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# Stability Results for Stochastic Programming Problems

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**Summary:** The paper deals with a statistical approach to stability analysis in nonlinear stochastic programming. Firstly the distribution function of the underlying random variable is estimated by the empirical distribution function, and secondly the problem of estimated parameters is considered. In both the cases the probability that the solution set of the approximate problem is not contained in an $\varepsilon$-neighbourhood of the solution set to the original problem is estimated, and under differentiability properties an asymptotic expansion for the density of the (unique) solution to the approximate problem is derived.

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## 1. Introduction

The aim of this paper is to present stability results for stochastic programming problems when either nothing is known about the probability measure and the distribution function is estimated by the empirical distribution function or the problem depends on an unknown parameter, for which a sequence of estimation functions is available.

At first we shall deal with the problem

$$\text{(P)} \qquad \min_{x \in \Gamma} \mathsf{E} f(x, Z)\,,$$

where $\Gamma$ may be given by deterministic inequality constraints, chance constraints, restrictions depending on the mean value of certain functions, or restrictions that have to be satisfied almost surely.

Let $[\Omega, \mathfrak{A}, \mathsf{P}]$ be the underlying probability space and $Z$ a random variable with values in the measurable space $[R^m, \mathfrak{B}^m]$, where $\mathfrak{B}^m$ denotes the $\sigma$-field of Borel sets of $R^m$. $\mathsf{P}$ induces the probability measure $\mathsf{P}_Z$ on $[R^m, \mathfrak{B}^m]$. $\mathsf{E}$ denotes the mean value with respect to $\mathsf{P}_Z$ and $F_Z$ the distribution function of $Z$. $\Gamma \neq \emptyset$ is a compact subset of $R^p$. $f: R^p \times R^m \to R^1$ is supposed to be continuous with respect to the first variable and integrable with respect to the second one. Furthermore we assume that there is a $\varkappa > 0$ and an $s > 1$ such that $\mathsf{E}|f(x, Z)|^s < \infty \ \forall x \in \Gamma + \varkappa B$. ($B = \{x \in R^p \mid \|x\| \leq 1\}$, $\|\cdot\|$ denotes the Euclidean norm.)

$\mathsf{P}_Z$ being unknown, we shall consider the problems

$$(\mathrm{P}_n) \qquad \min_{x \in \Gamma_n} \frac{1}{n} \sum_{i=1}^{n} f(x, Z_i) ,$$

where the $Z_i$ are independent, identically (like $Z$) distributed random variables and $\Gamma_n$ is a closed-valued measurable multifunction, which is defined on $[\Omega, \mathfrak{A}, \mathsf{P}]$ and approximates $\Gamma$ in a way that will be specified later on.

In the following we use the notations

$$X_0 = \arg \min_{x \in \Gamma} \mathsf{E} f(x, Z)$$

and

$$X_n(\omega) = \arg \min_{x \in \Gamma_n(\omega)} \frac{1}{n} \sum_{i=1}^{n} f(x, Z_i(\omega)) .$$

Under our assumptions $X_n$ is a measurable multifunction ([16, Theorem 2K and Proposition 2C]).

By $x_0$ we denote the elements of $X_0$ and by $x_n$ the measurable selection functions for $X_n$. In the same way we understand small letters in the following, when the corresponding capital letters denote sets or multifunctions.

Wets and Solis [19], [21] dealt with an unconstrained minimum and derived conditions that ensure asymptotic normality of $\sqrt{n}\,(x_n - x_0)$. Kăňková [13] estimated the rate of convergence of

$$\mathsf{P}\left\{\omega \;\middle|\; \left| \max_{x \in K} \frac{1}{n} \sum_{i=1}^{n} f(x, Z_i(\omega)) - \max_{x \in K} \mathsf{E} f(x, Z) \right| \geqq \varepsilon \right\}$$

(X a compact convex set, $f(\cdot, z)$ Lipschitz continuous with Lipschitz constant not depending on $z$) and showed that it is at least exponential.

Römisch and Wakolbinger [17] were concerned with approximations of probability measures in the framework of parametric programming. Applying their results to approximations of stochastic linear programming problems with complete fixed recourse via empirical measures, they obtained results on the convergence rate for the mean deviation of the optimal values. Recently Dupačová and Wets [8] extended the results on asymptotic normality of $\sqrt{n}\,(x_n - x_0)$. They allowed for constraints and imposed only weak differentiability assumptions.

We shall prove assertions on the convergence rate of $\mathsf{P}\{\omega \mid \exists x \in X_n(\omega)$ with $d(x, X_0) \geqq \varepsilon\}$, where $d$ denotes the Hausdorff distance (with $d(\emptyset, X_0) = \infty$). Furthermore, for the case of differentiable $f$ we shall derive an asymptotic expansion for the density of $\sqrt{n}\,(x_n - x_0)$ and in this way contribute to the results on asymptotic normality.

In the second part of our paper we shall consider the problem that the probability measure is determined up to an unknown parameter $\eta$. We are given the problem

$$\min_{x \in \Gamma} f(x, \eta) \quad \text{with} \quad \Gamma = \{x \in R^p \mid g_i(x, \eta) \leqq 0,\ i = 1, \ldots, q\} ,$$

where $\eta \in Y \subset R^m$, $Y$ open, and $f : R^p \times Y \to R^1$, $g_i : R^p \times Y \to R^1$, $i = 1\ ..., q$, are measurable with respect to the second variable.

We assume that a sequence of estimation functions $(Y^n)_{n \in N}$ for $\eta$, defined on the probability space $[\Omega, \mathfrak{A}, \mathsf{P}]$, is available. In such cases the necessity arises to study the stability behaviour of the solution sets $\hat{X}(y)$ to the programs

$$(\mathrm{P}_y) \qquad \min_{x \in \Gamma(y)} f(x, y) \quad \text{with} \quad \Gamma(y) = \{x \in R^p \mid g_i(x, y) \leqq 0,\ i = 1, ..., q\} .$$

DUPAČOVÁ [6], [7] showed for twice differentiable functions $f$ that asymptotic normality of $\sqrt{n}\,(Y^n - \eta)$ carries over to the sequence $\sqrt{n}\,(\hat{x}(Y^n) - \hat{x}(\eta))$ and estimated – for a special case – the accuracy of the normal approximation by means of the BERRY-ESSEÉN theorem.

In this paper we will estimate

$$\mathsf{P}\,\{\omega \mid \exists x \in \hat{X}(Y^n(\omega)) \text{ with } d(x, \hat{X}(\eta)) \geqq \varepsilon\}$$

and derive an asymptotic expansion for the density of $\sqrt{n}\,(\hat{x}(Y^n) - \hat{x}(\eta))$.

There are several related papers concerned with stability in stochastic programming but approaching the subject under another point of view (cf. [12], [15], [18], [20]).

## 2. Estimated Distribution Function

We consider the problems (P) and $(\mathrm{P}_n)$.

**Theorem 1:** *In addition to the presuppositions made in the introduction let the following assumptions be satisfied:*

(A 1.1) *There is a neighbourhood $U(X_0)$ of $X_0$ such that*

$$\mathsf{E} \mid \sup_{\bar{x} \in U(X_0)} f(\bar{x} \cdot Z)|^s < \infty .$$

(A 1.2) *To every $x \in \Gamma + \varkappa B$ there is a neighbourhood $\tilde{U}(x)$ such that*

$$\mathsf{E} \mid \inf_{\bar{x} \in \tilde{U}(x)} f(\bar{x} \cdot Z)|^s < \infty .$$

(A 1.3) *For all $\delta > 0$*

$$\mathsf{P}\,\{\omega \mid d(\Gamma_n(\omega), \Gamma) \geqq \delta\} = \boldsymbol{o}(n^{-s+1}) \quad (n \to \infty) .$$

*Then* $\mathsf{P}\,\{\omega \mid (\exists x \in X_n(\omega)$ *with* $d(x, X_0) \geqq \varepsilon)$ *or* $X_n(\omega) = \emptyset\} = \boldsymbol{o}(n^{-s+1})$ *for all* $\varepsilon > 0$.

Observe that the assumptions of Theorem 1 are not sufficient for $\mathsf{P}\,\{\omega \mid d(X_n(\omega), X_0) \geqq \varepsilon\} \to 0$, as the following example shows:

**Example 1:** Let $Z \mid [\Omega, \mathfrak{A}, \mathsf{P}] \to [R^1, \mathfrak{B}^1]$ be uniformly distributed on $[-1, 1]$, $x \in R^1$, $f(x, z) = zx$, $\Gamma_n(\omega) = \Gamma = [-1, 1]$ $\mathsf{P}$-a.s. Then $\mathsf{E} f(x\ Z) = 0\ \forall x$, hence $X_0 = [-1, 1]$.

On the other hand we have

$$\mathsf{P}\left\{\omega \mid X_n(\omega)=\{\overset{-}{{}_{(+)}}1\}\right\}=\mathsf{P}\left\{\omega \;\middle|\; \frac{1}{n}\sum_{i=1}^{n} Z_i(\omega) \underset{(<)}{>} 0\right\}=\frac{1}{2},$$

consequently $\mathsf{P}\{\omega \mid d(X_n(\omega), X_0)=2\}=1\;\forall n$.

