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Stabilization for a coupled PDE–ODE control system ☆

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Abstract

A control system of an ODE and a diffusion PDE is discussed in this paper. The novelty lies in that the system is coupled. The method of PDE backstepping as well as some special skills is resorted in stabilizing the coupled PDE–ODE control system, which is transformed into an exponentially stable PDE–ODE cascade with an invertible integral transformation. And a state feedback boundary controller is designed. Moreover, an exponentially convergent observer for anti-collocated setup is proposed, and the output feedback boundary control problem is solved. For both the state and output feedback boundary controllers, exponential stability analyses in the sense of the corresponding norms for the resulting closed-loop systems are given through rigid proofs.

1. Introduction

In control engineering, systems modeled by ordinary differential equations (ODE) are common. Over the past decades of years, systems modeled by partial differential equations (PDE) have been popular too. Recently, coupled systems have been active areas of research. Examples can be found in control problems of electromagnetic coupling, mechanical coupling and chemical reaction coupling. Some results on controllability of coupled PDE–PDE systems have been established (see, e.g., [12–14]). However, the problem of feasible controllers and

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observers designing for coupled PDE–PDE systems as well as coupled PDE–ODE systems is far from complete, and rather challenging. In fact, it is still an original area.

In this paper, the system considered couples an ODE with a heat equation. Physical background comes from, e.g., solid–gas interaction of chemical reaction and heat diffusion with insulated catalyst fixed at one point.

The most intuitive method to tackle coupling in the system is decoupling it directly. But this is not practicable for all the time. One of the most useful methods for boundary controller and observer designing of PDEs is PDE backstepping, which is introduced by Krstic. It is used to stabilize the cascaded PDE–ODE systems in [4–7,10,11] where the interconnection between the PDE and ODE is one-directional, and is employed here to stabilize the coupled PDE–ODE system where the interconnection between the PDE and ODE is two-directional. The papers [1–3,8,9] have also been referred to. In this paper, firstly, an invertible integral transformation is introduced to transform the original system into an exponentially stable target system. Since the kernel functions satisfy some conditions, which are also coupled, some special skills are also used in solving them. And a state feedback controller is designed. Secondly, an observer for anti-collocated setup is designed to achieve exponential convergence of the observer error, and an output feedback controller is established.

2. Problem formulation and analysis

Consider the following coupled PDE–ODE control system:

$$\dot{X}(t) = AX(t) + Bu(0,t) \tag{1}$$

$$u_t(x,t) = u_{xx}(x,t) + CX(t), \quad x \in (0,l) \tag{2}$$

$$u_x(0,t) = 0 \tag{3}$$

$$u(l,t) = U(t) \tag{4}$$

where $X(t) \in \mathbb{R}^n$ is the ODE state, and the pair (A, B) is assumed to be stabilizable; $u(x,t) \in \mathbb{R}$ is the PDE state, and C^T is a constant vector; $U(t)$ is the scalar input to the entire system. The coupled system is depicted in Fig. 1. The control objective is to exponentially stabilize the system (1)–(4).

The most intuitive method is to decouple the PDE and the ODE. After doing the decoupling directly, the system is transformed into an integral–differential system

$$X(t) = X(0)e^{At} + \int_0^t e^{A(t-\tau)} Bu(0,\tau)\, d\tau$$

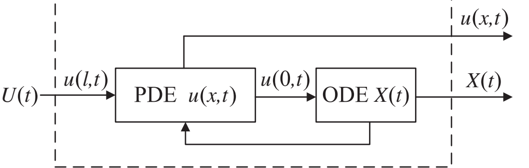

Fig. 1. The coupled control system of the heat equation PDE and the ODE.

$$u_t(x,t) = u_{xx}(x,t) + C\left(X(0)e^{At} + \int_0^t e^{A(t-\tau)}Bu(0,\tau)\,d\tau\right)$$

$$u_x(0,t) = 0$$

$$u(l,t) = U(t)$$

Intuitively, this system is stabilizable.

However, to achieve the stabilization of the system (1)–(4) in a strict manner, PDE backstepping is more effective.

The method of PDE backstepping is to seek an invertible integral transformation $(X,u)\mapsto(X,w)$ to convert the system (1)–(4) into an exponentially stable target system, e.g., the following system

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t) \tag{5}$$

$$w_t(x,t) = w_{xx}(x,t) \tag{6}$$

$$w_x(0,t) = 0 \tag{7}$$

$$w(l,t) = 0 \tag{8}$$

where K is chosen such that $A+BK$ is Hurwitz. Thus, with the invertibility of the transformation $(X,u)\mapsto(X,w)$, exponential stabilization of the original closed-loop system will be achieved.

3. State feedback controller design

The integral transformation $(X,u)\mapsto(X,w)$ is postulated in the following form:

$$X(t) = X(t) \tag{9}$$

$$w(x,t) = u(x,t) - \int_0^x \kappa(x,y)u(y,t)\,dy - \Phi(x)X(t) \tag{10}$$

where the gain functions $\kappa(x,y)\in\mathbb{R}$ and $\Phi(x)^T\in\mathbb{R}^n$ are to be determined.

