N76-10177

STABILIZATION BY MODIFICATION OF THE LAGRANGIAN*

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ABSTRACT

In order to reduce the error growth during a numerical integration, a method of stabilization of the differential equations of the Keplerian motion is offered. It is characterized by the use of the eccentric anomaly as an independent variable in such a way that the time transformation is given by a generalized Lagrange formalism. The control terms in the equations of motion obtained by this modified Lagrangian give immediately a completely Lyapunov-stable set of differential equations. In contrast to other publications, here the equation of time integration is modified by a control term which leads to an integral which defined the time element for the perturbed Keplerian motion.

INTRODUCTION

It is well known that the classical differential equations of the Keplerian motion are unstable in the sense of Lyapunov. In general, Lyapunov-unstable differential equations develop more unavoidable numerical errors during a numerical integration than Lyapunov-stable equations do. We consider here the stabilization of the differential equations of Keplerian motion with the aim of improving the accuracy and efficiency during the numerical integration. We propose a stabilization method which is purely conservative in contrast to other methods (Baumgarte, 1972a). It is characteristic for all conservative methods, that they make the revolution time independent of the initial conditions (Baumgarte, 1974).

GENERALIZED LAGRANGIAN

In this method the stabilization goes hand in hand with the introduction of a new independent variable s instead of the time t. This procedure is called time transformation and s is called fictitious time. Furthermore, we will require that our stabilized equations of motion be developed from the Lagrangian formalism. But, here we have to use an appropriately

* This paper was supported by the National Research Council and the National Aeronautics and Space Administration.

modified generalized Lagrangian formalism. In order to introduce, instead of t, the fictitious time s as a new independent variable, we have to use, instead of the original Lagrangian L

$$L = L(q_i, \dot{q}_i, t), \qquad i = 1, 2, \ldots, n, \tag{1}$$

where q_i are the coordinates and dot means differentiation with respect to t, the following generalized Lagrangian L*:

$$L^* = ht' + \mu \left\{ L\left(q_i, \frac{q_i'}{\mu}, t\right) - h \right\} \tag{2}$$

In equation (2) the prime means differentiation with respect to the independent variable s. The time t is now a dependent variable (time coordinate). Together with t appears its conjugated momentum h, which represents physically the negative total energy.

Furthermore, we consider only the case, where

$$\mu = \mu(q_i, h) > 0 \tag{3}$$

is a freely chosen positive function only dependent on q_i and h. Later we will see that μ may be interpreted as the local scale of the time transformation. More general dependences of the scale μ are also of interest but shall not be considered further here. Our special choice of μ retains the equivalence between Lagrangian and Hamiltonian formalism. This choice has two consequences:

1. The transformed kinetic energy in an inertial system is also a quadratic form in the velocity components q_i' with respect to the fictitious time s.
2. A conservative system remains conservative after the time transformation.

With these restrictions we obtain the differential equations of motion in the following form:

$$\frac{d}{ds}\left(\frac{\partial L^*}{\partial q_i'}\right) - \frac{\partial L^*}{\partial q_i} = 0 \tag{4a}$$

$$\frac{d}{ds}\left(\frac{\partial L^*}{\partial t'}\right) - \frac{\partial L^*}{\partial t} = 0 \tag{4b}$$

$$\frac{d}{ds}\left(\frac{\partial L^*}{\partial h'}\right) - \frac{\partial L^*}{\partial h} = 0 \longrightarrow \frac{\partial L^*}{\partial h} = 0 \tag{4c}$$

Relative to the system (4a, b, c) the following remarks are in order:

- Equation (4a) states the equations of motion, which are also given by the original Lagrangian $L(q_i, \dot{q}_i, t)$ after changing from t to s, but which are distinguished only by control terms (Baumgarte, 1972b). This fact will be observed later in the special case of the Kepler problem.

- Equation (4b) gives:

$$h' = \mu (q_i, h) \frac{\partial}{\partial t} L (q_i, q_i', h, t), \tag{5}$$

which asserts the energy relation. In the conservative case, where the original Lagrangian L does not depend explicitly on the time t, we get h = constant.

- From equation (4c) there follows:

$$t' = \mu + \left\{ h \frac{\partial \mu}{\partial h} - \frac{\partial}{\partial h} (\mu L) \right\}. \tag{6}$$

It will be obvious that the expression in the bracket is a control term, which represents the energy relation. This control term vanishes in the exact solution of the equations of motion, in such a way that $t' = \mu$. Therefore, it follows as previously promised that μ is the local scale of the time transformation.

EXAMPLE: KEPLERIAN MOTION

We consider now the Keplerian problem and use Cartesian coordinates. We will choose the scale μ in such a way that the fictitious time s will be for a pure Keplerian motion the eccentric anomaly. Only by doing this can we obtain both Lyapunov-stable differential equations and equivalence between Lagrange and Hamilton. We, therefore, choose:

$$\mu = \frac{r}{\sqrt{2h}}, r = |\underline{x}|, \tag{7}$$

where r is the distance.

