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STABILIZATION FOR THE 3D NAVIER-STOKES SYSTEM BY FEEDBACK BOUNDARY CONTROL

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Dedicated to Mark Iosifovich Vishik

Abstract. We study the problem of stabilization a solution to 3D Navier-Stokes system given in a bounded domain Ω. This stabilization is carried out with help of feedback control defined on a part Γ of boundary $\partial\Omega$. We assume that Γ is closed 2D manifold without boundary. Here we continuer investigation begun in [6],[7] where stabilization problem for parabolic equation and for 2D Navier-Stokes system was studied.

1. **Introduction.** This paper is devoted to study the problem of stabilization a solution $v(t, x)$ to boundary value problem for three-dimensional (3D) Navier-Stokes equations given in a bounded domain $\Omega \in \mathbb{R}^3$. This solution is stabilized near a steady- state solution to 3D Navier-Stokes system. We carry out this stabilization with help of control u defined on a part Γ of boundary $\partial\Omega$. Our assumption imposed on Γ is that Γ is a closed two-dimensional surface and therefore Γ is a separate component of $\partial\Omega$.

We require that control $u = u(t, x')$, $t > 0, x' \in \Gamma$ has to possess the following important property: u is a feedback control. This means that for each instant t $u(t, \cdot)$ is defined by fluid flow velocity vector field $v(t, \cdot)$ taken at the same instant t and therefore control u can react on unpredictable fluctuations of v suppressing their negative influence to fluid flow. There is a mathematical formalization for this physical notion of feedback, which was proposed long time ago. With help of this formalization number results of stabilization for equations described incompressible fluid flow were obtained: stabilization of 2D Navier-Stokes equation by distributed control supported on the whole Ω and written in an abstract form (Barbu, Sritharan [1]), and stabilization by boundary control of 2D Euler equations for incompressible fluid flow (Coron [3]).

In this paper we use a certain new formalization of feedback notion that we proposed in [6], [7], and that is more adequate (from our point of view) to study stabilization problem by boundary feedback control in the case of parabolic equations and Navier-Stokes system. Moreover, here we develop approach to stabilization problem from [6], [7] in order to get stabilization result for 3D Navier-Stokes equations. The main point of this approach is to construct a special operator that

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extends solenoidal vector fields from Ω on a certain domain G containing Ω. Extension operator connected with linearized Navier-Stokes equations (i.e. with Oseen system) is constructed below in Section 5 and extension operator corresponding to nonlinear case is built in Section 6. Construction of these extension operators is based on a linear independence property of finite systems of eigen and associated functions for adjoint steady- state Oseen system when these functions are regarded over arbitrary subdomain $\omega \subset G$. We prove this property in Section 4 with help of Carleman estimates using some abstract result from [6]. In Section 2 we formulate stabilization problem for Navier-Stokes equation and, besides, we compare two mathematical formalizations of feedback notion: classical one and the formalization that was proposed in [6], [7]. In Section 3 we recall certain well known results, used in the paper.

2. **Setting of the problem and the main idea of the method.**

2.1. **Setting of the stabilization problem.** Let $\Omega \subset \mathbb{R}^3$ be a bounded connected domain with C^∞ -boundary $\partial\Omega$ which consists of two nonintersecting parts Γ_0 and Γ:

$$\partial\Omega = \Gamma_0 \cup \Gamma, \qquad \Gamma_0 \cap \Gamma = \emptyset \tag{2.1}$$

where Γ_0, Γ are closed subsets of $\partial\Omega$, i.e. Γ_0, Γ are finite sets of connected C^∞-manifolds of dimension 2. We assume that $\Gamma \neq \emptyset$ but we admit that the set Γ_0 can be empty.

We set

$$Q = \mathbb{R}_+ \times \Omega, \quad \Sigma = \mathbb{R}_+ \times \Gamma, \quad \Sigma_0 = \mathbb{R}_+ \times \Gamma_0. \tag{2.2}$$

In space-time cylinder Q we consider the Navier-Stokes equations

$$\partial_t v(t,x) - \Delta v(t,x) + (v,\nabla)v + \nabla p(t,x) = f(x), \quad (t,x) \in Q \tag{2.3}$$

$$\operatorname{div} v = 0 \tag{2.4}$$

$(v = (v^1, v^2, v^3))$ with initial condition

$$v(t,x)|_{t=0} = v_0(x), \quad x \in \Omega \tag{2.5}$$

and boundary conditions

$$v|_{\Sigma_0} = 0, \quad v|_\Sigma = u, \tag{2.6}$$

where $u = (u^1, u^2, u^3)$ is a control defined on Σ.

We suppose also that a steady-state solution $(\hat{v}(x), \nabla\hat{p}(x))$ of Navier–Stokes system with the same right-hand side $f(x)$ as in (2.3) is given:

$$\Delta\hat{v}(x) + (\hat{v},\nabla)\hat{v} + \nabla\hat{p} = f(x), \quad \operatorname{div}\hat{v}(x) = 0, \quad x \in \Omega \tag{2.7}$$

$$\hat{v}|_{\Gamma_0} = 0. \tag{2.8}$$

Let $\sigma > 0$ be given. The problem of stabilization with the rate σ is to look for a control $u(t,x'), x' \in \Gamma$ such that the solution (v,p) of problem (2.3)–(2.6) with the boundary value u satisfies the inequality

$$\|v(t,\cdot) - \hat{v}\|_{(H^1(\Omega))^2} \leqslant ce^{-\sigma t} \quad \text{as } t \to \infty. \tag{2.9}$$

2.2. **Feedback control. Classical definition.** Our important additional requirement is that a stabilization problem should be solved with help of feedback control. From physical point of view feedback notion means that control function $u(t,\cdot)$ at instant t should be defined with help of state function $v(t,\cdot)$ taken just at the same time moment t. This has to give possibility for control to react on unpredictable fluctuations of the state function in order to suppress all undesirable effects of these fluctuations.

There is well known mathematical formalization of this physical feedback notion which was proposed in control theory for ordinary differential equations. To recall it we rewrite problem (2.3)–(2.6) in a form of abstract ordinary differential vector-valued equation

$$\partial_t v(t,\cdot) = F(v(t,\cdot), u(t,\cdot)), \qquad v(t,\cdot)|_{t=0} = v_0 \tag{2.10}$$

where $F(\cdot,\cdot) : V \times U \to W$ is a nonlinear operator acting from direct product of phase space V and control space U to W (V, U, W are Banach spaces). Then steady-state problem (2.7),(2.8) can be rewritten as follows:

$$F(\hat{v}(\cdot), \hat{u}(\cdot)) = 0 \tag{2.11}$$

Suppose that $(\hat{v}, \hat{u}) \in V \times U$ is a given solution of (2.11). Then the problem of stabilization for solution to (2.10) near steady-state solution $(\hat{v}, \hat{u})$ with a prescribed rate $\sigma > 0$ is to find $u(t,\cdot)$ such that the solution $v(t,\cdot)$ of problem (2.10) with this $u(t,\cdot)$ substituted into, satisfies the inequality

$$\|v(t,\cdot) - \hat{v}\|_V \leqslant ce^{-\sigma t} \quad \text{as} \quad t \longrightarrow \infty. \tag{2.12}$$

Definition 2.1. Control $u(t,\cdot)$ of stabilization problem (2.10)–(2.12) is called *feedback* if there exists a map $R : V \to U$ such that for each $t > 0$ $u(t,\cdot) = R(v(t,\cdot))$.

Recall that classical formulation of a stabilization problem by feedback control is as follows:

Given steady-state solution $(\hat{v}, \hat{u})$ of (2.10), find a map $R : V \to U$ such that the solution $v(t,\cdot)$ of problem

$$\dot{v}(t,\cdot) = F(v(t,\cdot), R(v(t,\cdot))), \qquad v(t,\cdot)|_{t=0} = v_0 \tag{2.13}$$

satisfies (2.12). (R does not depend on v_0)

Remark 2.1. Usual assumption imposed on the exponent σ in this formulation is that σ is a certain positive but it is not an arbitrary positive as in formulation (2.3)–(2.9). But usually a map R is looked for by such a way that boundary value problem (2.13) is well posed, i.e. inequality (2.12) remaines true after small fluctuations of data for this problem .

If stabilization problem by feedback control should be solved only for initial conditions v_0 belonging to a certain neighborhood of $\hat{v}$, then it is called a local stabilization problem. This classical setting of stabilization problem by feedback control was successfully applied not only for controlled ordinary differential equations but also for certain controlled PDE including 2D Navier-Stokes equations with distributed control supported on the whole domain containing liquid (see [1]). Nevertheless this approach did not give yet possibility to solve stabilization problem for general quasi linear parabolic equation or for Navier-Stokes system with feedback control supported on the boundary of the domain as our new setting proposed in [6], [7]. Properties of Euler equations regarded in [3] differ essentially from properties of parabolic equation or Navier-Stokes system.

2.3. **The main idea of construction.** Construction of feedback control which we proposed in [6], [7] is not included in the framework of Definition 2.1. Let recall this construction.

Let $\omega \subset \mathbb{R}^3$ be a bounded domain such that

$$\Omega \cap \omega = \emptyset, \quad \overline{\Omega} \cap \overline{\omega} = \Gamma \tag{2.14}$$

We set

$$G = \mathrm{Int}(\overline{\Omega} \cup \overline{\omega}) \tag{2.15}$$

(the notation $\mathrm{Int}\, A$ means, as always, the interior of the set A). We suppose that ∂G is a two-dimensional surface belonging to the smoothness class C^∞.

We extend problem (2.3)–(2.6) from $Q = \mathbb{R}_+ \times \Omega$ to $\Theta = \mathbb{R}_+ \times G$ For this end we forget for a while about the second boundary condition in (2.6) and write this extended problem as follows:

$$\partial_t w(t,x) - \Delta w + (w,\nabla)w + \nabla q(t,x) = g(x), \quad \mathrm{div}\, w(t,x) = 0 \tag{2.16}$$

$$w(t,x)|_{t=0} = w_0(x) \tag{2.17}$$

with additional condition

$$w|_S = 0 \tag{2.18}$$

where $S = \mathbb{R}_+ \times \partial G$. Moreover we assume that solution $(\hat{v}, \nabla\hat{p})$ of (2.7), (2.8) is extended on G in a pair $(a(x), \nabla\hat{q}(x))$, $x \in G$ such that

$$-\Delta a(x) + (a,\nabla)a + \nabla\hat{q}(x) = g(x), \quad \mathrm{div}\, a(x) = 0, \quad x \in G \tag{2.19}$$

$$a|_{\partial G} = 0 \tag{2.20}$$

where right side $g(x)$ is the same as in (2.16). (We show below how to construct such extension.) Note that, actually, w_0 from (2.17) will be a special extension of v_0 in (2.5) from Ω to $G:\ w_0 = \mathrm{Ext}_\sigma v_0$. More precisely, w_0 should belong to the stable manifold M_σ which is invariant with respect to the semigroup generated by the Navier-Stokes problem (2.16)–(2.18) and which contains solutions $w(t,\cdot)$ tending to a with the rate σ (as in (2.9)). More detailed definition of M_σ will be given in Section 6.

