of the form considered earlier and for which a fundamental solution has been defined. Thus we may also express the solution of problem (4.54) in terms of the fundamental solution $A(x,y;\xi,\eta)$ for Equation (4.63). The following lemma gives the connection between these three functions. This lemma along with Equation (4.62) are the results used in our discussion of the Gronwall inequality in Section 2.3.

<u>Lemma 4.13</u>. Suppose the functions $k_{2x}(x,y)$ and $k_{1y}(x,y)$ are continuous. Then,

$$A(x,y;\xi,\eta) = \overline{R}(x,y;\xi,\eta) = R(\xi,\eta;x,y).$$

Proof. We have already seen (Equation (4.60) that $\overline{R}(x,y;\xi,\eta) = R(\xi,\eta;x,y)$. Since $\overline{R}(x,y;\xi,\eta)$ is the solution of the characteristic value problem (4.59) we see by Remark 4.6 that $\overline{R}(x,y;\xi,\eta)$ satisfies the following:

$$\overline{R}(x,y;\xi,\eta) = \exp[\int_\xi^x k_1(\sigma,\eta)d\sigma] + \exp[\int_\eta^y k_2(\xi,\sigma)d\sigma] - 1$$

$$- \int_\xi^x k_1(r,\eta)\exp[\int_\xi^r k_1(\sigma,\eta)d\sigma]dr - \int_\eta^y k_1(\xi,s)\exp[\int_\eta^s k_1(\xi,\sigma)d\sigma]ds$$

$$+ \int_\xi^x k_2(r,y)\overline{R}(r,y;\xi,\eta)dr + \int_\eta^y k_1(x,s)\overline{R}(x,s;\xi,\eta)ds$$

$$+ \int_\xi^x\int_\eta^y [k_3(r,s) - k_{2r}(r,s) - k_{1s}(r,s)]\overline{R}(r,s;\xi,\eta)dsdr.$$

But

$$\exp[\int_\xi^x k_1(\sigma,\eta)d\sigma] = 1 + \int_\xi^x k_1(r,\eta)\exp[\int_\xi^r k_1(\sigma,\eta)d\sigma]dr$$

and

$$\exp[\int_\eta^y k_2(\xi,\sigma)d\sigma] = 1 + \int_\eta^y k_2(\xi,s)\exp[\int_\eta^s k_2(\xi,\sigma)d\sigma]ds.$$

Therefore, $\overline{R}(x,y;\xi,\eta)$ satisfies

$$\bar{R}(x,y;\xi,\eta) = 1 + \int_{\xi}^{x} k_1(r,y)\bar{R}(r,y;\xi,\eta)dr + \int_{\eta}^{y} k_2(x,s)\bar{R}(x,s;\xi,\eta)ds + \int_{\xi}^{x}\int_{\eta}^{y} [k_3(r,s) - k_{2r}(r,s) - k_{1s}(r,s)]\bar{R}(r,s;\xi,\eta)dsdr.$$

But this is the equation satisfied by the fundamental solution for Equation (4.63). Therefore $A(x,y;\xi,\eta) = \bar{R}(x,y;\xi,\eta)$. This completes the proof.

We see that the fundamental solution defined for integral equations may be thought of as a generalization of the Riemann function for the adjoint operator M. The Reimann function and integral representations for the solution of the Cauchy problem and the characteristic value problem associated with a hyperbolic partial differential equation in n variables have been studied by Sternberg [35]. He also considers the Riemann function for an adjoint operator and refers to this solution as the principle solution (he in turn attributes this terminology to duBois Reymond) for the original operator. The author conjectures that the result given in Lemma 4.13 may be generalized for the case of n independent variables to a relation between the fundamental solution for an equivalent integral equation and the functions considered by Sternberg.

Remark 4.7. From Lemma 4.13, it is now easy to see that the integral representations for the solution of the characteristic value problem (4.54) in terms of $\overline{R}(x,y;\xi,\eta)$ (or $R(x,y;\xi,\eta)$) is identical with the integral representation for the solution of the equivalent integral Equation (4.63) in terms of the fundamental solution. We note that there is a distinction when applying these representations. To give the representation in terms of $\overline{R}(x,y;\xi,\eta)$ we need only characteristic data while the representation in terms of the fundamental solution requires the initial function. We see from Equation (4.63) that these are not in general the same.

