STABILITY THEORY FOR COUNTABLY INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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Abstract. New stability results for a class of countably infinite systems of differential equations are established. We consider those systems which may be viewed as an interconnection of countably infinitely many free or isolated subsystems. Throughout, the analysis is accomplished in terms of simpler subsystems and in terms of the system interconnecting structure. This approach makes it often possible to circumvent difficulties usually encountered in the application of the Lyapunov approach to complex systems with intricate structure. Both scalar Lyapunov functions and vector Lyapunov functions are used in the analysis. The applicability of the present results is demonstrated by means of several motivating examples, including a *neural model.*

1. **Introduction.** In the present paper we establish new stability results for a class of countably infinite systems of ordinary differential equations. We consider those systems which may be viewed as an interconnection of countably many free or isolated subsystems (which are described by ordinary differential equations defined on finite dimensional spaces). Such systems are often called interconnected systems, composite systems, decentralized systems, large scale systems, and the like (see Michel and Miller [5]). As in [5], our objective will be to analyze interconnected systems in terms of their simpler subsystems and in terms of their interconnecting structure. In this way it is frequently possible to circumvent difficulties usually encountered when the Lyapunov approarch is applied to complex systems with intricate structure.

For existing results dealing with stability and well posedness of countably infinite systems of differential equations, the reader is referred to Persidskii [7, 8], Leung et al. [4], Shaw [10, 11] and Deimling [2].

This paper consists of several parts. First we introduce initial value

problems which are characterized by countably infinite systems of equa-
tions, which are well posed, and which may be viewed as interconnected
systems. Next, we establish several stability results which are obtained
via scalar Lyapunov functions (consisting of a weighted sum of Lyapunov
functions for the free subsystems). Next, we utilize infinite vector
Lyapunov functions in our analysis (whose components are Lyapunov
functions for the isolated subsystems). Throughout, several motivating
examples are provided, including a neural model, to demonstrate the
applicability of the method advanced.

2. **Countably infinite systems of differential equations.** A. *Initial
value problem.* In this paper we are concerned with the initial value
problem

(1) $\dot{z}_n = h_n(t, x)$, $z_n(0) = c_n$

for $n = 1, 2, 3, \cdots$. Here x is the infinite dimensional column vector
$x = (z_1^T, z_2^T, \cdots, z_n^T, \cdots)^T \in R^\omega$, $z_n \in R^{m_n}$, $D \subset R^\omega$ and $h_n: R^+ \times D \to R^{m_n}$. The
infinite product $R^\omega = R^{m_1} \times R^{m_2} \times \cdots \times R^{m_n} \times \cdots$ is given the usual
product topology. Since this topology is equivalent to introducing the
metric

(2) $\rho(x, \bar{x}) = \sum\limits_{n=1}^{\infty} |z_n - \bar{z}_n|(1 + |z_n - \bar{z}_n|)^{-1}2^{-n}$

then R^v is a convex Fréchet space (see, e.g., [9]).

A *solution* of (1) is a function $x: [0, b] \to D$ for some $b > 0$ such that
$z_n \in C_n^1[0, b]$, $z_n(0) = c_n$ and $\dot{z}_n(t) = h_n(t, x(t))$ for all $t \in [0, b]$ and for all
$n = 1, 2, 3, \cdots$.

REMARK 1. An alternate way to view the above initial value problem
is to fix a Banach space X of real sequences and to think of (1)

(1') $\dot{x} = h(t, x)$, $x(0) = c$

as a problem in X. In this case an X-solution is a function $x: [0, b] \to X$
which is continuously differentiable in the X-sense and satisfies (1').
Clearly, an X-solution for (1') is also a solution in our sense (for (1)).
However, in the present paper we shall not concern ourselves with
Banach space settings for this initial value problem. Our approach has
the advantage that well-posedness (i.e., existence, uniqueness and con-
tinuation results for (1)) poses no problems while it may pose serious
difficulties in a Banach space setting (for (1')). The disadvantage of our
approach is that before applying stability results, it is usually necessary
to argue separately that the solutions $x(t)$ of (1) remain in a bounded

set D.

B. *Interconnected systems.* Frequently we may regard system (1) as an *interconnected system* or a *composite system* of the form

$$(\Sigma) \qquad \dot{z}_n = f_n(t, z_n) + g_n(t, x) , \qquad z_n(0) = c_n ,$$

$n = 1, 2, 3, \cdots$, where in the notation of (1), $h_n(t, x) \overset{\Delta}{=} f_n(t, z_n) + g_n(t, x)$. We view (Σ) as an interconnection of countably infinitely many *isolated* or *free subsystems* described by equations of the form

$$(\mathscr{S}_n) \qquad \dot{w}_n = f_n(t, w_n) ,$$

$n = 1, 2, \cdots$. The terms $g_n(t, x)$, $n = 1, 2, \cdots$, comprise the *interconnecting structure* of system (Σ).

C. *Well-posedness.* We shall assume that for some constants $r_n > 0$ the set $D = D_k$ for some $k > 0$ where $D_k = \{x = (z_1^T, z_2^T, \cdots)^T \in R^{\omega} : |z_n| \leq kr_n$ for $n = 1, 2, 3, \cdots\}$. We shall also make the following additional assumptions.

(A-1): $f_n : R^+ \times R^{m_n} \to R^{m_n}$, $f_n(t, 0) = 0$ for all $t \geq 0$, f_n is locally Lipschitz continuous with Lipschitz constant independent of t and for $|z_n| \leq kr_n$, this Lipschitz constant is $L_{n0}(k)$.