The first two derivatives with respect to x of $w(x,t)$, as defined in (10), are given by

$$w_x(x,t) = u_x(x,t) - \kappa(x,x)u(x,t) - \int_0^x \kappa_x(x,y)u(y,t)\,dy - \Phi'(x)X(t) \tag{11}$$

$$\begin{aligned} w_{xx}(x,t) = {} & u_{xx}(x,t) - \kappa(x,x)u_x(x,t) - \left(\frac{d}{dx}\kappa(x,x) + \kappa_x(x,x)\right)u(x,t) \\ & - \int_0^x \kappa_{xx}(x,y)u(y,t)\,dy - \Phi''(x)X(t) \end{aligned} \tag{12}$$

The first derivative of $w(x,t)$ with respect of t is

$$\begin{aligned} w_t(x,t) = {} & u_{xx}(x,t) - \kappa(x,x)u_x(x,t) + \kappa_y(x,x)u(x,t) - \int_0^x \kappa_{yy}(x,y)u(y,t)\,dy \\ & + \kappa(x,0)u_x(0,t) - (\kappa_y(x,0) + \Phi(x)B)u(0,t) \\ & - \left(\Phi(x)A + \int_0^x \kappa(x,y)\,dy\,C - C\right)X(t) \end{aligned} \tag{13}$$

Let $x=0$ in the backstepping transformation (10) and Eq. (11) and subtract the two sides of (12) from the two sides of (13) separately, then the following identities

$$w(0,t)=u(0,t)-\Phi(0)X(t)$$

$$w_x(0,t)=-\kappa(0,0)u(0,t)-\Phi'(0)X(t)$$

$$\begin{aligned} w_t(x,t)-w_{xx}(x,t) &= 2\left(\frac{d}{dx}\kappa(x,x)\right)u(x,t) \\ &\quad+\int_0^x(\kappa_{xx}(x,y)-\kappa_{yy}(x,y))u(y,t)\,dy-(\kappa_y(x,0)+\Phi(x)B)u(0,t) \\ &\quad+\left(\Phi''(x)-\Phi(x)A-\int_0^x\kappa(x,y)\,dy\,C+C\right)X(t) \end{aligned}$$

are obtained, where the following notations

$$\kappa_x(x,x)=\frac{\partial}{\partial x}\kappa(x,y)|_{y=x}$$

$$\kappa_y(x,x)=\frac{\partial}{\partial y}\kappa(x,y)|_{y=x}$$

$$\frac{d}{dx}\kappa(x,x)=\kappa_x(x,x)+\kappa_y(x,x)$$

and the fact $u_x(0,t)=0$ have been used. A sufficient condition for Eqs. (5)–(7) to hold is that $\kappa(x,y)$ and $\Phi(x)$ satisfy

$$\kappa_{xx}(x,y)=\kappa_{yy}(x,y) \tag{14}$$

$$\kappa(x,x)=0 \tag{15}$$

$$\kappa_y(x,0)=-\Phi(x)B \tag{16}$$

which represents a hyperbolic PDE of second order and of Goursat type, and

$$\Phi''(x)-\Phi(x)A-\int_0^x\kappa(x,y)\,dy\,C+C=0 \tag{17}$$

$$\Phi(0)=K \tag{18}$$

$$\Phi'(0)=0 \tag{19}$$

What must be emphasized here is that the PDE (14)–(16) and the ODE (17)–(19) are weakly coupled, which can be decoupled and solved explicitly through some techniques of algebra and analytical mathematics.

Firstly, the solution to the PDE (14)–(16) can be obtained as

$$\kappa(x,y)=\int_0^{x-y}\Phi(\sigma)B\,d\sigma \tag{20}$$

Secondly, substituting (20) into (17), it is obtained that

$$\Phi''(x)-\Phi(x)A-\int_0^x\int_0^{x-y}\Phi(\sigma)B\,d\sigma\,dy\,C+C=0$$

which is a non-homogeneous linear ODE of second order. Changing the order of integration and differentiating the ODE twice, the following fourth order ODE:

$$\Phi^{(4)}(x) - \Phi''(x)A - \Phi(x)BC = 0 \tag{21}$$

and initial values

$$\Phi''(0) = KA - C, \quad \Phi^{(3)}(0) = 0$$

are obtained. Let I be a unit matrix, then the solution to the ODE (17)–(19) is

$$\Phi(x) = (K \ 0 \ KA - C \ 0)e^{Dx}E$$

where

$$D = \begin{pmatrix} 0 & 0 & 0 & BC \\ I & 0 & 0 & 0 \\ 0 & I & 0 & A \\ 0 & 0 & I & 0 \end{pmatrix}, \quad E = \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The transformation $(X,u) \mapsto (X,w)$ (9)–(10) is invertible, and the inverse transformation $(X,w) \mapsto (X,u)$ is postulated in the following form:

$$X(t) = X(t) \tag{22}$$

$$u(x,t) = w(x,t) + \int_0^x \iota(x,y)w(y,t)\,dy + \Psi(x)X(t) \tag{23}$$

where the kernel functions $\iota(x,y) \in \mathbb{R}$ and $\Psi(x)^T \in \mathbb{R}^n$ are to be driven.

