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Stabilization and regulator design for a one-dimensional unstable wave equation with input harmonic disturbance

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SUMMARY

This paper considers the parameter estimation and stabilization of a one-dimensional wave equation with instability suffered at one end and uncertainty of harmonic disturbance at the controlled end. The backstepping method for infinite-dimensional system is adopted in the design of the adaptive regulator. It is shown that the resulting closed-loop system is asymptotically stable. Meanwhile, the estimated parameter is shown to be convergent to the unknown parameter as time goes to infinity. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In the last two decades, much effort has been made for the generalization of traditional adaptive control algorithms to infinite-dimensional systems. The early results can be found in [1, 2]. The attempts on high-gain adaptive stabilization of infinite-dimensional systems are available in [1, 3, 4]. The non-identified high-gain adaptive stabilization of a general collocated first-order system with unbounded input and output is discussed in [5]. An exponential stabilization of high-gain adaptive direct strain feedback control for an Euler–Bernoulli beam is proved in [6]. For a nice survey on the topic, we refer to [7]. Some other efforts for the design of adaptive controllers for partial differential equation control systems (PDEs), particularly for parabolic PDEs with boundary control and unknown parameters that may cause instability of the system and affect the interior of domain are presented in [8–11]. The high gain adaptive regulator for undamped second-order hyperbolic systems with output disturbances and collocated control is designed in [12]. Moreover, the problems for nonlinear infinite-dimensional systems have also attracted attentions in recent years; for instance, a Burger's equation with various parametric uncertainties is discussed in [13, 14], and an adaptive exponential stabilizing controller for a Kirchhoff-type nonlinear beam is developed in [6].

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In this paper, we are concerned with a one-dimensional wave equation of the following:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0, \ x \in (0,1), \ t > 0, \\ y_x(0,t) = -qy(0,t), \ t \geqslant 0, \\ y_x(1,t) = u(t) + \bar{\theta}_1 \sin t + \bar{\theta}_2 \cos t, \ t \geqslant 0, \\ y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), \ 0 \leqslant x \leqslant 1, \end{cases} \tag{1.1}$$

where and henceforth y' or y_x denotes the derivative of y with respect to x and $\dot{y}$ or y_t the derivative with respect to t, $u(t)$ is the input, $\bar{\theta}_1$ and $\bar{\theta}_2$ are unknown amplitudes of harmonic disturbance vectors, q is a positive constant, (y_0, y_1) is the initial value. Obviously, the harmonic disturbance vector function $(\sin t, \cos t)^\top$ is the solution to the homogeneous equation with initial value $(\omega_{10}, \omega_{20}) = (0, 1)$:

$$\begin{cases} \dfrac{d}{dt}(\omega_1(t), \omega_2(t))^\top = B(\omega_1(t), \omega_2(t))^\top, \\ (\omega_1(0), \omega_2(0))^\top = (\omega_{10}, \omega_{20})^\top, \end{cases} \tag{1.2}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $q = 0$, (1.1) models a string, which is free at the end $x = 0$. For $q \neq 0$, the free end of the string is subject to a force proportional to the displacement, which physically may be the result of various phenomena with instability. We refer to [15] for the modeling and possible physical explanations in details. A recent progress is made in [15] where a more complicated backstepping controller, and a more complicated backstepping observer are employed to stabilize the string vibration, which is destabilized at $x = 0$ because of the Robin boundary condition $w_x(0,t) = -qw(0,t)$. However, when the system is subjected to some harmonic disturbances with unknown amplitudes, there is no report, to our best knowledge, on the design of adaptive regulator to achieve its stability and disturbance rejection simultaneously.

The objective of this paper is to construct an adaptive scheme to achieve both parameter estimation and stabilization under the state feedback.

We propose the following adaptive regulator for system (1.1):

$$\begin{cases} u(t) = -ky_t(1,t) - (c_0 + q)y(1,t) \\ -(c_0+q)\displaystyle\int_0^1 e^{q(1-\xi)}[qy(\xi,t) + ky_t(\xi,t)]d\xi - \theta_1(t)\sin t - \theta_2(t)\cos t, \\ \dot{\theta}_1(t) = r_1 \left[y_t(1,t) + (c_0+q)\displaystyle\int_0^1 e^{q(1-\xi)} y_t(\xi,t)d\xi \right] \sin t, t > 0, \\ \dot{\theta}_2(t) = r_2 \left[y_t(1,t) + (c_0+q)\displaystyle\int_0^1 e^{q(1-\xi)} y_t(\xi,t)d\xi \right] \cos t, t > 0, \\ \theta_1(0) = \theta_{10}, \ \theta_2(0) = \theta_{20}, \end{cases} \tag{1.3}$$

where k, r_1, r_2 and c_0 are positive constants. The recommended choice of the control gain is c_0 being relatively large [15]. Under the feedback controller (1.3), the closed-loop of system (1.1) becomes

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0, \ x \in (0,1), \ t > 0, \\ y_x(0,t) = -qy(0,t), \ t \geqslant 0, \\ y_x(1,t) = -ky_t(1,t) - (c_0+q)y(1,t) \\ -(c_0+q)\displaystyle\int_0^1 e^{q(1-\xi)}[qy(\xi,t) + ky_t(\xi,t)]d\xi + \widetilde{\theta}_1(t)\sin t + \widetilde{\theta}_2(t)\cos t, \ t \geqslant 0, \\ \dot{\widetilde{\theta}}_1(t) = -r_1 \left[y_t(1,t) + (c_0+q)\displaystyle\int_0^1 e^{q(1-\xi)} y_t(\xi,t)d\xi \right] \sin t, t > 0, \\ \dot{\widetilde{\theta}}_2(t) = -r_2 \left[y_t(1,t) + (c_0+q)\int_0^1 e^{q(1-\xi)} y_t(\xi,t)d\xi \right] \cos t, t > 0, \\ \widetilde{\theta}_1(0) = \widetilde{\theta}_{10} = \bar{\theta}_1 - \theta_{10}, \ \widetilde{\theta}_2(0) = \widetilde{\theta}_{20} = \bar{\theta}_2 - \theta_{20}, \\ y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), \ 0 \leqslant x \leqslant 1, \end{cases} \tag{1.4}$$

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where $\widetilde{\theta}_1(t) = \bar{\theta}_1 - \theta_1(t), \widetilde{\theta}_2(t) = \bar{\theta}_2 - \theta_2(t)$ are the parameter estimation errors. Make the invertible change of variable

$$w(x,t) = [(I+\mathbb{P})y](x,t) = y(x,t) + (c_0+q)\int_0^x e^{q(x-\xi)} y(\xi,t)d\xi, \tag{1.5}$$

where $\mathbb{P}$ is a Volterra transformation [15]. The inverse $(I+\mathbb{P})^{-1}$ is given by

$$y(x,t) = [(I+\mathbb{P})^{-1}w](x,t) = w(x,t) - (c_0+q)\int_0^x e^{-c_0(x-\xi)} w(\xi,t)d\xi. \tag{1.6}$$

It can be shown that (1.5) converts system (1.4) into

$$\begin{cases} w_{tt}(x,t) - w_{xx}(x,t) = 0,\ x\in(0,1),\ t>0, \\ w_x(0,t) = c_0 w(0,t),\ t\geqslant 0, \\ w_x(1,t) = -kw_t(1,t) + \widetilde{\theta}_1(t)\sin t + \widetilde{\theta}_2(t)\cos t,\ t\geqslant 0, \\ \dot{\widetilde{\theta}}_1(t) = -r_1 w_t(1,t)\sin t, t>0, \\ \dot{\widetilde{\theta}}_2(t) = -r_2 w_t(1,t)\cos t, t>0, \\ \widetilde{\theta}_1(0) = \widetilde{\theta}_{10},\ \widetilde{\theta}_2(0) = \widetilde{\theta}_{20}, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ 0\leqslant x\leqslant 1, \end{cases} \tag{1.7}$$

where

$$\begin{aligned} w_0(x) &= y_0(x) + (c_0+q)\int_0^x e^{q(x-\xi)} y_0(\xi)d\xi, \\ w_1(x) &= y_1(x) + (c_0+q)\int_0^x e^{q(x-\xi)} y_1(\xi)d\xi. \end{aligned} \tag{1.8}$$

We proceed as follows. In the next section, Section 2, the existence and uniqueness of classical solution to transformed system is given. The stability of transformed system and parameter estimation are discussed in Section 3. Section 4 is devoted to the well-posedness and stability of the closed-loop system. Finally, in Section 5, some numerical simulations are demonstrated to illustrate the theoretical results.

2. EXISTENCE AND UNIQUENESS OF CLASSICAL SOLUTION TO TRANSFORMED SYSTEM

Let $L^2(0,1)$ be the usual Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and inner product induced norm $\|\cdot\|$. We consider systems (1.7) and (1.2) in the energy state space $\mathcal{H} = H^1(0,1)\times L^2(0,1)\times\mathbb{R}^4 = \mathcal{V}\times\mathbb{R}^2$ with the inner product

$$\begin{aligned} &\langle (f_1,g_1,\xi_1,\eta_1,\omega_1,\varphi_1),(f_2,g_2,\xi_2,\eta_2,\omega_2,\varphi_2)\rangle_{\mathcal{H}} \\ &= \int_0^1 f_1'(x)f_2'(x)dx + \int_0^1 g_1(x)g_2(x)dx + c_0 f_1(0)f_2(0) + \frac{\xi_1\xi_2}{r_1} \\ &\quad + \frac{\eta_1\eta_2}{r_2} + \omega_1\omega_2 + \varphi_1\varphi_2,\ \forall\,(f_i,g_i,k_i,\xi_i,\eta_i,\omega_i,\varphi_i)\in\mathcal{H}, i=1,2, \end{aligned}$$

where $\mathcal{V} = H^1(0,1)\times L^2(0,1)\times\mathbb{R}^2$. Define the operator $\mathcal{A}: D(\mathcal{A})(\subset\mathcal{H})\to\mathcal{H}$ as follows:

$$\begin{cases} \mathcal{A}(u,v,\xi,\eta,\omega,\varphi) = (v,u'',-r_1 v(1)\omega,-r_2 v(1)\varphi,\varphi,-\omega), \\ D(\mathcal{A}) = \{(u,v,\xi,\eta,\omega,\varphi)\in(H^2(0,1)\cap H^1(0,1))\times H^1(0,1)\times\mathbb{R}^4| \\ u'(0) = c_0 u(0)\ u'(1) = -kv(1) + \xi\omega + \eta\varphi\}. \end{cases} \tag{2.1}$$