(A-2): $g_n : R^+ \times R^{\omega} \to R^{m_n}$, $g_n(t, 0) = 0$ for all $t \geq 0$, and for any $k > 0$ there exist constants $L_{nj}(k) \geq 0$ such that $\sum_{j=1}^{\infty} L_{nj}(k) r_j < \infty$ and such that $|g_n(t, x) - g_n(t, x^*)| \leq \sum_{j=1}^{\infty} L_{nj}(k) |z_j - z_j^*|$ for all $x, x^* \in D_k$, for all $t \geq 0$, and all $n = 1, 2, \cdots$.

THEOREM 1. (See Deimling [2, Corollary 6.2, p. 86]). *If* (A-1) *and* (A-2) *are true for some* k, *if* $|c_n| \leq \alpha kr_n$ *for some* $\alpha \in (0, 1)$ *and for all* $n \geq 1$ *and if*

(A-3): *there is an* $M > 0$ *such that*

$$(3) \qquad L_{n0}(k) r_n + \sum_{j=1}^{\infty} L_{nj}(k) r_j \leq Mr_n \qquad for \ all \quad n \geq 1 ,$$

then (Σ) *has a unique solution* $x(t, c)$ *which will exist at least on the interval* $0 \leq t \leq (1 - \alpha) M^{-1} = T$ *(where* $c^T = (c_1^T, c_2^T, \cdots)$ *and* $x(0, c) = c$*).*

COROLLARY 1. *If in Theorem 1 the constants* L_{nj} *can be chosen independent of* k, *then the solution will exist for all* $t \geq 0$ *and on any interval* $[0, T]$ *there will exist* k *such that* $x(t) \in D_k$ *for* $0 \leq t \leq T$.

PROOF. In this case $L_{nj}(k) = L_{nj}$ works for any D_k. Thus Equation (3) remains true when r_j is replaced by kr_j for any $k > 0$. Given $\{c_n\}$, compute $x(t, c)$ on $[0, T]$, $T = (1 - \alpha) M^{-1}$. Since $|x_n(t)| \leq kr_n$, we can

continue the solution $x(t, c) = x(t - T, x(T, c))$ on $[T, 2T]$ with $|x_n(2T)| \leq kr_n/\alpha$. This process can be continued indefinitely to define $x(t, c)$ on $[mT, (m + 1)T]$ with $|x_n(t)| \leq kr_n\alpha^{-m}$ on the interval $mT \leq t \leq (m + 1)T$.

<div align="right">q.e.d.</div>

We will also study *finite approximations* to (Σ) which are of the form

$$(\Sigma_N) \qquad \begin{cases} \dot{z}_n = f_n(t, z_n) + g_n(t, (z_1, z_2, \cdots, z_N, 0, \cdots)) \\ z_n(0) = c_n \end{cases}$$

$n = 1, 2, \cdots, N$, where $z_n(t) \equiv 0$ if $n > N$. In this case we let $x^N(t, c)$ denote the solution of (Σ_N) and let $c^N = (c_1^T, c_2^T, \cdots, c_N^T, 0, 0, \cdots)^T$.

THEOREM 2. (See Deimling [2, Theorem 7.5, p. 101]). *Under the hypotheses of Theorem 1 (or Corollary 1) we have*

$$\lim_{N \to \infty} x^N(t, c) = x(t, c)$$

uniformly for t in $[0, T]$ (or for t on compact subsets of R^+).

3. Analysis by scalar Lyapunov functions. A. *A stability result.* Subsequently we will require the following assumptions.

(L-1): For isolated subsystem $(\mathscr{S}_n)$ there is a continuously differentiable function $v_n: R^+ \times R^{m_n} \to R$ and three functions ψ_{1n}, ψ_{2n}, $\psi_{3n} \in K$ and a constant $\sigma_n \in R$ such that

$$\psi_{1n}(|z_n|) \leq v_n(t, z_n) \leq \psi_{2n}(|z_n|)$$
$$Dv_{n(\mathscr{S}_n)}(t, z_n) \leq \sigma_n\psi_{3n}(|z_n|)$$

for all $(t, z_n) \in R^+ \times R^{m_n}$. Here we define K as the class of all continuous functions with domain R^+ and range in R^+ having the properties that for any $\psi \in K$, $\psi(0) = 0$ and $\psi(r_1) > \psi(r_2)$ whenever $r_1 > r_2$. Also, $Dv_{(n)}$ denotes the derivative of v_n along the solutions of $(\mathscr{S}_n)$.

For composite system (Σ) we will utilize *scalar Lyapunov functions* of the form

$$(4) \qquad v(t, x) = \sum_{n=1}^{\infty} \lambda_n v_n(t, z_n)$$

where the functions v_n are defined in (L-1) and where the constants $\lambda_n > 0$ will be chosen later to ensure that v is defined and that along the solutions of composite system (Σ), the derivative of v with respect to t is non-positive, i.e., $Dv_{( )}(t, x) \leq 0$.

(L-2): Given ψ_{3n} in (L-1), there are constants $a_{nj} \in R$ such that $\nabla v_n(t, z_n)^T g_n(t, (z_1, \cdots, z_N, 0, 0, \cdots)) \leq \psi_{3n}(|z_n|)^{1/2} \sum_{j=1}^{N} a_{nj}\psi_{3j}(|z_j|)^{1/2}$ for all $t \geq 0$,

$z_n \in R^{m_n}$, $n = 1, 2, \cdots, N$ and $N = 1, 2, \cdots$.