Before we establish the generalized Lagrangian L*, we first write the original Lagrangian L. With the mass m = 1, K as the gravitational parameter, L has the form:

$$L = \frac{1}{2} |\dot{\underline{x}}|^2 + \frac{K^2}{r}. \tag{8}$$

With the help of equation (2) we now obtain as the generalized Lagrangian L*:

$$L^* = ht' + \frac{\sqrt{2h}}{2r} |\underline{x}'|^2 + \frac{K^2}{\sqrt{2h}} - \sqrt{\frac{h}{2}}\, r. \tag{9}$$

With (9) the stabilized equations of motion are:

$$\underline{x}'' = \frac{(\underline{x}, \underline{x}')}{r^2}\, \underline{x}' - \frac{1}{2}\left[\frac{|\underline{x}'|^2}{r^2} + 1\right]\underline{x} \tag{10a}$$

$$h' = 0 \tag{10b}$$

$$5 \quad t' = \frac{1}{2\sqrt{2h}}\left\{\frac{K^2}{h} + r - \frac{|\underline{x}'|^2}{r}\right\} \tag{10c}$$

(10a) requires that $h' = 0$. It is essential that in contrast to an earlier publication (Baumgarte, 1972b) the vector-equation for $\underline{x}''$ does not depend on h. This means the revolution time is the fixed number 2π and this implies Lyapunov-stability for the $\underline{x}''$ - equation.

From (10b) there follows h = constant. The constant h is computed from the initial conditions and is placed in the computer as a fixed value. This is the presupposition that now in contrast to earlier publications (Baumgarte, 1972b) the time integration (10c) is Lyapunov-stable. The expression:

$$\frac{1}{2\sqrt{2h}}\left\{r - \frac{|\underline{x}'|^2}{r}\right\},$$

a part of the right hand side of (10c), is proportional to the difference between potential and kinetic energy. Therefore, the mean value with respect to the fictitious time s of this difference vanishes because we have an oscillator problem in principle. Consequently, the expression $K^2/(2h)^{3/2}$ in (10c) represents the exact mean value of the right hand side of (10c). This fact implies Lyapunov-stability also for the time integration.

CONTROL TERMS AND LYAPUNOV-STABILITY

In order to show the effect of the stabilization by the control terms, in the classical Keplerian equations

$$\ddot{\underline{x}} = -\frac{K^2}{r^3}\,\underline{x}, \qquad \dot{} = \frac{d}{dt}, \tag{11}$$

we substitute in place of the independent variable t the fictitious time s by using $dt/ds = t' = r/\sqrt{2h}$. Because h = constant we obtain:

$$\underline{x}'' - \frac{(\underline{x}, \underline{x}')}{r^2}\,\underline{x}' + \frac{K^2}{2h}\,\frac{\underline{x}}{r} = 0 \tag{12a}$$

$$h' = 0 \tag{12b}$$

$$t' - \frac{r}{\sqrt{2h}} = 0. \tag{12c}$$

We now transpose system (10) in such a way that we can see clearly that system (12) differs from system (10) only by control terms which are the right hand sides of equations (13a) and (13c).

$$\underline{x}'' - \frac{(\underline{x}, \underline{x}')}{r^2}\underline{x}' + \frac{K^2}{2h}\frac{\underline{x}}{r} = -\left\{\frac{|\underline{x}'|^2}{2r^2} - \frac{K^2}{2hr} + \frac{1}{2}\right\}\underline{x} \tag{13a}$$

$$h' = 0 \tag{13b}$$

$$t' - \frac{r}{\sqrt{2h}} = -\left\{\frac{|\underline{x}'|^2}{2r^2} - \frac{K^2}{2hr} + \frac{1}{2}\right\}\frac{r}{\sqrt{2h}}. \tag{13c}$$

In the control terms in (13a) and (13c) the same bracket appears as a factor, which is analytically zero with respect to the energy relation.

Equation (13c) or (10c) can be integrated. We find as an integral of motion:

$$t = \frac{1}{\sqrt{2h}}\left[\frac{K^2}{2h}s - \frac{(\underline{x}, \underline{x}')}{r}\right] + C. \tag{14}$$

The control terms in (13a, c) produce the Lyapunov-stability under the supposition that the constant h is computed once and for all from the initial conditions. The proof for the stability of the complete system (13) or (10), respectively, (with respect to h = constant, whereby this constant is to be computed, finally, by the initial conditions) can easily be carried out by making the transformation into action and angle variables, because the equivalence between Lagrange and Hamilton exists. By doing this, the generalized Hamiltonian, obtained from the generalized Lagrangian by making a Legendre transformation, will be linear in the action variables, which implies Lyapunov-stability (Baumgarte, 1972b). Another proof follows from the equivalence of the system (10a, b) together with (14), to the corresponding equations of the KS-transformation (Stiefel and Scheifele, 1971).

We will call attention to the fact that the dependence of the time transformation $t' = r/\sqrt{2h}$ on the momentum h makes possible the elimination of the instability under the presupposition that $h' = 0$ is integrated exactly.

In the case of perturbed Keplerian motion, the stabilized system (13) or (10) is modified by additional perturbation terms. In equation (14), C is no longer constant but becomes the slowly varying time element (Baumgarte, 1972b; Stiefel and Scheifele, 1971).

Numerical experiments have always shown a reduction in error in the numerical integration. It appears that the positive effects of the stabilization of the pure Keplerian motion carry over to the perturbed problem.

We will finally remark that the KS-transformation can be performed directly in the generalized Lagrangian L* by inserting

$$r^2 = \sum x_i^2 = \left(\sum u_j^2\right)^2, \quad \frac{\sum x_i'^2}{r} = 4\sum u_j'^2, \quad \begin{matrix} i = 1,2,3 \\ j = 1,2,3,4, \end{matrix} \qquad (15)$$

thereby giving immediately the analogous complete, stabilized, set of differential equations which leads directly to the KS-elements (Stiefel and Scheifele, 1971).

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