We introduce the following space of solenoidal vector fields:

$$V^k(G) = \{v(x) \in (H^k(G))^3 :\ \mathrm{div}\, v(x) = 0\}$$

where $H^k(G)$ is Sobolev space of functions $f(x)$, $x \in G$ belonging to $L_2(\Omega)$ together with their derivatives of order not more than k. Definition of $H^k(G)$ with fractional or negative k see in [15]. $(H^k(G))^3$ is Sobolev space of three- dimensional vector fields $f(x) = (f_1(x), f_2(x), f_3(x))$ with $f_i(x) \in H^k(G)$. For vector fields defined on G we introduce the operator γ_Ω of restriction on Ω and the operator γ_Γ of restriction on Γ:

$$\gamma_\Omega : V^k(G) \longrightarrow V^k(\Omega),\ k \geqslant 0, \quad \gamma_\Gamma : V^k(G) \longrightarrow (H^{k-1/2}(\Gamma))^3,\ k > 1/2. \tag{2.21}$$

As well known (see, for instance, [15]), operators (2.21) are bounded.

Definition 2.2. A control $u(t,x)$ in stabilization problem (2.3)–(2.6) is called feedback if the solution $(v(t,x), u(t,x))$ of (2.3)–(2.6) is defined by the equality:

$$(v(t,x), u(t,x)) = (\gamma_\Omega w(t,\cdot), \gamma_\Gamma w(t,\cdot)) \tag{2.22}$$

where $w(t,x)$ satisfies to (2.16)–(2.18), and γ_Ω, γ_Γ are operators of restriction of a function defined on G to Ω and to Γ respectively.

This definition of feedback control is basic for us: we will use only it below. Now we show connection between Definitions 2.1 and 2.2.

Let $S(t,w_0)$ be the semigroup generated by boundary value problem (2.16)–(2.18), i.e. if $w(t,\cdot)$ is a solution to (2.16)–(2.18) with initial condition w_0, then $S(t,w_0)=w(t,\cdot)$. Then the map which acts initial condition v_0 from (2.5) to the solution $v(t,\cdot)$ of (2.3)–(2.5) is defined as follows:

$$v(t,\cdot)=\gamma_\Omega S(t,\mathrm{Ext}_\sigma v_0) \tag{2.23}$$

Using (2.23) we define the following extension operator from Ω to G:

$$E(t,v(t,\cdot))=S(t,\mathrm{Ext}_\sigma v_0) \tag{2.24}$$

where v_0 is the initial condition of $v(t,\cdot)$. Note that operator $E(t,\cdot)$ depends on t because it is defined on the set of functions which belong to the image of the operator $S(t,\mathrm{Ext}_\sigma\cdot)$ and this set depends on t. Now we can define the operator which acts solution $v(t,\cdot)$ to the correspondent control function $u(t,\cdot)$:

$$u(t,\cdot)=R(t,v(t,\cdot))\equiv\gamma_\Gamma E(t,v(t,\cdot)) \tag{2.25}$$

In contrast to operator R from Definition 2.1 operator (2.25) depends on t.

3. **Ozeen equations.** We begin investigation of stabilization problem from the case of linearized Navier-Stokes equations, i.e. from the Ozeen equations. Note that the results of this section connected with 3D Oseen equations as well as their proofs are absolutely identic to analogous results and proofs for 2D Oseen equations. That is why we give here only their short formulation. Detailed exposition of these results can be found in [7].

3.1. **Preliminaries.** Let G be domain (2.15). We consider in $\mathbb{R}_+\times G$ the Oseen equation which is written as follows:

$$\partial_t w(t,x)-\Delta w+(a(x),\nabla)w+(w,\nabla)a+\nabla p(t,x)=0 \tag{3.1}$$

$$\operatorname{div} w(t,x)=0 \tag{3.2}$$

$$w(t,x)|_{t=0}=w_0(x) \tag{3.3}$$

Moreover, we impose on w the zero Dirichlet boundary condition

$$w|_S=0, \tag{3.4}$$

where $S=\mathbb{R}_+\times\partial G$. We assume that

$$a(x)\in V^2(G)\cap\left(H_0^1(G)\right)^3 \tag{3.5}$$

where, recall, Sobolev spaces $V^k(G)$, $H^k(G)$ were defined above, and $H_0^1(G)=\{f(x)\in H^1(G): f(x)|_{x\in\partial G}=0\}$. In the case $k=0$ we define

$$V_0^0(G)=\{v(x)\in V^0(G): v\cdot\nu|_{\partial\Omega}=0\} \tag{3.6}$$

where $\nu(x)$ is the vector-field of outer normals to ∂G. In [16] it is established that restriction $v\cdot\nu|_{\partial\Omega}$ is well defined for $v\in V^0(G)$. Denote by

$$\pi:(L_2(G))^2\longrightarrow V_0^0(G) \tag{3.7}$$

the operator of orthogonal projection. We consider the Ozeen steady state operator

$$Av\equiv-\pi\Delta v+\pi[(a(x),\nabla)v+(v,\nabla)a]: V_0^0(G)\longrightarrow V_0^0(G) \tag{3.8}$$

where $a(x)$ is vector-field (3.5). This operator is closed and it has the domain:

$$\mathcal{D}(A)=V^2(G)\cap(H_0^1(G))^2 \tag{3.9}$$

which is dense in $V_0^0(G)$.

Assuming that spaces in (3.7), (3.9) are complex we denote by $\rho(A)$ the resolvent set of operator A, i.e. the set of $\lambda \in \mathbb{C}$ such that the resolvent operator

$$R(\lambda, A) \equiv (\lambda I - A)^{-1} : V_0^0(G) \longrightarrow V_0^0(G) \tag{3.10}$$

is defined and continuous. Here I is identity operator. Denote by $\Sigma(A) \equiv \mathbb{C}^1 \setminus \rho(A)$ the spectrum of operator A.

As well-known, Ozeen operator (3.8) is sectorial, i.e. there exist $\varphi \in (0, \pi/2)$, $M \geqslant 1, a \in \mathbb{R}$ such that

$$S_{a,\varphi} = \{\lambda \in \mathbb{C} : \quad \varphi \leqslant |\arg(\lambda - a)| \leqslant \pi, \quad \lambda \neq a\} \subset \rho(A) \tag{3.11}$$

and $\|(\lambda I - A)^{-1}\| \leqslant M/|\lambda - a|, \quad \forall \lambda \in S_{a,\varphi}$. Besides, for $\lambda \in \rho(A)$ resolvent (3.10) is a compact operator, and the spectrum $\Sigma(A)$ consists of a discrete set of points.

We decompose the resolvent $R(\lambda, -A)$ in a neighborhood of $-\lambda_j \in \Sigma(-A)$:

$$R(\lambda, -A) = \sum_{k=-m}^{\infty} (\lambda + \lambda_j)^k R_k, \quad R_k = (2\pi i)^{-1} \int\limits_{|\lambda + \lambda_j| = \varepsilon} (\lambda + \lambda_j)^{-k-1} R(\lambda, -A) d\lambda \tag{3.12}$$

Note that $m < \infty$.

Let us consider the adjoint operator A^* to Ozeen operator (3.8):

$$A^* w \equiv -\pi \Delta w - \pi[(a(x), \nabla) w - (\nabla a)^* w] : V_0^0(G) \longrightarrow V_0^0(G) \tag{3.13}$$

where

$$(\nabla a)^* w = ((\partial_1 a, w), (\partial_2 a, w), (\partial_3 a, w)), \quad (\partial_i a, w) = \sum_{j=1}^{3} \partial_i a_j w_j$$

Evidently, A^* is a closed operator with domain $\mathcal{D}(A^*) = V^2(G) \cap (H_0^1(G))^2$. Moreover, A^* is sectorial with a compact resolvent and

$$\rho(A^*) = \overline{\rho(A)} \quad \text{and} \quad R(\lambda, A)^* = R(\bar{\lambda}, A^*) \quad \forall\, \lambda \in \rho(A) \tag{3.14}$$

(Here the line above means complex conjugation.) Below we always assume that

$$\text{vector field } a(x) \text{ from } (3.5), (3.8), (3.13) \text{ is real valued.} \tag{3.15}$$

That is why we have $\rho(A) = \overline{\rho}(A) = \rho(A^*) = \overline{\rho}(A^*)$.

3.2. **Structure of R_k with $k < 0$.** Let $-\lambda_j \in \Sigma(-A)$ be an eigenvalue of $-A$, and $e \neq 0$, $e \in \ker(\lambda_0 I + A)$ be an eigenvector. Vector e_k is called associated vector of order k to e if e_k satisfies:

$$(\lambda_0 I + A) e = 0, \quad e + (\lambda_0 I + A) e_1 = 0, \ldots, \quad e_{k-1} + (\lambda_0 I + A) e_k = 0.$$

We say that $e, e_1, e_2, \ldots$ form a chain of associated vectors. The maximal order m of vectors, associated to e is finite and the number $m + 1$ is called multiplicity of the eigenvector e.

Definition 3.1. The set of eigenvectors and associated vectors

$$e^{(k)}(-\lambda_j), e_1^{(k)}(-\lambda_j), \ldots, e_{m_k}^{(k)}(-\lambda_j) \qquad (k = 1, 2, \ldots, N(-\lambda_j)) \tag{3.16}$$

corresponding to an eigenvalue $-\lambda_j$ is called canonical system if:

i) Vectors $e^{(k)}(-\lambda_j), k = 1, 2, \ldots, N(-\lambda_j)$ form a basis in the space of eigenvectors corresponding to the eigenvalue $-\lambda_j$.
ii) $e^{(1)}(-\lambda_j)$ is an eigenvector with maximal possible multiplicity.

iii) $e^{(k)}(-\lambda_j)$ is an eigenvector which can not be expressed by a linear combination of $e^{(1)}(-\lambda_j), \dots, e^{(k-1)}(-\lambda_j)$ and multiplicity of $e^{(k)}(-\lambda_j)$ achieves a possible maximum.

iy) Vectors (3.16) with fixed k form a complete chain of associated elements.

Besides canonical system (3.16) which corresponds to an eigenvalue $-\lambda_j$ of operator $-A$ we consider a canonical system

$$\varepsilon^{(k)}(-\overline{\lambda}_j), \varepsilon_1^{(k)}(-\overline{\lambda}_j), \dots, \varepsilon_{m_k}^{(k)}(-\overline{\lambda}_j) \qquad (k = 1, 2, \dots, N(-\overline{\lambda}_j)) \tag{3.17}$$

that corresponds to the eigenvalue $-\overline{\lambda}_j$ of the adjoint operator $-A^*$. Definition of canonical system (3.17) is absolutely analogous to Definition 3.1 of canonical system (3.16). We define canonical system (3.17) by $E^*(-\overline{\lambda}_j)$.

Theorem 3.1. *Let R_k are operators defined in (3.12). Then*

$$R_{-k}x = 0, \quad \forall k = 1, 2, \dots, m$$

if and only if

$$\langle x, \varepsilon_l^{(k)}(-\overline{\lambda}_j)\rangle = 0 \qquad \forall \varepsilon_l^{(k)}(-\overline{\lambda}_j) \in E^*(-\overline{\lambda_j})$$

This assertion follows immediately from one result of Keldysh [12] on structure of the main part of Laurent series for $R(\lambda, -A)$. The proof of Theorem 3.1 see in [7].