We will find it useful to define the matrix $R_N = [r_{ij}^N]$ and the matrix S_N by

(5)
$$r_{ij}^N = \begin{cases} \lambda_i(\sigma_i + a_{ii}) & \text{if } i = j \\ \lambda_i a_{ij} & \text{if } i \neq j \end{cases}$$

and

(6)
$$S_N = (R_N + R_N^T)/2 .$$

We shall call the trivial solution $x \equiv 0$ of the system (Σ) *stable* with respect to a set D if for any $\varepsilon > 0$ there is a $\delta > 0$ such that when $c \in D$ and $\rho(c, 0) < \delta$ then $\rho(x(t, c), 0) < \varepsilon$ for all $t \geq 0$. (Equivalently one can say that for any $\varepsilon > 0$ and any integer $J_0 \geq 1$ there exists $\delta > 0$ and an integer $J_1 \geq 1$ such that when $c \in D$ and $|c_n| \leq \delta$ for $1 \leq n \leq J_1$ then $|z_n(t, c)| < \varepsilon$ for all $t \geq 0$ and for $n = 1, 2, \cdots, J_0$.) We call the trivial solution $x \equiv 0$ *asymptotically stable with respect to* D if it is stable with respect to D and if for any initial condition $c \in D$ the solution $x(t, c) \to 0$ as $t \to \infty$.

LEMMA 1. *Suppose that* (A-1), (A-2), (A-3), (L-1) *and* (L-2) *are true and assume that there exist constants* $\lambda_n > 0$ *such that* $\sum_{n=1}^{\infty} \lambda_n \psi_{2n}(kr_n) < \infty$. *If matrix* S_N *is negative semi-definite for all* N *sufficiently large, then* $v(t, x(t, c))$ *is defined, continuous and nonincreasing in* $t \in [0, (1 - \alpha)M^{-1}]$ *for any initial condition* $c \in D_{\alpha k}$.

PROOF. Suppose that $x(t, c)$ exists on $[0, b]$. By Theorem 2 $x^N(t, c) \to x(t, c)$ as $N \to \infty$ uniformly on $[0, b]$. Thus, $\lim_{N \to \infty} v(t, x^N(t, c)) = v(t, x(t, c))$ uniformly on $[0, b]$. Moreover, $(d/dt)v(t, x^N(t, c)) = \sum_{n=1}^{N} \lambda_n Dv_{n(\mathscr{S}_n)}(t, z_n) + \lambda_n \nabla v_n(t, z_n)^T g_n(t, x^N(t, c)) \leq \sum_{n=1}^{N} \lambda_n[\sigma_n \psi_{3n}(|z_n|) + \sum_{j=1}^{N} \lambda_n a_{nj} \psi_{3n}(|z_n|)^{1/2} \psi_{3j}(|z_j|)^{1/2}] = \psi_N^T R_N \psi_N = \psi_N^T[(R_N + R_N^T)/2]\psi_N = \psi_N^T S_N \psi_N \leq 0$ where $\psi_n = (\psi_1(|z_1|)^{1/2}, \psi_2(|z_2|)^{1/2}, \cdots, \psi_N(|z_N|)^{1/2})$. Since $v(t, x^N(t, c))$ is nonincreasing in t (for all N sufficiently large), then so is the limit $v(t, x(t, c))$. q.e.d.

THEOREM 3. *Suppose that* (A-1), (A-2), *and* (A-3) *are true with Lipschitz constants* L_{nj} *independent of* k *and suppose that* (L-1) *and* (L-2) *are true. Assume that the constants* λ_n *can be picked so that the matrix* S_N *is negative semi-definite (see* (5) *and* (6)) *for all* N *sufficiently large and so that* $\sum_{n=1}^{\infty} \lambda_n \psi_{2n}(r_n) < \infty$. *Define* $D = D_k$ *for* $k = 1$. *Then the trivial solution* $x = 0$ *of composite system* (Σ) *is stable with respect to the set* D.

PROOF. Fix ε and J_0. Since $\psi_{1n} \in K$ for all $n \geq 1$ and since $v(t, x) \geq \sum_{n=1}^{\infty} \lambda_n \psi_{1n}(|z_n|) \geq \sum_{n=1}^{J_0} \lambda_n \psi_{1n}(|z_n|) \geq \max\{\lambda_n \psi_{1n}(|z_n|): 1 \leq n \leq J_0\}$, then there

exists $m > 0$ such that $v(t, x) \geq m$ if for some n between 1 and J_0 we have $|z_n| \geq \varepsilon$. Pick J_1 so large that $\sum_{n=J_1+1}^{\infty} \lambda_n \psi_{2n}(r_n) < m/2$ and pick $\delta > 0$ so small that $\sum_{n=1}^{J_1} \lambda_n \psi_{2n}(\delta) < m/2$. If $|c_n| \leq \delta$ for $1 \leq n \leq J_1$, then by Lemma 1 $v(t, x(t, c)) \leq v(0, c) < m$. Thus $|z_n(t, c)| < \varepsilon$ for all $t \geq 0$ and for $1 \leq n \leq J_0$.   q.e.d.

EXAMPLE 1. Consider the countable system of scalar equations

$$(7) \qquad \begin{cases} \dot{z}_1 = -z_1 \\ \dot{z}_n = -z_n + z_{n-1}\,, \qquad n \geq 2\,. \end{cases}$$

We can view (7) as an interconnection of countably infinite many free subsystems described by

$$(8) \qquad\qquad \dot{z}_n = -z_n\,,$$

$n = 1, 2, \cdots$, with interconnecting structure specified by $g_n(t, x) = z_{n-1}$, $n \geq 2$.

For any $r > 0$ we can set $r_n = r$ for all $n \geq 1$ and see that (A-1) and (A-2) are satisfied for (7).