Proof of Theorem 1. Parts of the following proof make use of ideas of Michel/Pfanzagl [14] and Čibisov [3].
If $X_n(\omega)=\emptyset$, $\Gamma_n(\omega)$ is either empty or unbounded, hence

$$\mathsf{P}\{\omega \mid X_n(\omega)=\emptyset\}=\boldsymbol{o}(n^{-s+1}).$$

Let $X_n(\omega)\neq\emptyset$, $\varepsilon>0$ fixed and define

$$X_0^n(\omega):=\{x\in\Gamma_n(\omega) \mid \mathsf{E}f(x,Z)=\inf_{\tilde{x}\in\Gamma_n(\omega)}\mathsf{E}f(\tilde{x},Z)\}.$$

Furthermore, for $\beta\in R^1$, $\beta>0$, we define

$$C_\beta:=\{x\in\Gamma \mid d(x, X_0)\geqq\beta\}.$$

Because of our assumptions $\mathsf{E}f(\cdot,Z)$ is continuous on $\Gamma+\varkappa B$. Consequently, if $C_\beta\neq\emptyset$, there is an $\alpha_\beta>0$ and a $\varkappa_\beta$, $0<\varkappa_\beta\leqq\varkappa$, such that

$$\mathsf{E}f(x,Z)>\mathsf{E}f(x_0,Z)+\alpha_\beta \quad \text{for all} \quad x\in C_\beta+\varkappa_\beta B \quad \text{and} \quad x_0\in X_0.$$

Now, according to Lemma 3 in [14], to every $x\in C_\beta+\varkappa_\beta B$ there is an open ball $\tilde{U}_\beta(x)$ such that

$$\mathsf{E}\inf_{x\in\tilde{U}_\beta(x)} f(x,Z)>\mathsf{E}f(x_0,Z)+\frac{\alpha_\beta}{2} \quad (x_0\in X_0).$$

The system $\tilde{U}_\beta(x)$, $x\in C_\beta+\varkappa_\beta B$, is an open cover of $C_\beta+\varkappa_\beta B$, hence there exists a finite cover $\tilde{U}_\beta(x_j^\beta)$, $j\in\{1,\dots,k_\beta\}$. In particular these statements hold for $\beta=\varepsilon$.

Further, because of $\mathsf{E}\,|\sup_{x\in U(X_0)} f(x,Z)|<\infty$, Lemma 3 in [14] is applicable to $-f$, consequently to every $x_0\in X_0$ there exists an open neighbourhood $\hat{U}(x_0)$ such that

$$\mathsf{E}\sup_{x\in\hat{U}(x_0)} f(x,Z)-\mathsf{E}f(x_0,Z)<\frac{\alpha_\varepsilon}{4}.$$

$X_0$ being compact, we find $x_{01},\dots,x_{0\hat{k}}\in X_0$ with

$$\bigcup_{i\in\{1,\dots,\hat{k}\}}\hat{U}(x_{0i})\supset X_0.$$

Let be $U_1(X_0):=\operatorname{cl}\big(\bigcup_{i\in\{1,\dots,\hat{k}\}}\hat{U}(x_{0i})\big)$, $\hat{d}:=d(U_1(X_0), X_0)$ and

$$\hat{\varkappa}:=\varkappa_\varepsilon.$$

Now define

$$S:=\{\omega \mid \exists x\in X_n(\omega) \quad \text{with} \quad d(x, X_0)\geqq\varepsilon\},$$

$$S_1:=\{\omega \mid X_0^n(\omega)\cup X_n(\omega)\subset\Gamma+\hat{\varkappa}B\} \quad \text{and} \quad S_2=\{\omega \mid X_0^n(\omega)\subset U_1(X_0)\}.$$

Obviously $S=(S\cap\bar{S}_1)\cup(S\cap S_1\cap\bar{S}_2)\cup(S\cap S_1\cap S_2)$.

At first we consider $\omega \in S \cap \bar{S}_1$. We have

$$S \cap \bar{S}_1 \subset \{\omega \mid d(\Gamma_n(\omega), \Gamma) > \hat{\varkappa}\}, \quad \text{hence} \quad \mathsf{P}\,(S \cap \bar{S}_1) = \boldsymbol{o}(n^{-s+1})\,.$$

Secondly, we assume $\omega \in S \cap S_1 \cap \bar{S}_2$ and look at $C_{\frac{\hat{d}}{2}}$. Let

$$C_1 := C_{\frac{\hat{d}}{2}} + \lambda B \quad \text{with} \quad \lambda = \min\left\{\frac{\hat{d}}{4}, \varkappa_{\frac{\hat{d}}{2}}\right\}.$$

In the case $X_0^n(\omega) \not\subset C_1 \cup U_1(X_0)$ we have $X_0^n(\omega) \not\subset \Gamma + \lambda B$, hence $d(\Gamma_n(\omega), \Gamma) > \lambda$. Let $\tilde{x} \in X_0^n(\omega)$ with $\tilde{x} \in C_1$. Then $\mathsf{E}f(\tilde{x}, Z) > \mathsf{E}f(x_0, Z) + \alpha_{\frac{\hat{d}}{2}}$, $(x_0 \in X_0)$.

Furthermore, there is a neighbourhood $U_2(X_0)$ of $X_0$, that does not depend on $\tilde{x}$, such that

$$\mathsf{E}f(\tilde{x}, Z) > \mathsf{E}f(x, Z) + \frac{\alpha_{\frac{\hat{d}}{2}}}{2} \quad \forall x \in U_2(X_0)\,.$$

Consequently, there is no $x \in \Gamma_n(\omega)$ with $x \in U_2(X_0)$. Therefore

$$d(\Gamma_n(\omega), \Gamma) > d(U_2(X_0), X_0)\,.$$

Finally we investigate the event $S \cap S_1 \cap S_2$. For $\omega \in S \cap S_1 \cap S_2$ we make use of the fact that there is a finite cover $\tilde{U}_\varepsilon(x_1^\varepsilon), \ldots, \tilde{U}_\varepsilon(x_{k_\varepsilon}^\varepsilon)$ of $C_\varepsilon + \hat{\varkappa} B$. Hence $x_n \in C_\varepsilon + \hat{\varkappa} B$ implies $x_n \in \tilde{U}_\varepsilon(x_j^\varepsilon)$ for a suitable $j$. Further,

$$\frac{1}{n} \sum_{i=1}^{n} f(x_n, Z_i(\omega)) \leqq \frac{1}{n} \sum_{i=1}^{n} f(\tilde{x}, Z_i(\omega)) \quad \forall \tilde{x} \in X_0^n(\omega),\ x_n \in X_n(\omega)\,.$$

Because of $X_0^n(\omega) \subset U_1(X_0)$ we find an $x_{0l} \in X_0$ such that $X_0^n(\omega) \cap \hat{U}(x_{0l}) \neq \emptyset$. Consequently,

$$\frac{1}{n} \sum_{i=1}^{n} u_j(Z_i(\omega)) \leqq \frac{1}{n} \sum_{i=1}^{n} h_l(Z_i(\omega))\,,$$

where

$$u_j(z) := \inf_{x \in \tilde{U}_\varepsilon(x_j^\varepsilon)} f(x, z) \quad \text{and} \quad h_l(z) := \sup_{x \in \hat{U}(x_{0l})} f(x, z)\,.$$

This implies

$$\frac{1}{n} \sum_{i=1}^{n} [u_j(Z_i(\omega)) - \mathsf{E}u_j(Z)] \leqq -\frac{1}{2}[\mathsf{E}u_j(Z) - \mathsf{E}h_l(Z)]$$

or

$$\frac{1}{n} \sum_{i=1}^{n} [h_l(Z_i(\omega)) - \mathsf{E}h_l(Z)] \leqq \frac{1}{2}[\mathsf{E}u_j(Z) - \mathsf{E}h_l(Z)]\,.$$

Now, taking into consideration how $\tilde{U}_\varepsilon(x_j^\varepsilon)$ and $\hat{U}(x_{0l})$ were introduced, we obtain (with $\tilde{\alpha} := \alpha_\varepsilon$)

$$\mathsf{E}u_j(Z) - \mathsf{E}h_l(Z) \geqq \mathsf{E}f(x_0, Z) + \frac{\tilde{\alpha}}{2} - \mathsf{E}h_l(Z) \geqq \frac{\tilde{\alpha}}{4}\,.$$

Consequently,

$$\mathsf{P}\,(S\cap S_1\cap S_2)\leqq\sum_{j=1}^{k_\varepsilon}\mathsf{P}\left\{\omega\,\middle|\,\frac{1}{n}\sum_{i=1}^{n}[u_j(Z_i(\omega))-\mathsf{E}u_j(Z)]\leqq-\frac{\tilde{\alpha}}{8}\right\}$$
$$+\sum_{j=1}^{\hat{k}}\mathsf{P}\left\{\omega\,\middle|\,\frac{1}{n}\sum_{i=1}^{n}[h_l(Z_i(\omega))-\mathsf{E}h_l(Z)]\geqq\frac{\tilde{\alpha}}{8}\right\}.$$

It remains to apply Lemma 2 in [3]. ■

If we assume that the random variables are bounded, we can derive a sharper result:

**Theorem 2:** *Let the following assumptions be satisfied:*

(A 2.1) *There is an $a\in R^1$ and a $b\in R^1$ such that*

$$a\leqq f(x,z)\leqq b\quad \mathsf{P}_Z-\text{a.e.}\quad \forall x\in\Gamma+\varkappa B\,.$$

(A 2.2) *There is a $C>0$ such that for all $\alpha>0$*

$$\mathsf{P}\,\{\omega\mid d(\Gamma_n(\omega),\Gamma)\geqq\delta\}=\boldsymbol{O}(\mathrm{e}^{-C\delta^2 n})\quad(n\to\infty)\,.$$

*Then for every $\varepsilon>0$ there is a $\lambda(\varepsilon)>0$ such that*

$$\mathsf{P}\,\{\omega\mid(\exists x\in X_n(\omega)\ \textit{with}\ d(x,X_0)\geqq\varepsilon)\ \textit{or}\ X_n(\omega)=\emptyset\}=\boldsymbol{O}(\mathrm{e}^{-n\lambda(\varepsilon)})\,.$$

Proof. The proof follows that of Theorem 1, but in the last step we use Theorem 2 in [10], from which we obtain

$$\mathsf{P}\left\{\omega\,\middle|\,\frac{1}{n}\sum_{i=1}^{n}[u_j(Z_i(\omega))-\mathsf{E}u_j(Z)]\leqq-\frac{\tilde{\alpha}}{8}\right\}\leqq 2\mathrm{e}^{-n\frac{\tilde{\alpha}^2}{32(b-a)^2}}$$

and

$$\mathsf{P}\left\{\omega\,\middle|\,\sum_{i=1}^{n}[h_l(Z_i(\omega))-\mathsf{E}h_l(Z)]\geqq\frac{\tilde{\alpha}}{8}\right\}\leqq 2\mathrm{e}^{-n\frac{\tilde{\alpha}^2}{32(b-a)^2}}.\quad ■$$

Now we turn to the investigation of the assumptions (A.1.3) and (A.2.2). Often the constraints of stochastic programming problems are given in the form

$$\Gamma:=\mathrm{cl}\,\{x\in G\mid F(x)\geqq\xi\}\,,$$

where $F$ denotes the distribution function of a certain random variable $\hat{Z}$, $G$ is a compact subset of $R^p$ and $0<\xi<1$. This case occurs, if we consider linear chance constraints with random right hand sides only.