Then the systems (1.2) and (1.7) can be written as a nonlinear evolution equation

$$\frac{d}{dt}z(\cdot,t) = \mathcal{A}z(\cdot,t), z(\cdot,0) = z_0(\cdot)\in\mathcal{H}, \tag{2.2}$$

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where

$$\begin{cases} z(x,t) = (w(x,t), w_t(x,t), \widetilde{\theta}_1(t), \widetilde{\theta}_2(t), \omega_1(t), \omega_2(t)), \\ z_0(x) = (w_0(x), w_1(x), \widetilde{\theta}_{10}, \widetilde{\theta}_{20}, \omega_{10}, \omega_{20}). \end{cases}$$

Obviously, (2.2) is a nonautonomous revolution system. However, it seems hard to use nonlinear semigroup to prove its well-posedness because of lack of dissipativity of $\mathcal{A}$ defined by (2.1) or $\mathcal{A} + \omega I$ for any constant $\omega \in \mathbb{R}$. In fact, let $Z_1, Z_2 \in D(\mathcal{A})$, where

$$Z_1 = (f_1, g_1, \xi_1, \eta_1, \omega_1, \varphi_1), Z_2 = (f_2, g_2, \xi_2, \eta_2, \omega_2, \varphi_2).$$

Then,

$$\begin{aligned} &\langle \mathcal{A}Z_1 - \mathcal{A}Z_2, Z_1 - Z_2\rangle_{\mathcal{H}} \\ =& \int_0^1 (g_1'(x) - g_2'(x))(f_1'(x) - f_2'(x))dx + \int_0^1 (g_1(x) - g_2(x))(f_1''(x) - f_2''(x))dx \\ &+ c_0(g_1(0) - g_2(0))(f_1(0) - f_2(0)) - (\xi_1 - \xi_2)[g_1(1)\omega_1 - g_2(1)\omega_2] \\ &+ (\eta_1 - \eta_2)[g_2(1)\varphi_2 - g_1(1)\varphi_1] + (\omega_1 - \omega_2)(\varphi_1 - \varphi_2) + (\omega_2 - \omega_1)(\varphi_1 - \varphi_2) \\ =& -k[g_1(1) - g_2(1)]^2 + [g_1(1) - g_2(1)](\xi_1\omega_1 + \eta_1\varphi_1 - \xi_2\omega_2 - \eta_2\varphi_2) \\ &- (\xi_1 - \xi_2)[g_1(1)\omega_1 - g_2(1)\omega_2] - (\eta_1 - \eta_2)[g_1(1)\varphi_1 - g_2(1)\varphi_2], \end{aligned}$$

from which it is not easy to get the dissipativity.

Instead of using nonlinear semigroup theory, we prove the existence and uniqueness of classical solution to system (1.7) by constructing a Galerkin scheme. Define operator A in $L^2(0,1)$ as follows:

$$A\phi = -\phi'', \ \forall\, \phi \in D(A) = \left\{\phi \in L^2(0,1), \phi'(0) = c_0\phi(0), \phi'(1) = 0\right\}. \tag{2.3}$$

Then A is an unbounded self-adjoint positive definite operator in $L^2(0,1)$ with compact resolvent. A simple computation shows that the eigenpairs $\{(\lambda_n, \phi_n)\}_{n=1}^{\infty}$ are

$$\begin{cases} \lambda_n = \omega_n^2, \ \ \omega_n = n\pi + \mathcal{O}(n^{-1}), \\ \phi_n(x) = \dfrac{c_0}{\omega_n}\sin\omega_n x + \cos\omega_n x = \cos n\pi x + \mathcal{O}(n^{-1}), \end{cases} \tag{2.4}$$

x where ω_n satisfies

$$\tan\omega_n = \frac{c_0}{\omega_n}.$$

Because $\{\phi_n\}_{n=1}^{\infty}$ defined by (2.4) is approximately normalized (i.e., $0 < c_1 < \|\phi_n\|_{L^2(0,1)} < c_2$ for some constants c_1, c_2 independent of n), it forms an orthogonal basis for $L^2(0,1)$.

Let $V = H^3(0,1) \cap D(A)$.

Theorem 2.1
Suppose that $(w_0, w_1, \widetilde{\theta}_{10}, \widetilde{\theta}_{20}) \in V \times V \times \mathbb{R}^2$ satisfies the following compatible condition:

$$-kw_1(1) + \widetilde{\theta}_{20} = 0 \tag{2.5}$$

and

$$-kw_0''(1) + \widetilde{\theta}_{10} - r_2 w_1(1) = 0. \tag{2.6}$$

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Then system (1.7) admits a unique (smoother) classical solution w in the sense that for any time $T > 0$,

$$\begin{cases} w \in L^\infty(0,T;H^3(0,1)),\ w_t \in L^\infty(0,T;H^2(0,1)),\ w_{tt} \in L^\infty(0,T;H^1(0,1)), \\ \widetilde{\theta}_1 \in C^1[0,T], \widetilde{\theta}_2 \in C^1[0,T], \\ w_{tt}(x,t) - w_{xx}(x,t) = 0\ in\ L^\infty(0,T;L^2(0,1)), \\ w_x(0,t) = c_0 w(0,t),\ t \geqslant 0, \\ w_x(1,t) = -kw_t(1,t) + \widetilde{\theta}_1(t)\sin t + \widetilde{\theta}_2(t)\cos t,\ t \geqslant 0, \\ \dot{\widetilde{\theta}}_1(t) = -r_1 w_t(1,t)\sin t, t > 0, \\ \dot{\widetilde{\theta}}_2(t) = -r_2 w_t(1,t)\cos t, t > 0, \\ \widetilde{\theta}_1(0) = \widetilde{\theta}_{10}, \widetilde{\theta}_2(0) = \widetilde{\theta}_{20}, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x). \end{cases}$$

By the Sobolev embedding theorem, it follows that $w \in C([0,1] \times [0,T])$.

Proof
The uniqueness of the classical solution is straightforward. Actually, suppose that there are two solutions $(w, w_t, \widetilde{\theta}_1, \widetilde{\theta}_2)$ and $(\widehat{w}, \widehat{w}_t, \widehat{\theta}_1^*, \widehat{\theta}_2^*)$. Set $p(x,t) = w(x,t) - \widehat{w}(x,t)$. Then p satisfies

$$\begin{cases} p_{tt}(x,t) - p_{xx}(x,t) = 0, \\ p_x(0,t) = c_0 p(0,t), \\ p_x(1,t) = -kp_t(1,t) + (\widetilde{\theta}_1(t) - \widehat{\theta}_1^*(t))\sin t + (\widetilde{\theta}_2(t) - \widehat{\theta}_2^*(t))\cos t, \\ p(x,0) = 0,\ p_t(x,0) = 0, \\ \dot{\widetilde{\theta}}_1(t) = -r_1 w_t(1,t)\sin t, \\ \dot{\widehat{\theta}}{}_1^*(t) = -r_1 \hat{w}_t(1,t)\sin t, \\ \dot{\widetilde{\theta}}_2(t) = -r_2 w_t(1,t)\cos t, \\ \dot{\widehat{\theta}}{}_2^*(t) = -r_2 \hat{w}_t(1,t)\cos t, \\ \widetilde{\theta}_1(0) = \widetilde{\theta}_{10}, \widetilde{\theta}_2(0) = \widetilde{\theta}_{20}, \widehat{\theta}_1^*(0) = \widetilde{\theta}_{10}, \widehat{\theta}_2^*(0) = \widetilde{\theta}_{20}. \end{cases} \tag{2.7}$$

Define the Lyapunov-like functional following:

$$V_p(t) = \int_0^1 p_t^2(x,t)dx + \int_0^1 p_x^2(x,t)dx + \frac{1}{r_1}[\widetilde{\theta}_1(t) - \widehat{\theta}_1^*(t)]^2 + \frac{1}{r_2}[\widetilde{\theta}_2(t) - \widehat{\theta}_2^*(t)]^2. \tag{2.8}$$

A direct computation shows that the time derivative of $V_p(t)$ along the solution of (2.7) satisfies

$$\begin{aligned} \dot{V}_p(t) &= 2p_t(1,t)p_x(1,t) - 2[\widetilde{\theta}_1(t) - \widehat{\theta}_1^*(t)]p_t(1,t)\sin t - 2[\widetilde{\theta}_2(t) - \widehat{\theta}_2^*(t)]p_t(1,t)\cos t \\ &= -2[p_t(1,t)]^2 \leqslant 0. \end{aligned} \tag{2.9}$$

Hence, $V_p(t) \leqslant V_p(0) = 0$ or $(w, w_t, \widetilde{\theta}_1, \widetilde{\theta}_2) \equiv (\widehat{w}, \widehat{w}_t, \widehat{\theta}_1^*, \widehat{\theta}_2^*)$.