As is done in the kernel functions seeking of the direct transformation, the derivatives w_x, w_{xx} and w_t are computed, and a sufficient condition for Eqs. (1)–(3) to hold is that $\iota(x,y)$ and $\Psi(x)$ satisfy

$$\iota_{xx}(x,y) = \iota_{yy}(x,y) \tag{24}$$

$$\iota(x,x) = 0 \tag{25}$$

$$\iota_y(x,0) = -\Psi(x)B \tag{26}$$

and

$$\Psi''(x) - \Psi(x)(A + BK) + C = 0 \tag{27}$$

$$\Psi(0) = K \tag{28}$$

$$\Psi'(0) = 0 \tag{29}$$

This cascade system can also be solved explicitly. Firstly, by employing the method of variable separating, the explicit solution to the ODE (27)–(29) can be obtained as follows:

$$\Psi(x) = F(x)e^{\begin{pmatrix} 0 & A+BK \\ 1 & 0 \end{pmatrix}x} \begin{pmatrix} I \\ 0 \end{pmatrix}$$

where

$$F(x) = (\, K \quad 0 \,) + (\, C(A+BK)^{-1} \quad 0 \,)(e^{-\begin{pmatrix} 0 & A+BK \\ 1 & 0 \end{pmatrix}x} - I)$$

Thus, through further calculation, the solution is

$$\Psi(x)=(K-C(A+BK)^{-1})G(x)+C(A+BK)^{-1}$$

where

$$G(x)=(\,I \quad 0\,)e^{Hx}\begin{pmatrix} I \\ 0 \end{pmatrix}$$

and

$$H=\begin{pmatrix} 0 & A+BK \\ I & 0 \end{pmatrix}$$

Secondly, the solution to the PDE (24)–(26) is

$$\imath(x,y)=\int_0^{x-y}\Psi(\sigma)B\,d\sigma$$

Write

$$\phi(s)=\int_0^s \Phi(\sigma)B\,d\sigma,\quad \psi(s)=\int_0^s \Psi(\sigma)B\,d\sigma$$

then the direct and inverse backstepping transformations are written into

$$w(x,t)=u(x,t)-\int_0^x \phi(x-y)u(y,t)\,dy-\Phi(x)X(t) \tag{30}$$

$$u(x,t)=w(x,t)+\int_0^x \psi(x-y)w(y,t)\,dy+\Psi(x)X(t) \tag{31}$$

Now, a controller is to be designed such that the boundary condition (8) is satisfied. Let $x=l$ in Eq. (10), then from Eqs. (4) and (8), a controller is chosen as

$$U(t)=\int_0^l \phi(l-y)u(y,t)\,dy+\Phi(l)X(t) \tag{32}$$

Furthermore, the explicit solution to the system (1)–(4), Eq. (32) can also be obtained. Firstly, the heat equation (6)–(8) is solved, and the solution

$$w(x,t)=\frac{2}{l}\sum_{m=1}^{\infty} e^{(-(m+1/2)^2\pi^2/l^2)t}\cos\left(\frac{(m+\frac{1}{2})\pi}{l}x\right)\mu_m \tag{33}$$

is obtained, where

$$\mu_m=\int_0^l w_0(\xi)\cos\left(\frac{(m+\frac{1}{2})\pi}{l}\xi\right)d\xi$$

and the initial condition $w_0(x)$ can be calculated since it is related to the initial state $u(x,0)$ via (10). Secondly, the solution to the closed-loop system (1)–(4), Eq. (32) can be obtained from

$$X(t)=X(0)e^{(A+BK)t}+\int_0^t e^{(A+BK)(t-\tau)}Bw(0,\tau)\,d\tau \tag{34}$$

and (23).

Theorem 1. *For any initial data* $X(0) \in \mathbb{R}$ *and* $u(\cdot,0) \in H^1(0,l)$, *the closed-loop system consisting of the plant* (1)–(4) *and the control law* (32) *has a unique classical solution and is exponentially stabilized in the sense of the norm*

$$\|(X(t),u(\cdot,t))\|^2 = |X(t)|^2 + \|u(\cdot,t)\|^2_{H^1(0,l)}$$

Proof. A Lyapunov function

$$V(t) = X^T P X + \frac{a}{2}\|w(\cdot,t)\|^2_{L^2(0,l)} + \frac{1}{2}\|w_x(\cdot,t)\|^2_{L^2(0,l)}$$

is employed, where the matrix $P = P^T > 0$ is the solution to the Lyapunov equation

$$P(A+BK) + (A+BK)^T P = -Q$$

for some $Q = Q^T > 0$, and the parameter $a > 0$ is to be chosen later.For simplicity, in the sequel, the symbol $\|\cdot\|$ stands for the norm in $L^2(0,l)$.