For $(\mathscr{S}_n)$ we choose $v_n(z_n) = z_n^2/2$. In the notation of (L-1) and (L-2) we have $\psi_{1n}(|z_n|) = \psi_{2n}(|z_n|) = z_n^2/2$, $\psi_{3n}(|z_n|) = z_n^2$, $\sigma_n = -1$, $a_{n,n-1} = 1$ for $n \geq 2$, and $a_{ij} = 0$ for all other i and j. Thus, (L-1) and (L-2) are satisfied as well.

To show that the remaining hypotheses of Theorem 3 are satisfied, we let

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & & \cdots & \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

If $0 < a_1 < a_2 < \cdots < a_N$ and $p = (a_1, a_2, \cdots, a_N)^T$, then $Ap > 0$ (i.e., each component of vector Ap is positive). If $b_1 > b_2 > \cdots > b_N > 0$ and $q = (b_1, b_2, \cdots, b_N)^T$ then $A^T q > 0$. Define $e_j = b_j/a_j$, $1 \leq j \leq N$, $W = \text{diag}\,[e_1, e_2, \cdots, e_N]$, and $B = [b_{ij}] = (WA + A^T W)$. Then $b_{ij} \leq 0$ for all $i \neq j$ and $Bp = (WA + A^T W)p = W(Ap) + A^T q > 0$. It follows that B is an M-matrix [3, 6] (i.e., $b_{ij} \leq 0$ for all $i \neq j$ and all successive principal minors of B are positive) and $B = B^T$ is positive definite.

Using the notation of Equations (4)-(6) we choose $\lambda_j = e_j$. Then the matrix B constructed above equals the matrix $-2S_N$. Thus S_N is negative definite. The only remaining question is whether the resulting λ_n are summable. However, this is easy to arrange. For example, if $a_n = 2^{n/2}$, $b_n = 2^{-n/2}$, then $\lambda_n = 2^{-n}$ will do.

All hypotheses of Theorem 3 are now satisfied and the trivial solution $x = 0$ of system (7) is stable with respect to D_k for any $k > 0$.

The system of equations (Σ) can in this particular case be solved explicitly to obtain $z_n(t) = e^{-t}[c_n + c_{n-1}t + c_{n-2}t^2/2! + \cdots + c_1 t^{n-1}/(n-1)!]$. Thus, in the present example all solutions tend to zero as $t \to \infty$. We develop below some theory to predict this asymptotic stability.

B. *An invariance result.* The purpose of the next example is to provide motivation for the next result.

EXAMPLE 2. Consider the system of scalar equations

$$(9) \qquad \dot{z}_n = -z_n + 2 \sum_{m=n+1}^{\infty} (g_{nm}/\lambda_n)z_m \, ,$$

$n = 1, 2, \cdots$, where $g_{nm} \geq 0$, $g_{n,n+1} > 0$, and the g are assumed to be such that $\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm} < \infty$ and where $\lambda_n = (\sum_{m=1}^{n-1} g_{mn} + \sum_{m=n+1}^{\infty} g_{nm}) > 0$. Reversing the order of summation in the first sum below we see that $\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} (\sum_{m=1}^{n-1} g_{mn} + \sum_{m=n+1}^{\infty} g_{nm}) = 2 \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm} < \infty$.

For each isolated subsystem $(\mathscr{S}_n)$ of (Σ), given by

$$(10) \qquad \dot{z}_n = -z_n$$

we choose $v_n(z_n) = z_n^2/2$ and for composite system (9) we choose the Lyapunov function $v(x) = \sum_{n=1}^{\infty} \lambda_n v_n(z_n)$. Since the system of equations (9) is upper triangular, rather than using finite approximations, we can equivalently compute directly that $-Dv_{(9)}(x) = \sum_{n=1}^{\infty} [(\sum_{m=1}^{n-1} g_{mn} + \sum_{m=n+1}^{\infty} g_{nm})z_n^2 - 2 \sum_{m=n+1}^{\infty} g_{nm}z_n z_m] = \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} g_{mn}z_n^2 + \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm}(z_n^2 - 2z_n z_m) = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm}(z_n - z_m)^2$. Thus, $Dv_{(9)}(x) \leq 0$ and with $r_n = r$ for all $n \geq 1$ we obtain stability for system (9) by Theorem 3.

REMARK 2. In the above example it is natural to ask if more can be asserted. Specifically, do the solutions (of (9) in Example 2) tend to the set $E = \{(z_1, z_2, \cdots, z_n, \cdots) : z_1 = z_2 = \cdots = z_n = \cdots\}$? This is a natural question, since above $Dv_{(9)}(x)$ is negative definite with respect to the set E. Is it possible to prove an invariance theorem (see [13] or [1, Section 5.4]) for this system (i.e., do the solutions tend to an invariant subset of E?)? The next result provides some answers along such lines.

THEOREM 4. *Assume that the hypotheses of Theorem 3 are true. Assume there exists $k > 0$ such that $x(t, c) \in D_k$ for all $t \geq 0$ and for all $c \in D$. Suppose that $Dv_{(\Sigma)}(t, x) \leq -W(x)$ where $W(x)$ is positive definite with respect to a set $E \subset D_k$ (i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in D_k$ and $\rho(x, E) \geq \varepsilon$ then $W(x) \geq \delta$). If the functions f_n, g_n and v_n are all independent of t, then for any $c \in D$*

(i) $x(t, c)$ *tends to* E *as* $t \to \infty$, *and*

(ii)   *the positive limit set* $\Omega(x(\cdot, c))$ *is an invariant set with respect to* (Σ).