We mainly show the existence of solution. Transform (1.7) into an equivalent problem with zero initial value by the following transformation:

$$v(x,t) = w(x,t) - u(x,t), \tag{2.10}$$

$$u(x,t) = w_0(x) + tw_1(x). \tag{2.11}$$

Then v satisfies

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$$\begin{cases} v_{tt}(x,t) - v_{xx}(x,t) - u_{xx}(x,t) = 0, \ x \in (0,1), \ t > 0, \\ v_x(0,t) = c_0 v(0,t), \ t \geqslant 0, \\ v_x(1,t) = -k[v_t(1,t) + w_1(1)] + \widetilde{\theta}_1(t)\sin t + \widetilde{\theta}_2(t)\cos t, \ t \geqslant 0, \\ \dot{\widetilde{\theta}}_1(t) = -r_1[v_t(1,t) + w_1(1)]\sin t, t > 0, \\ \dot{\widetilde{\theta}}_2(t) = -r_2[v_t(1,t) + w_1(1)]\cos t, t > 0, \\ \widetilde{\theta}_1(0) = \widetilde{\theta}_{10}, \widetilde{\theta}_2(0) = \widetilde{\theta}_{20}, \\ v(x,0) = 0, \ v_t(x,0) = 0, \ 0 \leqslant x \leqslant 1. \end{cases} \tag{2.12}$$

v is a solution of (2.12) if and only if $w = v + u$ is a solution of (1.7). For sufficiently large $N \in Z$, let $V_N = \text{Span}\{\phi_1, \phi_2, \cdots, \phi_N\}$. Find the Galerkin approximation solution to (2.12):

$$v^N(x,t) = \sum_{n=1}^{N} g_{nN}(t)\phi_n(x), \tag{2.13}$$

which satisfies

$$\begin{cases} \langle v_{tt}^N(\cdot,t), \phi \rangle + \langle v_x^N(\cdot,t), \phi_x \rangle - \langle u_{xx}(\cdot,t), \phi \rangle \\ = \left[-k[v_t^N(1,t) + w_1(1)] + \widetilde{\theta}_1^N(t)\sin t + \widetilde{\theta}_2^N(t)\cos t \right] \phi(1) - c_0 v^N(0,t)\phi(0), \ \forall \phi \ \in V_N, \\ \dot{\widetilde{\theta}}_1^N(t) = -r_1[v_t^N(1,t) + w_1(1)]\sin t, \ t > 0, \\ \dot{\widetilde{\theta}}_2^N(t) = -r_2[v_t^N(1,t) + w_1(1)]\cos t, t > 0, \\ \widetilde{\theta}_1^N(0) = \widetilde{\theta}_{10}, \widetilde{\theta}_2^N(0) = \widetilde{\theta}_{20}, \\ v^N(x,0) = 0, \ v_t^N(x,0) = 0, \ 0 \leqslant x \leqslant 1. \end{cases} \tag{2.14}$$

The existence and uniqueness of the solution to (2.14) in some interval $[0, t_N), t_N > 0$ are ensured by the local Lipschitz condition. The lemma 2.1 next shows that $t_N = T$. The proof for the existence of solution will be split into several lemmas.

Lemma 2.1

$$\sup_N [\|\dot{v}^N(\cdot,t)\| + \|v_x^N(\cdot,t)\| + c_0[v^N(0,t)]^2 + |\widetilde{\theta}_1^N(t)| + |\widetilde{\theta}_2^N(t)|] < \infty \ for \ t \in [0,T] \text{ a.e..} \tag{2.15}$$

Proof
Integrate the first equation of (2.14) with $\phi = v_t^N(\cdot,t)$, to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{ \|\dot{v}^N(\cdot,t)\|^2 + \|v_x^N(\cdot,t)\|^2 + c_0[v^N(0,t)]^2 + \frac{1}{r_1}[\widetilde{\theta}_1^N(t)]^2 + \frac{1}{r_2}[\widetilde{\theta}_2^N(t)]^2 \right\} \\ &= v_t^N(1,t)\left[-k[v_t^N(1,t) + w_1(1)] + \widetilde{\theta}_1^N(t)\sin t + \widetilde{\theta}_2^N(t)\cos t \right] \\ &\quad + \frac{1}{r_1}\widetilde{\theta}_1^N(t)\dot{\widetilde{\theta}}_1^N(t) + \frac{1}{r_2}[\widetilde{\theta}_2^N(t)]\dot{\widetilde{\theta}}_2^N(t) + \langle u_{xx}(\cdot,t), v_t^N(\cdot,t) \rangle \\ &= -k[v_t^N(1,t) + w_1(1)]^2 - k[w_1(1)]^2 + kw_1(1)v_t^N(1,t) \\ &\quad - w_1(1)[\widetilde{\theta}_1^N(t)\sin t + \widetilde{\theta}_2^N(t)\cos t] + \langle u_{xx}(\cdot,t), v_t^N(\cdot,t) \rangle \\ &\leqslant kw_1(1)v_t^N(1,t) - w_1(1)[\widetilde{\theta}_1^N(t)\sin t + \widetilde{\theta}_2^N(t)\cos t] + \langle u_{xx}(\cdot,t), v_t^N(\cdot,t) \rangle. \end{aligned}$$

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Hence,

$$
\begin{aligned}
&\|\dot{v}^N(\cdot,t)\|^2+\|v_x^N(\cdot,t)\|^2+\frac{1}{r_1}[\widetilde{\theta}_1^N(t)]^2+\frac{1}{r_2}[\widetilde{\theta}_2^N(t)]^2+c_0[v^N(0,t)]^2\\
&\leqslant 2kw_1(1)\int_0^t v_t^N(1,s)ds-2w_1(1)\int_0^t[\widetilde{\theta}_1^N(s)\sin s+\widetilde{\theta}_2^N(s)\cos s]ds\\
&\quad+2\int_0^t\langle u_{xx}(\cdot,s),v_t^N(\cdot,s)\rangle ds+\frac{\widetilde{\theta}_{10}^2}{r_1}+\frac{\widetilde{\theta}_{20}^2}{r_2}\\
&\leqslant 2kw_1(1)v^N(1,t)+|w_1(1)|\int_0^t\frac{1}{r_1}[\widetilde{\theta}_1^N(s)]^2ds+|w_1(1)|\int_0^t[r_1\sin^2 s+r_2\cos^2 s]ds\\
&\quad+|w_1(1)|\int_0^t\frac{1}{r_2}[\widetilde{\theta}_2^N(s)]^2ds+\int_0^t\|u_{xx}(\cdot,s)\|^2ds+\int_0^t\|\dot{v}^N(\cdot,s)\|^2ds+\frac{\widetilde{\theta}_{10}^2}{r_1}+\frac{\widetilde{\theta}_{20}^2}{r_2}\\
&\leqslant 4\delta_1k^2w_1^2(1)+\frac{4}{\delta_1}\left[[v^N(0,t)]^2+\|v_x^N(\cdot,t)\|^2\right]+|w_1(1)|\int_0^t\frac{1}{r_1}[\widetilde{\theta}_1^N(s)]^2ds\\
&\quad+|w_1(1)|\int_0^t[r_1\sin^2 s+r_2\cos^2 s]ds+|w_1(1)|\int_0^t\frac{1}{r_2}[\widetilde{\theta}_2^N(s)]^2ds\\
&\quad+\int_0^t\|u_{xx}(\cdot,s)\|^2ds+\int_0^t\|\dot{v}^N(\cdot,s)\|^2ds+\frac{\widetilde{\theta}_{10}^2}{r_1}+\frac{\widetilde{\theta}_{20}^2}{r_2}.
\end{aligned}
\tag{2.16}
$$

Taking $\delta_1>\max\left\{4,\frac{4}{c_0}\right\}$ and applying Gronwall's inequality to 2.16 to obtain inequality (2.15). Therefore, (2.14) admits a unique global classical solution in whole $[0,T]$. □

Lemma 2.2

$$\sup_N\|\ddot{v}^N(\cdot,0)\|<\infty. \tag{2.17}$$

Proof
Let $t=0$ in the first equation of (2.14). Then for any $\phi\in V_N$, we have

$$\langle v_{tt}^N(\cdot,0),\phi\rangle+\langle w_0''(\cdot),\phi\rangle=[-kw_1(1)+\widetilde{\theta}_{20}]\phi(1). \tag{2.18}$$

By (2.18) and the compatible condition (2.5), it follows that

$$\langle v_{tt}^N(\cdot,0),\phi\rangle=\langle w_0''(\cdot),\phi\rangle,\ \ \forall\phi\in V_N. \tag{2.19}$$

Now take $\phi_i, i=1,2,\cdots,N$ in (2.19) to produce

$$\ddot{g}_{nN}(0)=\langle w_0''(\cdot),\phi_n\rangle,\ n=1,2,\cdots,N. \tag{2.20}$$

And then multiply (2.20) by ϕ_n and sum for $n=1,2,\cdots,N$ to obtain

$$v_{tt}^N(\cdot,0)=\sum_{n=1}^N\langle w_0''(\cdot),\phi_n\rangle\phi_n=w_{xx}^N(\cdot,0). \tag{2.21}$$

Thus, we have $||v_{tt}^N(\cdot,0)||=\|w_{xx}^N(\cdot,0)\|=\|Aw^N(\cdot,0)\|\leqslant\|w_0''\|$. □

Lemma 2.3

$$\sup_N\left[\|\ddot{v}^N(\cdot,t)\|+\|\dot{v}_x^N(\cdot,t)\|+c_0[v_t^N(0,t)]^2\right]<\infty\ for\ t\in[0,T]\text{ a.e..} \tag{2.22}$$

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Proof
Differentiate the first equation of (2.14) with respect to t, to obtain

$$\begin{aligned}&\langle v_{ttt}^N(\cdot,t),\phi\rangle+\langle v_{xt}^N(\cdot,t),\phi_x\rangle-\langle w_1''(\cdot),\phi\rangle=[-kv_{tt}^N(1,t)+\widetilde{\theta}_1^N(t)\cos t-\widetilde{\theta}_2^N(t)\sin t]\phi(1)\\&\quad-(r_1\sin^2 t+r_2\cos^2 t)[v_t^N(1,t)+w_1(1)]\phi(1)-c_0v_t^N(0,t)\phi(0),\ \forall\phi\in V_N.\end{aligned}\tag{2.23}$$

Take $\phi=v_{tt}^N(\cdot,t)$ in (2.23), to yield

$$\begin{aligned}&\langle v_{ttt}^N(\cdot,t),v_{tt}^N(\cdot,t)\rangle+\langle v_{xt}^N(\cdot,t),v_{ttx}^N(\cdot,t)\rangle-\langle w_1''(\cdot),v_{tt}^N(\cdot,t)\rangle\\&\quad=-k[v_{tt}^N(1,t)]^2+v_{tt}^N(1,t)[\widetilde{\theta}_1^N(t)\cos t-\widetilde{\theta}_2^N(t)\sin t]\\&\quad-(r_1\sin^2 t+r_2\cos^2 t)[v_t^N(1,t)+w_1(1)]v_{tt}^N(1,t)-c_0v_t^N(0,t)v_{tt}^N(0,t).\end{aligned}\tag{2.24}$$