From the backstepping transformations (30) and (31), it can be obtained that

$$\|w\|^2 \le \alpha_1\|u\|^2 + \alpha_2|X|^2, \quad \|w_x\|^2 \le \alpha_3\|u_x\|^2 + \alpha_4\|u\|^2 + \alpha_5|X|^2 \tag{35}$$

$$\|u\|^2 \le \beta_1\|w\|^2 + \beta_2|X|^2, \quad \|u_x\|^2 \le \beta_3\|w_x\|^2 + \beta_4\|w\|^2 + \beta_5|X|^2 \tag{36}$$

where

$$\alpha_1 = 3(1 + l\|\phi\|^2), \quad \alpha_2 = 3\|\Phi\|^2, \quad \alpha_3 = 3, \quad \alpha_4 = 3l\|\phi_x\|^2, \quad \alpha_5 = 3\|\Phi'\|^2$$

$$\beta_1 = 3\left(1 + l\|\psi\|^2\right), \quad \beta_2 = 3\|\Psi\|^2, \quad \beta_3 = 3, \quad \beta_4 = 3l\|\psi_x\|^2, \quad \beta_5 = 3\|\Psi'\|^2$$

From Eqs. (35) and (36), it can be obtained that

$$\underline{\delta}(|X|^2 + \|u\|^2_{H^1(0,l)}) \le V \le \overline{\delta}(|X|^2 + \|u\|^2_{H^1(0,l)})$$

where

$$\overline{\delta} = \max\left\{\lambda_{\max}(P) + \frac{a\alpha_2}{2} + \frac{\alpha_5}{2}, \frac{a\alpha_1}{2} + \frac{\alpha_4}{2}, \frac{\alpha_3}{2}\right\}$$

$$\underline{\delta} = \frac{\min\left\{\lambda_{\min}(P), \frac{a}{2}, \frac{1}{2}\right\}}{\max\{\beta_2 + \beta_5 + 1, \beta_1 + \beta_4, \beta_3\}}$$

Taking a derivative of the Lyapunov function along the solutions to the system (5)–(8), then

$$\dot{V} \le -\frac{\lambda_{\min}(Q)}{2}|X|^2 + 2\frac{|PB|^2}{\lambda_{\min}(Q)}w(0,t)^2 - a\|w_x\|^2 - \|w_{xx}\|^2$$

From Agmon's inequality, the following inequality:

$$-\|w_{xx}\|^2 \le \frac{1+l}{l}\|w_x\|^2 - w_x(0,t)^2$$

can be proved, and thus

$$\dot{V} \le -\frac{\lambda_{\min}(Q)}{2}|X|^2 - \left(a - 8\frac{|PB|^2 l}{\lambda_{\min}(Q)} - \frac{1+l}{l}\right)\|w_x\|^2 - w_x(0,t)^2$$

By taking

$$a > 8\frac{|PB|^2 l}{\lambda_{\min}(Q)} + \frac{1+l}{l}$$

and using Poincaré inequality, it can be shown that

$$\dot{V} \le -bV$$

where

$$b = \min\left\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{2}{1+4l^2}\left(1 - 8\frac{|PB|^2 l}{a\lambda_{\min}(Q)} - \frac{1+l}{al}\right)\right\}$$

Therefore, $V(t) \le V(0)e^{-bt}$. Let $\delta = \overline{\delta}/\underline{\delta}$, then

$$(|X(t)|^2 + \|u(\cdot,t)\|^2_{H^1(0,l)}) \le \delta(|X(0)|^2 + \|u(\cdot,0)\|^2_{H^1(0,l)})e^{-bt}$$

for all $t \ge 0$, which completes the proof. □

4. Observer design and output feedback

To implement the control law (32), the signals $u(x,t)$ and $X(t)$ are supposed to be measurable. Sometimes, the information of the signal $u(x,t)$ is measurable only at one of the ends, or for economic considerations, is measured only at one end. In this situation, an observer is necessary to track the signal $u(x,t)$. Consider the case that only $u(0,t)$ is available for measurement, and the input is at the other end $x = l$.