PROOF. Assume that (i) is not true. Then there is a monotone sequence $\{t_n\}$ such that $t_n \to \infty$ and an $\varepsilon > 0$ such that $\rho(x(t_n, c), E) \geqq 2\varepsilon$. Since D_k is a compact set and $h_n = f_n + g_n$ is continuous there, then h_n is bounded on D_k. Thus $z_n(t)$ is uniformly continuous on $0 \leqq t < \infty$ for each $n \geqq 1$, i.e., there is an $\eta > 0$ such that $\rho(x(t, c), E) \geqq \varepsilon$ for $|t - t_n| \leqq \eta$. Also, there is a $\delta > 0$ such that $W(x) \geqq \delta$ if $\rho(x, E) \geqq \varepsilon$. Thus for $t \geqq t_N + \eta$ we have

$$ 0 \leqq v(x(t)) \leqq v(c) - \int_0^t W(x(s))ds \leqq v(c) - \sum_{j=1}^N \int_{t_n - \eta}^{t_n + \eta} \delta ds . $$

The last expression is negative when $N > v(c)/(2\delta\eta)$. This contradiction completes the proof of (i).

To prove (ii) note that each component $z_n(t)$ is uniformly continuous and bounded on $0 \leqq t < \infty$. Let $d \in \Omega(x(\cdot, c))$ and let $t_n \to \infty$ be a sequence such that $x(t_n, c) \to d = (d_1, d_2, \cdots)$. Then $z_m(t + t_m, c)$ is an equicontinuous, uniformly bounded sequence of functions defined on $-T \leqq t \leqq T$ for any $T > 0$. By possibly taking a subsequence we can assume that $\lim_{n \to \infty} z_m(t + t_n, c) = y_m(t)$ exists uniformly for t on compact subsets of $-\infty < t < \infty$. Thus, as $n \to \infty$ we get $z_m(t + t_n, c) = z_m(t_n, c) + \int_0^t h_m(x(s + t_n, c))ds$. In the limit this is $y_m(t) = d_m + \int_0^t h_m(y(s))ds$. This means that $y(t) = x(t, d)$ for $-\infty < t < \infty$.   q.e.d.

REMARK 3.   Part (i) of Theorem 4 covers Example 1 above. Indeed in that example it is easy to compute that $Dv_{(7)}(t, x) \leqq (-3/8)z_1^2 - (1/4)\sum_{n=2}^\infty z_n^2/2^n \leqq (-1/8)v(t, x)$ if we choose $\lambda_n = 2^{-n}$. Since $v(t, x) = v(x)$ is positive definite, then we can take $E = $ the origin and assert asymptotic stability of (7) with respect to D_k for any $k > 1$.

REMARK 4.   In Example 2 we see that given any solution $x(t, c)$ with $|z_n(t, c)| \leqq k$ for some k and for all $t \geqq 0$, then by Theorem 4, $x(t, c) \to E = \{(z_1, z_2, \cdots)^T : z_1 = z_2 = \cdots\}$. Indeed $x(t, c)$ tends to the largest invariant subset of E, which we denote by M. On M the system of equations (9) reduces to $\dot{z}_n = (-1 + (2/\lambda_n)\sum_{m=n+1}^\infty g_{nm})z_n = \eta z_n$, $n = 1, 2, 3, \cdots$, where $\eta = -1 + (2/\lambda_n)\sum_{m=n+1}^\infty g_{nm}$ must be independent of n. Indeed, since the components z_n are bounded on $-\infty < t < \infty$, either $\eta = 0$ or $z_n \equiv 0$. Now $\eta = -1 + (2/\lambda_n)\sum_{m=n+1}^\infty g_{nm} = (\sum_{m=n+1}^\infty g_{nm} - \sum_{m=1}^{n-1} g_{nm})/(\sum_{m=n+1}^\infty g_{nm} + \sum_{m=1}^{n-1} g_{nm})$. Therefore either M is just the origin or else $\sum_{m=n+1}^\infty g_{nm} = \sum_{m=1}^{n-1} g_{mn}$ $(n = 1, 2, 3, \cdots)$. This is impossible since for $n = 1$, $\sum_{m=1}^0 g_{mn}$ means zero

while $\sum_{m=2}^{\infty} g_{1m} = \lambda_1 > 0$.

In Theorem 4 the hypothesis that $x(t, c) \in D_k$ for all $t \geq 0$ for some $k > 0$ is needed only to ensure that $x(t, c)$ is uniformly continuous on $[0, \infty)$. This can also be assured in other ways, e.g., Theorem 5. We say that system (Σ) is *row-finite* if for each $n \geq 1$ there is an integer $m(n) \geq 1$ such that $h_n(t, x) = h_n(t, x_1, \cdots, x_{m(n)})$ depends only on t and x_j for $1 \leq j \leq m(n)$. The proof of the following result is similar to that of Theorem 4 above.

THEOREM 5. *Assume that the hypotheses of Theorem 3 are true. Suppose that $Dv_{(\Sigma)}(t, x) \leq - W(x)$ where $W(x)$ is positive definite with respect to a set E. If f_n, g_n and v_n are all independent of t and (Σ) is row finite, then for any $c \in D$ conclusions (i) and (ii) of Theorem 4 remain true.*

REMARK 5. In Example 2 if for each n there exists $m(n)$ such that $g_{nm} = 0$ when $m \geq m(n)$ and if $\sup_n |c_n| < \infty$, then $x(t, c) \to 0$ as $t \to \infty$.