Hence,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\|\ddot v^N(\cdot,t)\|^2+\|\dot v_x^N(\cdot,t)\|^2+c_0[v_t^N(0,t)]^2\right\}\\&=-k[v_{tt}^N(1,t)]^2+v_{tt}^N(1,t)[\widetilde{\theta}_1^N(t)\cos t-\widetilde{\theta}_2^N(t)\sin t]\\&\quad-(r_1\sin^2 t+r_2\cos^2 t)[v_t^N(1,t)+w_1(1)]v_{tt}^N(1,t)+\langle w_1''(\cdot),v_{tt}^N(\cdot,t)\rangle\\&\leqslant-\left(1-\frac{\delta_2}{2}-\frac{\delta_3}{2}\right)k[v_{tt}^N(0,t)]^2+\frac{1}{2\delta_2k}[\widetilde{\theta}_1^N(t)\cos t-\widetilde{\theta}_2^N(t)\sin t]^2\\&\quad+\frac{1}{2\delta_3k}(r_1+r_2)^2[v_t^N(1,t)+w_1(1)]^2+\langle w_1''(\cdot),v_{tt}^N(\cdot,t)\rangle.\end{aligned}\tag{2.25}$$

Integrate both sides of (2.25) over $[0,t]$ in t and take $\delta_2>0,\delta_3>0,\delta_2+\delta_3<2$ to get

$$\begin{aligned}&\|\ddot v^N(\cdot,t)\|^2+\|\dot v_x^N(\cdot,t)\|^2+c_0[v_t^N(0,t)]^2\\&\leqslant\frac{1}{\delta_2k}\int_0^t[\widetilde{\theta}_1^N(s)\cos s-\widetilde{\theta}_2^N(s)\sin s]^2ds\\&\quad+\frac{(r_1+r_2)^2}{\delta_3k}\int_0^t[v_t^N(1,s)+w_1(1)]^2ds+\|\ddot v^N(\cdot,0)\|^2+2\int_0^t\langle w_1''(\cdot),v_{tt}^N(\cdot,s)\rangle ds\\&\leqslant\frac{1}{\delta_2k}\int_0^t[\widetilde{\theta}_1^N(s)\cos s-\widetilde{\theta}_2^N(s)\sin s]^2ds+\frac{4(r_1+r_2)^2}{\delta_3k}\int_0^t\|\dot v_x^N(\cdot,t)\|^2\\&\quad+\frac{4(r_1+r_2)^2}{\delta_3k}\int_0^t[v_t^N(0,s)]^2ds+\frac{2(r_1+r_2)^2}{\delta_3k}w_1^2(1)T\\&\quad+\|\ddot v^N(\cdot,0)\|^2+T\|w_1''\|^2+\int_0^t\|\ddot v^N(\cdot,s)\|^2ds.\end{aligned}\tag{2.26}$$

Once again, applying Grownwall's inequality to (2.26) and taking (2.15) and (2.17) into account produces (2.22). □

Lemma 2.4

$$\sup_N\left\{\|\ddot v_t^N(\cdot,t)\|^2+\|\ddot v_x^N(\cdot,t)\|^2+c_0[v_{tt}^N(0,t)]^2\right\}<\infty\ for\ t\in[0,T]\text{ a.e..}\tag{2.27}$$

Proof
Differentiating the first equation of (2.23) with respect to t leads to

$$\begin{aligned}&\langle v_{tttt}^N(\cdot,t),\phi\rangle+\langle v_{xtt}^N(\cdot,t),\phi_x\rangle=\{-kv_{ttt}^N(1,t)-[\widetilde{\theta}_1^N(t)\sin t+\widetilde{\theta}_2^N(t)\cos t]\\&\quad-(3r_1-3r_2)[v_t^N(1,t)+w_1(1)]\sin t\cos t\\&\quad-(r_1\sin^2 t+r_2\cos^2 t)v_{tt}^N(1,t)\}\phi(1)-c_0v_{tt}^N(0,t)\phi(0).\end{aligned}\tag{2.28}$$

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Again, substitution of ϕ by $\ddot{v}_t^N(\cdot,t)$ in (2.28) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\|\ddot{v}_t^N(\cdot,t)\|^2+\|\ddot{v}_x^N(\cdot,t)\|^2+c_0[v_{tt}^N(0,t)]^2\}=-k[v_{ttt}^N(1,t)]^2\\&-[\widetilde{\theta}_1^N(t)\sin t+\widetilde{\theta}_2^N(t)\cos t]v_{ttt}^N(1,t)-\frac{3r_1-3r_2}{2}\sin 2t[v_t^N(1,t)+w_1(1)]v_{ttt}^N(1,t)\\&-(r_1\sin^2 t+r_2\cos^2 t)v_{tt}^N(1,t)v_{ttt}^N(1,t).\end{aligned} \tag{2.29}$$

Integrate over $[0,t]$ to give

$$\begin{aligned}&\|\ddot{v}_t^N(\cdot,t)\|^2+\|\ddot{v}_x^N(\cdot,t)\|^2+c_0[v_{tt}^N(0,t)]^2=-2k\int_0^t[\ddot{v}_t^N(0,s)]^2ds\\&-2\int_0^t[\widetilde{\theta}_1^N(s)\sin s+\widetilde{\theta}_2^N(s)\cos s]\ddot{v}_t^N(1,s)ds-(3r_1-3r_2)\int_0^t\sin 2s[\dot{v}^N(1,s)+w_1(1)]\ddot{v}_t^N(1,s)ds\\&-2\int_0^t(r_1\sin^2 s+r_2\cos^2 s)\ddot{v}^N(1,s)\ddot{v}_t^N(1,s)ds\\&+\|\ddot{v}_x^N(\cdot,0)\|^2+\|\ddot{v}_t^N(\cdot,0)\|^2+c_0[v_{tt}^N(0,0)]^2=\sum_{i=1}^7 I_i.\end{aligned} \tag{2.30}$$

Now, we estimate each term on the right-hand side of (2.30).

$$\begin{aligned}I_2&=-2\int_0^t[\widetilde{\theta}_1^N(s)\sin s+\widetilde{\theta}_2^N(s)\cos s]\ddot{v}_t^N(1,s)ds\\&\leqslant\delta_4 k\int_0^t[\ddot{v}_t^N(0,s)]^2ds+\frac{1}{\delta_4 k}\int_0^t[\widetilde{\theta}_1^N(s)\sin s+\widetilde{\theta}_2^N(s)\cos s]^2ds.\end{aligned}$$

$$\begin{aligned}I_3&=-\int_0^t(3r_1-3r_2)\sin 2s[\dot{v}^N(1,s)+w_1(1)]\ddot{v}_t^N(1,s)ds\\&\leqslant\frac{1}{2}\delta_5 k\int_0^t[\ddot{v}_t^N(1,s)]^2ds+\frac{1}{2\delta_5 k}(3r_1-3r_2)^2\int_0^t[\dot{v}^N(1,s)+w_1(1)]^2ds\\&+\frac{1}{2}\delta_5 k\int_0^t[\ddot{v}_t^N(1,s)]^2ds+\frac{2}{\delta_5 k}(3r_1-3r_2)^2\int_0^t\|\dot{v}_x^N(\cdot,t)\|^2ds\\&+\frac{2}{\delta_5 k}(3r_1-3r_2)^2\int_0^t\|\dot{v}^N(0,s)\|^2ds+\frac{1}{\delta_5 k}(3r_1-3r_2)^2w_1^2(1)T.\end{aligned}$$

$$\begin{aligned}I_4&=-2\int_0^t(r_1\sin^2 s+r_2\cos^2 s)\ddot{v}^N(1,s)\ddot{v}_t^N(1,s)ds\\&=-(r_1\sin^2 t+r_2\cos^2 t)[\ddot{v}^N(1,t)]^2+r_2[\ddot{v}^N(1,0)]^2+\int_0^t[\ddot{v}^N(1,s)]^2(r_1-r_2)\sin 2s ds\\&\leqslant r_2[\ddot{v}^N(1,0)]^2+2(r_1+r_2)\int_0^t\|\ddot{v}_x^N(x,s)\|^2ds+2(r_1+r_2)\int_0^t[v_{tt}^N(0,s)]^2ds\\&\leqslant r_2[Aw_0^N(1)]^2+2(r_1+r_2)\int_0^t\|\ddot{v}_x^N(x,s)\|^2ds+2(r_1+r_2)\int_0^t[v_{tt}^N(0,s)]^2ds.\end{aligned}$$

$$I_5=\|\ddot{v}_x^N(\cdot,0)\|^2\leqslant\|w_0^{(3)}\|^2.$$

From (2.23), it follows that

$$\langle v_{ttt}^N(\cdot,0),\phi\rangle-\langle w_1''(\cdot),\phi\rangle=[-k\ddot{v}^N(1,0)+\widetilde{\theta}_{10}-r_2w_1(1)]\phi(1),\ \forall\phi\in V_N. \tag{2.31}$$

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Take $\phi = \phi_n,\ n = 1, 2, \cdots, N$ in (2.31) to give

$$g_{ttt}^N(0) - \langle w_1''(\cdot), \phi_n\rangle = [-k\ddot{v}^N(1,0) + \widetilde{\theta}_{10} - r_2 w_1(1)]\phi_n(1). \tag{2.32}$$

Multiply by ϕ_n and sum for $n = 1, 2, \cdots, N$ in (2.31) to produce

$$v_{ttt}^N(\cdot,0) - \sum_{n=1}^{N}\langle w_1''(\cdot), \phi_n\rangle\phi_n = [-k\ddot{v}^N(1,0) + \widetilde{\theta}_{10} - r_2 w_1(1)]\sum_{n=1}^{N}\phi_n(1)\phi_n \tag{2.33}$$