3.3. **Holomorphic semigroups.** We regard boundary value problem (3.1)–(3.4) for Ozeen equations written in the form

$$\frac{dw(t)}{dt} + Aw(t) = 0, \quad w|_{t=o} = w_0. \tag{3.18}$$

where A is operator (3.8). Then for each $w_0 \in V_0^0(G)$ the solution $w(t, \cdot)$ of (3.18) is defined by $w(t, \cdot) = e^{-At}w_0$ and

$$e^{-At} = (2\pi i)^{-1} \int_\gamma (\lambda I + A)^{-1} e^{\lambda t} d\lambda, \tag{3.19}$$

where γ is a contour belonging to $\rho(-A)$ such that $\arg \lambda = \pm\theta$ for $\lambda \in \gamma, |\lambda| \geqslant N$ for certain $\theta \in (\pi/2, \pi)$ and for sufficiently large N. Moreover, γ surrounds $\Sigma(-A)$ from the right. Such contour γ exists, of course, because we can choose γ belonging to set $-S_{a,\varphi}$ from (3.11).

Let $\sigma > 0$ satisfy:

$$\Sigma(-A) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda = -\sigma\} = \emptyset \tag{3.20}$$

The case when there are certain points of $\Sigma(-A)$ placed righter than the line $\{\mathrm{Re}\lambda = -\sigma\}$ will be interesting for us. By γ_σ we denote the continuous contour that is placed in $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leqslant -\sigma\}$ and constructed from an interval of the line $\{\mathrm{Re}\lambda = -\sigma\}$ and from two branches of contour γ that transform to $\{\arg \lambda = \theta\}$ and $\{\arg \lambda = -\theta\}$, $\theta \in (\pi/2, \pi)$ for sufficiently large $|\lambda|$.

In virtue of Cauchy Theorem we reduce integration over γ in (3.19) to integration over γ_σ and integration around poles $-\lambda_j$ from (3.12) for λ_j satisfying $\mathrm{Re}\lambda_j < \sigma$ After calculation corresponding residues we transform (3.19) to the equality:

$$e^{-At} = (2\pi i)^{-1} \int\limits_{\gamma_{\sigma_0}} (\lambda E + A)^{-1} e^{\lambda t} d\lambda + \sum_{\mathrm{Re}\lambda_j < \sigma} e^{-\lambda_j t} \sum_{n=1}^{m(-\lambda_j)} \frac{t^{n-1}}{(n-1)!} R_{-n}(-\lambda_j). \tag{3.21}$$

We denote

$$V^1_0(G) = \{v \in V^1(G) : v|_{\partial G} = 0\}.$$

Equality (3.21) implies

Theorem 3.2. *Suppose that A is operator* (3.8) *and $\sigma > 0$ satisfies* (3.20). *Then for each $w_0 \in V^1_0(G)$ that satisfies*

$$\langle w_0, \varepsilon^{(k)}_l(-\bar{\lambda}_j) >= 0, \quad \forall l = 0, 1, \dots, m_k,\ k = 1, 2, \dots, N(-\lambda_j),\ \mathrm{Re}(\lambda_j) < \sigma$$

(*here by definition $\varepsilon^{(k)}_0(-\lambda_j) = \varepsilon^{(k)}(-\lambda_j)$*) *the following inequality holds:*

$$\|e^{At}w_0\|_{V^1_0(G)} \leqslant ce^{-\sigma t}\|w_0\|_{V^1_0(G)} \quad \text{for } t \geqslant 0 \tag{3.22}$$

Proof see in [6], [7].

4. **unique continuation property.** To solve stabilization problem we will use unique continuation property for solution of adjoint Oseen equation, i.e. for solution of equation $(\overline{\mu}_0 - A)^* w = 0$ where A^* is operator (3.13). Unique continuation property for the Stokes equations has been established in [9] with help of Carleman estimates derived in [11]. That Stokes equation differs from adjoint Oseen equation indicated above. That is why we give here complete proof of the unique continuation property for a solution of $(\overline{\mu}_0 - A)^* w = 0$. As in [11], [9], to do it we use Carleman estimate, but our technology differ from techniques of [11] and it is close to methods from [5, Ch.7, §7].

So we consider equality $(\overline{\mu}_0 - A^*)w = 0,\ x \in G$, where $\overline{\mu}_0$ is an eigenvalue of the operator A^* and w is a corresponding eigenvector. Note that generally speaking $\overline{\mu}_0$ is a complex number and w is a complex-valued vector field. As usual, the bar over notation of a complex number means the operation of complex conjugation. By (3.13) and by definition of operator π this equality can be rewritten as follows:

$$\Delta v(x) + (a(x), \nabla)v(x) - (\nabla a(x))^* v(x) + \overline{\mu}_0 v + \nabla\tilde{p}(x) = 0, \quad \mathrm{div} v = 0, \tag{4.1}$$

where $x \in G$. Applying to the first equality in (4.1) operator div and taking into account the second equality $\mathrm{div}\, v = 0$ we get

$$\Delta\tilde{p}(x) = -(\partial_i a_j(x))\partial_j v_i + (\Delta a, v) + (\partial_j a_i)(\partial_j v_i) \tag{4.2}$$

First of all we prove some Carleman estimate for solution of (4.1), (4.2).

4.1. **Carleman estimate.** We consider the following analog of (4.1),(4.2):

$$\Delta z(x) + (a(x), \nabla)z - (\nabla a)^* z + \overline{\mu}_0 z + \nabla p(x) = f(x) \quad \mathrm{div} v = 0, \tag{4.3}$$

$$\Delta p(x) = \mathrm{div}\, f(x) - (\partial_i a_j(x))\partial_j z_i + (\Delta a, z) + (\partial_j a_i)(\partial_j z_i). \tag{4.4}$$

We suppose also that $z(x)$ and $p(x)$ satisfy on ∂G the following equalities:

$$z|_{\partial G} = 0, \quad \nabla z|_{\partial G} = 0, \quad p|_{\partial G} = 0, \quad \nabla p|_{\partial G} = 0. \tag{4.5}$$

Let $\varepsilon > 0$ be sufficiently small. Denote

$$G_{-\varepsilon} = \{x \in G : \mathrm{dist}(x, \partial G) \equiv \min_{y \in \partial G} |x - y| > \varepsilon\}$$

(the sign "minus" in lower index in $G_{-\varepsilon}$ means that $G_{-\varepsilon}$ is a subset of G in contrast of notation Ω_ε used below in Theorem 5.1 when Ω_ε contains Ω)

Let ω be a subdomain compactly embedded to $G_{-\varepsilon}:\ \omega \subset\subset G_{-\varepsilon}$. We consider a function $\beta(x) \in C^2(G)$ which has no critical points outside ω , i.e.

$$\min_{x \in \overline{G \setminus \omega}} |\nabla\beta(x)| > 0 \tag{4.6}$$

and

$$\beta(x) \geqslant 1,\ x \in G; \qquad \max_{x\in\overline{G}\setminus G_{-\varepsilon/2}} \beta(x) < 2 \min_{x\in\overline{G}_{-\varepsilon}} \beta(x) \tag{4.7}$$

Lemma 4.1. . *A function $\beta(x) \in C^2(G)$ satisfying (4.6),(4.7) exists.*

Proof. We consider $\beta_0(x) \in C^2(G)$ such that

$$\beta_0(x) \geqslant 2,\ x \in \overline{G}, \qquad \max_{x\in\overline{G}\setminus G_{-\varepsilon/2}} \beta_0(x) < 4 \min_{x\in\overline{G}_{-\varepsilon}} \beta_0(x), \qquad \min_{x\in\overline{G}\setminus G_{-\varepsilon}} |\nabla\beta_0(x)| \geqslant 1. \tag{4.8}$$

Existence of $\beta_0(x)$ satisfying these properties is evident. After that we include the domain $G \subset \mathbb{R}^3$ in a cube and identify opposite 2-faces of this cube. In other words we include G into 3-dimensional torus Π. We extend β_0 to a function $\beta_1 \in C^2(\Pi)$. As well known (see [4, Part 2, Ch.2, §10.4]), β_1 can be approached in $C^2(\Pi)$ by a Morse function $\beta_2 \in C^2(\Pi)$, (i.e. $\nabla\beta_2(x) = 0$ not more than in finite number of points x, called critical points). Let β_3 be the restriction of β_2 on G. Since β_0 satisfies (4.8), the function β_3 satisfies (4.7) and $\min_{x\in\overline{G\setminus G_{-\varepsilon}}} |\nabla\beta_3(x)| > 0$. Now we "transform" critical points of β_3 to ω as we did it in [5, Ch. 7, §7.4], and obtain the desired function β. □

We also introduce the function:

$$\varphi(x) \equiv \varphi_\lambda(x) = e^{\lambda\beta(x)} \tag{4.9}$$

where $\lambda > 0$ is a parameter.

Recall that coefficient $a(x)$ from (4.3), (4.4) satisfies (3.5).

Theorem 4.1. *Let $z \in (H^3(G))^3,\ p \in H^2(G),\ f \in (H^1(G))^3$ satisfy (4.3)–(4.5). Then there exists a magnitude $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$ the following Carleman estimate holds:*

$$\int_G e^{2\varphi_\lambda(x)}[\lambda^4\varphi_\lambda^3(x)(|z(x)|^2 + |p(x)|^2) + \lambda^2\varphi_\lambda(x)(|\nabla z(x)|^2 + |\nabla p(x)|^2]dx \leqslant$$
$$c(\int_G e^{2\varphi_\lambda(x)}(|f(x)|^2 + |\operatorname{div} f(x)|^2)dx + \tag{4.10}$$
$$\int_\omega e^{2\varphi_\lambda(x)}[\lambda^4\varphi_\lambda^3(x)(|z(x)|^2 + |p(x)|^2) + \lambda^2\varphi_\lambda(x)(|\nabla z(x)|^2 + |\nabla p(x)|^2)]dx)$$

where the constant $c > 0$ does not depend on z, p, f, and $\lambda > \lambda_0$.