If we don't know $F$, we can approximate $\Gamma$ by

$$\Gamma_n(\omega):=\mathrm{cl}\,\{x\in G\mid F_n(x,\omega)\geqq\xi\}$$

with the empirical distribution function $F_n$. For the accuracy of this approximation we have Theorem 3. To simplify matters, we claim that $G$ is a closed cuboid the edges of which are parallel to the axises of our system of co-ordinates. Furthermore, we assume that all cubes used in the following have this property. By dia $(\Gamma)$ we denote the diameter of the greatest ball that is contained in $\Gamma$.

**Theorem 3:** *Let* $\mathrm{dia}\,(\Gamma):=D>0$. *Assume further that $F$ is absolutely continuous* *and that the distribution density $\vartheta$ has the property $\vartheta(x) > \varrho > 0\ \forall x \in G$. Then* $\mathsf{P}\{\omega \mid d(\Gamma_n(\omega), \Gamma) \geqq \delta\} = \boldsymbol{O}\left(\mathrm{e}^{-2n\varrho^2\left(\frac{\min\{\delta, D\}}{2\sqrt{p}}\right)^{2p}}\right)$ *for all $\delta > 0$.*

The proof makes use of the following lemma:

**Lemma 1:** *Let $E_n(l, \varrho_0) := \{\omega \mid$ there exists a (closed) cube $M(l)$ with edge length $l$ and a constant $\varrho_0$ such that $|F_n(x, \omega) - F(x)| \geqq \varrho_0\ \forall x \in M(l)\}$.*

*Then $\mathsf{P}(E_n(l, \varrho_0)) = \boldsymbol{O}(\mathrm{e}^{-2n\varrho_0^2})$.*

Proof. $G$ being compact, it can be covered by a finite number of cubes $M_1, \ldots, M_{\hat{k}}$ with edge length $l$. We investigate the events $\{\omega \mid |F_n(x_i, \omega) - F(x_i)| \geqq \varrho_0\}$, where the $x_i$ denote the centres of $M_i$. According to Theorem 2 in [10]

$$\mathsf{P}\{\omega \mid |F_n(x_i, \omega) - F(x_i)| \geqq \varrho_0\} \leqq 2\mathrm{e}^{-2n\varrho_0^2} \quad \forall i \in \{1, \ldots, \hat{k}\}$$

holds.

Because of $|F_n(x, \omega) - F(x)| \geqq \varrho_0\ \forall x \in M(l)$, we find an $i_0 \in \{1, \ldots, \hat{k}\}$ with $|F_n(x_{i_0}, \omega) - F(x_{i_0})| \geqq \varrho_0$. Hence

$$\mathsf{P}(E_n(l, \varrho_0)) \leqq \sum_{i=1}^{\hat{k}} \mathsf{P}\{\omega \mid |F_n(x_i, \omega) - F(x_i)| \geqq \varrho_0\} \leqq 2\hat{k}\mathrm{e}^{-2n\varrho_0^2}. \quad \blacksquare$$

Proof. of Theorem 3. Let $\delta$, $\omega$ and $n$ be fixed and assume that $d(\Gamma_n(\omega), \Gamma) \geqq \delta$. We abbreviate $\Gamma_n := \Gamma_n(\omega)$ and $F_n(x) := F_n(x, \omega)$. By "$\leqq$" we denote the partial ordering generated by $R_+^p$.

If int $\Gamma_n = \emptyset$, we have $F_n(x) < \xi\ \forall x \in G$. Because of our assumptions on $\Gamma$ and $\vartheta$ we find an $x_1 \in \Gamma$ and a cube $M_1 = M_1\left(\frac{D}{2\sqrt{p}}\right)$ with edge length $\frac{D}{2\sqrt{p}}$ such that $M_1 \subset \Gamma$, $x_1 \leqq x\ \forall x \in M_1$ and $d(x_1, M_1) \geqq \frac{D}{2}$. Hence

$$F(x) - F(x_1) > \varrho\left(\frac{D}{2\sqrt{p}}\right)^p \quad \text{and} \quad |F_n(x) - F(x)| > \varrho\left(\frac{D}{2\sqrt{p}}\right)^p \quad \forall x \in M_1\left(\frac{D}{2\sqrt{p}}\right).$$

Now we turn to the case int $\Gamma_n \neq \emptyset$. At first we assume that $d(\Gamma_n, \Gamma) = \sup_{x \in \Gamma} \inf_{y \in \Gamma_n} d(x, y) := \hat{d} \geqq \delta$. Then there exists an $x_1 \in \Gamma$, $x_1 \notin \Gamma_n$, and an $x_2 \in bd\Gamma_n$ such that $\|x_1 - x_2\| = \hat{d}$. Furthermore, $x_1$ and $x_2$ have the property that $x_1 \leqq x_2$. (Because of the monotonicity of $F_n$ for every $x_1 \in \Gamma$ the vector $x_2(x_1) \in \Gamma_n$ with minimal distance to $x_1$ is greater than $x_1$.)

Now, as $F_n$ is continuous on the left, $F_n(x) < \xi\ \forall x \leqq x_2$. On the other hand, to $x_2$ we find a cube $M_2 = M_2\left(\frac{\delta}{2\sqrt{p}}\right) \subset \Gamma$ and a point $x_3 \in \Gamma$ with $x_3 \leqq x \leqq x_2\ \forall x \in M_2$ and $d(x_3, M_2) \geqq \frac{\delta}{2}$. Consequently $F(x) - F(x_3) > \varrho\left(\frac{\delta}{2\sqrt{p}}\right)^p$ and, because of $F_n(x) < F(x_3)$, $|F_n(x) - F(x)| > \varrho\left(\frac{\delta}{2\sqrt{p}}\right)^p\ \forall x \in M_2\left(\frac{\delta}{2\sqrt{p}}\right)$.

If $d(\Gamma_n, \Gamma) = \sup_{x \in \Gamma_n} \inf_{y \in \Gamma} d(x, y) := \hat{d}$, we find an $x_1 \in bd\Gamma_n$, $x_1 \notin \Gamma$, and an $x_2 \in bd\Gamma$ such that $x_1 \leqq x_2$ and $\|x_1 - x_2\| = \hat{d}$. Obviously $F_n(x) \geqq \xi \ \forall x > x_1$. (" $>$ " means " $>$ " in every component.)

Furthermore, to $x_1$ we find a cube $M_3 = M_3\left(\frac{\delta}{2\sqrt{p}}\right)$, $M_3 \cap \Gamma = \emptyset$, and a point $x_3 \in bd\Gamma$ with $x_1 \leqq x \leqq x_3 \ \forall x \in M_3$ and $d(x_3, M_3) \geqq \frac{\delta}{2}$. Consequently there is a cube $M_4\left(\frac{\delta}{2\sqrt{p}}\right)$ with $F(x_3) - F(x) > \varrho \left(\frac{\delta}{2\sqrt{p}}\right)^p \ \forall x \in M_4$, hence

$$F_n(x) - F(x) \geqq F(x_3) - F(x) > \varrho \left(\frac{\delta}{2\sqrt{p}}\right)^p \quad \forall x \in M_4\left(\frac{\delta}{2\sqrt{p}}\right).$$

It remains to apply Lemma 1. ■

For discrete distributions we have a similar result:

**Proposition 1:** *Let $\hat{Z}$ be such that*

$$\mathsf{P}\{\omega \mid \hat{Z}(\omega) = z_j\} = p_j, \quad j \in \{1, 2, \ldots\}; \quad \sum_{j=1}^{\infty} p_j = 1 .$$

*Assume further that* $\operatorname{dia}(\Gamma) =: D > 0$ *and* $0 < \varrho := \min_{x \in R^p} |F(x) - \xi|$.

*Then* $\mathsf{P}\{\omega \mid d(\Gamma_n(\omega), \Gamma) \geqq \delta\} = \boldsymbol{O}(\mathrm{e}^{-2n\varrho^2})$ *for all* $\delta > 0$.