4.6 Stability for a Characteristic Value Problem

In this section we will consider stability for the following characteristic value problem,

$$\left.\begin{aligned} &Lu = f(x,y,u) \\ &u(x,\overline{b}) = \phi_1(x), \quad u(\overline{a},y) = \phi_2(y), \quad \phi_1(\overline{a}) = \phi_2(\overline{b}) \end{aligned}\right\} \qquad (4.64)$$

where L is defined in Section 4.5, $\phi_1(x)$, $\phi_2(y)$ are continuously differentiable, k_{1y}, k_{2x} are continuous, and $\overline{a}$, $\overline{b}$ are fixed. Under these assumptions this problem is equivalent to the Volterra integral equation

$$u(x,y) = \phi_1(x) + \phi_2(y) - \phi_1(\bar{a}) - \int_{\bar{a}}^{x} k_1(r,\bar{b})\phi_1(r)dr$$

$$- \int_{\bar{b}}^{y} k_2(\bar{a},s)\phi_2(s)ds + \int_{\bar{a}}^{x} k_1(r,y)u(r,y)dr + \int_{\bar{b}}^{y} k_2(x,s)u(x,s)ds$$

$$+ \int_{\bar{a}}^{x}\int_{\bar{b}}^{y} [k_3(r,s) - k_{2r}(r,s) - k_{1s}(r,s)]u(r,s)dsdr$$

$$+ \int_{\bar{a}}^{x}\int_{\bar{b}}^{y} f(r,s,u(r,s))dsdr. \qquad (4.65)$$

We define stability for problem (4.64) in terms of the characteristic data $\phi_1(x)$ and $\phi_2(y)$ and then study spaces of characteristic data for which the initial function

$$\phi(x,y) = \phi_1(x) + \phi_2(y) - \phi_1(\bar{a}) - \int_{\bar{a}}^{x} k_1(r,\bar{b})\phi_1(r)dr$$

$$- \int_{\bar{b}}^{y} k_2(\bar{a},s)\phi_2(s)ds \qquad (4.66)$$

for the equivalent integral Equation (4.65) remains in one of the spaces considered previously. Thus, the results obtained earlier for integral equations will yield stability results for the characteristic value problem (4.64).

Let $G = \{\phi_1 | \phi_1 : [\bar{a},\infty) \to R\} \times \{\phi_2 | \phi_2 : [\bar{b},\infty) \to R\}$ where G is normed in some appropriate way. We now make a stability definition for problem (4.64).

<u>Definition 4.2</u>. Suppose $(\phi_1,\phi_2) \in G$ and $u(x,y)$ is a solution of problem (4.64) for $x \geq \bar{a}$, $y \geq \bar{b}$ with the characteristic data $\phi_1(x)$ and $\phi_2(y)$. The solution $u(x,y)$ is <u>stable on G</u> if given $\varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that if $(\hat{\phi}_1,\hat{\phi}_2) \in G$ and $\|(\hat{\phi}_1,\hat{\phi}_2) - (\phi_1,\phi_2)\| < \delta$ then any solution $\hat{u}(x,y)$ of problem (4.64) with data $\hat{\phi}_1(x)$, $\hat{\phi}_2(y)$ exists for $x \geq \bar{a}$, $y \geq \bar{b}$ and satisfies $\|\hat{u} - u\|_{0,(\bar{a},\bar{b})} < \varepsilon$. We say that $u(x,y)$ is <u>asymptotically stable</u> if it is stable and if there is a δ_1 so that when $\|(\hat{\phi}_1,\hat{\phi}_2) - (\phi_1,\phi_2)\| < \delta_1$ we have $\lim_{|(x,y)|\to\infty} |\hat{u}(x,y) - u(x,y)| = 0$.

As before, we assume that $f(x,y,0) \equiv 0$ and say that problem (4.64) is stable or asymptotically on G if $u \equiv 0$ is a stable or asymptotically stable solution on G. We will be interested in the following spaces.

$$G_1 = \{\phi_1 | \phi_1 : [\bar{a},\infty) \to R, \quad \int_{\bar{a}}^{\infty} |\phi_1'(r)|dr < \infty\}$$

$$\times \{\phi_2 | \phi_2 : [\bar{b},\infty) \to R, \quad \int_{\bar{b}}^{\infty} |\phi_2'(s)|ds < \infty\}.$$

G_1 will be normed by

$$\|(\phi_1,\phi_2)\| = \|\phi_1\|_{0,\bar{a}} + \|\phi_2\|_{0,\bar{b}} + \int_{\bar{a}}^{\infty} |\phi_1'(r)|dr + \int_{\bar{b}} |\phi_2'(s)|ds.$$

$$G_2 = \{\phi_1 | \phi_1 : [\bar{a},\infty) \to R,$$

$$\int_{\bar{a}}^{\infty} |\phi_1(r)|dr < \infty, \int_{\bar{a}}^{\infty} |\phi_1'(r)|dr < \infty\}$$

$$\times \{\phi_2 | \phi_2 : [\bar{b},\infty) \to R \, , \int_{\bar{b}}^{\infty} |\phi_2(s)|ds < \infty, \int_{\bar{b}}^{\infty} |\phi_2'(s)|ds < \infty\}.$$