Proof. We do in (4.3),(4.4) the change of functions:

$$z(x) = e^{-\varphi}w(x), \quad p(x) = e^{-\varphi}q(x) \tag{4.11}$$

Evidently (4.5) imply:

$$w|_{\partial G} = 0, \quad \nabla w|_{\partial G} = 0, \quad q|_{\partial G} = 0, \quad \nabla q|_{\partial G} = 0, \tag{4.12}$$

We substitute (4.11) into (4.3). Then taking into account that

$$\nabla\varphi = \lambda\varphi\nabla\beta, \quad \Delta\varphi = \lambda^2\varphi|\nabla\beta|^2 + \lambda\varphi\Delta\beta$$

we obtain the equality

$$A_1w + A_2w = e^{\varphi}f + L_1(w, q) \tag{4.13}$$

where

$$A_1w = \Delta w + \varphi^2\lambda^2|\nabla\beta|^2 w, \qquad A_2w = -2\varphi\lambda(\nabla\beta, \nabla)w, \tag{4.14}$$

$$L_1(w,q) = (\lambda^2\varphi|\nabla\beta|^2 + \lambda\varphi\Delta\beta)w +$$
$$\lambda\varphi(a,\nabla\beta)w - (a,\nabla)w + (\nabla a)^* w - \overline{\mu}_0 w - \nabla q + \lambda\varphi q\nabla\beta \tag{4.15}$$

It follows from (4.13) that

$$\|A_1 w\|^2_{L_2(G)} + \|A_2 w\|^2_{L_2(G)} + 2\mathrm{Re}(A_1 w, A_2 w)_{L_2(G)} = \int_G |e^{-\varphi} f + L_1(w,q)|^2 dx \tag{4.16}$$

where Rez is the real part of a complex number z. In virtue of (4.14)

$$2\mathrm{Re}(A_1 w, A_2 w)_{L_2(G)} = I_1 + I_2 \tag{4.17}$$

where

$$I_1 = -4\mathrm{Re}\int_G \varphi\lambda(\Delta w, (\nabla\beta,\nabla)w)dx, \quad I_2 = -4\int_G \varphi^3\lambda^3|\nabla\beta|^2\mathrm{Re}(w,(\nabla\beta,\nabla)w)dx$$

Integration by parts with help of (4.12) yields:

$$I_2 = -2\int_G \varphi^3\lambda^3|\nabla\beta|^2(\nabla\beta,\nabla|w|^2)dx =$$
$$\int_G (6\varphi^3\lambda^4|\nabla\beta|^4|w|^2 + 2\varphi^3\lambda^3|w|^2(|\nabla\beta|^2\Delta\beta + (\nabla\beta,\nabla|\nabla\beta|^2))dx, \tag{4.18}$$

$$I_1 = 4\int_G \varphi\lambda\mathrm{Re}[\partial_i w_j \partial_k\beta\partial_k\partial_i\overline{w}_j]dx +$$
$$\int_G \{4\varphi\lambda^2((\nabla\beta,\nabla)w,(\nabla\beta,\nabla)w) + 4\varphi\lambda\mathrm{Re}[\partial_i w_j\partial_i\partial_k\beta\partial_k\overline{w}_j]\}dx =$$
$$-2\int_G \varphi\lambda^2|\nabla\beta|^2|\nabla w|^2 dx + \tag{4.19}$$
$$\int_G 2\varphi\lambda(-\Delta\beta|\nabla w|^2 + 2\lambda((\nabla\beta,\nabla)w,(\nabla\beta,\nabla)w) + 2\mathrm{Re}[\partial_i w_j\partial_i\partial_k\beta\partial_k\overline{w}_j]\}dx$$

Besides, estimating (4.15) we get for $\lambda \geqslant 1$:

$$\int_G |L_1(w,q)|^2 dx \leqslant c\int_G (\lambda^4\varphi^2|w|^2 + |\nabla w|^2 + |\nabla q|^2 + \lambda^2\varphi^2 q^2)dx \tag{4.20}$$

where c does not depend on $w, q, \lambda \geqslant 1$.

Now we substitute (4.17)–(4.19) into (4.16) and do simple transformations taking into account (4.20) and inequalities $((\nabla\beta,\nabla w),(\nabla\beta,\nabla w)) \geqslant 0,\ \varphi_\lambda \geqslant \lambda$ (see (4.7),(4.9)). As a result we obtain:

$$\|A_1 w\|^2_{L_2(G)} + \|A_2 w\|^2_{L_2(G)} + \int_G 6\varphi^3\lambda^4|\nabla\beta|^4|w|^2 - 2\varphi\lambda^2|\nabla\beta|^2|\nabla w|^2 dx \leqslant$$
$$c_1\int_G (e^{2\varphi}|f(x)|^2 + \lambda^3\varphi^3|w(x)|^2 + \lambda\varphi|\nabla w(x)|^2 + |\nabla q(x)|^2 + \lambda^2\varphi^2|q(x)|^2)dx \tag{4.21}$$

where c_1 does not depend on w, q, f and $\lambda \geqslant 1$.

Substitution (4.11) into (4.4) implies the equality

$$\hat{A}_1 q + \hat{A}_2 q = L_2(w,q) + e^{\varphi}\mathrm{div} f \tag{4.22}$$

where

$$\hat{A}_1 q = \Delta q + \varphi^2\lambda^2|\nabla\beta|^2 q, \quad \hat{A}_2 q = -2\varphi\lambda(\nabla\beta, \nabla q) \tag{4.23}$$

$$L_2(w,q) = (\lambda^2\varphi|\nabla\beta|^2 + \lambda\varphi\Delta\beta)q - (\partial_i a_j)\partial_j w_i + \lambda\varphi(\partial_i a_j)\partial_j\beta w_i +$$
$$(\Delta a, w) + (\partial_j a_i)(\partial_j w_i) - \lambda\varphi(\partial_j a_i)(\partial_j\beta)w_j \tag{4.24}$$

Repeating arguments which were used to derive (4.21) from (4.13),(4.15), we derive from (4.22)–(4.24) the following inequality:

$$\|\hat{A}_1 q\|^2_{L_2(G)} + \|\hat{A}_2 q\|^2_{L_2(G)} + \int\limits_G 6\varphi^3\lambda^4|\nabla\beta|^4|q|^2 - 2\varphi\lambda^2|\nabla\beta|^2|\nabla q|^2 dx \leqslant \tag{4.25}$$

$$c_2 \int\limits_G (e^{2\varphi}|\mathrm{div} f(x)|^2 + \lambda^3\varphi^3|q(x)|^2 + \lambda\varphi|\nabla q(x)|^2 + |\nabla w(x)|^2 + \lambda^2\varphi^2|w(x)|^2)dx$$

where c_2 does not depend on w, q, f and $\lambda \geqslant 1$.

Now we scale in $L_2(G)$both parts of (4.13) on $\lambda^2\varphi|\nabla\beta|^2 w$. Taking real part of obtained equality and using the first equality in (4.14) we obtain:

$$\int\limits_G \lambda^2\varphi \mathrm{Re}(\Delta w, w)|\nabla\beta|^2 + \lambda^4\varphi^3|\nabla\beta|^4|w|^2 dx = R_1 \tag{4.26}$$

where

$$R_1 = \mathrm{Re} \int\limits_G (e^{\varphi} f + L_1(w,q) - A_2 w, w)\lambda^2\varphi|\nabla\beta|^2 dx \tag{4.27}$$

Integration by parts in left side of (4.26) yields:

$$\int\limits_G (\lambda^4\varphi^3|\nabla\beta|^4|w|^2 - \lambda^2\varphi|\nabla w|^2|\nabla\beta|^2)dx = R_1 +$$

$$\int\limits_G (\frac{\lambda^3\varphi}{2}(\nabla\beta, \nabla|w|^2)|\nabla\beta|^2 + \frac{\lambda^2\varphi}{2}(\nabla|\nabla\beta|^2, \nabla|w|^2))dx = R_1 -$$

$$\int\limits_G \frac{\lambda^3\varphi}{2}(\lambda|\nabla\beta|^4 + |\nabla\beta|^2\Delta\beta + (\nabla\beta, \nabla|\nabla\beta|^2) + (\nabla|\nabla\beta|^2, \nabla\beta) + \lambda^{-1}\Delta|\nabla\beta|^2)|w|^2 dx$$

Multiplying this equation on -1 and estimating its right side with help of (4.27) and (4.20) we get:

$$\int\limits_G (\lambda^2\varphi|\nabla w|^2|\nabla\beta|^2 - \lambda^4\varphi^3|\nabla\beta|^4|w|^2)dx \leqslant \frac{1}{4}\|A_2 w\|^2_{L_2(G)} + \tag{4.28}$$

$$\frac{1}{2}\int\limits_G e^{2\varphi}|f|^2 dx + c_3 \int\limits_G (\lambda^4\varphi^2|w|^2 + |\nabla w|^2 + |\nabla q|^2 + \lambda^2\varphi^2 q^2)dx$$

Analogously, scaling (4.22) on $\lambda^2\varphi|\nabla\beta|^2 q$ in $L_2(G)$ and taking real part, we obtain after some transformations the inequality:

$$\int\limits_G (\lambda^2\varphi|\nabla q|^2|\nabla\beta|^2 - \lambda^4\varphi^3|\nabla\beta|^4|q|^2)dx \leqslant \frac{1}{4}\|\hat{A}_2 q\|^2_{L_2(G)} + \tag{4.29}$$

$$\frac{1}{2}\int\limits_G e^{2\varphi}|\mathrm{div} f|^2 dx + c_3 \int\limits_G (\lambda^4\varphi^2|q|^2 + |\nabla q|^2 + |\nabla w|^2 + \lambda^2\varphi^2|w|^2)dx$$

Note that constant c_3 in (4.28), (4.29) does not depend on $w, q, f, \lambda \geqslant 1$.

Now we multiply both parts of (4.28), (4.29) on 3, add these inequalities and add obtained inequality with the sum of inequalities (4.21) and (4.25). As the result we obtain

$$\|A_1 w\|^2_{L_2(G)} + \|\hat{A}_1 q\|^2_{L_2(G)} + \frac{1}{4}\|A_2 w\|^2_{L_2(G)} + \frac{1}{4}\|\hat{A}_2 q\|^2_{L_2G(G)} +$$

$$\int\limits_G 3\varphi^3\lambda^4|\nabla\beta|^4(|w|^2+|q|^2) + \varphi\lambda^2|\nabla\beta|^2(|\nabla w|^2+|\nabla q|^2)dx \leqslant$$

$$c\int\limits_G [e^{2\varphi}(|f|^2+|\mathrm{div} f|^2) + \lambda^3\varphi^3(|w|^2+|q|^2) + \lambda\varphi(|\nabla w|^2+|\nabla q|^2)]dx \tag{4.30}$$

In virtue of (4.6) inequality (4.30) implies

$$\int\limits_G (\varphi^3\lambda^4|(|w|^2+q^2) + \varphi\lambda^2(|\nabla w|^2+|\nabla q|^2)dx \leqslant$$

$$c\Bigg[\int\limits_G e^{2\varphi}(|f|^2+|\mathrm{div} f|^2)dx + \int\limits_G (\lambda^3\varphi^3(|w|^2+|q|^2) + \lambda\varphi(|\nabla w|^2+|\nabla q|^2))dx +$$

$$\int\limits_\omega (\varphi^3\lambda^4(|w|^2+q^2) + \varphi\lambda^2(|\nabla w|^2+|\nabla q|^2)dx\Bigg] \tag{4.31}$$

Evidently for $\lambda > \lambda_0$ with enough large λ_0 inequalities $\lambda^3\varphi^3 < \lambda^4\varphi^3/2$ and $\lambda\varphi \leqslant \lambda^2\varphi/2$ are true. Therefore (4.31) implies inequality

$$\int\limits_G (\varphi^3\lambda^4(|w|^2+q^2) + \varphi\lambda^2(|\nabla w|^2+|\nabla q|^2)dx \leqslant$$

$$2c\Bigg[\int\limits_G e^{2\varphi}(|f|^2+|\mathrm{div} f|^2)dx + \int\limits_\omega (\varphi^3\lambda^4(|w|^2+q^2) + \varphi\lambda^2(|\nabla w|^2+|\nabla q|^2)dx\Bigg] \tag{4.32}$$

that holds for each $\lambda \geqslant \lambda_0$ and with a constant c which does not depend on λ. Now we substitute $w = e^\varphi z,\ q = e^\varphi p$ into (4.32) and after simple transformations we get that for sufficiently large λ equality (4.10) is true. □

4.2. **Unique continuation property.** We consider now equations (4.1) with coefficient $a(x) \in V^2(G) \cap V^1_0(G)$ which possesses a solution $(v(x), \tilde{p}(x))$ satisfying the boundary condition

$$v|_{\partial G} = 0 \tag{4.33}$$

It is easy to see that if $(v(x), \tilde{p}(x)) \in V^1_0(G) \times L_2(G)$ satisfies (4.1), (4.33) then $(v(x), \tilde{p}(x)) \in (V^1_0(G) \cap H^3(G)) \times H^2(G)$

Theorem 4.2. . *Suppose that a solution $(v(x), \tilde{p}(x)) \in (V^1_0(G) \cap H^3(G)) \times H^2(G)$ of (4.1), (4.33) satisfies the condition*

$$v(x) \equiv 0 \quad for \quad x \in \omega \tag{4.34}$$

where ω is a subdomain of G. Then $v(x) \equiv 0,\ \tilde{p}(x) \equiv const$ for $x \in G$.