Proof. We consider an $\omega$ with $d(\Gamma_n(\omega), \Gamma) \geqq \delta$ and abbreviate $\Gamma_n := \Gamma_n(\omega)$ and $F_n(x) := F_n(x, \omega)$. If $\operatorname{int} \Gamma_n = \emptyset$, we have $F_n(x) < \xi \ \forall x \in G$. On the other hand $F(x) \geqq \xi + \varrho \ \forall x \in \Gamma$, consequently there is a cube $M_1\left(\frac{D}{\sqrt{p}}\right)$ such that

$$|F_n(x) - F(x)| \geqq \varrho \quad \forall x \in M_1\left(\frac{D}{\sqrt{p}}\right).$$

Now we turn to case $\operatorname{int} \Gamma_n \neq \emptyset$. Then there exists either a cube $M_2\left(\frac{\delta}{2\sqrt{p}}\right)$ with $M_2 \subset \operatorname{int} \Gamma$, $M_2 \cap \Gamma_n = \emptyset$, or a cube $M_3\left(\frac{\delta}{2\sqrt{p}}\right)$ with $M_3 \subset \operatorname{int} \Gamma_n$, $M_3 \cap \Gamma = \emptyset$. In the first case we have $F(x) \geqq \xi$, hence $F(x) \geqq \xi + \varrho$, but $F_n(x) < \xi \ \forall x \in M_2$; in the second case $F_n(x) \geqq \xi$ and $F(x) \leqq \xi - \varrho \ \forall x \in M_3$.

Application of Lemma 1 yields the desired result. ■

Now we consider constraints of the form

$$\hat{\Gamma} := \operatorname{cl}\{x \in R^p \mid g(x, z) \leqq 0 \text{ for } \mathsf{P}_Z\text{-almost all } z\}$$

and

$$\hat{\Gamma}_n(\omega) := \operatorname{cl}\{x \in R^p \mid g(x, Z_i(\omega)) \leqq 0,\ i = 1, \ldots, n\}$$

with a (possibly vector-valued) function $g$ that is measurable with respect to the second variable.

We restrict ourselves to discrete distributions, because in the case of absolutely continuous distributions such an approximation would make sense only in special cases.

**Proposition 2:** *Let $\hat{\Gamma} \neq \emptyset$ be a compact set and let $Z$ be such that*

$$\mathsf{P}\{\omega \mid Z(\omega) = z_j\} = p_j > 0, \quad j = 1, \ldots, j_0; \quad \sum_{i=1}^{j_0} p_j = 1 .$$

*Then* $\mathsf{P}\{\omega \mid d(\hat{\Gamma}_n(\omega), \hat{\Gamma}) \geqq \delta\} = \boldsymbol{O}(\mathrm{e}^{n \cdot \ln[\max_{j \in \{1, \ldots, j_0\}} (1-p_j)]})$ *for all* $\delta > 0$.

Proof. Let $\omega$ be fixed and $d(\hat{\Gamma}_n(\omega), \hat{\Gamma}) \geqq \delta$. Then there exists a $z_k$ such that $Z_i(\omega) \neq z_k$ $\forall i \in \{1, \ldots, n\}$. Hence

$$\mathsf{P}\{\omega \mid d(\hat{\Gamma}_n(\omega), \hat{\Gamma}) \geqq \delta\} \leqq \sum_{j=1}^{j_0} \mathsf{P}\{\omega \mid Z_i(\omega) \neq z_j \ \forall i \in \{1, \ldots, n\}\}$$
$$= \sum_{j=1}^{j_0} (1-p_j)^n \leqq j_0 [\max_{j \in \{1, \ldots, j_0\}} (1-p_j)]^n = j_0 \cdot \mathrm{e}^{n \cdot \ln[\max_{j \in \{1, \ldots, j_0\}} (1-p_j)]} . \quad \blacksquare$$

If we are given chance constraints in a more general form than that considered in Theorem 3, we can use the equation

$$\mathsf{P}\{\omega \mid \tilde{g}_j(x, Z(\omega)) \leqq 0\} = \mathsf{E}\chi_{\{z \mid \tilde{g}_j(x,z) \leqq 0\}}(Z) ,$$

where $\chi_A(z) := \begin{cases} 1, \text{ if } z \in A, \\ 0 \text{ otherwise,} \end{cases}$

and approximate the set

$$\tilde{\tilde{\Gamma}} := \mathrm{cl}\,\{x \in G \mid \mathsf{P}\{\omega \mid \tilde{g}_j(x, Z(\omega)) \leqq 0\} \geqq \xi_j, \ j = 1, \ldots, q\}$$

by

$$\tilde{\tilde{\Gamma}}_n(\omega) = \mathrm{cl}\left\{x \in G \mid \frac{1}{n} \sum_{i=1}^{n} \chi_{\{z \mid \tilde{g}_j(x,z) \leqq 0\}}(Z_i(\omega)) \geqq \xi_j, \ j = 1, \ldots, q\right\} .$$

In a more general setting, we are given

$$\tilde{\Gamma} := \mathrm{cl}\,\{x \in G \mid \mathsf{E}g_j(x, Z) \leqq 0, \ j = 1, \ldots, q\}$$

and

$$\tilde{\Gamma}_n(\omega) := \mathrm{cl}\left\{x \in G \mid \frac{1}{n} \sum_{i=1}^{n} g_j(x, Z_i(\omega)) \leqq 0, \ j = 1, \ldots, q\right\} ,$$

where the $g_j : R^p \times R^m \to R^1$ are supposed to be $\mathsf{P}_Z$-as continuous in every point $x \in \Gamma + \varkappa B$ and integrable with respect to the second variable. $G$ is compact, but not necessarily a cuboid.

The application of Proposition 3 to chance constraints will be given elsewhere.

**Proposition 3:** *Let the following assumptions be satisfied:*

(P 3.1) int $(B(x, \delta) \cap \tilde{\Gamma}) \neq \emptyset$ *for all balls* $B(x, \delta)$ *with centre* $x \in \tilde{\Gamma}$ *and radius* $\delta$.

(P 3.2) $\mathsf{E}g_j(\cdot, Z)$ *is continuous on* int $\tilde{\Gamma}$ *and l.s.c. on* $R^p$, $j = 1, \ldots, q$;

(P 3.3) $\mathsf{E}g_j(x, Z) < 0$ $\forall x \in \mathrm{int}\,\tilde{\Gamma}$, $j \in \{1, \ldots, q\}$,

(P 3.4) *There exists an* $s > 1$ *such that*

$$\mathsf{E}\,|\sup_{\bar{x} \in \tilde{\Gamma}} g_j(\bar{x}, Z)|^s < \infty \quad j \in \{1, \ldots, q\} ,$$

(P 3.5) *To every* $x \notin \tilde{\Gamma}$ *there exists a neighbourhood* $U(x)$ *such that*

$$\mathsf{E}\,|\inf_{\bar{x} \in U(x)} g_j(\bar{x}, Z)|^s < \infty \quad j \in \{1, \ldots, q\} .$$

*Then* $\mathsf{P}\{\omega \mid d(\tilde{\Gamma}_n(\omega), \tilde{\Gamma}) \geqq \delta\} = \boldsymbol{o}(n^{-s+1})$ *for all* $\delta > 0$.

Proof. In the first step we show that $\mathsf{P}\{\omega \mid \exists x \in \tilde{\Gamma}_n(\omega) \text{ with } d(x, \tilde{\Gamma}) \geqq \delta\} = \boldsymbol{o}(n^{-s+1})$ holds. Define $\Gamma_j := \text{cl}\{x \in G \mid \mathsf{E}g_j(x, Z) \leqq 0\}$. Now, as $\bigcap_{j \in \{1, \dots, q\}} (\Gamma_j + \tilde{\delta}B) \subset \tilde{\Gamma} + \delta B$ for sufficiently small $\tilde{\delta}$, to each $\delta > 0$ there exists a $\tilde{\delta}$ (depending on $\delta$ and the form of $\tilde{\Gamma}$) such that $d(x, \tilde{\Gamma}) \geqq \delta$ implies $d(x, \Gamma_{j_0}) \geqq \tilde{\delta}$ for at least one $j_0 \in \{1, \dots, q\}$.

Let $\delta > 0$ be fixed, choose $\tilde{\delta} = \tilde{\delta}(\delta)$ and define

$$C_{\tilde{\delta},j} := \{x \in R^p \mid d(x, \Gamma_j) \geqq \tilde{\delta}\} \cap G .$$

$\mathsf{E}g_j(\cdot, Z)$ being l.s.c., there exists an $\alpha > 0$ such that $\mathsf{E}g_j(x, Z) > \alpha \ \forall x \in C_{\tilde{\delta},j}$, $j \in \{1, \dots, q\}$. To every $x \in C_{\tilde{\delta},j}$ we find a neighbourhood $U_j(x)$ with

$$\mathsf{E} \inf_{\bar{x} \in U_j(x)} g_j(\bar{x}, Z) > \frac{\alpha}{2} \quad \text{(slight modification of Lemma 3 in [14])} .$$

The system $U_j(x)$, $x \in C_{\tilde{\delta},j}$, is an open cover of $C_{\tilde{\delta},j}$, hence there exists a finite cover $U_j(x_1^j), \dots, U_j(x_{k_j}^j)$.

Now, let $\omega$ be such that there is an $\hat{x} \in \tilde{\Gamma}_n(\omega)$ with $d(\hat{x}, \tilde{\Gamma}) \geqq \delta$. To $\hat{x}$ we find a $j_0 \in \{1, \dots, q\}$ such that $\hat{x} \in C_{\tilde{\delta},j_0}$ and an $l \in \{1, \dots, k_{j_0}\}$ with $\hat{x} \in U_{j_0}(x_l^{j_0})$ and $\mathsf{E} \inf_{\bar{x} \in U_{j_0}(x_l^{j_0})} g_{j_0}(\bar{x}, Z) > \frac{\alpha}{2}$.