Observer with Dirichlet actuation of the following form:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + Bu(0,t) + P_0(u(0,t) - \hat{u}(0,t)) \tag{37}$$

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + C\hat{X}(t) + p_1(x)(u(0,t) - \hat{u}(0,t)) \tag{38}$$

$$\hat{u}_x(0,t) = p_2(u(0,t) - \hat{u}(0,t)) \tag{39}$$

$$\hat{u}(l,t) = U(t) \tag{40}$$

where P_0 is a constant vector, $p_1(x)$ is a function and p_2 is a constant, which is to be designed to achieve exponential stabilization of the error system

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - P_0\tilde{u}(0,t) \tag{41}$$

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) + C\tilde{X}(t) - p_1(x)\tilde{u}(0,t) \tag{42}$$

$$\tilde{u}_x(0,t) = -p_2\tilde{u}(0,t) \tag{43}$$

$$\tilde{u}(l,t) = 0 \tag{44}$$

where

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t), \quad \tilde{X}(t) = X(t) - \hat{X}(t)$$

A transformation of the form

$$\tilde{w}(x,t) = \tilde{u}(x,t) - \Theta(x)\tilde{X}(t) \tag{45}$$

is also to be looked for to convert the system (41)–(44) into an exponentially stable target system, e.g.,

$$\dot{\tilde{X}}(t)=(A-P_0\Theta(0))\tilde{X}(t)-P_0\tilde{w}(0,t) \tag{46}$$

$$\tilde{w}_t(x,t)=\tilde{w}_{xx}(x,t) \tag{47}$$

$$\tilde{w}_x(0,t)=0 \tag{48}$$

$$\tilde{w}(l,t)=0 \tag{49}$$

where $A-P_0\Theta(0)$ is a Hurwitz matrix. Thus, the output injection functions P_0,$p_1(x)$ and p_2, together with $\Theta(x)$, are to be determined.

According to the transformation (45), the first two derivatives with respect to x and the first derivative with respect to t of $\tilde{w}(x,t)$ are given by

$$\tilde{w}_x(x,t)=\tilde{u}_x(x,t)-\Theta'(x)\tilde{X}(t) \tag{50}$$

$$\tilde{w}_{xx}(x,t)=\tilde{u}_{xx}(x,t)-\Theta''(x)\tilde{X}(t) \tag{51}$$

$$\tilde{w}_t(x,t)=\tilde{u}_{xx}(x,t)+(\Theta(x)P_0-p_1(x))\tilde{u}(0,t)-(\Theta(x)A-C)\tilde{X}(t) \tag{52}$$

By matching the systems (41)–(44) and (46)–(49), a sufficient condition for Eqs. (46)–(49) to hold is obtained as follows:

$$\Theta''(x)-\Theta(x)A+C=0 \tag{53}$$

$$\Theta'(0)=0 \tag{54}$$

$$\Theta(l)=0 \tag{55}$$

and

$$p_1(x)=\Theta(x)P_0 \tag{56}$$

$$p_2=0 \tag{57}$$

To construct the solution to the ODE (53)–(55), a lemma is shown firstly.

Lemma 1. *Write*

$$J=\begin{pmatrix}0 & A\\ I & 0\end{pmatrix},\quad L=(\,I\quad 0\,)e^{Jl}\begin{pmatrix}I\\ 0\end{pmatrix}$$

then L *is a nonsingular matrix if and only if the matrix* A *has no eigenvalues of the form* $-(2k+1)^2\pi^2/(4l^2)$ *for* $k\in\mathbb{N}$.

Proof. Firstly, there exists an invertible matrix M such that $M^{-1}AM$ is the Jordan's canonical form of A, that is

$$M^{-1}AM=\mathrm{diag}(\,N_1\quad\cdots\quad N_p\,)$$

where each Jordan block $N_q,1\leq q\leq p$, is a square matrix of lower-triangular type, and all the elements on its main diagonal are the eigenvalues of A, which are denoted by $\varsigma_j,j=1,2,\ldots,n$.

Secondly, a simple calculation gives that

$$L = \sum_{i=0}^{\infty} \frac{(l^2 A)^i}{(2i)!}$$

Thus

$$S := M^{-1} L M = \sum_{i=0}^{\infty} \frac{l^{2i}}{(2i)!} \operatorname{diag}(N_1^i \quad \cdots \quad N_p^i)$$

$$= \begin{pmatrix} \sum_{i=0}^{\infty} \frac{(l^2 \varsigma_1)^i}{(2i)!} & & 0 \\ & \ddots & \\ * & & \sum_{i=0}^{\infty} \frac{(l^2 \varsigma_n)^i}{(2i)!} \end{pmatrix} = \begin{pmatrix} \cosh(l\varsigma_1^{1/2}) & & 0 \\ & \ddots & \\ * & & \cosh(l\varsigma_n^{1/2}) \end{pmatrix}$$

Therefore S is singular if and only if $l\varsigma_j^{1/2} = ((2k+1)\pi/2)i_u$ (here i_u stands for the imaginary unit) for some ς_j, $j = 1,2,\ldots,n$ and $k \in \mathbb{N}$. Thus, L is a nonsingular matrix if and only if A has no eigenvalues of the form $-(2k+1)^2\pi^2/(4l^2)$ for $k \in \mathbb{N}$. □