EXAMPLE 3. As in Example 2 suppose that $g_{nm} \geq 0$ with $g_{n,n+1} > 0$ and $\sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm} < \infty$, $\lambda_n \overset{\Delta}{=} \sum_{m=1}^{n-1} g_{mn} + \sum_{m=n+1}^{\infty} g_{nm} > 0$ and consider the system of equations

$$\dot{z}_1 = - z_1 + \sum_{m=2}^{\infty} (g_{1n}/\lambda_1) z_m$$

$$\dot{z}_n = - z_n + \sum_{m=n+1}^{\infty} (g_{nm}/\lambda_n) z_m + \sum_{m=1}^{n-1} (g_{mn}/\lambda_n) z_m , \qquad n \geq 2 .$$

If $v(x) = \sum_{n=1}^{\infty} \lambda_n z_n^2/2$, then by a calculation much like that in the last example we obtain $- Dv_{(\Sigma)}(x) = \sum_{n=1}^{\infty} [\lambda_n z_n^2 - \sum_{m=n+1}^{\infty} g_{nm} z_n z_m - \sum_{m=1}^{n-1} g_{mn} z_n z_m] = \sum_{n=1}^{\infty} [\sum_{m=n+1}^{\infty} g_{nm} z_n^2 - \sum_{m=n+1}^{\infty} g_{nm} z_n z_m] + \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} g_{mn} z_n^2 - \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} g_{mn} z_n z_m = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm}(z_n^2 - z_n z_m) + \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} g_{mn}(z_n^2 - z_n z_m) = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm}(z_n^2 - z_n z_m + z_m^2 - z_m z_n) = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} g_{nm}(z_n - z_m)^2$. If $\sup_n |c_n| < \infty$, then the solution $x(t, c) \to E$ as $t \to \infty$ where $E = \{(z_1, z_2, z_3, \cdots): z_1 = z_2 = z_3 = \cdots\}$, provided either $\sup_{t \geq 0, n \geq 1} |z_n(t, c)| < \infty$ or the system is row-finite. An invariant subset of E has solutions with the property $z_1 \equiv z_2 \equiv z_3 \equiv \cdots$ so that $\dot{z}_n = -z_n + \sum_{m=n+1}^{\infty} (g_{nm}/\lambda_n) z_n + \sum_{m=1}^{n-1} (g_{mn}/\lambda_n) z_n = -(1/\lambda_n)(\lambda_n - \sum_{m=n+1}^{\infty} g_{nm} - \sum_{m=1}^{n-1} g_{mn}) z_n$. By the definition of λ_n, these coefficients are all zero. Thus all points of the form $x^T = (z_1, z_2, \cdots)$ with $z_1 = z_2 = z_3 = \cdots = K$ constant are solutions. (The origin is not an attractor.)

EXAMPLE 4. Consider the system of equations

(11) $$\begin{cases} \dot{z}_1 = - 2z_1 + z_2 \\ \dot{z}_n = z_{n-1} - 2z_n + z_{n+1} , \qquad n \geq 2 . \end{cases}$$

Let $\sum_{n=1}^{\infty} r_n^2 < \infty$, $\sup_n (r_{n+1} + r_{n-1})/r_n = S < \infty$ and choose $v(x) = (1/2) \sum_{n=1}^{\infty} z_n^2$. Then (A-3) is true with $M = S + 2$ and $Dv_{(11)}(x) = -z_1^2 - \sum_{n=1}^{\infty} (z_n - z_{n+1})^2 \leqq 0$. The solutions are uniformly bounded, $\sup_{n,t} |z_n(t,c)| < \infty$ and $x(t, c) \to 0$ as $t \to \infty$ by invariance.

C. *Weak-coupling conditions.* In our next result we consider some alternate hypotheses.

(L-3): For each free subsystem $(\mathscr{S}_n)$ there is a function $v_n \colon R^+ \times R^{m_n} \to R^+$, a constant $\sigma_n \in R$, a constant $L_n \geqq 0$, and functions $\psi_{jn} \in K$, $j = 1, 2, 3$, such that

$$\psi_{1n}(|z_n|) \leqq v_n(t, z_n) \leqq \psi_{2n}(|z_n|)$$
$$Dv_{n(\mathscr{S}_n)}(t, z_n) \leqq \sigma_n \psi_{3n}(|z_n|)$$
$$|v_n(t, z_n) - v_n(t, z_n^*)| \leqq L_n |z_n - z_n^*|$$

for all $t \geqq 0$ and for all $z_n, z_n^* \in R^{m_n}$.

(L-4): There are constants $a_{nm} \geqq 0$ such that $|g_n(t, x)| \leqq \sum_{m=1}^{N} a_{nm} \psi_{3m}(|z_m|)$ whenever $x^T = (z_1^T, z_2^T, \cdots, z_N^T, 0, 0, \cdots)$ for any $N \geqq 1$.

Also, in our next result we will employ Lyapunov functions for (Σ) which are of the form $v(t, x) = \sum_{n=1}^{\infty} \lambda_n v_n(t, z_n)$ with constants $\lambda_n > 0$ which we will further specify later. We will also utilize $N \times N$ matrices $R_N = [r_{nm}]$ specified by

$$r_{nm} = \begin{cases} -(\sigma_n + L_n a_{nn}) & \text{if } n = m \\ -L_n a_{nm} & \text{if } n \neq m . \end{cases}$$

Finally, we will also require the following additional hypothesis:

(L-5): For each N sufficiently large, $\Lambda_N R_N \geqq 0$, i.e., $\sum_{j=1}^{N} \lambda_j r_{jn} \geqq 0$ for $1 \leqq n \leqq N$.