Proof. We reduce our problem to such one that all conditions of Theorem 4.1 are fulfilled. Recall that $G_{-\delta} = \{x \in G : \operatorname{dist}(x, \partial G) > \delta\}$. We can suppose that for a sufficiently small $\delta > 0$ the set ω in (4.34) satisfies condition $\omega \subset\subset G_{-\delta}$, i.e. $\operatorname{dist}(\omega, \partial G) > \delta$. Otherwise we change ω on its a certain open subset.

We reduce now problem (4.1), (4.33) to problem (4.3), (4.5). To do this we consider a function $\psi(x) \in C^\infty(\overline{G})$ satisfying

$$\psi(x) = \begin{cases} 1, & x \in G_{-\delta/2} \\ 0, & x \in G \setminus G_{-\delta/4} \end{cases}$$

Let us consider the boundary value problem

$$\operatorname{rot} w(x) = v(x), \quad \operatorname{div} w(x) = 0, \quad x \in G$$

$$(w(x), n(x))|_{\partial G} = 0$$

where $n(x)$ is the vector field of outer normals to ∂G. As well-known (see, for instance [16]) since $v(x) \in V_0^1(G) \cap H^3(G)$, there exists a solution $w(x)$ of this problem and $w(x) \in H^4(G)$. We introduce the vector field

$$z(x) = \operatorname{rot}(\psi(x) w(x)) \tag{4.35}$$

Note that in virtue of (4.34) the component $\tilde{p}(x)$ of solution $(v, \tilde{p})$ to (4.1), (4.33) satisfies $\nabla \tilde{p}(x) \equiv 0$ for $x \in \omega$. Since $\tilde{p}(x)$ is defined from (4.1) to within arbitrary constant we can choose this constant such that

$$\tilde{p}(x) = 0, \qquad x \in \omega \tag{4.36}$$

We define

$$p(x) = \psi(x) \tilde{p}(x) \tag{4.37}$$

So we have $(z(x), p(x)) \in (V_0^1(G) \cap H^3(G)) \times H^2(G), \operatorname{div} z(x) = 0$, for $x \in G$ and z, p satisfy (4.5). Besides, for $x \in G_{-\delta}$ $(z(x), p(x)) = (v(x), \tilde{p}(x))$. Therefore if we substitute $(z(x), p(x))$ in the left side of (4.1), we obtain equations (4.3) with a right side $f(x)$ which satisfies

$$f(x) = 0 \quad \text{for} \quad x \in G_{-\delta/2} \tag{4.38}$$

As a result we see that the triplet (z, p, f) satisfies all condition of Theorem 4.1 and therefore estimate (4.10) is true. In virtue of (4.9), (4.34)– (4.37) this estimate implies the following upper bound:

$$\int_{G_{-\delta}} \exp(2e^{\lambda\beta(x)}) \lambda^4 e^{3\lambda\beta(x)} (|z(x)|^2 + |p(x)|^2) dx \leqslant$$

$$c \int_G \exp(2e^{\lambda\beta(x)}) (|f(x)|^2 + |\operatorname{div} f(x)|^2) dx \tag{4.39}$$

which is true for each $\lambda \geqslant \lambda_0$ and c in (4.39) does not depend on $\lambda \geqslant \lambda_0$.

Assume that there exists a set $\Lambda \subset G_{-\delta}$ of positive Lebesgue measure such that $|z(x)|^2 + |p(x)|^2 > 0$ for $x \in \Lambda$. Then (4.39) is not true for sufficiently large λ because f satisfies (4.38), and for $\beta(x)$ the second equality in (4.7) is true. Hence $|z(x)|^2 + |p(x)|^2 = 0$ for $x \in G_{-\delta}$. In virtue of (4.35),(4.37) the solution $(v(x), \tilde{p}(x))$ of (4.1) also satisfies the equality

$$|v(x)|^2 + |\tilde{p}(x)|^2 \equiv 0, \qquad x \in G_{-\delta}$$

Since $\delta > 0$ can be chosen arbitrary small, desired assertion of the Theorem 4.2 has been proved. $\square$

4.3. **On linear independence of $\varepsilon_l^{(k)}(x,-\overline{\lambda}_j)$.** We set some strengthening of well-known result on linear independence of eigenvectors and associated vectors for operator A^* which is defined in (3.13). To prove this result we use Theorem 4.2

Theorem 4.3. *Consider the set*

$$E_\sigma^* \equiv \bigcup_{\mathrm{Re}\lambda_j<\sigma} E^*(-\overline{\lambda}_j)$$

of canonical systems (3.17) for operator $-A^*$ with σ satisfying (3.20)*.Then for an arbitrary subdomain $\omega\subset G$ vector fields $\varepsilon_l^{(k)}(x,-\overline{\lambda}_j)\in E_\sigma^*$ regarded for $x\in\omega$ are linear independent.*

*Proof.*The main part of Theorem 4.3 is to prove that eigenvectors $\varepsilon^{(k)}(-\overline{\lambda}_j,x)\equiv \varepsilon_0^{(k)}(-\overline{\lambda}_j,x), k=1,...,N$ of operator A^* with fixed eigenvalue $-\overline{\lambda}_j$ are linear independent if they are regarded for $x\in\omega$. Indeed, suppose that

$$v(x)\equiv\sum_{k=1}^N c_k\varepsilon^{(k)}(-\overline{\lambda}_j,x)=0 \quad \text{for} \quad x\in\omega.$$

Since this $v(x)$ with a certain $p(x)$ satisfies (4.1),(4.33), in virtue of Theorem 4.2 equality $v(x)=0, x\in\omega$,imply that $v(x)=0$ for $x\in G$. Since by Definition 3.1 eigenvectors $\varepsilon^{(k)}(-\overline{\lambda}_j,x)$ are linear independent on G, the last equality implies that $c_k=0, k=1,...,N$. Note that only in this part of proof we use specific of equation (4.1). The general assertion of Theorem 4.3 is derived from the property proved above with help of some general arguments written in [6]. □

Impose on canonical systems (3.17) the following condition

$$\varepsilon^{(k)}(-\lambda_j)=\bar\varepsilon^{(k)}(-\bar\lambda_j); \quad \varepsilon_l^{(k)}(-\bar\lambda_j)=\bar\varepsilon_l^{k}(-\lambda_j) \tag{4.40}$$

Condition (4.40) can be realized with help of (3.15).

In virtue of (4.40) canonical system corresponding to real $-\lambda_j$ consists of real-valued vector fields. If $\mathrm{Im}\lambda_j\neq 0$, instead of vector fields $\varepsilon^{(k)}(-\bar\lambda_j)$, $\varepsilon_l^{(k)}(-\lambda_j)$, $l=0,1,\dots$, we consider real valued vector fields

$$\mathrm{Re}\varepsilon_l^{(k)}(-\bar\lambda_j),\ \mathrm{Im}\varepsilon_l^{(k)}(-\bar\lambda_j), \quad l=0,\dots, \quad k=1,2\dots \tag{4.41}$$

(with $\varepsilon_0^{(k)}(-\bar\lambda_j)=\varepsilon^{(k)}(-\bar\lambda_j)$ by definition). We renumber all functions (4.41) with $\mathrm{Re}\lambda_j<\sigma$ (including fields with $\mathrm{Im}\lambda_j=0$) as follows:

$$\varepsilon_1(x),\dots,\varepsilon_K(x) \tag{4.42}$$

Lemma 4.2. *For an arbitrary subdomain $\omega\subset G$ vector fields* (4.42) *restricted on ω are linear independent over the field $\mathbb{R}$ of real numbers.*

Lemma 4.2 follows easily from Theorem 4.3 (see details in [6]).

Note that Theorem 3.2 and Lemma 4.2 imply immediately the following assertion.

Corollary 4.1. *Assume, that A is operator (3.8) and $\sigma>0$ satisfies* (3.20)*. Then for each $w_0\in V_0^0(G)$ satisfying*

$$\int_G (w_0(x),\varepsilon_j(x))\partial x=0, \qquad j=1,..,K$$

with ε_j from (4.42)*, inequality* (3.22) *is true.*

5. **Stabilization of Oseen equations.**

5.1. **Setting of the problem.** As in section 2 we suppose that $\Omega \subset \mathbb{R}^3$ is a bounded connected domain with C^∞-boundary $\partial\Omega$, which is decomposed on two parts:

$$\partial\Omega = \Gamma \cup \Gamma_0, \quad \Gamma \neq \emptyset \tag{5.1}$$

where Γ, Γ_0 are closed subsets of $\partial\Omega$ and $\Gamma \cap \Gamma_0 = \emptyset$. The case $\Gamma_0 = \emptyset$ is possible. In other words if

$$\partial\Omega = \bigcup_{j=1}^{J} \partial\Omega_j \tag{5.2}$$

where $\partial\Omega_j$ are closed connected components of $\partial\Omega$ then (possibly after renumeration of $\partial\Omega_j$)

$$\Gamma = \bigcup_{j=1}^{l} \partial\Omega_j, \quad \Gamma_0 = \bigcup_{j=l+1}^{J} \partial\Omega_j \tag{5.3}$$

Let $Q = \mathbb{R}_+ \times \Omega, \Sigma = \mathbb{R}_+ \times \Gamma, \Sigma_0 = \mathbb{R}_+ \times \Gamma_0$. In space-time cylinder Q we consider the Ozeen equations

$$\partial_t v(t,x) - \Delta v + (a(x), \nabla)v + (v, \nabla)a + \nabla p(t,x) = 0 \tag{5.4}$$

$$\operatorname{div} v(t,x) = 0 \tag{5.5}$$

with initial and boundary conditions

$$v(t,x)|_{t=0} = v_0(x). \tag{5.6}$$

$$v|_{\Sigma_0} = 0, \quad v|_{\Sigma} = u \tag{5.7}$$

where $a(x) = (a_1(x), a_2(x), a_3(x))$ is a solenoidal vector field ($\operatorname{div} a = 0$) and $u = (u_1, u_2, u_3)$ is a control. Besides solenoidalness of initial condition $v_0(x)$ we suppose that

$$\int\limits_{\partial\Omega_j} (v_0(x'), n(x'))dx' = 0, \qquad j = 1, \dots, l. \tag{5.8}$$

Analogously to section 2 stabilization problem for Oseen equations is formulated as follows:

Given $\sigma > 0$ find a control u on Σ such that the solution $v(t,x)$ of problem (4.2)–(4.5) satisfies the inequality

$$\|v(t,x)\|_{L_2(\Omega)} \leqslant ce^{-\sigma t} \tag{5.9}$$

where $c > 0$ depends on v_0, σ and Γ. Moreover, we require that this control u satisfies the *feedback* property in the meaning analogous to Definition 2.2: firstly we extend by a special way problem (5.4)–(5.7) (without second equality from (5.7)) to the problem (3.1)–(3.4) defined on a domain $G \supset \Omega$, solve the last problem, and after that we define the solution (v, u) of stabilization problem (5.4)–(5.9) by the formula (2.22) Details of this definition will be given simultaneously with the construction of feedback control.