When A has no eigenvalues of the form $-(2k+1)^2\pi^2/(4l^2)$ for $k \in \mathbb{N}$, according to Lemma 1, L is nonsingular and thus the solution to the non-homogeneous linear ODE two-point-boundary-value problem (53)–(55) is as

$$\Theta(x) = \Upsilon(x) e^{Jx} \begin{pmatrix} I \\ 0 \end{pmatrix} \tag{58}$$

where

$$\Upsilon(x) = (\Theta(0) \quad 0) - \int_0^x (0 \quad C) e^{-J\xi} \, d\xi$$

$$\Theta(0) = \int_0^l (0 \quad C) e^{-J\xi} \, d\xi \cdot e^{Jl} \begin{pmatrix} I \\ 0 \end{pmatrix} L^{-1}$$

Choose P_0 such that $A - P_0\Theta(0)$ is Hurwitz, then all the quantities needed to implement the observer (37)–(40) are determined.

The system (46)–(49) is a cascade of the exponentially stable heat equation (47)–(49) and the exponentially stable ODE (46). The entire observer error system is exponentially stable.

Theorem 2. *Assume that the matrix A has no eigenvalues of the form $-(2k+1)^2\pi^2/(4l^2)$ for $k \in \mathbb{N}$, then the observer* (37)–(40), *with gains defined through Eqs.* (56)–(58), *guarantees that observer error exponentially converges to zero, that is, $\hat{X}(t)$ and $\hat{u}(t)$ exponentially track $X(t)$ and $u(t)$ in the sense of the norm*

$$\|(\tilde{X}(t), \tilde{u}(\cdot,t))\|^2 = |\tilde{X}(t)|^2 + \|\tilde{u}(\cdot,t)\|^2_{H^1(0,l)}$$

Proof. From the transformation (45), the following relations

$$\|\tilde{w}\|^2 \le 2\|\tilde{u}\|^2 + 2|\Theta|^2|\tilde{X}|^2, \quad \|\tilde{w}_x\|^2 \le 2\|\tilde{u}_x\|^2 + 2|\Theta'|^2|\tilde{X}|^2$$

$$\|\tilde{u}\|^2 \le 2\|\tilde{w}\|^2 + 2|\Theta|^2|\tilde{X}|^2, \quad \|\tilde{u}_x\|^2 \le 2\|\tilde{w}_x\|^2 + 2|\Theta'|^2|\tilde{X}|^2$$

are obtained. With a Lyapunov function

$$\tilde{V}(t) = \tilde{X}^T \tilde{P}\tilde{X} + \frac{\tilde{a}}{2}\|\tilde{w}(\cdot,t)\|^2 + \frac{1}{2}\|\tilde{w}_x(\cdot,t)\|^2$$

where $\tilde{P} = \tilde{P}^T > 0$ is the solution to the Lyapunov equation

$$\tilde{P}(A - P_0\Theta(0)) + (A - P_0\Theta(0))^T\tilde{P} = -\tilde{Q}$$

for some $\tilde{Q} = \tilde{Q}^T > 0$ and $\tilde{a}$ is a constant to be determined, it can be obtained that

$$\underline{\varrho}(|\tilde{X}|^2 + \|\tilde{u}\|^2_{H^1(0,l)}) \le \tilde{V} \le \overline{\varrho}(|\tilde{X}|^2 + \|\tilde{u}\|^2_{H^1(0,l)})$$

where

$$\underline{\varrho} = \frac{\min\left\{\frac{\tilde{a}}{2}, \frac{1}{2}, \lambda_{\min}(\tilde{P})\right\}}{\max\{2, 2|\Theta'|^2 + 2|\Theta|^2 + 1\}}$$

$$\overline{\varrho} = \max\{\tilde{a}, 1, |\Theta'|^2 + \tilde{a}|\Theta|^2 + \lambda_{\max}(\tilde{P})\}$$

Take the time derivative of the Lyapunov function along the solution to the system (46)–(49), then

$$\dot{\tilde{V}} \le -\frac{\lambda_{\min}(\tilde{Q})}{2}|\tilde{X}|^2 - \left(\tilde{a} - 8\frac{|\tilde{P}P_0|^2 l}{\lambda_{\min}(\tilde{Q})} - \frac{1+l}{l}\right)\|\tilde{w}_x\|^2 - \tilde{w}_x(0,t)^2$$

where the last line is obtained by using Agmon's inequality and the following inequality:

$$-\|\tilde{w}_{xx}\|^2 \le \frac{1+l}{l}\|\tilde{w}_x\|^2 - \tilde{w}_x(0,t)^2$$

Take

$$\tilde{a} > 8\frac{|\tilde{P}P_0|^2 l}{\lambda_{\min}(\tilde{Q})} + \frac{1+l}{l}$$

and use Poincaré inequality, then

$$\dot{\tilde{V}} \le -\tilde{b}\tilde{V} \tag{59}$$

where

$$\tilde{b} = \min\left\{\frac{\lambda_{\min}(\tilde{Q})}{2\lambda_{\max}(\tilde{P})}, \frac{2}{1+4l^2}\left(1 - 8\frac{|\tilde{P}P_0|^2 l}{\tilde{a}\lambda_{\min}(\tilde{Q})} - \frac{1+l}{\tilde{a}l}\right)\right\} > 0$$