It follows from (2.20), (2.4) and the compatible condition (2.6) that

$$|-k\ddot{v}^N(1,0) + \widetilde{\theta}_{10} - r_2 w_1(1)| = k|\sum_{n=N+1}^{\infty}\langle w_0'', \varphi_n\rangle\phi_n(1)| \leqslant \frac{C_1\|w_0'''\|}{N},$$

where C_1 is a constant independent of N. This together with the fact $\|\sum_{n=1}^N \phi_n(1)\phi_n\| = C_2\sqrt{N}$ gives that

$$\|v_{ttt}^N(\cdot,0) - \sum_{n=1}^{N}\langle w_1''(\cdot), \phi_n\rangle\phi_n\| \leqslant C_2 C_1\|w_0'''\|.$$

Therefore,

$$I_6 = \|\ddot{w}_t^N(\cdot,0)\|^2 \leqslant \|w_1''\|^2 + C_2C_1\|w_0'''\|,\ I_7 = c_0[v_{tt}^N(0,0)]^2 = c_0[Aw_0^N(0)]^2.$$

The estimates for $I_i, i = 2, 3, \cdots, 7$ together with (2.15) and (2.22) show that

$$\begin{aligned}\|\ddot{w}_t^N(\cdot,t)\|^2 + \|\ddot{v}_x^N(\cdot,t)\|^2 + c_0[v_{tt}^N(0,t)]^2 &\leqslant C - (2 - \delta_4 - \frac{1}{2}\delta_5)k\int_0^t[\ddot{v}_t^N(0,s)]^2 ds\\ &+ 2(r_1+r_2)\int_0^t\|\ddot{v}_x^N(x,s)\|^2 ds + 2(r_1+r_2)\int_0^t[v_{tt}^N(0,s)]^2 ds,\end{aligned}$$

where C is a constant independent of N but depending on the initial date and T. Take $\delta_i > 0, i = 4, 5$ and $C_3 = \max\left\{2(r_1+r_2), \frac{2}{c_0}(r_1+r_2)\right\}$ so that $2 - \delta_4 - \frac{1}{2}\delta_5 > 0$ to get

$$\begin{aligned}\|\ddot{w}_t^N(\cdot,t)\|^2 + \|\ddot{w}_x^N(\cdot,t)\|^2 + c_0[w_{tt}^N(0,t)]^2 &\leqslant C\\ + C_3\int_0^t\{\|\ddot{w}_t^N(\cdot,s)\|^2 + \|\ddot{w}_x^N(\cdot,s)\|^2 &+ c_0[w_{tt}^N(0,s)]^2\}ds.\end{aligned} \tag{2.34}$$

Apply Gronwall's inequality to (2.34) to produce $\sup_N\|\ddot{w}_x^N(\cdot,t)\| < \infty$ uniformly for almost every $t \in [0,T]$. The proof is complete. □

Remark 2.1
In Theorem 2.1, the condition (2.5) is the natural compatible condition for the classical solution of (1.7), and the condition (2.6) is for the existence of the more smoother solution that we shall need in the proof of Theorem 3.1.

The Lemmas 2.1, 2.2, and 2.4 enable us to obtain four subsequences of $\{v^N\}, \{\widetilde{\theta}_1^N\}$ and $\{\widetilde{\theta}_2^N\}$, which will still be denoted by $\{v^N\}, \{\widetilde{\theta}_1^N\}$ and $\{\widetilde{\theta}_2^N\}$ without confusion, and a function $v, \widetilde{\theta}_1$ and $\widetilde{\theta}_2$ so that

$$v^N \rightharpoonup v \text{ in } L^\infty(0,T;H^1(0,1)) \text{ weak}^*, \tag{2.35}$$

$$\dot{v}^N \rightharpoonup \dot{v} \text{ in } L^\infty(0,T;H^1(0,1)) \text{ weak}^*, \tag{2.36}$$

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$$\ddot{v}^N \rightharpoonup \ddot{v} \ \text{ in } \ L^\infty(0,T;L^2(0,1)) \ \text{ weak}^*, \tag{2.37}$$

$$v_t^N(0,t) \rightharpoonup v_t(0,t) \ \text{ in } \ L^2(0,T) \ \text{ weakly}, \tag{2.38}$$

$$\widetilde{\theta}_1^N(t) \rightharpoonup \widetilde{\theta}_1(t) \ \text{ in } \ L^2(0,T) \ \text{ weakly}, \tag{2.39}$$

$$\widetilde{\theta}_2^N(t) \rightharpoonup \widetilde{\theta}_2(t) \ \text{ in } \ L^2(0,T) \ \text{ weakly}, \tag{2.40}$$

$$-k[v_t^N(1,t)+w_1(1)] + \widetilde{\theta}_1^N(t)\sin t + \widetilde{\theta}_2^N(t)\cos t \rightharpoonup \chi \ \text{ in } \ L^2(0,T) \ \text{ weakly}. \tag{2.41}$$

Because $\{\phi_n\}_{n=1}^\infty$ is a basis of $L^2(0,1)$, for all $\eta \in \mathcal{D}(0,T)$ and all $\varphi \in L^2(0,1)$, pass to the limit in (2.14) as $N \to \infty$ to obtain

$$\begin{aligned} &\int_0^T \langle v_{tt}(\cdot,t),\phi\rangle \eta(t)dt + \int_0^T \langle v_x(\cdot,t),\phi_x\rangle\eta(t)dt - \int_0^T \langle u_{xx}(\cdot,t),\phi\rangle\eta(t)dt \\ &= \int_0^T \chi(t)\eta(t)dt\phi(1) - \int_0^T c_0 v(0,t)\eta(t)dt\phi(0). \end{aligned} \tag{2.42}$$

Taking $\phi \in \mathcal{D}(0,1)$ in (2.42), we get

$$v_{tt} = v_{xx} + u_{xx} \text{ in } \mathcal{D}'((0,1)\times(0,T)).$$

Because $v_{tt} \in L^2(0,T;L^2(0,1))$, one has $v_{xx} \in L^2(0,T;L^2(0,1))$. Therefore,

$$v_{tt} = v_{xx} + u_{xx} \text{ in } \ L^2(0,T;L^2(0,1)). \tag{2.43}$$

By integration by parts, we deduce that

$$v_x(0,t) = c_0 v(0,t) \text{ in } \ L^2(0,T), \tag{2.44}$$

$$v_x(1,t) = \chi(t) \text{ in } \ L^2(0,T). \tag{2.45}$$

Now we show that

$$\chi(t) = -k[v_t(1,t)+w_1(1)] + \widetilde{\theta}_1(t)\sin t + \widetilde{\theta}_2(t)\cos t.$$

Actually, from $v_t^N(1,t) = v_t^N(0,t) + \int_0^1 v_{xt}^N(x,t)dx$, we conclude, by (2.36) and (2.38), that $\dot{v}^N(1,t) \rightharpoonup \dot{v}(1,t)$ in $L^2(0,T)$. On the other hand, by Lemmas 2.1, 2.3, and 2.4, it follows that $\max\limits_{t\in[0,T]}\sup\limits_N |\dot{v}^N(1,t)| = \max\limits_{t\in[0,T]}\sup\limits_N \left\{|\dot{v}^N(0,t) + \int_0^1 v_{xt}^N(x,t)dx|\right\} < \infty$ and $\max\limits_{0\leqslant t\leqslant T}|\ddot{v}^N(1,t)| = \max\limits_{0\leqslant t\leqslant T}|\ddot{v}^N(0,t) + \int_0^1 v_{xtt}^N(x,t)dx| < \infty$. So, the sequence $\{\dot{v}^N(1,t)\}_{N=1}^\infty$ is uniformly bounded and equicontinuous. By the Ascoli–Arzelá theorem, we conclude that

$$\dot{v}^N(1,t) \to \dot{v}(1,t) \ \text{ in } \ C[0,T], \tag{2.46}$$

and so $\dot{v}(1,t)$ is continuous in t. Similarly,

$$\begin{cases} \widetilde{\theta}_1^N(t) \to \widetilde{\theta}_1(t) \ \text{ in } \ C[0,T], \\ \widetilde{\theta}_2^N(t) \to \widetilde{\theta}_2(t) \ \text{ in } \ C[0,T]. \end{cases} \tag{2.47}$$

Therefore,

$$\begin{aligned} &-k[v_t^N(1,t)+w_1(1)] + \widetilde{\theta}_1^N(t)\sin t + \widetilde{\theta}_2^N(t)\cos t \\ &\to -k[v_t(1,t)+w_1(1)] + \widetilde{\theta}_1(t)\sin t + \widetilde{\theta}_2(t)\cos t \ \text{ in } \ C[0,T]. \end{aligned} \tag{2.48}$$

Combine (2.41) and (2.48) to get

$$\chi(t) = -k[v_t(1,t)+w_1(1)] + \widetilde{\theta}_1(t)\sin t + \widetilde{\theta}_2(t)\cos t.$$

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From (2.46) and (2.47), it is seen that

$$\begin{cases} \dot{\widetilde{\theta}}_1^N(t) = -r_1[v_t^N(1,t) + w_1(1)]\sin t \to \dot{\widetilde{\theta}}_1(t) \\ \quad = -r_1[v_t(1,t) + w_1(1)]\sin t, \text{ in } C[0,T], \\ \dot{\widetilde{\theta}}_2^N(t) = -r_2[v_t^N(1,t) + w_1(1)]\cos t \to \dot{\theta}_2(t) \\ \quad = -r_2[v_t(1,t) + w_1(1)]\cos t \text{ in } C[0,T]. \end{cases}$$

The existence is thus proved.

3. STABILITY OF THE TRANSFORMED SYSTEM

In this section, we establish the convergence of the transformed system (1.7). To do this, we need the weak solution of (1.7).