Under the above assumptions we can easily compute $(d/dt)v(t, x^N(t,c)) \leqq \sum_{n=1}^{N} \lambda_n \{\sigma_n \psi_{3n}(|z_n|) + \sum_{m=1}^{N} L_n a_{nm} \psi_{3m}(|z_m|)\} = -\Lambda_N R_N \psi_N(|x|)$ where $\Lambda_N = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ and $\psi_N(|x|) \overset{\Delta}{=} (\psi_{31}(|z_1|), \cdots, \psi_{3N}(|z_N|))^T$. By assumption (L-5) it now follows that $\Lambda_N R_N \geqq 0$. Thus, $Dv_{(\Sigma)}(t, x) \leqq 0$. Indeed, we have $Dv_{(\Sigma)}(t, x) \leqq -\sum_{n=1}^{\infty} (\sum_{j=1}^{\infty} \lambda_j r_{jn}) \psi_{3n}(|z_n|) = -W(x)$. Moreover, $W(x)$ is positive definite with respect to the set $E^* = \{x = (z_1, z_2, z_3, \cdots) \colon z_n = 0 \text{ when } \sum_{j=1}^{\infty} \lambda_j r_{jn} > 0\}$.

Summarizing, we now have

THEOREM 6. *If assumptions* (A-1), (A-2) *and* (A-3) *are true with Lipschitz constant* L_{nj} *independent of* $k > 0$, *if assumptions* (L-4) *and* (L-5) *are true, and if* $\sum_{n=1}^{\infty} \lambda_n \psi_{2n}(r_n) < \infty$, *then*

(i) *the trivial solution* $x = 0$ *of system* (Σ) *is stable with respect to* D_k *for* $k = 1$.

(ii) *Let* M *be the largest invariant subset of* E^*. *If* $c \in D$ *and*

either $x(t, c) \in D_K$ *for all* $t \geq 0$ *and some fixed* $k > 1$ *or* (Σ) *is row finite, then* $x(t, c) \to M$ *as* $t \to \infty$.

EXAMPLE 5. We reconsider Example 4,

(12)
$$\begin{cases} \dot{z}_1 = -2z_1 + z_2 \\ \dot{z}_n = z_{n-1} - 2z_n + z_{n+1} \quad (n \geq 2) \ . \end{cases}$$

For each isolated subsystem (13),

(13)
$$\dot{z}_n = -2z_n \ ,$$

$n \geq 1$, we choose Lyapunov functions of the form $v_n(z_n) = |z_n|$. Then $\psi_{1n}(|z_n|) = \psi_{2n}(|z_n|) = |z_n|$, $\psi_{3n}(|z_n|) = |z_n|$, $\sigma_n = -2$ and $L_n = 1$. Thus, (L-3) is satisfied. Hypothesis (L-4) is also satisfied with $a_{n,n-1} = a_{n,n+1} = 1$ for $n \geq 2$, $a_{12} = 1$, and $a_{ij} = 0$ for all other indices. Choose c such that $\sum_{n=1}^{\infty} |c_n| < \infty$ and let $r_n \geq |c_n|$ be such that $\sup_n (r_{n+1} + r_{n-1})/r_n = S < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$ (e.g., $r_n = K/2^n$). For system (12) we choose $v(x) = \sum_{n=1}^{\infty} v_n(z_n) = \sum_{n=1}^{\infty} |z_n|$. Then $Dv_{(12)}(x) = -2|x_1| + |x_2| + \sum_{n=2}^{\infty} (|x_{n-1}| - 2|x_n| + |x_{n+1}|) = -|x_1| \leq 0$. Thus $\sup_n \sup_{t \geq 0} |z_n(t, c)| < \infty$. Moreover, $E^* = \{x : z_1 = 0\}$ and if $x(t, c) \in M$, then $z_1(t, c) = 0$ implies that $\dot{z}_1 + 2z_1 = z_2 = 0$. Similarly, $\dot{z}_2 + 2z_2 = z_3 = 0$, and so forth. Thus $M = \{(0, 0, \cdots, 0, \cdots)\}$.

All the hypotheses of Theorem 6 are satisfied. The trivial solution of (12) is asymptotically stable with respect to D_k for any $k > 1$.

4. Analysis by vector Lyapunov functions. An (infinite) vector Lyapunov function approach can also be used to analyze certain stability problems for system (Σ). For example, if $Dv_{n(\mathscr{S}_n)} \leq \sigma_n v_n$ and if $|g_n(t, x)| \leq \sum_{m=1}^{\infty} a_{nm} v_m(t, z_m)$ for some sequence $a_{nm} \geq 0$, then we can consider for system (Σ) the infinite vector Lyapunov function

$$v(t, x) = (v_1(t, z_1), v_2(t, z_2), \cdots)^T \ .$$

This vector satisfies for (Σ) the differential inequality $Dv_n \leq \sigma_n v_n + L_n \sum_{m=1}^{\infty} a_{nm} v_m$, $n = 1, 2, \cdots$. By the comparison results in [12] and Theorem 2 above we have $v_n(t, z_n(t, c)) \leq w_n(t)$ for all $n \geq 1$ and all $t \geq 0$, where w_n, $n = 1, 2, \cdots$, are the solutions of the linear equations

(σ)
$$\dot{w}_n = \sigma_n w_n + \sum_{m=1}^{\infty} L_n a_{nm} w_m \quad (n \geq 1)$$

with $w_n(0) \geq v_n(0, z_n(0, c))$. Thus, if the stability properties of system (σ) can be determined, then the stability properties of system (Σ) can be inferred.

In our next example we consider a *neural model* which has been studied previously (see [4]).