G_2 will be normed by

$$\|(\phi_1,\phi_2)\| = \|\phi_1\|_{0,\bar{a}} + \|\phi_2\|_{0,\bar{b}} + \int_{\bar{a}}^{\infty} |\phi_1(r)|dr$$

$$+ \int_{\bar{b}}^{\infty} |\phi_2(s)|ds + \int_{\bar{a}}^{\infty} |\phi_1'(r)|dr + \int_{\bar{b}}^{\infty} |\phi_2'(s)|ds$$

$$G_3 = \{\phi_1 | \phi_1 : [\bar{a},\infty) \to R, \quad \phi_1, \quad \phi_1' \text{ are bounded,}$$

$$\int_{\bar{a}}^{\infty} |\phi_1(r)|dr < \infty\} \times \{\phi_2 | \phi_2 : [\bar{b},\infty) \to R \, , \quad \phi_2, \quad \phi_2' \text{ are bounded,}$$

$$\int_{\bar{b}}^{\infty} |\phi_2(s)|ds < \infty\}.$$

The norm on G_3 will be

$$\|(\phi_1,\phi_2)\| = \|\phi_1\|_{0,\bar{a}} + \|\phi_2\|_{0,\bar{b}} + \|\phi_1'\|_{0,\bar{a}} + \|\phi_2'\|_{0,\bar{b}}$$

$$+ \int_{\bar{a}}^{\infty} |\phi_1(r)|dr + \int_{\bar{b}}^{\infty} |\phi_2(s)|ds.$$

We observe that $G_1 \supset G_2$.

We have the following stability theorem for problem (4.64).

Theorem 4.7. Assume $k_1(x,y) \equiv k_1(x)$ and $k_2(x,y) \equiv k_2(y)$. Suppose $f(x,y,z)$ in problem (4.64) satisfies (H1)' and (H2)'.

i) Suppose any one of the hypotheses (H6), (H7) or (H10) hold. Then problem (4.64) is stable on G_1.

ii) Suppose any one of the hypotheses (H5.a), (H8.a), (H9.a), (H11.a) or (H12.a) hold. Suppose in the hypotheses (H5.a), (H8.a), and (H11.a) the functions $k_1(r)$, $k_2(s)$ are bounded. Then problem (4.64) is asymptotically stable on G_2.

iii) Suppose any one of the hypotheses (H8.b), (H9.b), (H11.b) or (H12.b) hold. Suppose in (H8.b) and (H11.b) the functions $k_1(r)$, $k_2(s)$ are bounded. Then problem (4.64) is stable on G_3.

Proof. i) By Theorem 4.2 we see that Equation (4.65) is uniformly stable on N_2. We need only show that if $(\phi_1,\phi_2) \in G_1$ then the function

$$\phi(x,y) = \phi_1(x) + \phi_2(y) - \phi_1(\bar{a}) - \int_{\bar{a}}^{x} k_1(r)\phi_1(r)dr - \int_{\bar{b}}^{y} k_2(s)\phi_2(s)ds \tag{4.67}$$

is in N_2 and that if the norm of (ϕ_1,ϕ_2) in G_1 is small then so is the N_2 norm of $\phi(x,y)$. For any $(\phi_1,\phi_2) \in G$, we have

$$\left.\begin{aligned}\phi_x(x,y) &= \phi_1'(x) - k_1(x)\phi_1(x)\\ \phi_y(x,y) &= \phi_2'(y) - k_2(y)\phi_2(y)\\ \phi_{xy}(x,y) &\equiv 0\end{aligned}\right\}. \qquad (4.68)$$

Take any $(\phi_1,\phi_2) \in G_1$. Each of the hypotheses implies $\int_{\bar{a}}^{\infty}|k_1(r)|dr < \infty$ and $\int_{\bar{b}}^{\infty}|k_2(s)|ds < \infty$. Thus we have

$$|\phi(x,y)| \leq \|\phi_1\|_{0,\bar{a}} + \|\phi_2\|_{0,\bar{b}} + \|\phi_1\|_{0,\bar{a}} + \|\phi_1\|_{0,\bar{a}}\int_{\bar{a}}^{\infty}|k_1(r)|dr$$
$$+ \|\phi_2\|_{0,\bar{b}}\int_{\bar{b}}^{\infty}|k_2(s)|ds$$

and ϕ is bounded for $x \geq \bar{a}$, $y \geq \bar{b}$. Using Equations (4.68) we see that

$$\int_{\bar{a}}^{\infty} \sup_{\bar{b}\leq y<\infty}|\phi_r(r,y)|dr \leq \int_{\bar{a}}^{\infty}|\phi_1'(r)|dr + \|\phi_1\|_{0,\bar{a}}\int_{\bar{a}}^{\infty}|k_1(r)|dr$$

and

$$\int_{\bar{b}}^{\infty} \sup_{\bar{a}\leq x<\infty}|\phi_s(x,s)|ds \leq \int_{\bar{b}}^{\infty}|\phi_2'(s)|ds + \|\phi_2\|_{0,\bar{b}}\int_{\bar{b}}^{\infty}|k_2(s)|ds.$$

Thus $\phi(x,y)$ given by Equation (4.67) is in N_2.