In the following we investigate $\frac{1}{n} \sum_{i=1}^{n} g_{j_0}(\hat{x}, Z_i(\omega))$. Because of $\hat{x} \in \tilde{\Gamma}_n(\omega)$, $\frac{1}{n} \sum_{i=1}^{n} g_{j_0}(\hat{x}, Z_i(\omega)) \leqq 0$; hence

$$\frac{1}{n} \sum_{i=1}^{n} \inf_{\bar{x} \in U_{j_0}(x_l^{j_0})} g_{j_0}(\bar{x}, Z_i(\omega)) \leqq 0 .$$

Summarizing,

$$\begin{aligned} &\mathsf{P}\{\omega \mid \exists x \in \tilde{\Gamma}_n(\omega) \text{ with } d(x, \tilde{\Gamma}) \geqq \delta\} \\ &\leqq \sum_{j=1}^{q} \left[ \sum_{l=1}^{k_j} \mathsf{P}\left\{\omega \mid -\frac{1}{n} \sum_{i=1}^{n} \inf_{\bar{x} \in U_j(x_l^j)} g_j(\bar{x}, Z_i(\omega)) + \mathsf{E} \inf_{\bar{x} \in U_j(x_l^j)} g_j(\bar{x}, Z) > \frac{\alpha}{2}\right\} \right. . \end{aligned}$$

It remains to apply Lemma 2 in [3].

Now we turn to the investigation of

$$S = \{\omega \mid (\exists x \in \tilde{\Gamma} \text{ with } d(x, \tilde{\Gamma}_n(\omega)) \geqq \delta) \wedge (\sup_{\bar{x} \in \tilde{\Gamma}_n(\omega)} d(\bar{x}, \tilde{\Gamma}) < \delta)\} .$$

Let $\beta > 0$ and define

$$CI_\beta := \{x \in \tilde{\Gamma} \mid d(x, \bar{x}) \geqq \beta \ \forall \bar{x} \in bd\tilde{\Gamma}\} .$$

$\tilde{\Gamma}$ being compact, $CI_\beta$ is compact. Consequently we find an $\alpha_\beta$ with

$$\sup_{x \in CI_\beta} \mathsf{E}g_j(x, Z) < \alpha_\beta < 0 \quad \forall j \in \{1, \dots, q\} .$$

According to Lemma 3 in [14] to every $x \in CI_\beta$ there exists a neighbourhood $\tilde{U}_\beta(x)$ such that

$$\mathsf{E} \sup_{\bar{x} \in \tilde{U}_\beta(x)} g_j(\bar{x}, Z) < \frac{\alpha_\beta}{2} \quad j \in \{1, \dots, q\}.$$

The system $\tilde{U}_\beta(x)$, $x \in CI_\beta$, being an open cover of $CI_\beta$, we can select a finite cover $\tilde{U}_\beta(x_1^\beta), \dots, \tilde{U}_\beta(\bar{x}_{k_\beta}^\beta)$.

Let $\omega \in S$. Then, due to (P 3.1), there must be a $\beta_0$ and an $\tilde{x} \in \Gamma$, $\tilde{x} \notin \Gamma_n(\omega)$ such that $d(\tilde{x}, x) \geqq \beta_0 \; \forall x \in bd\Gamma$. Consequently $\tilde{x} \in CI_{\beta_0}$ and $\tilde{x} \in \tilde{U}_{\beta_0}(x_{i_0}^{\beta_0})$ for some $i_0 \in \{1, \dots, k_{\beta_0}\}$. Because of $\tilde{x} \notin \Gamma_n(\omega)$ there exists a $j_0$ with $\frac{1}{n} \sum_{i=1}^{n} g_{j_0}(x, Z_i(\omega)) > 0$, hence

$$\frac{1}{n} \sum_{i=1}^{n} \sup_{\bar{x} \in \tilde{U}_{\beta_0}(x_{i_0}^{\beta_0})} g_{j_0}(\bar{x}, Z_i(\omega)) > 0.$$

On the other hand

$$\mathsf{E} \sup_{\bar{x} \in \tilde{U}_{\beta_0}(x_{i_0}^{\beta_0})} g_{j_0}(\bar{x}, Z) < \frac{\alpha_{\beta_0}}{2} < 0.$$

Thus

$$\mathsf{P}(S) \leqq \sum_{j=1}^{q} \sum_{l=1}^{k_{\beta_0}} \mathsf{P} \left\{ \omega \mid \frac{1}{n} \sum_{i=1}^{n} \sup_{\bar{x} \in \tilde{U}_{\beta_0}(x_l^{\beta_0})} g_j(\bar{x}, Z_i(\omega)) - \mathsf{E} \sup_{\bar{x} \in \tilde{U}_{\beta_0}(x_l^{\beta_0})} g_j(\bar{x}, Z) > -\frac{\alpha_{\beta_0}}{2} \right\},$$

and we can apply Lemma 2 in [3]. ∎

If the $g_j(x, z)$ are bounded, Proposition 3 can be sharpened, making use of HOEFFDING's inequality (Theorem 2 in [10]).

Now we shall assume that $f$ is differentiable and $\Gamma_n = \Gamma$ is given by equality constraints. In this case we can make use of papers by ČIBISOV and derive an asymptotic expansion for the density of $\sqrt{n}\,(x_n - x_0)$, ($x_0$ is supposed to be unique.) For the description of necessary differentiability properties we shall use the following notations:

Let $\alpha = (\alpha_1, \dots, \alpha_j)$ be a vector with nonnegative integer components. Then

$$|\alpha| := \alpha_1 + \dots + \alpha_j,$$
$$\alpha! := \alpha_1! \cdot \dots \cdot \alpha_j!,$$
$$\vartheta^\alpha := \vartheta_1^{\alpha_1} \cdot \dots \cdot \vartheta_j^{\alpha_j} \quad (\vartheta \in R^j).$$

For functions $u: R^j \to R^1$ we define $u^{(\alpha)} = u$ for $|\alpha| = 0$ and

$$u^{(\alpha)}(\vartheta) := \frac{\partial^{|\alpha|}}{(\partial\vartheta)^\alpha} u(\vartheta) = \frac{\partial^{\alpha_1 + \dots + \alpha_j}}{\partial^{\alpha_1}\vartheta_1 \cdot \dots \cdot \partial^{\alpha_j}\vartheta_j} u(\vartheta) \text{ otherwise.}$$

If $u$ is a mapping into $R^l$, by $u^{(\alpha)}(\vartheta)$ we denote the vector $(u_1^{(\alpha)}(\vartheta), \dots, u_l^{(\alpha)}(\vartheta))^\mathsf{T}$.

Now we assume that every program

$$(P_n) \qquad \min_{x\in\Gamma} \frac{1}{n}\sum_{i=1}^{n} f(x, Z_i) \quad \text{with} \quad \Gamma=\{x \mid g_i(x)=0,\ i=1, \dots, q\}$$
$$(g_i \mid R^p \to R^1,\ q<p)$$

has a (random) solution $x_n$. Furthermore, we claim that there exist vectors of LAGRANGE multipliers $\pi_n$, such that $(x_n, \pi_n)$ satify the optimality equations

$$L_l^{(n)}(x_n, \pi_n)=0, \quad l=1, \dots, p+q, \quad \mathsf{P}\text{-a.s.}$$

where

$$\begin{aligned} L_l^{(n)}(x(\omega), \pi(\omega)) &= \frac{1}{n}\sum_{i=1}^{n} L_l(x(\omega), \pi(\omega), Z_i(\omega)) \\ &= \frac{1}{n}\sum_{i=1}^{n} f^{(\tilde{\alpha}_l)}(x(\omega), Z_i(\omega)) + \sum_{j=1}^{q} g_j^{(\tilde{\alpha}_l)}(x(\omega))\cdot \pi_j(\omega) \\ &\quad l=1, \dots, p \quad (\tilde{\alpha}_l=(\delta_{1l}, \dots, \delta_{pl})),\ \delta_{il} \text{ is KRONECKER's delta}) \\ L_l^{(n)}(x(\omega), \pi(\omega)) &= g_{l-p}(x(\omega)), \quad l=p+1, \dots, p+q\,. \end{aligned}$$

A corresponding assumption is to be fulfilled with respect to (P):

$$L_l^0(x_0, \pi_0)=0, \quad l=1, \dots, p+q\,,$$

where

$$L_l^0(x_0, \pi_0)=[\mathsf{E}f(x_0, Z)]^{(\tilde{\alpha}_l)} + \sum_{j=1}^{q} g_j^{(\tilde{\alpha}_l)}(x_0)\cdot \pi_{0,j}, \quad l=1, \dots, p\,,$$
$$L_l^0(x_0, \pi_0)=g_{l-p}(x_0), \quad l=p+1, \dots, p+q\,.$$

Finally, we introduce the notations $\vartheta=(\vartheta_l)_{l=1}^{p+q}:=(x^{(1)}, \dots, x^{(p)}, \pi_1, \dots, \pi_q)^{\mathsf{T}}=\begin{pmatrix} x \\ \pi \end{pmatrix}$,

$$\vartheta_n:=\begin{pmatrix} x_n \\ \pi_n \end{pmatrix}, \quad n=0, 1, \dots, L^{(n)}(\vartheta):=(L_l^{(n)}(\vartheta))_{l=1}^{p+q}\,,$$

$$\tilde{A}=\nabla_{xx}^2 \mathsf{E}f(x_0, Z) + \sum_{j=1}^{q} \nabla_{xx}^2 g_j(x_0)\cdot \pi_{0,j}\,,$$

$$A:=\begin{bmatrix} \tilde{A} & \nabla_x g_1(x_0) & \dots & \nabla_x g_q(x_0) \\ \nabla_x^{\mathsf{T}} g_1(x_0) & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \nabla_x^{\mathsf{T}} g_q(\bar{x}_0) & 0 & \dots & 0 \end{bmatrix}$$