Let $\varrho = \overline{\varrho}/\underline{\varrho}$, then

$$|\tilde{X}(t)|^2 + \|\tilde{u}(\cdot,t)\|^2_{H^1(0,l)} \le \varrho(|\tilde{X}(0)|^2 + \|\tilde{u}(\cdot,0)\|^2_{H^1(0,l)})e^{-\tilde{b}t}$$

for all $t \ge 0$, which means that the error system (41)–(44) is exponentially stable in the sense of the norm

$$\|(\tilde{X}(t), \tilde{u}(\cdot,t))\|^2 = |\tilde{X}(t)|^2 + \|\tilde{u}(\cdot,t)\|^2_{H^1(0,l)}$$

and thus completes the proof. □

Replace $u(y,t)$ and $X(t)$ with $\hat{u}(y,t)$ and $\hat{X}(t)$ in Eq. (32) respectively, an output feedback control law is obtained as follows:

$$U(t) = \int_0^l \phi(l-y)\hat{u}(y,t)\,dy + \Phi(l)\hat{X}(t) \tag{60}$$

Theorem 3. *Assume that the matrix A has no eigenvalues of the form $-(2k+1)^2\pi^2/(4l^2)$ for $k \in \mathbb{N}$, then for any initial data $X(0),\hat{X}(0) \in \mathbb{R}$ and $u(\cdot,0),\hat{u}(\cdot,0) \in H^1(0,l)$, the closed-loop system consisting of the plant* (1)–(3), *the controller* (60) *and the observer* (37)–(40) *has a unique classical solution and is exponentially stable in the sense of the norm*

$$\|(X(t),u(\cdot,t),\hat{X}(t),\hat{u}(\cdot,t))\|^2 = |X(t)|^2 + \|u(\cdot,t)\|^2_{H^1(0,l)} + |\hat{X}(t)|^2 + \|\hat{u}(\cdot,t)\|^2_{H^1(0,l)}$$

Proof. The transformation

$$\hat{w}(x,t) = \hat{u}(x,t) - \int_0^x \phi(x-y)\hat{u}(y,t)\,dy - \Phi(x)\hat{X}(t) \tag{61}$$

converts Eqs. (37)–(40) into the system

$$\dot{\hat{X}}(t) = (A+BK)\hat{X}(t) + B\hat{w}(0,t) + (B+P_0)\big(\tilde{w}(0,t) + \Theta(0)\tilde{X}(t)\big) \tag{62}$$

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t) + \left(p_1(x) - \Phi(x)(B+P_0) - \int_0^x \phi(x-y)p_1(y)\,dy\right)(\tilde{w}(0,t) + \Theta(0)\tilde{X}(t)) \tag{63}$$

$$\hat{w}_x(0,t) = 0 \tag{64}$$

$$\hat{w}(l,t) = 0 \tag{65}$$

The $(\tilde{X},\tilde{w})$ system (46)–(49) and the homogeneous part of the $(\hat{X},\hat{w})$ system (62)–(65) (without $\tilde{X}(t),\tilde{w}(0,t)$) are exponentially stable. The interconnection of the two systems $(\hat{X},\hat{w},\tilde{X},\tilde{w})$ is a cascade. The combined $(\hat{X},\hat{w},\tilde{X},\tilde{w})$ system is exponentially stable. In fact, this can be proved by taking the weighted Lyapunov function

$$E(t) = \hat{X}^T\hat{P}\hat{X} + \frac{\hat{a}}{2}\|\hat{w}(\cdot,t)\|^2 + \frac{1}{2}\|\hat{w}_x(\cdot,t)\|^2 + e\tilde{V}(t) \tag{66}$$

where the matrix $\hat{P} = \hat{P}^T > 0$ is the solution to the Lyapunov equation

$$\hat{P}(A+BK) + (A+BK)^T\hat{P} = -\hat{Q}$$

for some $\hat{Q} = \hat{Q}^T > 0$, the constant $\hat{a}$ and the weighting constant e are to be chosen later. Taking the time derivative of Eq. (66),

$$\begin{aligned}
\dot{E} \le & -\hat{X}^T\hat{Q}\hat{X} + 2\hat{X}^T P(B\hat{w}(0,t) + (B+P_0)(\tilde{w}(0,t) + \Theta(0)\tilde{X}(t))) \\
& -\hat{a}\|\hat{w}_x\|^2 + \hat{a}\int_0^l \hat{w}(x)\Bigg(p_1(x) - \Phi(x)(B+P_0) \\
& - \int_0^x \phi(x-y)p_1(y)\,dy\Bigg)(\tilde{w}(0,t) + \Theta(0)\tilde{X}(t))\,dx \\
& -\|\hat{w}_{xx}\|^2 + \int_0^l \hat{w}_x(x)\Bigg(p_1'(x) - \Phi'(x)(B+P_0)
\end{aligned}$$