5.2. **Theorem on extension.** First of all we define the set ω from (2.14), (2.15) which is used to extend the domain Ω to the set G. Note that being closed each component $\partial\Omega_j$ of $\partial\Omega$ separates $\mathbb{R}^3$ on two parts: Ω_{j-} and Ω_{j+}. By definition points of Ω_{j-} which are close enough to $\partial\Omega_j$ belongs to Ω. Taking a sufficiently small magnitude $\kappa > 0$ we define the set ω as follows:

$$\omega_j = \{x \in \Omega_{j+} : \mathrm{dist}(x, \partial\Omega_j) < \kappa\}; \qquad \omega = \bigcup_{j=1}^{l} \omega_j \tag{5.10}$$

where l is defined in (5.3). Now we define the domain G by the formula:

$$G = \mathrm{Int}(\overline{\Omega} \cup \overline{\omega}) \tag{5.11}$$

We introduce the following spaces

$$V_0^1(G) = \{u(x) = (u_1(x), u_2(x)) \in V^1(G) \,:\, u|_{\partial G} = 0\}$$

$$\hat{V}^k(G) = \{u(x) = (u_1(x), u_2(x), u_3(x)) \in V^k(G) \,:\,$$
$$\int_{\partial G_j} (u(x'), n(x')dx' = 0, \quad j = 1, ..., J\} \tag{5.12}$$

where k is nonnegative integer, and ∂G_j, $j = 1, ..., J)$ are all connected components of the boundary ∂G, and n is outer normal to ∂G. Recall that operation of restriction onto the boundary for $(u(x), n(x))$ is well defined for $u \in V^k(G)$ with each $k \geqslant 0$ (see [16]).

Below we use well-known operator rot which is defined by the formula:

$$\mathrm{rot} v(x) = \left(\frac{\partial v_2(x)}{\partial x_3} - \frac{\partial v_3(x)}{\partial x_2}, \frac{\partial v_3(x)}{\partial x_1} - \frac{\partial v_1(x)}{\partial x_3}, \frac{\partial v_1(x)}{\partial x_2} - \frac{\partial v_2(x)}{\partial x_1} \right)$$

It is clear that for each gradient vector field $\nabla p(x)$, $p \in H^1(G)$ the inclusion $\nabla p(x) \in \ker \mathrm{rot}$ is true. Note that, generally speaking, for the functions space $V_0^0(G)$ defined in (3.6) the inequality $V_0^0(G) \cap \ker \mathrm{rot} \neq \{0\}$ holds. Indeed, the following orthogonal decomposition with respect to the scalar product in $L_2(G)$ is true (see, e.g. [10] and [16, Appendix 1,pp.458-471]):

$$V_0^0(G) = W^0(G) \bigoplus H_c, \tag{5.13}$$

where $H_c = V_0^0(G) \cap \ker \mathrm{rot}$ is a finite-dimensional space of C^∞-vector fields isomorphic to the space of the first cohomologies of G. H_c consists of vector fields $\nabla p(x)$, where $p(x)$ are multi-valued functions satisfying $\Delta p = 0$ and $(\partial p/\partial n)|_{\partial G} = 0$; for details see [16, Appendix 1]. Now the functions space $W^0(G)$ is well defined by equality (5.13). For each integer $k \geqslant 0$ we define

$$W^k(G) = W^0(G) \cap (H^k(G))^3 \tag{5.14}$$

where $(H^k(G))^3$ is usual Sobolev space of vector valued functions of smoothness k.

Lemma 5.1. . *Let $k \geqslant 0$. Then the operator*

$$\mathrm{rot} \,:\, W^{k+1} \longrightarrow \hat{V}^k(G) \tag{5.15}$$

is an isomorphism.

The proof of this Lemma see in [16, Appendix 1], [8].

Lemma 5.2. . *The operator*

$$\mathrm{rot}^{-1}\ :\ V_0^1(G) \longrightarrow H^2(G) \tag{5.16}$$

satisfying $\mathrm{rot}\,\mathrm{rot}^{-1}v = v\ \forall v \in V_0^1(G)$ *is well defined on* $V_0^1(G)$.

Proof. In virtue of Lemma 5.1 operator (5.15) has the inverse operator rot^{-1}. Restriction of this operator on $V_0^1(G) \subset \hat{V}^1(G)$ defines the right inverse operator (5.16). □

We prove now the extension theorem. In the space of real valued vector fields $V_0^1(G)$ we introduce the subspace

$$X_\sigma^1(G) = \{v(x) \in V_0^1(G)\ :\ \int\limits_G v(x)\cdot \varepsilon_j(x)\,dx = 0, \quad j = 1,\dots,K\} \tag{5.17}$$

where $\varepsilon_j(x)$ are functions (4.42). Let also

$$V^1(\Omega,\Gamma_0) = \{v \in V^1(\Omega)\ :\ v|_{\Gamma_0} = 0,\ \int_{\partial\Omega_j} (v(x'), n(x'))dx' = 0,\ j = 1,...,J\} \tag{5.18}$$

where $\partial\Omega_j$ are closed connected component of $\partial\Omega$, and $n(x')$ is the vector fields of outer normals to $\partial\Omega$.

Theorem 5.1. *There exists a linear bounded extension operator*

$$E_\sigma^1 : V^1(\Omega,\Gamma_0) \to X_\sigma^1(G) \tag{5.19}$$

(*i.e.* $E_\sigma^1(v)(x) \equiv v(x)$ *for* $x \in \Omega$).

Proof. *Step 1.* Recall firstly that there exists a linear continuous extension operator

$$L : V^1(\Omega,\Gamma_0) \to V_0^1(G). \tag{5.20}$$

Indeed let $v \in V^1(\Omega,\Gamma_0) \subset \hat{V}^1(\Omega)$. Then $\mathrm{rot}^{-1}v \in H^2(\Omega)$ is the vector field well defined in virtue of Lemma 5.2. Existence of bounded extension operator

$$E\ :\ H^2(\Omega) \longrightarrow H^2(G)$$

is well known (see e.g. [15]). Set

$$\Omega_\varepsilon = \{x \in G\ :\ \mathrm{dist}(x,\Omega) < \varepsilon\}$$

where $\mathrm{dist}(x,\Omega)$ is the distance from x to Ω. Suppose that ε is so small that for each $j = 1,\dots,l$ $\omega_j \setminus \Omega_\varepsilon \neq \emptyset$. Let $\psi(x) \in C^\infty(\bar{G})$, $0 \leqslant \psi(x) \leqslant 1$,

$$\psi(x) = \begin{cases} 1, & x \in G \cap \Omega_{\varepsilon/2}, \\ 0, & x \in G \setminus \Omega_\varepsilon. \end{cases} \tag{5.21}$$

Then we denote operator (5.20) by the formula $L = \mathrm{rot}\circ\psi\circ\mathrm{rot}^{-1}$.

*Step 2.*Introduce now an open subset $\hat{\Omega} = G \setminus \overline{\Omega}_{\varepsilon/2}$. We look for extension operator E_σ^1 in a form

$$E_\sigma^1 v(x) = (Lv)(x) + \hat{w}(x), \tag{5.22}$$

where L is operator (5.20) and $\hat{w}(x)$ is a vector field which satisfies:

$$\hat{w} \in V_0^1(G), \qquad \mathrm{supp}\,\hat{w} \subset \hat{\Omega} = G \setminus \overline{\Omega}_{\varepsilon/2}. \tag{5.23}$$

By virtu of (5.17) to establish inclusion $E_K^1 v \subset X_K^1(G)$ we have to assume that

$$\int\limits_G \varepsilon_k(x)\hat{w}(x)\,dx = -\int\limits_G \varepsilon_k(x)(Lv)(x)\,dx \tag{5.24}$$

where $k = 1, \dots, K$. At last, to determine $\hat{w}$ uniquely we suppose that

$$\|\hat{w}\|^2_{V_0^1(G)} = \inf_{w\in\mathcal{A}} \|w(\cdot)\|^2_{V_0^1(G)} \quad \text{where } \mathcal{A} = \{w:\ w \,\text{satisfies (5.23), (5.24)}\}. \tag{5.25}$$

(Recall that $\|v\|_{V_0^1(G)} = \|\nabla v\|_{L_2(G)}$.)

*Step 3.*We have to show that there exists unique vector field $\hat{w}$ that satisfies (5.25). To do this we define the operator R by the formula:

$$R:\ V_0^1(\hat{\Omega}) \to \mathbb{R}^K, \qquad Rv = \Big(\int_G \varepsilon_1(x)v(x)\,dx, \dots, \int_G \varepsilon_K(x)v(x)\,dx\Big). \tag{5.26}$$

We claim that $\operatorname{Im} R = \mathbb{R}^K$. Indeed, if this is not true, there exists a vector $p = (p_1, \dots, p_K) \neq 0$ such that

$$\int_G \sum_{j=1}^K p_j \varepsilon_j(x) v(x)\,dx = 0 \qquad \forall v \in V_0^1(\hat{\Omega}).$$

This equality implies that

$$\sum_{j=1}^K p_j \varepsilon_j(x) = \nabla q(x) \qquad x \in \hat{\Omega} \tag{5.27}$$

Since in virtue of (3.5) $\varepsilon_j(x) \in V^3(G)$, we get that $q(x) \in H^4(\hat{\Omega})$. Therefore, equality $\varepsilon_j|_{\partial G} = 0$ implies that $\nabla q|_{\partial G\setminus\Gamma_0} = 0$. As a result we obtain:

$$q|_{(\partial G\setminus\Gamma_0)\cap\overline{\omega}_j} = c_j, \quad \partial_n q|_{(\partial G\setminus\Gamma_0)\cap\overline{\omega}_j} = 0 \tag{5.28}$$

where c_j are constants and ∂_n is the derivative with respect to the vector field n of outer normals to ∂G. Applying to both parts of (5.27) operator div we get that

$$\Delta q(x) = 0, \qquad x \in \hat{\Omega} \tag{5.29}$$

In virtue of uniqueness of solution for Cauchy problem (5.29), (5.28), $q|_{\omega_j} = c_j$ and therefore (5.27) implies

$$\sum_{j=1}^K p_j \varepsilon_j(x) = 0 \qquad x \in \hat{\Omega}$$

This equality and Lemma 4.2 imply $p_j = 0,\ j = 1, \dots, K$ that contradicts to the assumption $(p_1, \dots, p_K) \neq 0$.