Definition 1
For any initial data $(w_0, w_1, \widetilde{\theta}_{10}, \widetilde{\theta}_{20}) \in \mathcal{V}$, the weak solution $(w, w_t, \widetilde{\theta}_1, \widetilde{\theta}_2)$ of (1.7) is defined as the limit of any convergent subsequence of $(w^n, w_t^n, \widetilde{\theta}_1^n, \widetilde{\theta}_2^n)$ in the space $L^\infty(0,\infty;\mathcal{V})$ where $(w^n, w_t^n, \widetilde{\theta}_1^n, \widetilde{\theta}_2^n)$ is the classical solution ensured by Theorem 3.1 with the initial condition

$$(w^n(x,0), w_t^n(x,0), \widetilde{\theta}_1^n(0), \widetilde{\theta}_2^n(0)) = (w_0^n(x), w_1^n(x), \widetilde{\theta}_{10}^n, \widetilde{\theta}_{20}^n) \in V \times V \times \mathbb{R}^2, \forall x \in (0,1)$$

satisfying

$$\lim_{n\to\infty} \|(w_0^n(x), w_1^n(x), \widetilde{\theta}_{10}^n, \widetilde{\theta}_{20}^n) - (w_0, w_1, \widetilde{\theta}_{10}, \widetilde{\theta}_{20})\|_{\mathcal{V}} = 0.$$

Definition (1) makes sense because from (2.8) and (2.9), we know that $\{(w^n, w_t^n, \widetilde{\theta}_1^n, \widetilde{\theta}_2^n)\}$ must be a Cauchy sequence in $L^\infty(0,\infty;\mathcal{V})$, and its limit does not depend on the choice of the initial values.

Theorem 3.1
For any initial value $(w_0, w_1, \widetilde{\theta}_{10}, \widetilde{\theta}_{20}) \in \mathcal{V}$, the solution of the system (1.7) is asymptotically stable in the sense that

$$\lim_{t\to\infty} \left[\frac{1}{2}\int_0^1 [w_t^2(x,t) + w_x^2(x,t)]dx + c_0 w^2(0,t) \right] = 0$$

and

$$\lim_{t\to\infty} \theta_1(t) = \bar{\theta}_1, \ \lim_{t\to\infty} \theta_2(t) = \bar{\theta}_2.$$

Proof
By density argument, we may regard without loss of generality that the initial value $(w_0, w_1, \widetilde{\theta}_{10}, \widetilde{\theta}_{20})$ belongs to $V \times V \times \mathbb{R}^2$ and satisfies compatible conditions (2.5) and (2.6). Construct Lyapunov functional $V_w(t)$ for the system (2.2) following:

$$V_w(t) = \frac{1}{2}\int_0^1 [w_t^2(x,t) + w_x^2(x,t)]dx + \frac{c_0}{2}w^2(0,t) + \frac{1}{2r_1}\widetilde{\theta}_1(t)^2 + \frac{1}{2r_2}\widetilde{\theta}_2(t)^2 + \omega_1^2(t) + \omega_2^2(t),$$

where $\omega_1(t) = \sin t, \omega_2(t) = \cos t$. The time derivative of $V_w(t)$ along the solution of system (2.2) is found to be

$$\dot{V}_w(t) = -k[w_t(1,t)]^2.$$

This shows that $V_w(t) \leqslant V_w(0)$ and hence

$$\sup_{t\geqslant 0} \left[\frac{1}{2}\int_0^1 [w_t^2(x,t) + w_x^2(x,t)]dx + c_0 w^2(0,t) + |\widetilde{\theta}_1(t)| + |\widetilde{\theta}_2(t)| \right] < \infty. \tag{3.1}$$

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In particular, one has

$$w_t(1,t) \in L^2(0,\infty). \tag{3.2}$$

Similarly, let

$$U(t) = \frac{1}{2}\int_0^1 \left[w_{xx}^2(x,t) + w_{tx}^2(x,t)\right] dx + c_0 w_t^2(0,t).$$

It is found that the time derivative of $U(t)$ along the solution of transformed system (1.7) can be found as

$$\begin{aligned}\dot{U}(t) = & -k[w_{tt}(1,t)]^2 - w_{tt}(1,t)[\widetilde{\theta}_1(t)\cos t - \widetilde{\theta}_2(t)\sin t] \\ & - (r_1 \sin^2 t + r_2 \cos^2 t) w_t(1,t) w_{tt}(1,t).\end{aligned} \tag{3.3}$$

Integrating over $[0,t]$ on both sides of (3.3) gives

$$\begin{aligned}U(t) = & -k\int_0^t [\ddot{w}(1,s)]^2 ds - \int_0^t \ddot{w}(1,s)[\widetilde{\theta}_1(s)\cos s - \widetilde{\theta}_2(s)\sin s] ds \\ & - \int_0^t (r_1 \sin^2 s + r_2\cos^2 s)\dot{w}(1,s)\ddot{w}(1,s) ds + U(0) \\ = & -k\int_0^t [\ddot{w}(1,s)]^2 ds - \dot{w}(1,t)[\widetilde{\theta}_1(t)\cos t - \widetilde{\theta}_2(t)\sin t] + w_1(1)\widetilde{\theta}_{10} \\ & - \int_0^t \dot{w}(1,s)[\widetilde{\theta}_1(s)\sin s + \widetilde{\theta}_2(s)\cos s] ds + \frac{r_2 - r_1}{2}\int_0^t [\dot{w}(1,s)]^2 \sin 2s ds \\ & - \int_0^t (r_1 \sin^2 s + r_2\cos^2 s)\dot{w}(1,s)\ddot{w}(1,s) ds + U(0) \\ = & -k\int_0^t [\ddot{w}(1,s)]^2 ds - \dot{w}(1,t)[\widetilde{\theta}_1(t)\cos t - \widetilde{\theta}_2(t)\sin t] + w_1(1)\widetilde{\theta}_{10} \\ & + \frac{\widetilde{\theta}_1^2(t)}{2r_1} - \frac{\widetilde{\theta}_{10}^2}{2r_1} + \frac{\widetilde{\theta}_2^2(t)}{2r_2} - \frac{\widetilde{\theta}_{20}^2}{2r_2} + \frac{r_2 - r_1}{2}\int_0^t [\dot{w}(1,s)]^2 \sin 2s ds \\ & - \int_0^t (r_1 \sin^2 s + r_2\cos^2 s)\dot{w}(1,s)\ddot{w}(1,s) ds + U(0).\end{aligned} \tag{3.4}$$

From (3.4), we have

$$\begin{aligned}U(t) \leqslant & -(1 - \frac{\delta_6}{2})k\int_0^t [\ddot{w}(1,s)]^2 ds + \delta_7 c_0 w_t^2(0,t) + \delta_7 c_0 \int_0^1 w_{tx}^2(x,t) dx \\ & + \frac{1}{2\delta_7 c_0}[\widetilde{\theta}_1(t)\cos t - \widetilde{\theta}_2(t)\sin t]^2 + \frac{\widetilde{\theta}_1^2(t)}{2r_1} + \frac{\widetilde{\theta}_2^2(t)}{2r_2} \\ & + \frac{1}{2\delta_6}[(r_1 + r_2)^2 + (r_1 + r_2)]\int_0^\infty [\dot{w}(1,s)]^2 ds + U(0).\end{aligned} \tag{3.5}$$

Take $0 < \delta_6 < 2$ and $0 < \delta_7 < \min\left\{\frac{1}{4}, \frac{1}{4c_0}\right\}$ to give

$$\begin{aligned}U(t) \leqslant & \frac{1}{\delta_7 c_0}[\widetilde{\theta}_1(t)\cos t - \widetilde{\theta}_2(t)\sin t]^2 + \frac{\widetilde{\theta}_1^2(t)}{r_1} + \frac{\widetilde{\theta}_2^2(t)}{r_2} \\ & + \frac{1}{\delta_6}[(r_1 + r_2)^2 + (r_1 + r_2)]\int_0^\infty [\dot{w}(1,s)]^2 ds + 2U(0).\end{aligned} \tag{3.6}$$

It is found from (3.1), (3.2), and (3.6) that

$$\sup_{t \geqslant 0} U(t) < \infty,$$

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which implies that the trajectory of system (2.2)

$$\gamma(z_0) = \{(w, w_t, \widetilde{\theta}_1(t), \widetilde{\theta}_2(t), \omega_1(t), \omega_2(t)) | t \geqslant 0\}$$

is precompact in $\mathcal{H}$. In light of Lasalle's invariance principle [16], any solution of system (2.2) tends to the maximal invariant set of the following:

$$S = \{(w, w_t, \widetilde{\theta}_1(t), \widetilde{\theta}_2(t), \sin t, \cos t) \in H^1(0,1) \times L^2(0,1) \times \mathbb{R}^4 | \dot{V}_w(t) = 0\}.$$

Now, by $\dot{V}_w(t) = 0$, it follows that $w_t(1,t) = 0, \widetilde{\theta}_1 \equiv \widetilde{\theta}_{10}$ and $\widetilde{\theta}_2 \equiv \widetilde{\theta}_{20}$. So, the solution reduces to

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t),\ x \in (0,1), \\ w_x(0,t) = c_0 w(0,t),\ w_t(1,t) = 0, \\ w_x(1,t) = \widetilde{\theta}_{10} \sin t + \widetilde{\theta}_{20} \cos t. \end{cases} \tag{3.7}$$

We show that (3.7) admits zero solution only. To this end, we first consider the equation

$$\begin{cases} w_{tt}(x,t) = w_{xx},\ x \in (0,1), \\ w_x(0,t) = c_0 w(0,t), \\ w_t(1,t) = 0. \end{cases} \tag{3.8}$$

Introduce a Hilbert space $\mathbf{H} = H^1(0,1) \times L^2(0,1)$ with the inner product

$$\langle (y_1, z_1), (y_2, z_2) \rangle_{\mathbf{H}} = \int_0^1 [y_1'(x)\overline{y_2'(x)} + z_1(x)\overline{z_2(x)}]dx + c_0 y_1(0)\overline{y_2(0)}.$$

Define a linear operator A associated to the system (3.7)

$$\begin{cases} \mathbf{A}(y,z) = (z, y''), \\ \quad D(\mathbf{A}) = \{(y,z) \in H^2(0,1) \times H^1(0,1) | y'(0) = c_0 y(0), z(1) = 0\}. \end{cases} \tag{3.9}$$