Let $M = \max\{1, \int_{\bar{a}}^{\infty}|k_1(r)|\,dr, \int_{\bar{b}}^{\infty}|k_2(s)|\,ds\}$. Take any δ and any $(\phi_1,\phi_2) \in G_1$ so that $\|(\phi_1,\phi_2)\| < \frac{\delta}{9M}$. Then from the estimates above we have

$$\|\phi(x,y)\| \le \frac{5\delta}{9}, \quad \int_{\bar{a}}^{\infty} \sup_{\bar{b}\le y<\infty} |\phi_r(r,y)|\,dr \le \frac{2\delta}{9} \text{ and}$$

$$\int_{\bar{b}}^{\infty} \sup_{\bar{a}\le x<\infty} |\phi_s(x,s)|\,ds \le \frac{2\delta}{9}.$$

These calculations show that $\|\phi\| \le \delta$ and stability on G_1 follows.

ii) Theorem 4.2 shows that Equation (4.65) is asymptotically stable on N_2. Take $(\phi_1,\phi_2) \in G_2$. In each case there are positive constants M_1, M_2 such that $|k_1(r)| \le M_1$ and $|k_2(s)| \le M_2$. Thus

$$|\phi(x,y)| \le \|\phi_1\|_{0,\bar{a}} + \|\phi_2\|_{0,\bar{b}} + \|\phi_1\|_{0,\bar{a}}$$

$$+ M_1\int_{\bar{a}}^{\infty}|\phi_1(r)|\,dr + M_2\int_{\bar{b}}^{\infty}|\phi_2(s)|\,ds.$$

Also

$$\int_{\bar{a}}^{\infty} \sup_{\bar{b}\le y<\infty} |\phi_x(x,y)|\,dx \le \int_{\bar{a}}^{\infty}|\phi_1'(r)|\,dr + M_1\int_{\bar{a}}^{\infty}|\phi_1(r)|\,dr,$$

and

$$\int_{\bar{b}}^{\infty} \sup_{\bar{a} \le x < \infty} |\phi_y(x,y)| dy \le \int_{\bar{b}}^{\infty} |\phi_2'(s)| ds + M_2 \int_{\bar{b}}^{\infty} |\phi_2(s)| ds.$$

Therefore ϕ given by Equation (4.67) is in N_2. For any $(\phi_1, \phi_2) \in G_2$ so that $\|(\phi_1, \phi_2)\| \le \frac{\delta}{9M}$ with $M = \max\{1, M_1, M_2\}$ an argument similar to that used in part i) shows that $\|\phi\| \le \delta$ and the result follows.

iii) By Theorem 4.2 Equation (4.65) is stable on N_1. The result then follows from an argument similar to that used in parts i) and ii). This completes the proof.

With regards to part ii) of the previous theorem we point out that asymptotic stability of problem (4.64) implies that $\phi_1(x)$, $\phi_2(y)$ must go to zero as $x \to \infty$, $y \to \infty$. Although this is not explicit in the assumption that $(\phi_1, \phi_2) \in G_2$ it may be shown that this assumption is sufficient for $\lim_{x \to \infty} \phi_1(x) = 0$ and $\lim_{y \to \infty} \phi_2(y) = 0$.

We now give a theorem for problem (4.64) where the function $f(x,y,z)$ satisfies (H3)'. The proof follows from Theorem 4.3 and arguments similar to those given in Theorem 4.7.

Theorem 4.8. Suppose $k_1(x,y) \equiv k_1(x)$ and $k_2(x,y) \equiv k_2(y)$. Suppose $f(x,y,z)$ in problem (4.64) satisfies (H3)'. Let any one of the hypotheses (H8.b), (H9.b), (H11.b), or (H12.b) hold.

i) Suppose the functions $k_1(r)$, $k_2(s)$ in (H8.b) and H(11.b) are also bounded. Then problem (4.64) is stable on G_2 and G_3.

ii) Suppose the functions $k_1(r)$, $k_2(s)$ in (H8.b) and (H11.b) are negative constants. Then problem (4.64) is asymptotically stable on G_2.

We now give a theorem for problem (4.64) under the assumptions $k_{2x} - k_1k_2 - k_3 = 0$ (or $k_{1y} - k_1k_2 - k_3 = 0$). Under these assumptions the problem (4.64) is equivalent to the Volterra equation

$$u(x,y) = \phi(x,y) + \int_{\bar{a}}^{x} k_1(r,y)u(r,y)dr + \int_{\bar{b}}^{y} k_2(x,s)u(x,s)ds$$

$$- \int_{\bar{a}}^{x}\int_{\bar{b}}^{y} \bar{k}_3(r,s)u(r,s)dsdr + \int_{\bar{a}}^{x}\int_{\bar{b}}^{y} f(r,s,u(r,s))dsdr \qquad (4.69)$$

where $\phi(x,y)$ is given by (4.66) and $\bar{k}_3(r,s) = k_1(r,s)k_2(r,s) + k_{1s}(r,s)$ (or $\bar{k}_3(r,s) = k_1(r,s)k_2(r,s) + k_{2r}(r,s)$). Then using Theorem 4.4 and arguments similar those used in Theorem 4.7 we have the following.