Since $\operatorname{Im} R = \mathbb{R}^K$, the set $\mathcal{A}$ of admissible elements for problem (5.25) is not empty. In virtue of definition (5.25) $\mathcal{A}$ is a closed convex subset of $V_0^1(G)$, and (5.25),actually, is the problem to determine the distance from origin to the set $\mathcal{A}$ in the Hilbert space $V_0^1(G)$. As well known, this problem has unique solution $\hat{w}(x)$.

Step 4. Existence and uniqueness of solution for problem (5.25) implies that the operator E_1 that transforms a vector field $v \in V^1(\Omega, \Gamma_0)$ to the solution $\hat{w}$ of problem (5.25): $E_1 v = \hat{w}$, is well defined. To finish the proof of the Theorem we have to show that the operator

$$E_1:\ V^1(\Omega, \Gamma_0) \to V_0^1(\hat{\Omega}) \tag{5.30}$$

is linear and bounded.

We derive optimality system for minimization problem (5.25) with help of Lagrange principle. As one can see, for instance, in [5], the relation $\operatorname{Im} R = \mathbb{R}^K$ which was proved for the operator R defined in (5.26) guarantees, that we can apply to

(5.25) Lagrange principle when Lagrange multiplier before minimized functional equals one. This Lagrange function has a form:

$$\mathcal{L}(w,p_1,\dots,p_K)=\frac{1}{2}\|w\|^2_{V_0^1(\hat\Omega)}+\sum_{j=1}^K p_j\left(\int\limits_{\hat\Omega}(\varepsilon_j(x),w(x))\,dx+\int\limits_G(\varepsilon_j(x),Lv(x))\,dx\right)$$

By Lagrange principle for solution $\hat w$ of (5.25) there exists a vector $p=(p_1,\dots,p_K)$ such that for each $h(x)\in V_0^1(\hat\Omega)$

$$<\mathcal{L}'_w(\hat w,p_1,\dots,p_K),h>=\int\limits_{\hat\Omega}[(\nabla w,\nabla h)+\sum_{j=1}^K p_j(\varepsilon_j,h)]\,dx=0 \qquad (5.31)$$

In the set $\hat\Omega$ we consider the Stokes problem:

$$-\Delta w(x)+\nabla p(x)=v(x),\quad \operatorname{div} w(x)=0,\quad x\in\hat\Omega;\quad w|_{\partial\hat\Omega}=0$$

As well known, for each $v\in V^0(\hat\Omega)$ there exists unique solution $w\in V_0^1(\hat\Omega)\cap V^2(\hat\Omega)$ of this problem. The resolving operator of this problem we denote as follows: $(-\pi\Delta)^{-1}_{\hat\Omega}v=w$. Extension of $(-\pi\Delta)^{-1}_{\hat\Omega}v$ from $\hat\Omega$ to G by zero we also denote as $(-\pi\Delta)^{-1}_{\hat\Omega}v$. Evidently, $(-\pi\Delta)^{-1}_{\hat\Omega}v\in V_0^1(G)$.

Since (5.31) means that $\hat w$ is the solution of the Stokes problem with right side $v=-\sum_{j=1}^K p_j\varepsilon_j$, we get

$$\hat w=-(-\pi\Delta)^{-1}_{\hat\Omega}\sum_{j=1}^K p_j\varepsilon_j \qquad (5.32)$$

Substitution of (5.32) into (5.24) yields the linear system of equations:

$$\sum_{j=1}^K a_{kj}p_j=b_k,\quad \text{where } a_{kj}=\int\limits_{\hat\Omega}(\varepsilon_k,(-\pi\Delta)^{-1}_{\hat\Omega}\varepsilon_j)\,dx,\ b_k=\int\limits_G(\varepsilon_k,Lv)\,dx \qquad (5.33)$$

Relations (5.33), (5.32) implies that operator E_1 from(5.30) is linear.

To prove the boundedness of operator (5.30) we show that the matrix $A=\|a_{kj}\|$ is positively defined. Note that

$$a_{kj}=\int\limits_{\hat\Omega}\varepsilon_k(x)\cdot[(-\pi\Delta)^{-1}_{\hat\Omega}\varepsilon_j(x)]\,dx=\int\limits_{\hat\Omega}(\nabla w_k(x)):(\nabla w_j(x))\,dx. \qquad (5.34)$$

where $w_j(x)=(-\pi\Delta)^{-1}_{\hat\Omega_1}\varepsilon_j(x)$ and : is the sign of scalar product between two tensors. Let $\alpha=(\alpha_1,\cdots,\alpha_K)$, $f(x)=\sum_{j=1}^K\alpha_j w_j(x)$. Then

$$(A\alpha,\alpha)=\sum_{k,j=1}^K\alpha_k\alpha_j\int\limits_{w_1}\nabla w_k(x)):\nabla w_j(x)\,dx=\int\limits_{\omega_1}|\nabla f(x)|^2dx\geqslant 0.$$

Moreover, if for some α $(A\alpha,\alpha)=\int\limits_{\omega_1}|\nabla f(x)|^2dx=0$ then $\Delta f(x)=\operatorname{div}\nabla f(x)=0$ for $x\in\hat\Omega$. By definition of $w_j(x)$ we have: $-\Delta w_j(x)=\varepsilon_j(x)-\nabla q_j(x)$, $x\in\hat\Omega$, where $q_j(x)$ is a harmonic function in $\hat\Omega$ (to see this one can apply the operator div to both parts of previous equality). Hence, $0=-\Delta f(x)=\sum_{j-1}^K\alpha_j\varepsilon_j-\nabla q(x)$ where $q(x)$ is a harmonic function in $\hat\Omega$ and by previous equality we get that $\nabla q|_{\partial\hat\Omega\cap\partial G}=0$ (because $\varepsilon_j|_{\partial\hat\Omega\cap\partial G}=0$). In virtue of uniqueness for solution to Cauchy problem for

Laplace operator we get that $\nabla q(x) = 0$ for $x \in \hat{\Omega}$. Hence $\sum_{j-1}^{K} \alpha_j \varepsilon_j = 0$ and by Lemma 4.2 $\alpha_1 = \cdots = \alpha_K = 0$.

So, $\det A \neq 0$. In virtue of (5.31), (5.32)

$$\hat{w} = E_1 v = -(-\pi\Delta)^{-1}_{\hat{\Omega}}(A^{-1}b, \varepsilon)$$

where $b = (b_1, \dots, b_K)$ (see (5.32)), $\varepsilon = (\varepsilon_1, \dots, \varepsilon_K)$, and therefore operator (5.30) is bounded. □

Remark 5.1. In fact in the Theorem's 5.1 proof we showed that operator (5.19) can be defined by formulas (5.22), (5.32) where $(p_1, \dots, p_K)$ is the solution of the system from (5.33). This definition is equivalent to the definition of operator (5.19) given in the proof of Theorem 5.1

5.3. **Result on stabilization.** We prove now the main theorem of this section on stabilization of 3D Ozeen equations by feedback boundary control.

Theorem 5.2. *Let domains Ω and G satisfy (5.10), (5.11). Then for each initial value $v_0(x) \in V^1(\Omega, \Gamma_0)$ and for each $\sigma > 0$ there exists a feedback control u defined on Σ such that the solution $v(t,x)$ of* (5.4)–(5.7) *satisfies the inequality*

$$\|v(t,\cdot)\|_{(H^1(\Omega))^2} \leqslant ce^{-\sigma t} \quad \text{as} \quad t \to \infty. \tag{5.35}$$

Proof. We can assume that σ satisfies to condition (3.20), otherwise we make it a little bit more. We act to initial condition $v_0 \in V^1(\Omega, \Gamma_0)$ by the operator E^1_σ from (5.19), (5.22) and by Theorem 5.1 we obtain that $w_0 = E^1_\sigma v_0 \in X^1_\sigma(G)$. Since $X^1_\sigma(G) \subset V^1_0(G) \subset V^0_0(G)$, the solution $w(t,x)$ of problem (3.1)–(3.4) can be written in the form $w(t,\cdot) = e^{-At}w_0$ where A is operator (3.8). By Theorem 3.2 $w(t,\cdot)$ satisfies estimate (3.22). Now we define the solution $(v(t,x), u(t,x'))$ of stabilization problem for (5.4)–(5.7) by formula (2.22). Then (3.22) implies (5.35) □

6. **Stabilization of 3D Navier–Stokes equations.** In this section we study the problem of stabilization a solution to the Navier–Stokes equations which is formulated in subsection 2.1. In particular, the boundary $\partial\Omega$ of the space component Ω to space-time cylinder $Q = \mathbb{R}_+ \times \Omega$ where the Navier-Stokes system is determined, satisfies condition (2.1). We do this stabilization with help of control determined on the part $\Sigma = \mathbb{R}_+ \times \Gamma$ of the lateral surface to cylinder Q, and we consider only feedback control in the meaning of Definition 2.2.

6.1. **Invariant manifolds.** Let $g(x)$ from (2.16) satisfies the condition:

$$g(x) \in (L_2(G))^2. \tag{6.1}$$

Then as well-known (see, for instance [17]) equations (2.16) are equivalent to the following equation with respect to one unknown vector field $w(t,x)$:

$$\partial_t w(t,x) - \pi\Delta w + \pi(w,\nabla)w = \pi g(x) \tag{6.2}$$

where π is orthoprojector (3.7) on $V^0_0(G)$ (see (3.6)). We set an initial condition for this equation:

$$w(t,x)|_{t=0} = w_0(x), \qquad w_0 \in V^1_0(G). \tag{6.3}$$

We look for a solution w of (6.2) (as well as solution w of (2.16)) in the space

$$V^{1,2}_0(\Theta_T) \equiv \{w(t,x) \in L_2(0,T; V^2(G) \cap (H^1_0(G))^3) \,:\, \partial_t w \in L_2(0,T; V^0_0(G)\} \tag{6.4}$$

for each $T > 0$, where $\Theta_T = (0,T) \times G$.

Note that we can rewrite (2.19) in the form analogous to (6.2):

$$-\pi\Delta a(x)+\pi(a,\nabla)a=\pi g,\quad a(x)\in V^2(G)\cap V_0^1(G). \tag{6.5}$$

It is known (see [14], [17]) that if for each $T>0$ there exists a small enough $\varepsilon=\varepsilon(T)$ such that $\|a-w_0\|_{V_0^1(G)}<\varepsilon$ then unique solution $w(t,x)\in V_0^{1,2}(Q_T)$ of problem (6.2), (6.3) exists. Solution $w(t,x)$ of (6.2), (6.3) taken at time moment t we denote as $S(t,w_0)(x)$:

$$w(t,x)=S(t,w_0)(x). \tag{6.6}$$

Since embedding $V^{1,2}(Q_T)\subset C(0,T;V_0^1(G))$ is continuous, the family of operators $S(t,w_0)$ is continuous semigroup on the space $V_0^1(G):S(t+\tau,w_0)=S(t,S(\tau,w_0))$.