**Theorem 4.** *Let in addition to the assumptions* (A 1.1)–(A 1.3) *of Theorem* 1 $\left(\textit{with } s=\frac{k+1}{2},\ k\in N,\ k>1\right)$ *the following assumptions be satisfied:*

(A 4.1) *There exists a neighbourhood $\hat{U}(x_0)$ of $x_0$ where the derivatives*

$$f^{(\alpha)}(x, z):=\frac{\partial^{|\alpha|}}{(\partial x)^\alpha} f(x, z) \quad \textit{and}$$

$$g_i^{(\alpha)}(x):=\frac{\partial^{|\alpha|}}{(\partial x)^\alpha} g_i(x), \quad i=1, \dots, q, \quad 1\le|\alpha|\le k+1, \quad k>1\,,$$

*for $\mathsf{P}_Z$-almost all $z$ exist and are continuous.*

(A 4.2) $\sup_{x\in\hat{U}(x_0)} \mathsf{E}\,|f^{(\alpha)}(x,Z)|^{k+1}<\infty, \quad |\alpha|\leqq k+1.$

(A 4.3) *To every $x\in\Gamma$ there exists a neighbourhood $\tilde{U}(x)$ such for all $x_1, x_2\in\tilde{U}(x)$ and $|\alpha|=k+1$*

$$|f^{(\alpha)}(x_1\ z)-f^{(\alpha)}(x_2,z)|\leqq R_f(x,z)\cdot\|x_1-x_2\|\,,$$
$$|g_i^{(\alpha)}(x_1)-g_i^{(\alpha)}(x_2)|\leqq R_g(x)\cdot\|x_1-x_2\| \quad \mathsf{P}_Z\text{-a.e.},\quad i=1,\dots,q\,,$$

*and* $\mathsf{E}\big(R_f(x,Z)^{\frac{k+1}{2}}\big)<\infty$.

(A 4.4) *The matrices $A$ and $\tilde{A}$ are non-singular.*

(A 4.5) *The vector* $\vec{f}(x,Z):=\begin{pmatrix}\vec{f}^{(\hat{\alpha}_1)}(x,Z)\\ \vdots\\ \vec{f}^{(\hat{\alpha}_k)}(x,Z)\end{pmatrix}$ *with* $\vec{f}^{(\hat{\alpha}_i)}:=(f^{(\hat{\alpha}_i)})_{|\hat{\alpha}_i|=i}$ *allows a representation of the form*

$$\vec{f}(x,Z)=\Phi(x)\cdot\begin{pmatrix}\psi_1(x,Z)\\ \vdots\\ \psi_l(x,Z)\end{pmatrix}+\check{\Phi}(x)\,,$$

*where $\Phi(x)$ and $\check{\Phi}(x)$ are given matrices, and the probability distribution $\mathsf{P}_\psi$ of* $\begin{pmatrix}\psi_1(x_0,Z)\\ \vdots\\ \psi_1(x_0,Z)\end{pmatrix}$ *satisfies the following condition* (D):

$\mathsf{P}_\psi(A)=aW_1(A)+(1-a)\,W_2(A)$, $A\in\mathfrak{B}^1$, $a>0$, *with probability measures $W_1$ and $W_2$ such that for a certain $n_0\geqq 1$ the ($n_0$-fold) convolution $W_1^{*n_0}$ has a density function $p_{n_0}(\psi)\leqq c<\infty$.*

(A 4.6) *The covariance matrix $\Phi_1(x_0)\,\Sigma\Phi_1(x_0)^{\mathsf{T}}$ of $\vec{f}^{(\hat{\alpha}_1)}(x_0,Z)$ is non-singular.*

*Then $\sqrt{n}\,(x_n-x_0)$ has the property that*

$$\mathsf{P}\,\{\omega\mid\sqrt{n}\,(x_n(\omega)-x_0)\in M\}=\int_M\left[1+\sum_{j=1}^{k-1}n^{-\frac{j}{2}}Q_j(y)\right]\varphi(y)\,\mathrm{d}y+\mathbf{o}(n^{-\frac{k-1}{2}}),$$
$$(n\to\infty)\,,$$

*uniformly with respect to the convex* BOREL *subsets $M\subset R^p$. The $Q_j$ are computable polynomials, the components of which depend on $\Phi(x_0)$ and the moments of* $\begin{pmatrix}\psi_1(x_0,Z\\ \vdots\\ \psi_l\ x_0,Z\end{pmatrix}$, *and $\varphi$ is the density function of the normal distribution with parameters 0 and $\tilde{A}^{-1}\Phi_1(x_0)\,\Sigma\Phi_1(x_0)^{\mathsf{T}}\,(\tilde{A}^{\mathsf{T}})^{-1}$.*

Proof. We make use of Theorem 1 (of this paper) and the Theorem in [3] and obtain

$$\sqrt{n}\,(\vartheta_n-\vartheta_0)=h_1+\frac{1}{\sqrt{n}}h_2+\dots+\frac{1}{(\sqrt{n})^{k-1}}h_k+\frac{1}{(\sqrt{n})^k}\varrho_n\,.$$

Here the $h_j$, $j=1,\dots,k$, are vectors, the components of which are homogeneous polynomials of the order $j$ of

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n[L_l(\vartheta_0,Z_i)-\mathsf{E}L_l(\vartheta_0,Z)]^{(\alpha)}\quad|\alpha|\leqq j-1,\quad l=1,\dots,p+q\,,$$

and $(\varrho_n)_{n\in N}$ is a sequence of random vectors such that for an arbitrary sequence

$$(v_n)_{n\in N} \quad \text{with} \quad v_n \in R^1, \quad v_n \to \infty, \quad \frac{v_n}{(\sqrt{n})^{k+1}} \to 0, \quad \frac{v_n}{(\log n)^{k+1}} \to \infty,$$

$$\mathsf{P}\,\{\omega \mid \|\varrho_n(\omega)\| > v_n\} = \boldsymbol{o}(n^{-\frac{k-1}{2}}).$$

Now

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} [L_l(\vartheta_0, Z_i) - \mathsf{E} L_l(\vartheta_0, Z)]^{(\alpha)}$$
$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [f^{(\alpha_l)}(x_0, Z_i) - \mathsf{E} f^{(\alpha_l)}(x_0, Z)]^{(\alpha)}, \quad l = 1, \ldots, p; \quad (\alpha_l) = (\delta_{1l}, \ldots, \delta_{pl}),$$

hence the components of $h_j$ are in particular polynomials of

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} [f^{(\alpha_l)}(x_0, Z_i) - \mathsf{E} f^{(\alpha_l)}(x_0, Z)]^{(\alpha)}, \quad |\alpha| \in \{1, \ldots, k-1\},$$

and – because of (A 4.5) and the uniform integrability of $f^{(\alpha)}(x, z)$ with respect to $x \in U(x_0)$ – polynomials of

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_u(x_0, Z_i) - \mathsf{E}\psi_u(x_0, Z), \quad u = 1, \ldots, l.$$

Now we can apply Theorem 2 in [2] by putting $S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi(x_0, Z_i) - \mathsf{E}\psi(x_0, Z)$, and obtain the assertion of our theorem. ■

**Remarks:** 1. Deterministic inequality constraints in the description of $\Gamma$ can be allowed if the strict complementarity condition holds.

2. We use condition (A 4. 5), because the probability distribution of $\vec{f}(x_0, Z)$ in general does not satisfy condition (D). However, it is often possible to find a vector $\psi(x_0, Z)$ that fulfils the sufficient conditions for (D), given in [2], part 8.

3. If the covariance matrix of $\vec{f}^{(\hat{\alpha}_1)}(x_0, Z)$ is singular, we find a linear transformation $T: R^p \to R^{l_0}$ $(l_0 < p)$ such that $\sqrt{n}T(x_n - x_0)$ converges to a random variable with non-singular normal distribution. Then we can derive an asymptotic expansion for the density of $\sqrt{n}T(x_n - x_0)$.

## 3. Estimated Parameters

We consider the problems

$$(\mathrm{P}_y) \qquad \min_{x \in \Gamma(y)} f(x, y) \quad \text{with} \quad \Gamma(y) = \{x \in R^p \mid g_i(x, y) \leqq 0,\ i = 1, \ldots, q\},$$

where $y$ belongs to a neighbourhood $U(\eta)$ of $\eta \in Y$. To simplify notations, let $X_0 := \hat{X}(\eta)$.

**Proposition 4:** *Let the following assumptions be satisfied:*

(P 4.1) *The multifunction $\hat{X}(\cdot)$ is u.s.c. (according to* HAUSDORFF*) in $\eta$.*

(P 4.2) *There exists a sequence of estimation functions $(Y^n)_{n\in N}$ for $\eta$ such that for all $\delta>0$ and an $r>0$*

$$\mathsf{P}\,\{\omega \mid \|Y^n(\omega)-\eta\|\geqq\delta\}=\boldsymbol{o}(n^{-r}) \quad (n\to\infty)\,.$$

*Then $\mathsf{P}\,\{\omega \mid \exists\hat{x}\in\hat{X}(Y^n(\omega))$ with $d(\hat{x}, X_0)\geqq\varepsilon\}=\boldsymbol{o}(n^{-r})$ for all $\varepsilon>0$.*

Proof. Let $\varepsilon$ be fixed and $y$ be such that there is an $\hat{x}\in\hat{X}(y)$ with $d(\hat{x}, X_0)\geqq\varepsilon$. Consequently, $\hat{X}(y)\not\subset X_0+\frac{\varepsilon}{2}B$. Because of (P 4.1) we then find a $\delta(\varepsilon)>0$ with $\|y-\eta\|\geqq\delta$. That means

$$\{y \mid \exists\hat{x}\in\hat{X}(y) \text{ with } d(\hat{x}, X_0)\geqq\varepsilon\}\subset\{y \mid \|y-\eta\|\geqq\delta\}\,,$$

and the proposition is proved. ■

Sufficient conditions for (P 4.2) are given for instance in [2], [3] and [14]. As for the upper semicontinuity of $\hat{X}(\cdot)$ compare [1].