We claim that $\mathbf{A}$ is skew-symmetric in $\mathbf{H}$. In fact, for any $(f,g), (\phi,\psi) \in D(\mathbf{A})$,

$$\begin{aligned} \langle \mathbf{A}(f,g), (\phi,\psi) \rangle_{\mathbf{H}} &= \int_0^1 [g_1'(x)\overline{\phi_2'(x)} + f''(x)\overline{\psi(x)}]dx + cg(0)\overline{\phi}(0) \\ &= -c_0 f(0)\overline{\psi}(0) + c_0 g(0)\overline{\phi}(0) - \int_0^1 f'(x)\overline{\psi'}(x)dx + \int_0^1 g'(x)\overline{\phi'}(x)dx \\ &= -\langle (f,g), \mathbf{A}(\phi,\psi) \rangle_{\mathbf{H}}. \end{aligned} \tag{3.10}$$

So, all eigenvalues are located on the imaginary axis. Now we claim that each eigenvalue of $\mathbf{A}$ is geometrically simple and hence algebraically simple from general functional analysis theory. To this end, for any $\lambda \in \sigma_p(\mathbf{A})$, solving the eigenvalue problem

$$\mathbf{A}(\phi,\psi) = \lambda(\phi,\psi),$$

one has $\psi = \lambda\phi$ with $\phi \neq 0$ satisfying

$$\begin{cases} \lambda^2\phi(x) - \phi''(x) = 0, \\ \phi'(0) = c_0\phi(0),\ \lambda\phi(1) = 0. \end{cases} \tag{3.11}$$

Solve (3.11) in the case $\lambda = 0$ to give

$$\phi(x) = c_0 x + 1. \tag{3.12}$$

When $\lambda \neq 0$,

$$\phi(x) = \frac{c_0 + \lambda}{\lambda - c_0} e^{\lambda x} + e^{-\lambda x} \tag{3.13}$$

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with

$$\frac{c_0+\lambda}{\lambda-c_0}e^{\lambda}+e^{-\lambda}=0. \tag{3.14}$$

So, λ is geometrically simple.

Next, we claim that the spectrum of $\mathbf{A}$ consists of isolated eigenvalues only. In fact, for a given $(f,g)\in\mathbf{H}$ and $\mu\in\rho(\mathbf{A}),\mu\neq 0$, solve $(\mu I-\mathbf{A})(\phi,\psi)=(f,g)$, that is,

$$\begin{cases}\phi''(x)=\mu^2\phi(x)-\mu f(x)-g(x),\\ \phi'(0)=c_0\phi(0),\ \phi(1)=\dfrac{f(1)}{\mu},\\ \psi(x)=\mu\phi(x)-f(x),\end{cases}$$

to give

$$\begin{cases}\phi(x)=c_1e^{\mu x}+c_2e^{-\mu x}-\dfrac{1}{2\mu}\displaystyle\int_0^x[e^{\mu(x-\xi)}-e^{-\mu(x-\xi)}](\mu f(\xi)+g(\xi))d\xi,\\ \psi(x)=\mu\phi(x)-f(x),\end{cases} \tag{3.15}$$

where c_1,c_2 satisfy the following algebraic equations:

$$\begin{cases}(\mu-c_0)c_1-(\mu+c_0)c_2=0,\\ c_1e^{\mu}+c_2e^{-\mu}=\dfrac{f(1)}{\mu}+\dfrac{1}{2\mu}\displaystyle\int_0^1[e^{\mu(1-\xi)}-e^{-\mu(1-\xi)}](\mu f(\xi)+g(\xi))d\xi.\end{cases} \tag{3.16}$$

This, together with (3.14), shows that the determinant of coefficient of (3.16) $e^{\mu}(c_0+\mu)+(\mu-c_0)e^{-\mu}\neq 0$, which implies that

$$(\mu I-\mathbf{A})^{-1}(f,g)=(\phi,\psi),\ \forall\ (f,g)\in\mathcal{H},$$

and hence

$$\|(\mu I-\mathbf{A})^{-1}(f,g)\|_{H^2(0,1)\times H^1(0,1)}\leqslant C_4\|(f,g)\|_{\mathcal{H}}$$

for some constant $C_4>0$. By the Sobolev embedding theorem, $(\mu I-\mathbf{A})^{-1}$ is compact on $\mathcal{H}$. That is, A is a skew-adjoint operator with compact resolvent on $\mathcal{H}$. Consequently, the spectrum of A consists of isolated eigenvalues only.

Furthermore, from (3.13) and (3.14), we can obtain the following asymptotic expressions of eigenpairs of $\mathbf{A}$.

$$\lambda_n=\left(n\pi-\frac{\pi}{2}\right)i+\mathcal{O}(n^{-1}),\ \ \phi_n(x)=\cos\left(n-\frac{1}{2}\right)\pi x+\mathcal{O}(n^{-1}). \tag{3.17}$$

Define

$$\begin{cases}\lambda_n=\left(n\pi-\dfrac{\pi}{2}\right)i+\mathcal{O}(n^{-1}),\ \lambda_{-n}=\overline{\lambda}_n,\\ \Phi_n=(\lambda_n^{-1}\phi_n,\phi_n),\ \Phi_{-n}=(\lambda_{-n}^{-1}\phi_n,\phi_n),\ n=1,2,\cdots.\end{cases} \tag{3.18}$$

By general theory of functional analysis, $\{\Phi_n\}_{n\in\mathbb{Z}}$ forms an orthogonal basis for $\mathcal{H}$. Therefore, the solution of (1.7) can be represented as

$$(w(\cdot,t),\dot{w}(\cdot,t))=a_0(c_0x+1,0)+\sum_{n=1}^{\infty}a_ne^{\lambda_nt}\Phi_n+\sum_{n=1}^{\infty}a_{-n}e^{\lambda_{-n}t}\Phi_{-n},$$

where the constants $\{a_n\}_{n\in\mathbb{Z}}$ are determined by the initial condition. That is,

$$w_0=a_0(c_0x+1)+\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n}\phi_n+\sum_{n=1}^{\infty}\frac{a_{-n}}{\lambda_{-n}}\phi_n,\ \ w_1=\sum_{n=1}^{\infty}a_n\phi_n+\sum_{n=1}^{\infty}a_{-n}\phi_n.$$

Hence,

$$w_x(1,t)=a_0c_0+\sum_{n=1}^{\infty}a_n\frac{\phi_n'(1)}{\lambda_n}e^{\lambda_n t}+\sum_{n=1}^{\infty}a_{-n}\frac{\phi_n'(1)}{\lambda_{-n}}e^{\lambda_{-n}t}=\widetilde{\theta}_{10}\sin t+\widetilde{\theta}_{20}\cos t.$$

Therefore,

$$a_0c_0+\sum_{n=1}^{\infty}a_n\frac{\phi_n'(1)}{\lambda_n}e^{\lambda_n t}+\sum_{n=1}^{\infty}a_{-n}\frac{\phi_n'(1)}{\lambda_{-n}}e^{\lambda_{-n}t}-[\widetilde{\theta}_{20}-i\widetilde{\theta}_{10}]e^{it}-[\widetilde{\theta}_{20}+i\widetilde{\theta}_{10}]e^{-it}=0. \quad (3.19)$$

We claim that $a_{\pm n}=0$, for all $n\geqslant 1$. Because otherwise, if there exists a n_0 such that $\left|a_{n_0}\frac{\phi_{n_0}'(1)}{\lambda_{n0}}\right|\neq 0$. The smoothness of the initial value guarantees that $\sum_{n\in\mathbb{Z},n\neq 0}\left|a_n\frac{\phi_n'(1)}{\lambda_n}\right|<\infty$, which implies that there exists an integer $N>n_0$ such that

$$\sum_{n=N}^{\infty}\left|a_n\frac{\phi_n'(1)}{\lambda_n}\right|<\frac{1}{4}\left|a_{n_0}\frac{\phi_{n_0}'(1)}{\lambda_{n0}}\right| \text{ and } \sum_{n=N}^{\infty}\left|a_{-n}\frac{\phi_{-n}'(1)}{\lambda_{-n}}\right|<\frac{1}{4}\left|a_{n_0}\frac{\phi_{n_0}'(1)}{\lambda_{n0}}\right|. \quad (3.20)$$

Because $\lambda_n\neq\lambda_m$ for any $n,m\in Z, n\neq m$ and $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n|=\pi, n\in Z$, one has, for $t>0$, that

$$\begin{aligned}&a_{n_0}\frac{\phi_{n_0}'(1)}{\lambda_{n0}}+\sum_{n=N+1}^{\infty}a_n\frac{\phi_n'(1)}{\lambda_n}e^{(\lambda_n-\lambda_{n_0})t}+\sum_{n=1,n\neq n_0}^{N}a_n\frac{\phi_n'(1)}{\lambda_n}e^{(\lambda_n-\lambda_{n_0})t}\\&\quad+\sum_{n=N+1}^{\infty}a_{-n}\frac{\phi_{-n}'(1)}{\lambda_{-n}}e^{(\lambda_{-n}-\lambda_{n_0})t}+\sum_{n=1}^{N}a_{-n}\frac{\phi_{-n}'(1)}{\lambda_{-n}}e^{(\lambda_{-n}-\lambda_{n_0})t}\\&\quad+a_0c_0e^{-\lambda_{n_0}t}-[\widetilde{\theta}_{20}-i\widetilde{\theta}_{10}]e^{(i-\lambda_{n_0})t}-[\widetilde{\theta}_{20}+i\widetilde{\theta}_{10}]e^{-(i+\lambda_{n_0})t}=0.\end{aligned} \quad (3.21)$$

Integrating both sides of (3.21) and using the fact $\mathrm{Re}\lambda_n=0$ and (3.20), we obtain

$$\begin{aligned}\left|a_{n_0}\frac{\phi_{n_0}'(1)}{\lambda_{n0}}\right|t&\leqslant 2\left|\int_0^t\sum_{n=1,n\neq n_0}^{N}a_n\frac{\phi_n'(1)}{\lambda_n}e^{(\lambda_n-\lambda_{n_0})s}ds\right|+2\left|\int_0^t\sum_{n=1}^{N}a_{-n}\frac{\phi_{-n}'(1)}{\lambda_{-n}}e^{(\lambda_{-n}-\lambda_{n_0})s}ds\right|\\&+2\left|\int_0^t a_0c_0e^{-\lambda_{n_0}s}ds\right|+2\left|\int_0^t[\widetilde{\theta}_{20}-i\widetilde{\theta}_{10}]e^{(i-\lambda_{n_0})s}+[\widetilde{\theta}_{20}+i\widetilde{\theta}_{10}]e^{-(i+\lambda_{n_0})s}ds\right|.\end{aligned}$$

Because the right side of the earlier equation has an upper bound for all $t\geqslant 0$, we get that $a_{n_0}=0$, which is a contraction. By (3.19), $a_{\pm n}=0, n=1,2,\cdots, a_0=0, \widetilde{\theta}_{10}=0$ and $\widetilde{\theta}_{20}=0$.