$$-\int_0^x \phi'(x-y)p_1(y)\,dy\Big)(\tilde{w}(0,t)+\Theta(0)\tilde{X}(t))\,dx$$
$$+e\left(-\frac{\lambda_{\min}(\tilde{Q})}{2}|\tilde{X}|^2-\left(\tilde{a}-8\frac{|\tilde{P}P_0|^2 l}{\lambda_{\min}(\tilde{Q})}-\frac{1+l}{l}\right)\|\tilde{w}_x\|^2\right)$$

Using Poincaré, Agmon's and Young inequalities, taking

$$\theta=\max\left\{p_1(x)-\Phi(x)(B+P_0)-\int_0^x \phi(x-y)p_1(y)\,dy\right\}$$

$$\vartheta=\max\left\{p_1'(x)-\Phi'(x)(B+P_0)-\int_0^x \phi'(x-y)p_1(y)\,dy\right\}$$

then it can be obtained that

$$\dot{E}\le -e_1|\hat{X}|^2-e_2\|\hat{w}_x\|^2-e_3|\tilde{X}|^2-e_4\|\tilde{w}_x\|^2$$

where

$$e_1=\frac{\lambda_{\min}(\hat{Q})}{2}-\varepsilon|P(B+P_0)|^2$$

$$e_2=\frac{\hat{a}}{2}-\frac{1}{2}-16\frac{|PB|^2 l}{\lambda_{\min}(\hat{Q})}-\frac{1+l}{l}$$

$$e_3=\frac{\lambda_{\min}(\tilde{Q})}{2}e-\left(\frac{1}{\varepsilon}+4\hat{a}\theta^2 l^3+\vartheta^2 l\right)|\Theta(0)|^2$$

$$e_4=e\left(\tilde{a}-8\frac{|\tilde{P}P_0|^2 l}{\lambda_{\min}(\tilde{Q})}-\frac{1+l}{l}\right)-16\frac{|P(B+P_0)|^2 l}{\lambda_{\min}(\hat{Q})}-16\hat{a}\theta^2 l^4-4\vartheta^2 l^2$$

and $\varepsilon>0$. Take

$$\hat{a}>32\frac{|PB|^2 l}{\lambda_{\min}(\hat{Q})}+\frac{3l+2}{l},\quad \varepsilon<\frac{\lambda_{\min}(\hat{Q})}{2|P(B+P_0)|^2}$$

$$e>\frac{2}{\lambda_{\min}(\tilde{Q})}\left(\frac{1}{\varepsilon}+4\hat{a}\theta^2 l^3+\vartheta^2 l\right)|\Theta(0)|^2$$

$$\tilde{a}>8\frac{|\tilde{P}P_0|^2 l}{\lambda_{\min}(\tilde{Q})}+\frac{1+l}{l}+\frac{1}{e}\left(16\frac{|P(B+P_0)|^2 l}{\lambda_{\min}(\hat{Q})}+16\hat{a}\theta^2 l^4+4\vartheta^2 l^2\right)$$

then it can be obtained that

$$\dot{E}\le -fE$$

where

$$f=\min\left\{\frac{e_1}{\lambda_{\max}(\hat{P})},\frac{2e_2}{\hat{a}(1+4l^2)},\frac{e_3}{e\lambda_{\max}(\tilde{P})},\frac{2e_4}{e\tilde{a}(1+4l^2)}\right\}$$

Hence, the system $(\hat{X},\hat{w},\tilde{X},\tilde{w})$ is exponentially stable. Since the transformations (45) and (61) are invertible, exponential stability of the system $(\hat{X},\hat{w},\tilde{X},\tilde{w})$ ensures exponential stability of the system $(\hat{X},\hat{u},\tilde{X},\tilde{u})$. This directly implies the closed-loop stability of the system $(X,u,\hat{X},\hat{u})$. □

5. Conclusions and comments

In this paper boundary controller and observer for a coupled PDE–ODE control system are developed through PDE backstepping. Meanwhile, state and output feedback boundary control problems are solved.

Firstly, the method of PDE backstepping is employed here. For PDE backstepping, difficulties generally come from seeking for the kernel functions, and here the equations of kernel functions are still coupled. By using some skills, it is feasible to decouple and then solve them. Secondly, the systems are generally considered whether to be stabilized in the L^2 norm, but they are stabilized in the H^1 norm in this paper.

Stabilization for coupled PDE–ODE control systems with boundary control is an original area with so many problems to be considered. Coupled PDE–ODE control systems with delays are also being worked on. More interesting areas, such as stabilization for coupled PDE–PDE systems with boundary control, are also subjects of the ongoing research.

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