<u>Theorem 4.9</u>. Suppose $k_{2x} - k_1k_2 - k_3 = 0$ (or $k_{1y} - k_1k_2 - k_3 = 0$).

a) Let $f(x,y,z)$ satisfy (H1)' and (H2)'.

i) Suppose (H13) holds. If k_1, k_2 are also bounded then problem (4.64) is stable on G_2.

ii) Suppose (H13.a) holds and k_1, k_2 are bounded. Then problem (4.64) is asymptotically stable on G_2.

iii) Suppose (H14) holds and k_1, k_2 are bounded. Then problem (4.64) is stable on G_3.

b) Let $f(x,y,z)$ satisfy (H3)'. Suppose (H14) holds and k_1, k_2 are bounded. Then problem (4.64) is stable on G_3 and asymptotically stable on G_2

As an example we will now discuss briefly a physical problem whose mathematical formulation can be put in the form of a characteristic value problem. We will apply the results of this section and then give physical interpretations for our stability conclusions. We will give the equations describing the process and indicate how we obtain a characteristic value problem. For a more complete discussion the reader is referred to [37, pp. 176-179].

We consider a semi-infinite tube filled with an absorbant. Let the axis of the tube be the x-axis. Suppose a gas and air mixture is passed through the tube with a constant velocity v. We assume that the amount of gas $\bar{u}(t)$ in the gas-air mixture at the open end of the tube $(x = 0)$ is given. Let $a(x,t)$ be the amount of gas absorbed per unit volume of absorbant and let $u(x,t)$ be the concentration of gas in the pores of the absorbent in the layer x.

The conservation of mass law yields the equation

$$-v\frac{\partial u}{\partial x} = \frac{\partial a}{\partial t} + \frac{\partial u}{\partial t}.$$

We must also satisfy the equation of kinetics of absorption

$$\frac{\partial u}{\partial t} = \beta(u - y)$$

where β is the kinetic coefficient (β is a positive constant) and y is the concentration of the gas in equilibrium with the quantity a of gas absorbed. The quantities a and y are related by the equation

$$a = f(y)$$

which is determined by the absorbent under consideration. We will assume that $a = \frac{1}{\gamma}y$ (γ is a positive constant) which is valid for small concentrations. We will also assume that $a(x,0) \equiv 0$. The problem now is to find a an u satisfying

$$\left.\begin{aligned} & -v\,\frac{\partial u}{\partial x} = \frac{\partial a}{\partial t} + \frac{\partial u}{\partial t} \\ & \frac{\partial a}{\partial t} = \beta(u - \gamma a) \\ & u(0,t) = \overline{u}(t) \quad a(x,0) \equiv 0 \end{aligned}\right\}. \tag{4.70}$$

The function $a(x,t)$ may be eliminated from this system and the new condition for $u(x,t)$ is a hyperbolic equation whose characteristics are $x = c_1$, $t - \frac{x}{v} = c_2$. (It is also possible to eliminate $u(x,t)$ to get a hyperbolic equation.) It may then be shown that the solution for $x \geq 0$, $t - \frac{x}{v} < 0$ is $u \equiv 0$ and that along the characteristic $t = \frac{x}{v}$, u is discontinuous. Writing the hyperbolic equation for u in terms of the characteristic variables (ξ,τ), with $\xi = x$, $\tau = t - \frac{x}{v}$, we obtain the following equation:

$$u_{\xi\tau} = -\frac{\beta}{v}u_\tau - \beta\gamma u_\xi \qquad \xi > 0, \quad \tau > 0. \tag{4.71}$$

It may also be shown that u along the characteristic $\xi = 0$, $\tau = 0$ must be given by:

$$\left.\begin{aligned} u(\xi,0) &= \bar{u}(0)\exp(-\frac{\beta}{v}\xi) \\ u(0,\tau) &= \bar{u}(\tau) \end{aligned}\right\}. \tag{4.72}$$

From Theorem 4.7, part ii) it follows that the characteristic value problem given by (4.71) and (4.72) is asymptotically stable on G_2. We see that $(\bar{u}(0)\exp(-\frac{\beta}{v}\xi),$ $\bar{u}(\tau))$ will be in G_2 if $\bar{u}(\tau)$ is continuous,

$\int_0^\infty |\bar{u}(\tau)|d\tau < \infty$, and if $\bar{u}'(\tau)$ is continuous with $\int_0^\infty |\bar{u}'(\tau)|d\tau < \infty$. We also note that the norm of $(\bar{u}(0)\exp(-\frac{\beta}{V}\xi),\bar{u}(\tau))$ will be small if the quantities

$$|\bar{u}(0)|, \quad \int_0^\infty |\bar{u}(\tau)|d\tau, \quad \int_0^\infty |\bar{u}'(\tau)|d\tau \tag{4.73}$$

are small.