We have thus proved that $S=\{(0,0,0,0)\}$, that is

$$\lim_{t\to\infty}\left[\frac{1}{2}\int_0^1[w_t^2(x,t)+w_x^2(x,t)]dx+c_0w^2(0,t)+\frac{1}{2r_1}\widetilde{\theta}_1^2(t)+\frac{1}{2r_2}\widetilde{\theta}_2^2(t)\right]=0.$$

The proof is complete. □

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4. WELL-POSEDNESS AND ASYMPTOTIC STABILITY OF CLOSED-LOOP SYSTEM

We go back to the closed-loop system (1.4), which is rewritten here:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0,\ x \in (0,1),\ t > 0, \\ y_x(0,t) = -qy(0,t),\ t \geqslant 0, \\ y_x(1,t) = -ky_t(1,t) - (c_0+q)y(1,t) \\ -(c_0+q)\displaystyle\int_0^1 e^{q(1-\xi)}[qy(\xi,t) + ky_t(\xi,t)]d\xi + \widetilde{\theta}_1(t)\sin t + \widetilde{\theta}_2(t)\cos t,\ t \geqslant 0, \\ \dot{\widetilde{\theta}}_1(t) = -r_1[y_t(1,t) + (c_0+q)\displaystyle\int_0^1 e^{q(1-\xi)} y_t(\xi,t)d\xi]\sin t, t > 0, \\ \dot{\widetilde{\theta}}_2(t) = -r_2[y_t(1,t) + (c_0+q)\displaystyle\int_0^1 e^{q(1-\xi)} y_t(\xi,t)d\xi]\cos t, t > 0, \\ \widetilde{\theta}_1(0) = \widetilde{\theta}_{10},\ \widetilde{\theta}_2(0) = \widetilde{\theta}_{20}, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x),\ 0 \leqslant x \leqslant 1, \end{cases} \tag{4.1}$$

Notice the invertible transformation

$$\begin{pmatrix} w \\ w_t \\ \widetilde{\theta}_1 \\ \widetilde{\theta}_2 \end{pmatrix} = \begin{pmatrix} I+\mathbb{P} & 0 & 0 & 0 \\ 0 & I+\mathbb{P} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ y_t \\ \widetilde{\theta}_1 \\ \widetilde{\theta}_2 \end{pmatrix}. \tag{4.2}$$

Because we have proved the stability of weak solution $(w, w_t, \widetilde{\theta}_1, \widetilde{\theta}_2)$ in Theorem 3.1, we can discuss the stability of weak solution of (4.1) in $\mathcal{V} = H^1(0,1) \times L^2(0,1) \times \mathbb{R}^2$.

Theorem 4.1
For any initial value $(y_0, y_1, \widetilde{\theta}_{10}, \widetilde{\theta}_{20}) \in \mathcal{V}$,, there exists a unique (weak) solution to (4.1) such that $(y(\cdot,t), y_t(\cdot,t), \widetilde{\theta}_1(t), \widetilde{\theta}_2(t)) \in C([0,\infty); \mathcal{V})$. Moreover, the closed-loop solution $(y, y_t, \widetilde{\theta}_1(t), \widetilde{\theta}_2(t))$ of (4.1) is asymptotically stable in the sense that

$$\lim_{t\to\infty}\left[\int_0^1 [y_x^2(x,t) + y_t^2(x,t)]dx + c_0 y^2(0,t)\right] = 0$$

and

$$\lim_{t\to\infty} \theta_1(t) = \bar{\theta}_1,\ \lim_{t\to\infty} \theta_2(t) = \bar{\theta}_2.$$

Proof
For any initial value $(y_0, y_1, \widetilde{\theta}_{10}, \widetilde{\theta}_{20}) \in \mathcal{V}$, it follows from 1.8 that $(w_0, w_1, \widetilde{\theta}_{10}, \widetilde{\theta}_{20}) \in \mathcal{V}$.

Let

$$y(x,t) = w(x,t) - (c_0+q)\int_0^x e^{-c_0(x-\xi)} w(\xi,t)d\xi. \tag{4.3}$$

Then a direct computation shows that such a defined $(y, y_t, \widetilde{\theta}_1, \widetilde{\theta}_2)$ satisfies (4.1) with initial value $(y_0, y_1, \widetilde{\theta}_{10}, \widetilde{\theta}_{20})$. This solution is unique by the invertible transformation (4.2) and the uniqueness of solution to (1.7). The asymptotic stability follows from (4.2) and Theorems 3.1. □

5. NUMERICAL SIMULATION

In this section, we present some numerical simulations to demonstrate the convergence. The second-order equations in time are first converted into a system of two first-order equations in time, and then backward Euler method in time with Chebyshev spectral method in space is used. The numerical

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code is programmed in MATLAB. Here we take the grid size $N = 20$ and time step $dt = 10^{-3}$. The parameter values are set to be $r_1 = 1,\ r_2 = 0.5,\ c_0 = 100,\ q = 0.4, k = 0.3, \bar{\theta}_1 = 0.8$ and $\bar{\theta}_2 = 0.2$.

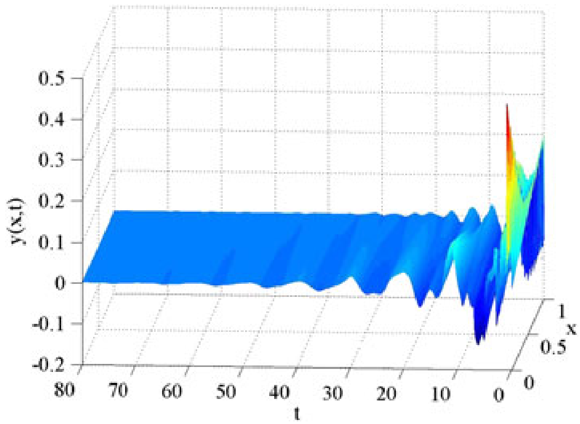

Figure 1. Amplitude $y(x,t)$.

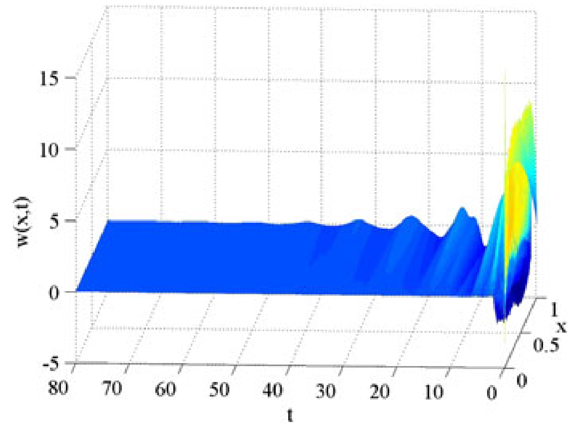

Figure 2. Amplitude $w(x,t)$.

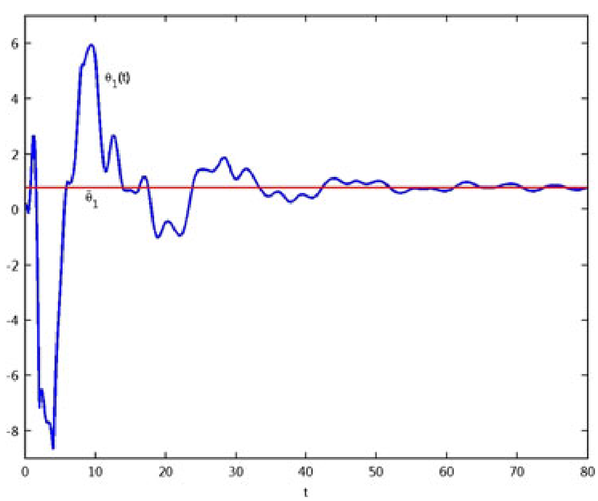

Figure 3. Parameter estimation $\theta_1(t)$.

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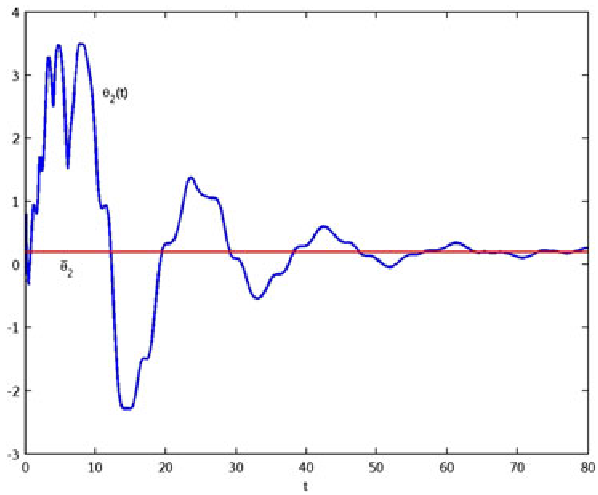

Figure 4. Parameter estimation $\theta_2(t)$.

The initial values are taken:

$$\theta_{10} = 0.2,\ \theta_{20} = 0.8\ ,\ y_0(x) = 0.2(1-x),\ y_1(x) = \begin{cases} \dfrac{1}{5}, 0 < x \leqslant \dfrac{1}{2}, \\ 0, \text{otherwise} \end{cases}$$

Figures 1 and 2 show that system (4.1) and system (1.7) are both asymptotically stable.

Figures 3 and 4 show the estimates of the parameters. It is seen that the estimate $\theta_1(t)$ with initial value $\theta_{10} = 0.2$ and $\theta_2(t)$ with initial value $\theta_{20} = 0.8$, respectively, approximate the unknown parameter values $\bar{\theta}_1 = 0.8$ and $\bar{\theta}_2 = 0.2$ quite satisfactorily.

□

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