There are several stability results in deterministic programming for the case of twice differentiable $f$ and $g_i$ or at least LIPSCHITZ continuous $\nabla_x f$ (cf. [4], [9], [20]). The authors mentioned obtain LIPSCHITZ continuity of $\hat{x}(\cdot)$ in a neighbourhood of $\eta$ and derive (in part) bounds for the LIPSCHITZ constant. In view of our problem this, however, would not yield much more than a result like Proposition 4, for in general we don't have exact bounds for the difference between $Y^n$ and $\eta$.

Now we turn to the derivation of an asymptotic expansion for the density of $\sqrt{n}(\hat{x}(Y^n)-x_0)$. As in part I, we assume that we have equality constraints only, $x_0$ is unique, and for every $y\in Y$ the program $(P_y)$ has a solution $\hat{x}(y)$. Furthermore we claim that there exists a vector of LAGRANGE multipliers $\hat{\pi}(y)$ such that $(\hat{x}(y), \hat{\pi}(y))$ satisfies the optimality equations

$$L_l(y, \hat{x}(y), \hat{\pi}(y))=0 \quad l=1, \ldots, p+q$$

where

$$L_l(y, x_l, \pi)=f^{(\alpha l)}(x, y)+\sum_{j=1}^{q} g_j^{(\alpha l)}(x, y)\cdot\pi_j, \quad l=1, \ldots, p\,,$$
$$L_l(y, x, \pi)=g_{l-p}(x, y), \quad l=p+1, \ldots, p+q\,.$$

Finally we introduce the notations

$$\vartheta=(\vartheta_l)_{l=1}^{p+q}:=\begin{pmatrix}x\\ \pi\end{pmatrix}, \hat{\vartheta}(y):=\begin{pmatrix}\hat{x}(y)\\ \hat{\pi}(y)\end{pmatrix}, \quad \vartheta_0:=\begin{pmatrix}\hat{x}(\eta)\\ \hat{\pi}(\eta)\end{pmatrix}=\begin{pmatrix}x_0\\ \pi_0\end{pmatrix},$$
$$L(y, \vartheta):=(L_l(y, \vartheta))_{l=1}^{p+q}\,,$$
$$\tilde{A}:=\nabla_{xx}^2 f(x_0, \eta)+\sum_{j=1}^{q}\nabla_{xx}^2 g_j(x_0, \eta)\cdot\pi_{0,j}\,,$$

$$A := \nabla_{\vartheta} L(\eta, \vartheta_0) = \begin{bmatrix} \tilde{A} & \nabla_x g_1(x_0, \eta) & \dots & \nabla_x g_q(x_0, \eta) \\ \nabla_x^{\mathsf{T}} g_1(x_0, \eta) & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \nabla_x^{\mathsf{T}} g_q(x_0, \eta) & 0 & \dots & 0 \end{bmatrix},$$

$$\tilde{B} := \left[\frac{\partial^2}{\partial y_j \partial x_i} f(x_0, \eta) + \sum_{l=1}^{q} \frac{\partial^2}{\partial y_j \partial x_i} \; g_l(x_0, \eta) \cdot \pi_{0,l}\right]_{i,j},$$

$$B := \nabla_y L(\eta, \vartheta_0) = \begin{bmatrix} \tilde{B} \\ \nabla_y^{\mathsf{T}} g_1(x_0, \eta) \\ \vdots \\ \nabla_y^{\mathsf{T}} g\ (x_0, \eta) \end{bmatrix}.$$

Notations that are not explained here are defined as in part II. In the following we shall distinguish between derivatives with respect to $\vartheta$, denoted by the subscript $(\alpha)$, and derivatives with respect to $y$, denoted by $(\beta)$; $(\alpha, \beta)$ we mean the vector $(\alpha_1, \dots, \alpha_{p+q}, \beta_1, \dots, \beta_m)$.

**Theorem 5:** *Let the following assumptions be satisfied:*

(A 5.1) *There exists a neighbourhood $U(x_0) \times U(\eta)$ of $(x_0, \eta)$ where the derivatives*

$$f^{(\alpha,\beta)}(x, y) := \frac{\partial^{|(\alpha,\beta)|}}{(\partial x)^\alpha (\partial y)^\beta} f(x, y), \quad |(\alpha, \beta)| = k+1, \quad |\beta| \leqq k,$$

*and*

$$g_i^{(\alpha,\beta)}(x, y) := \frac{\partial^{|(\alpha,\beta)|}}{(\partial x)^\alpha (\partial y)^\beta} g_i(x, y), \quad |(\alpha, \beta)| = k, \quad i = 1, \dots, q, \quad k > 1,$$

*exist and are* LIPSCHITZ *continuous.*

(A 5.2) *$A$ and $\tilde{A}$ are non-singular.*

(A 5.3) *The random vector $\sqrt{n}\,(Y^n - \eta)$ allows a respresentation*

$$\sqrt{n}\,(Y^n - \eta) = H_0 \cdot S_n + \sum_{j=1}^{k-1} \left(\frac{1}{\sqrt{n}}\right)^j H_j(S_n) + \left(\frac{1}{\sqrt{n}}\right)^k \varrho_n,$$

*where $S_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} W_i$ and the $W_i$ are independent identically distributed random vectors, the distribution of which satisfies condition* D) (c.p. 281); *$W_1$ has finite moments of the order $k+1$, $\mathsf{E} W_1 = 0$, and the covariance matrix $\Sigma$ of $W_1$ is nonsingular. $H_0$ is a matrix; the $H_j(S_n)$ are vectors the components of which are polynomials; $(\varrho_n)_{n \in N}$ is a sequence of random vectors such that for an arbitrary sequence $(v_n)_{n \in N}$ with $v_n \in R^1$, $\frac{v_n}{n^\mu} \to \infty$ $(\mu > 0)$*

$$\mathsf{P}\,\{\omega \mid \|\varrho_n(\omega)\| > v_n\} = \boldsymbol{o}(n^{-\frac{k-1}{2}})$$

*holds.*

(A 5.4) *The matrix $\tilde{B} H_0 \Sigma H_0^{\mathsf{T}} \tilde{B}^{\mathsf{T}}$ is non-singular. Then $\sqrt{n}\,(\hat{x}(Y^n) - x_0)$ has the property that*

$$\mathsf{P}\,\{\omega \mid \sqrt{n}\,(\hat{x}(Y^n(\omega)) - x_0) \in M\} = \int_M \left[1 + \sum_{j=1}^{k-1} n^{-\frac{j}{2}} Q_j(y)\right] \varphi(y)\,\mathrm{d}y + \boldsymbol{o}(n^{-\frac{k-1}{2}})$$

*uniformly with respect to the convex* BOREL *subsets $M \subset R^p$. The $Q_j$ are computable polynomials the coefficients of which depend on the coefficients of $H_l$ and $L^{(\alpha,\beta)}(\eta, \vartheta_0)$; $\varphi$ is the density function of the normal distribution with parameters* 0 *and* $\tilde{A}^{-1}\tilde{B}H_0\Sigma H_0^{\mathsf{T}}\tilde{B}^{\mathsf{T}}\tilde{A}^{\mathsf{T}-1}$.

As pointed out by ČIBISOV [2], [3], condition (A 5.3) is fulfilled by a broad class of estimation functions.

Before we prove the theorem we shall give a lemma.

**Lemma 2:** *Let condition* (A 5.3) *be satisfied and let* $(u_n)_{n\in N}$ *be a sequence with* $u_n \in R^1$, $\frac{u_n}{n^\mu} \to \infty$ $(\mu > 0)$. *Then*

$$\mathsf{P}\,\{\omega \mid \sqrt{n}\,\|Y^n(\omega) - \eta\| > u_n\} = \boldsymbol{o}(n^{-\frac{k-1}{2}})\,.$$

Proof. Because of (A 5.3) the assumptions of Theorem 2 in [2] are fulfilled. Hence

$$\mathsf{P}\,\{\omega \mid \sqrt{n}\,(Y^n(\omega) - \eta) \in M\} = \int_M \left[1 + \sum_{j=1}^{k-1} n^{-\frac{j}{2}}\hat{Q}_j(y)\right] \hat{\varphi}(y)\,\mathrm{d}y + \boldsymbol{o}(n^{-\frac{k-1}{2}})$$

uniformly with respect to all convex sets $M \in \mathfrak{B}^m$. Taking into consideration that

$$\int_{\{y \mid \|y-\eta\| > u_n\}} \left[1 + \sum_{j=1}^{k-1} n^{-\frac{j}{2}}\hat{Q}_j(y)\right] \hat{\varphi}(y)\,\mathrm{d}y = \boldsymbol{o}(n^{-r})$$

for arbitrary $r \in R^1$, the estimation immediately follows. ■

Proof of Theorem 5. Because of Lemma 2 we can concentrate our investigation on $U(x_0) \times U(\eta)$. According to the implicit-function theorem there is a (bounded) neighbourhood $U_1(\eta) \subset U(\eta)$ such that for arbitrary $y \in U_1(\eta)$ the system $L(y, \vartheta) = 0$ has a unique solution $(\hat{x}(y), \hat{\pi}(y))$ and the components of $(\hat{x}(\cdot), \hat{\pi}(\cdot))$ are $k$-times continuously differentiable functions of $y$. Consequently we have the TAYLOR's expansion

$$\hat{x}_l(y) - x_{0,l} = \sum_{1 \le |\alpha| \le k} \frac{1}{\alpha!}\,\hat{x}_l^{(\alpha)}(\eta)\,(y - \eta)^\alpha + \sum_{|\alpha| = k} \frac{1}{\alpha!}\,[\hat{x}_l^{(\alpha)}(y^{(l)}) - \hat{x}_l^{(\alpha)}(\eta)]\,(y - \eta)^\alpha$$

with $\|y^{(l)} - \eta\| \le \|y - \eta\|$, $l = 1, \ldots, p$.