Stability of the characteristic value problem means that if the gas supplied at $x = 0$, that is $\bar{u}(\tau)$, is small in the sense that the quantities in (4.73) are sufficiently small then $u(\xi,\tau)$ will be small for $\xi = x \geq 0$, $0 \leq t - \frac{x}{V} = \tau$. Asymptotic stability implies, in addition, that for each $\varepsilon > 0$ there is a T so that if $|(\xi,\tau)|_\infty \geq T$ then $|u(\xi,\tau)| < \varepsilon$. To give further physical interpretation to asymptotic stability we note that each characteristic line $\tau = t - \frac{x}{V} = c$ may be though of as the wave front of the gas-air mixture entering the tube at time $t = c$. Our solution $u(\xi,\tau)$ will be less than ε independent of the manner in which (ξ,τ) becomes large. We may fix a position $x_0 = \xi_0$ in the tube and allow the number of wave fronts passing this position to become large. This means $|(\xi_0,\tau)|_\infty$ will be large by moving along the curve C_1 shown in Figure 1A. Once the number of wave fronts is large enough so $|(\xi_0,\tau)|_\infty \geq T$ (C_1 crosses the parallelogram $|(\xi,\tau)|_\infty = T$),

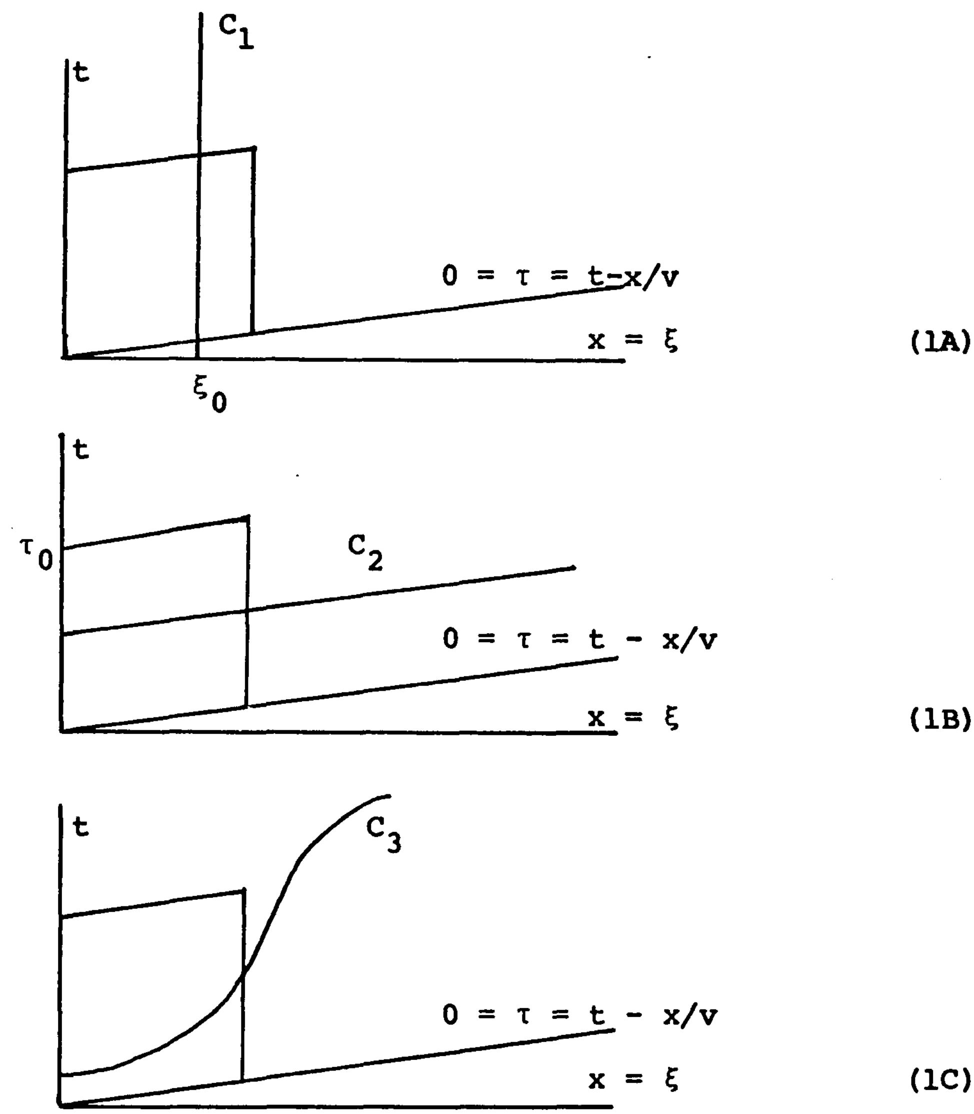

Figure 1. Illustrations Associated With Asymptotic Stability of Problem (4.71), (4.72). (The parallelogram in each figure is the set of (ξ,τ) such that $|(\xi,\tau)|_{\infty} < T$.)

then $|u(\xi_0,\tau)| < \varepsilon$ for all future waves passing ξ_0. We may fix a wave τ_0 and follow it until ξ is large enough so that $|(\xi,\tau_0)|_\infty \geq T$. We are then passing out of the parallelogram $|(\xi,\tau)|_\infty < T$ along the curve C_2 as shown in Figure 1B. Once ξ is sufficiently large the concentration on the wave τ_0 will be less than ε. We may also move through the tube at a variable velocity less than v. We are then at a different position and on a different wave front at each time and following a typical path C_3 as shown in Figure 1C. Again once the (ξ,τ) associated with our movement is such that $|(\xi,\tau)|_\infty \geq T$, then any reading of the concentration as we move will be less than ε.