EXAMPLE 6. Consider the neural model given by the scalar equations

(14)
$$\begin{cases} \dot{z}_n = -\alpha_n z_n + f_n(x) \\ z_n(0) = c_n \end{cases}$$

where $\alpha_n > 0$, $0 < c_n < 1$, $f_n(x) = [1 + \exp(-\gamma_n - \sum_{m=1}^{\infty} \beta_{nm} z_m)]^{-1}$, and $\sum_{m=1}^{\infty} |\beta_{nm}| \leqq b < \infty$ for each $n \geqq 1$. We will accomplish our analysis of (14) in several steps. In doing so we shall make the assumptions that $\beta_{nm} \geqq 0$, $\alpha_n[1 + \exp(-\gamma_n - \sum_{m=1}^{\infty} \beta_{nm})] > 1$, and there exists $\lambda_n \geqq 0$ such that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{m=1}^{\infty} \lambda_m \alpha_m \beta_{mn} < \lambda_m \alpha_m$ $(n \geqq 1)$.

(a) CLAIM. *There is a critical point $x^* = (z_1^*, z_2^*, \cdots)^T$ with $0 < z_n^* < 1$.*

To verify this we define S by $S = \{x = (z_1, z_2, \cdots)^T : 0 \leqq z_n < 1\}$ and we define Tx by $T_n x = \alpha_n^{-1} f_n(x)$. Let $z^0 = (0, \cdots, 0, \cdots)^T$ and $z^{n+1} = T(z^n)$. By monoticity, $z^0 \leqq z^1 \leqq z^2 \leqq \cdots < (1, 1, \cdots, 1, \cdots) = e$. This and continuity imply that $z^m \nearrow z^*$ for some z^* with $z^0 < z^* < e$ and $z^* = \lim_{m \to \infty} z^{m+1} = \lim_{m \to \infty} T(z^m) = T(z^*)$.

(b) CLAIM. *The solution $x(t, c)$ satisfies $0 < z_n(t, c) < 1$ for all $t \geqq 0$ and all $n \geqq 1$.*

To verify this, we note that $\dot{z}_n = -\alpha_n z_n + f_n(x) \geqq -\alpha_n z_n$ so that $z_n(t) \geqq c_n \exp(-\alpha_n t) > 0$. Let $y_n = 1 - z_n$ so that $\dot{y}_n = -\alpha_n y_n + g_n(y)$, where $g_n(y) = \alpha_n - (1 + \exp(-\gamma_n - \sum_{m=1}^{\infty} \beta_{nm} + \sum_{m=1}^{\infty} \beta_{nm} y_m))^{-1}$. Define $\tilde{g}_n(y) = g_n(w)$, where $w_n = \max\{0, y_n\}$. Let $\dot{w}_n = -\alpha_n w_n + \tilde{g}_n(w) \geqq -\alpha_n w_n + \tilde{g}_n(0)$. Thus $w_n(t) \geqq e^{-\alpha_n t} + \tilde{g}_n(0)/\alpha_n \geqq \tilde{g}_n(0)/\alpha_n > 0$ for as long as $w_n(t)$ exists. This means that $\tilde{g}_n(w(t)) = g_n(w(t))$, so by uniqueness $0 < w_n(t) = y_n(t) = 1 - z_n(t)$. This proves the claim.

(c) CLAIM. *The critical point x^* is asymptotically stable with respect to $\{x = (z_1, z_2, \cdots) : 0 < z_n < 1\}$.*

To prove this claim we define for stability of x^*, $u_n = z_n - z_n^*$ and $F_n(u) = f_n(u + x^*) - f_n(x^*)$ for $u = (u_1, u_2, u_3, \cdots)^T$. Then

(15)
$$\begin{cases} \dot{u}_n = -\alpha_n u_n + F_n(u) \\ u_n(0) = u_{n0} = c_n - z_n^* \end{cases}$$

and $-1 < -z_n^* < u_n(t) < 1 - z_n^* < 1$ for all $t \geqq 0$, $n \geqq 1$. Let $v_n(u_n) = |u_n|$ and $v(u) = (v_1(u_1), v_2(u_2), \cdots)^T$. Then for some numbers $\delta_n \in [0, 1)$ $(d/dt)v_n(u_n(t)) \leqq -\alpha_n |u_n| + |F_n(u)| \leqq -\alpha_n v_n + \sum_{m=1}^{\infty} (1 + \exp(-\gamma_n - \sum_{j=1}^{\infty} \beta_{nj}))^{-1} \delta_n \beta_{nm} v_m$. The Lyapunov function $v(t, x) = \sum_{n=1}^{\infty} \lambda_n |v_n|$ for the linear system $v_n' = -\alpha_n v_n + \sum_{m=1}^{\infty} \delta_n \alpha_n \beta_{nm} v_m$, and Theorem 6 imply that $u(t) \to 0$ as $t \to \infty$.

REMARK 6. The analysis of this system in [4] proceeds under some-what different assumptions. The assumptions $\beta_{nm} \geqq 0$, and $\sum_{m=1}^{\infty} \lambda_m \alpha_m \beta_{mn} < \lambda_n \alpha_n$ are not used. It is assumed that $\sup |\gamma_n| = \gamma < \infty$, that $\sum_{m=1}^{\infty} |\beta_{nm}| \leqq 1$ for each $n \geq 1$ and there is a δ such that $\alpha_n^{-1}(1 + \exp(-1 - \gamma))^{-1} \leqq \delta < 1$ for all $n \geq 1$. Under these conditions it is shown that there is a critical point x^* which is stable (in the sense of the sup norm $\|x\| = \sup_n |z_n|$). Asymptotic stability is studied in [4] only for the finite approximations (Σ_N).

5. Concluding remarks. Countably infinite systems of differential equations were viewed as an interconnection of countably infinite many free or isolated subsystems (described by ordinary differential equations defined on finite dimensional spaces). Well posedness results and several new stabilty results were established using scalar Lyapunov functions as well as vector Lyapunov functions. These were applied to several specific examples, including a neural model.

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