Now let be $Y^n(\omega) \in U_1(\eta)$. Then, according to (A 5.3),

$$\begin{aligned}&\sqrt{n}\,(\hat{x}\,(Y^n(\omega) - x_0)\\ &= \sum_{1 \le |\alpha| \le k} \frac{1}{\alpha!}\,\hat{x}^{(\alpha)}(\eta)\,\tau^{|\alpha|-1}\left(\sum_{j=0}^{k-1} \tau^j \cdot H_j(S_n(\omega)) + \tau^k \varrho_n(\omega)\right)^\alpha + \tau^k \zeta(Y^n(\omega), \eta)\,,\end{aligned}$$

where $\tau = \frac{1}{\sqrt{n}}$, $\zeta(Y^n(\omega), \eta) = \sqrt{n}^{k+1} \sum_{|\alpha|=k} \frac{1}{\alpha!}\,[\hat{x}^{(\alpha)}(\hat{Y}^n(\omega)) - \hat{x}^{(\alpha)}(\eta)]\,(Y^n(\omega) - \eta)^\alpha$ and $\hat{x}^{(\alpha)}(\hat{Y}^n(\omega))$ stands for $\begin{pmatrix}\hat{x}_1^{(\alpha)}(Y^{n,(1)}(\omega))\\ \vdots\\ \hat{x}_p^{(\alpha)}(Y^{n,(p)}(\omega))\end{pmatrix}$, $\|Y^{n,(l)}(\omega) - \eta\| \le \|Y^n(\omega) - \eta\|$.

Now we rearrange the terms on the right hand side in the following way:

$$\sqrt{n}\,(\hat{x}(Y^n)-x_0)=\sum_{j=0}^{k-1}\tau^j G_j(S_n)+\tau^k\xi_n(S_n,\varrho_n)+\tau^k\zeta(Y^n,\eta)\,. \qquad (*)$$

Here the components of the $G_j$ are polynomials again.

We shall show that the summands on the right hand side of $(*)$ satisfy the assumptions of Theorem 2 in [2]. In the first step we consider $\zeta(Y^n,\eta)$. We have

$$\left(\text{with}\quad \tilde{A}(y)=\left[\frac{\partial L_j}{\partial x_k}(\hat{x}(y),y)\right]_{j,k=1,\dots,p}\right)$$

$$\left[\frac{\partial \hat{x}_j}{\partial y_k}(y)\right]_{\substack{j=1,\dots,p\\ k=1,\dots,m}}=-\tilde{A}(y)^{-1}\cdot\left[\frac{\partial L_j}{\partial y_k}(\hat{x}(y),y)\right]_{\substack{j=1,\dots,p\\ k=1,\dots,m}}.$$

Hence $\frac{\partial \hat{x}_i}{\partial y_k}(y)=-\sum_{j=1}^{p}\frac{\tilde{A}_{ji}(y)}{\det\tilde{A}(y)}\cdot\frac{\partial L_j}{\partial y_k}(\hat{x}(y),y)$, where the $\tilde{A}_{ij}(y)$ denote the algebraic adjuncts of $\tilde{A}(y)$. Therefore $\hat{x}_i^{(\alpha)}(y)$, $|\alpha|=k$, can be written as $\hat{x}_i^{(\alpha)}(y)=\sum_{j=1}^{p}\frac{\varkappa_{ij}(y)}{(\det\tilde{A}(y))^{2k-1}}$. The $\varkappa_{ij}(y)$ are sums of products and the factors are partial derivatives of $L$ up to the order $k$.

As $\det\tilde{A}(y)$ is a continuous function of the elements of $\tilde{A}(y)$ and $\det A(\eta)\neq 0$, we find a neighbourhood $U_2(\eta)\subset U_1(\eta)$ and a $\lambda>0$ such that

$$(\det\tilde{A}(y))^{2k-1}>\lambda\quad\forall y\in U_2(\eta)\,.$$

Hence, because of the Lipschitz continuity of the partial derivatives of $f$ and $g_i$, we find a constant $c$ such that

$$|\hat{x}_i^{(\alpha)}(y)-\hat{x}_i^{(\alpha)}(\eta)|\leqq c\,\|y-\eta\|\quad\forall y\in U_2(\eta)\,.$$

Now, for $Y^n(\omega)\in U_2(\eta)$, $\|\zeta(Y^n(\omega),\eta)\|\leqq c_1\sqrt{n}^{k+1}\|Y^n(\omega)-\eta\|^{k+1}$. Consequently,

$$\{\omega\mid\|\zeta(Y^n(\omega),\eta)\|>n^{\frac{1}{4}}\}$$

$$\subset\{\omega\mid Y^n(\omega)\notin U_2(\eta)\}\cup\left\{\omega\mid\sqrt{n}\,\|Y^n(\omega)-\eta\|>\frac{n^{\frac{1}{4(k+1)}}}{c_2}\right\}.$$

Obviously

$$\{\omega\mid Y^n(\omega)\notin U_2(\eta)\}\subset\{\omega\mid\sqrt{n}\,\|Y^n(\omega)-\eta\|>c_3\sqrt{n}\}$$

for a suitable $c_3$, thus (Lemma 2)

$$\mathsf{P}\{\omega\mid Y^n(\omega)\notin U_2(\eta)\}=\boldsymbol{o}(n^{-\frac{k-1}{2}})\,.$$

As $\mathsf{P}\left\{\omega\mid\sqrt{n}\,\|Y^n(\omega)-\eta\|>\frac{n^{\frac{1}{4(k+1)}}}{c_2}\right\}=\boldsymbol{o}(n^{-\frac{k-1}{2}})$ too, we have

$$\mathsf{P}\{\omega\mid\|\zeta(Y^n(\omega),\eta)\|>n^{\frac{1}{4}}\}=\boldsymbol{o}(n^{\frac{k-1}{2}})\,.$$

Now we shall show that

$$\mathsf{P}\{\omega\mid\|\xi_n(S_n(\omega),\varrho_n(\omega))\|>n^{\frac{1}{4}}\}=\boldsymbol{o}(n^{-\frac{k-1}{2}})\,.$$

$\xi_n(S_n, \varrho_n)$ can be written in the following form:

$$\xi_n(S_n, \varrho_n) = \sum_{j=k}^{k^2-1} \tau^{j-k} \bar{G}_j(S_n) + \sum_{|\gamma|=1}^{k} \tau^{k(|\gamma|-1)} \varrho_n^\gamma + \sum_{j=1}^{k^2+k-2} \tau^{j-k} \sum_{\substack{1 \leq |\gamma|, 0 \leq r \\ r+(k+1)|\gamma|=j}} \hat{G}_r(S_n) \varrho_n^\gamma ,$$

where the $\bar{G}_j$ and $\hat{G}_r$ are vectors the components of which are polynomials.

We have $|\varrho_n^\gamma| \leq \|\varrho_n\|^{|\gamma|}$, hence for $n > n_0$ and $\tilde{v}_n = n^{\frac{1}{8}}$

$$\mathsf{P}\{\omega \mid |\tau^{k(|\gamma|-1)} \varrho_n(\omega)^\gamma| > \tilde{v}_n\} \leq \mathsf{P}\{\omega \mid \|\varrho_n(\omega)\| > \tilde{v}_n^{\frac{1}{|\gamma|}}\} = \boldsymbol{o}(n^{-\frac{k-1}{2}}) .$$

Furthermore, $\mathsf{P}\{\omega \mid \|\tau^{j-k} \bar{G}_j(S_n(\omega))\| > \tilde{v}_n\} \leq \mathsf{P}\left\{\omega \mid \|S_n(\omega)\| > \frac{\tilde{v}_n^{\frac{1}{l}}}{c_4}\right\}$,

where $l$ denotes the maximal degree of the polynomials $\bar{G}_j$.

$S_n$ can be regarded as a "special case" of $\sqrt{n}\,(Y^n - \eta)$, thus we can apply Lemma 2 and obtain

$$\mathsf{P}\{\omega \mid \|\tau^{j-k} \bar{G}_j(S_n(\omega))\| > \tilde{v}_n\} = \boldsymbol{o}(n^{-\frac{k-1}{2}}) .$$

Finally, $\{\omega \mid \|\tau^{j-k} \hat{G}_r(S_n(\omega)) \cdot \varrho_n(\omega)^\gamma\| > \tilde{v}\} \subset \{\omega \mid \|\tau^{j-k|\gamma|} \hat{G}_r(S_n(\omega))\| > \tilde{v}_n\}$
$\cup \{\omega \mid |\tau^{k(|\gamma|-1)} \varrho_n(\omega)^\gamma| > \tilde{v}_n\}$, hence

$$\mathsf{P}\{\omega \mid \|\tau^{j-k} \hat{G}_r(S_n(\omega)) \cdot \varrho_n(\omega)^\gamma\| > \tilde{v}_n^2\} = \boldsymbol{o}(n^{-\frac{k-1}{2}})$$

and

$$\mathsf{P}\{\omega \mid \|\xi_n(S_n(\omega), \varrho_n(\omega))\| > n^{\frac{1}{4}}\} = \boldsymbol{o}(n^{-\frac{k-1}{2}}) .$$

It remains to apply Theorem 2 in [2]. ■

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