APPENDIX A

VERIFICATION OF INTERCHANGING THE ORDER OF INTEGRATION

In the body of the dissertation we have used (without verification) the fact that the order of integration in a multiple integral may be interchanged. The proposition given below is sufficiently general to verify all cases of interchanging the order of integration occurring in the dissertation. We will make use of Fubini's Theorem in the following two forms [32].

1) Suppose $a, b \in R^n$, $c, d \in R^m$ and $f:[a,b] \times [c,d] \to R$ such that f is integrable in the Lebesque sense. Then $\int_a^b \int_c^d f(r,s)dsdr = \int_c^d \int_a^b f(r,s)dsdr$.
2) Suppose $a, b \in R$ and $g(x,y)$ is real valued and integrable on $a \leq x \leq y \leq b$. Then $\int_a^b \int_a^x g(x,y)dydx = \int_a^b \int_y^b g(x,y)dxdy$.

We return to the notation introduced in Chapter 1 and have the following.

<u>Proposition</u>. Let α_p, γ_q be any two combinations of the integers $\{1,2,...,n\}$. Suppose for each $x \geq a$ the

function $h(x,r_{\alpha_p},s_{\gamma_q})$, having values in R^m, satisfies the following:

i) for each fixed r_{α_p} with $a_{\alpha_p} \le r_{\alpha_p} \le x_{\alpha_p}$, $h(x,r_{\alpha_p},s_{\gamma_q})$ is continuous in s_{γ_q} for $a_{\gamma_q} \le s_{\gamma_q} \le w_{\gamma_q}(x,r;\alpha_p)$.

ii) For each fixed s_{γ_q} with $a_{\gamma_q} \le s_{\gamma_q} \le x_{\gamma_q}$, $h(x,r_{\alpha_p},s_{\gamma_q})$ is continuous in r_{α_p} for $w_{\alpha_p}(a,s;\gamma_q) \le r_{\alpha_p} \le x_{\alpha_p}$.

iii) Suppose there exists a constant $M(x)$ such that $|h(x,r_{\alpha_p},s_{\gamma_q})| \le M(x)$ for $a_{\alpha_p} \le r_{\alpha_p} \le x_{\alpha_p}$, $a_{\gamma_q} \le s_{\gamma_q} \le w_{\gamma_q}(x,r;\alpha_p) \le x_{\gamma_q}$.

Then for each x we have

$$\int_{a_{\alpha_p}}^{x_{\alpha_p}}\int_{a_{\gamma_q}}^{w_{\gamma_q}(x,r;\alpha_p)} h(x,r_{\alpha_p},s_{\gamma_q})\,ds_{\gamma_q}\,dr_{\alpha_p}$$

$$= \int_{a_{\gamma_q}}^{x_{\gamma_q}}\int_{w_{\alpha_p}(a,s;\gamma_q)}^{x_{\alpha_p}} h(x,r_{\alpha_p},s_{\gamma_q})\,dr_{\alpha_p}\,ds_{\gamma_q}.$$

<u>Proof</u>. Let $h(x,r_{\alpha_p},s_{\gamma_q}) = (h_1(x,r_{\alpha_p},s_{\gamma_q}), \ldots, h_m(x,r_{\alpha_p},s_{\gamma_q}))$. Take any i such that $1 \le i \le m$ and consider the function $h_i(x,r_{\alpha_p},s_{\gamma_q})$. Let

$\beta_t = \alpha_p \cap \gamma_q = \{k_1, k_2, \ldots, k_t\}$ (we assume β_t is ordered) and let $\alpha_p^* = \alpha_p - \beta_t$, $\gamma_q^* = \gamma_q - \beta_t$. Then by repeated application of 1) above and Lemma 2.2 we have

$$\int_{a_{\alpha_p}}^{x_{\alpha_p}} \int_{a_{\gamma_q}}^{w_{\gamma_q}(x,r;\alpha_p)} h_i(x, r_{\alpha_p}, s_{\gamma_q}) ds_{\gamma_q} dr_{\alpha_p}$$

$$= \int_{a_{\alpha^*_p}}^{x_{\alpha^*_p}} \int_{a_{\beta_t}}^{x_{\beta_t}} \int_{a_{\gamma^*_q}}^{x_{\gamma^*_q}} \int_{a_{\beta_t}}^{r_{\beta_t}} h_i(x, r_{\alpha_p}, s_{\gamma_q}) ds_{\beta_t} ds_{\gamma^*_q} dr_{\beta_t} dr_{\alpha^*_p}$$

$$= \int_{a_{\gamma^*_q}}^{x_{\gamma^*_q}} \int_{a_{\alpha^*_p}}^{x_{\alpha^*_p}} \int_{a_{\beta_t}}^{x_{\beta_t}} \int_{a_{\beta_t}}^{r_{\beta_t}} h_i(x, r_{\alpha_p}, s_{\gamma_q}) ds_{\beta_t} dr_{\beta_t} dr_{\alpha^*_p} ds_{\gamma^*_q}$$

$$= \int_{a_{\gamma^*_q}}^{x_{\gamma^*_q}} \int_{a_{\alpha^*_p}}^{x_{\alpha^*_p}} \int_{a_{k_1}}^{x_{k_1}} \int_{a_{k_1}}^{r_{k_1}} \int_{a_{k_2}}^{x_{k_2}} \int_{a_{k_2}}^{r_{k_2}}$$

$$\ldots \int_{a_{k_t}}^{x_{k_t}} \int_{a_{k_t}}^{r_{k_t}} h_i(x, r_{\alpha_p}, s_{\gamma_q}) ds_{k_t} dr_{k_t}$$

$$\ldots ds_{k_2} dr_{k_2} ds_{k_1} dr_{k_1} dr_{\alpha^*_p} ds_{\gamma^*_q}.$$

Using the fact (see [10], p. 644) that if a function $f(x_1, x_2, \ldots, x_n)$ is continuous in each variable x_k holding the others fixed then f is measurable and 2) we obtain

$$\int_{a_{\alpha_p}}^{x_{\alpha_p}} \int_{a_{\gamma_q}}^{w_{\gamma_q}(x,r;\alpha_p)} h_i(x, r_{\alpha_p}, s_{\gamma_q}) ds_{\gamma_q} dr_{\alpha_p}$$

$$= \int_{a_{\gamma^*_q}}^{x_{\gamma^*_q}} \int_{a_{\alpha^*_p}}^{x_{\alpha^*_p}} \int_{a_{k_1}}^{x_{k_1}} \int_{s_{k_1}}^{x_{k_1}} \int_{a_{k_2}}^{x_{k_2}} \int_{s_{k_2}}^{x_{k_2}}$$

$$\cdots \int_{a_{k_t}}^{x_{k_t}} \int_{s_{k_t}}^{x_{k_t}} h_i(x, r_{\alpha_p}, s_{\gamma_q}) dr_{k_t} ds_{k_t}$$

$$\cdots dr_{k_2} ds_{k_2} dr_{k_1} ds_{k_1} dr_{\alpha^*_p} ds_{\gamma^*_q} .$$

Again through repeated use of 1) and Lemma 2.2 we see that

$$\int_{a_{\alpha_p}}^{x_{\alpha_p}} \int_{a_{\gamma_q}}^{w_{\gamma_q}(x,r;\alpha_p)} h_i(x, r_{\alpha_p}, s_{\gamma_q}) ds_{\gamma_q} dr_{\alpha_p}$$

$$= \int_{a_{\gamma^*_q}}^{x_{\gamma^*_q}} \int_{a_{\alpha^*_p}}^{x_{\alpha^*_p}} \int_{a_{\beta_t}}^{x_{\beta_t}} \int_{s_{\beta_t}}^{x_{\beta_t}} h_i(x, r_{\alpha_p}, s_{\gamma_q}) dr_{\beta_t} ds_{\beta_t} dr_{\alpha^*_p} ds_{\gamma^*_q}$$

$$= \int_{a_{\gamma^*_q}}^{x_{\gamma^*_q}} \int_{a_{\beta_t}}^{x_{\beta_t}} \int_{a_{\alpha^*_p}}^{x_{\alpha^*_p}} \int_{s_{\beta_t}}^{x_{\beta_t}} h_i(x, r_{\alpha_p}, s_{\gamma_q}) dr_{\beta_t} dr_{\alpha^*_p} ds_{\beta_t} ds_{\gamma^*_q}$$

$$= \int_{a_{\gamma_q}}^{x_{\gamma_q}} \int_{w_{\alpha_p}(a,s;\gamma_q)}^{x_{\alpha_p}} h_i(x, r_{\alpha_p}, s_{\gamma_q}) dr_{\alpha_p} ds_{\gamma_q}.$$

Hence it follows that

$$\int_{a_{\alpha_p}}^{x_{\alpha_p}} \int_{a_{\gamma_q}}^{w_{\gamma_q}(x,r;\alpha_p)} h(x, r_{\alpha_p}, s_{\gamma_q}) ds_{\gamma_q} dr_{\alpha_p}$$

$$= \int_{a_{\gamma_q}}^{x_{\gamma_q}} \int_{w_{\alpha_p}(a,s;\gamma_q)}^{x_{\alpha_p}} h(x, r_{\alpha_p}, s_{\gamma_q}) dr_{\alpha_p} ds_{\gamma_q}$$

and the proof is complete.

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