Since $a(x)$ is steady-state solution of (6.2), $S(t,a)=a$ for each $t\geqslant 0$. We can decompose semigroup $S(t,w_0)$ in a neighborhood of a in the form

$$S(t,w_0+a)=a+L_tw_0+B(t,w_0) \tag{6.7}$$

where $L_tw_0=S'_w(t,a)w_0$ is derivative of $S(t,w_0)$ with respect to w_0 at point a, and $B(t,w_0)$ is nonlinear operator with respect to w_0. Differentiability of $S(t,w_0)$ is proved, for instance in [2, Ch. 7. Sect. 5]. Therefore

$$B(t,0)=0,\quad B'_w(t,0)=0. \tag{6.8}$$

Moreover in [2, Ch. 7. Sect. 5] is proved that $B'(t,w)$ belongs to class C^α with $\alpha=1/2$ with respect to w. This means that for each $w_0\in V_0^1(G)$ such that $\|a-w_0\|_{V_0^1(G)}<\varepsilon(t)$

$$\|B'_w(t,w_0)\|_{C^\alpha}\equiv \sup_{\substack{\|u-w_0\|_{V_0^1(G)}\leqslant 1\\ \|u-a\|_{V_0^1(G)}\leqslant\varepsilon(t)}}\frac{\|B'_w(t,u)-B'_w(t,w_0)\|_{V_0^1(G)}}{\|u-w_0\|^\alpha_{V_0^1(G)}}<\infty$$

and left side is a continuous function with respect to w_0.

We study now semigroup $L_tw_0=S'_w(t,a)w_0$ of linear operators. First of all note that $w(t,x)=L_tw_0$ is the solution of problem (3.1)–(3.4) in which the coefficient a is the solution of (6.5). Therefore

$$L_tw_0=e^{-At}w_0 \tag{6.9}$$

where A is Ozeen operator (3.8).

Below we suppose that $r_0\in(0,1)$ satisfies the property:

$$\{\zeta\in\mathbb{C}\,:\,|\zeta|=r_0\}\cap\Sigma(e^{-At_0})=\emptyset \tag{6.10}$$

where, recall, $\Sigma(e^{-At})$ is the spectrum of operator (6.9).

It is clear, that $\zeta_j\in\Sigma(e^{-At_0})$ if and only if $\zeta_j=e^{-\lambda_jt_0}$ and $-\lambda_j\in\Sigma(-A)$. That is why condition (6.10) is equivalent to condition (3.20) where $\sigma=-\ln r_0/t_0$. Besides, if $|\zeta_j|>r_0$ then $-\mathrm{Re}\lambda_j>-\sigma$.

The following assertion holds:

Theorem 6.1. *Family of operators $e^{-At}:V_0^1(G)\to V_0^1(G)$ where A is operator (3.8) is well defined for each $t\geqslant 0$. Let*

$$\sigma_+=\{\zeta_1,\dots,\zeta_N\,:\,\zeta_j\in\Sigma(e^{-At_0}),\quad |\zeta_j|>r_0,\ j=1,\dots,N\} \tag{6.11}$$

where $r_0\in(0,1)$ and satisfies (6.10). Let $X_+\subset V_0^1(G)$ be the invariant subspace for e^{-At_0} corresponding to σ_+, $\Pi_+:V_0^1(G)\to X_+$ be the projector on X_+ (i.e., $\Pi_+V_0^1(G)=X_+$, $\Pi_+^2=\Pi_+$) and $X_-=(I-\Pi_+)V_0^1(G)$ be complementary invariant *subspace. Let* $L^+_{t_0} = e^{-At_0}|_{X_+} : X_+ \to X_+$, $L^-_{t_0} = e^{-At_0}|_{X_-} : X_- \to X_-$. *Then operator* $L^+_{t_0}$ *has inverse operator* $(L^+_{t_0})^{-1}$. *For some* t_0 *there exist constants* $\hat{r}$, ε_+, $\varepsilon_- \in (0,1)$ *such that*

$$\|L^-_{t_0}\| \leqslant \hat{r}(1-\varepsilon_-), \quad \|(L^+_{t_0})^{-1}\| \leqslant \hat{r}^{-1}(1-\varepsilon_+). \tag{6.12}$$

The proof of this Theorem is absolutely the same as in the case of space dimension two (see [7]).

Generally speaking eigenvalues of operators A and e^{-At} are complex-valued. That is why all spaces in Theorem 6.1 are complex. But to apply obtained results to (nonlinear) Navier—Stokes equation we need to have analogous results for the real spaces of the same type. Actually, for this we have to define the projector of Π_+ in real spaces.

Lemma 6.1. *Restriction of operator* Π_+ *on the real space* $V^1_0(G)$ *can be written in the form*

$$(\Pi_+)(x) = \sum_{j=1}^{K} e_j(x) \int\limits_G v(x)\varepsilon_j(x)\,dx \tag{6.13}$$

where $\{\varepsilon_j\}$ *is the set of functions (4.42) which are suitably renumbered and renormalized functions (4.41) and* $\{e_j\}$ *is set of Real and Imaginary parts of functions (3.16) analogously renumbered and renormalized.*

The proof of this simple lemma one can find in [6].

Lemma 6.2. *For an arbitrary subdomain* $\omega \subset G$ *vector fields* $\{e_j(x), j = 1, \dots, K\}$ *from* (6.13) *restricted on* ω *are linear independent over* $\mathbb{R}$.

To prove this Lemma we first establish analog of Theorem 4.1 for functions (3.16). After that we derive Lemma 6.2 from this Theorem by the same way as Lemma 4.2 was derived from Theorem 4.1.

Using (6.13) we can easily restrict spaces X_+ and X_- as well as operators $L^+_{t_0}$, $L^-_{t_0}$ defined in formulation of Theorem 6.1 on the real subspaces of $V^1_0(G)$. We denote this new real spaces and operators also by X_+, X_-, $L^+_{t_0}$, $L^-_{t_0}$. This will not lead to misunderstanding because below we do not use their complex analogs.

In a neighborhood of steady-state solution a of (6.5) we establish existence of a manifold M_- which is invariant with respect to semigroup $S(t,w)$ (i.e., $\forall w \in M$ $\forall t > 0$, $S(t,w)$ is well defined and for each $t > 0$, $S(t,w) \in M_-$). This manifold can be represented as the graph:

$$M_- = \{u \in V^1_0(G) : u = a + u_- + g(u_-), \quad u \in X_- \cap \mathcal{O}\} \tag{6.14}$$

where $\mathcal{O}$ is a neighborhood of origin in $V^1_0(G)$, $g : X_- \cap \mathcal{O} \to X_+$ is an operator-function of class $C^{3/2}$ and

$$g(0) = 0, \quad g'(0) = 0. \tag{6.15}$$

Note that condition (6.15) means that manifold (6.14) is tangent to X_- at point a.

The following theorem is true.

Theorem 6.2. *Let* a *satisfy* (6.5), $\sigma > 0$ *satisfy* (3.20), $\mathcal{O} = \mathcal{O}_\varepsilon = \{v \in V^1_0(G) : \|v\|_{V^1_0(G)} < \varepsilon\}$ *and* ε *is sufficiently small. Then there exists unique operator-function* $g : X_- \cap \mathcal{O} \to X_+$ *of class* $C^{3/2}$ *satisfying* (6.15) *such that the manifold* M_- *defined in* (6.14) *is invariant with respect to resolving semigroup* $S(t, w_0)$

of problem (6.2),(6.3). *There exists a constant $c > 0$ such that*

$$\|S(t, w_0) - a\|_{V_0^1(G)} \leqslant c\|w_0 - a\|_{V_0^1(G)} e^{-\sigma t} \quad \text{as } t \geqslant 0 \tag{6.16}$$

for each $w_0 \in M_-$.

This theorem follows form results of [2, Ch. 5, Sect. 2; Ch. 7,Sect. 5], from Theorem 6.1, and Lemma 6.1.

6.2. **Extension operator.** Here we construct extension operator for Navier—Stokes equations. This operator is nonlinear analog of extension operator (5.19) constructed for Ozeen equations.

Recall that the domain Ω and its extension G satisfy (5.10), (5.11). Besides, the space $V^1(\Omega, \Gamma_0)$ is defined in (5.18).

Theorem 6.3. *Suppose that $a(x)$ is a steady-state solution of* (6.5), *$\hat{v}(x) = \gamma_\Omega a$, and M_- is the invariant manifold constructed in a neighborhood $a+\mathcal{O}$ of a in $V_0^1(G)$ in Theorem 6.2. Let $B_{\varepsilon_1} = \{v_0 \in V^1(\Omega, \Gamma_0) : \|v_0 - \hat{v}\|_{V^1(\Omega)} < \varepsilon_1\}$. Then for sufficiently small ε_1 there exists a continuous operator*

$$\mathrm{Ext}_\sigma : \hat{v} + B_{\varepsilon_1} \to M_-, \tag{6.17}$$

which is operator of extension for vector fields from Ω to G:

$$(\mathrm{Ext}_\sigma v)(x) \equiv v(x), \quad x \in \Omega. \tag{6.18}$$

Proof. Let $L : V^1(\Omega, \Gamma_0) \to V_0^1(G)$ be the extension operator constructed in *Step 1* of Theorem's 5.1 proof. Similarly to (5.22) we introduce the following operator of extension:

$$Qv(x) = Lv(x) + w(x) \tag{6.19}$$

where $w(x)$ is a vector field concentrated in $\hat{\Omega} = G \setminus \overline{\Omega}_{\varepsilon/2}$ which is constructed by $v(x)$. We describe its construction below. At last we define the desired operator Ext_σ by the formula

$$\mathrm{Ext}_\sigma v = \Pi_- Qz + g(\Pi_- Qz) + a, \quad \text{with } z = v - a, \tag{6.20}$$

where $\Pi_- = I - \Pi_+$, Π_+ is operator (6.13) of projection on $X_+ = \Pi_+ V_0^1(G)$, $X_- = \Pi_- V_0^1(G)$, and $g : X_- \to X_+$ is the operator constructed in Theorem 6.2. By definition (6.14) of M_- we have $\mathrm{Ext}_\sigma v \in M_-$. Hence we have to ensure that the equality

$$(\mathrm{Ext}_\sigma v)(x) \equiv v(x), \quad x \in \Omega \tag{6.21}$$

is true, that shows that Ext_σ is an extension operator. By (6.13) $\{e_j(x)\}$ generates X_+ and therefore the map $g(u)$ can be written in the form

$$g(u) = \sum_{j=1}^{K} e_j g_j(u).$$

That is why taking into account (6.13) we can rewrite (6.20) in the form

$$\mathrm{Ext}_\sigma v = a(x) + Qz(x) - \sum_{j=1}^{K} e_j(x) \int\limits_Q Qz(y)\varepsilon_j(y)\, dy + \sum_{j=1}^{K} e_j(x) g_j(\Pi_- Qz), \tag{6.22}$$

$(z = v - a)$.

In virtue of Lemma 6.2 $\{e_j(x),\ x\in\Omega\}$ are linear independent and therefore (6.21), (6.22) imply

$$\int\limits_G Qz(x)\varepsilon_j(x)\,dx = g_j(\Pi_- Qz),\quad j=1,\dots,K. \tag{6.23}$$

Similarly to (5.32) we look for the vector field $w(x)$ from (6.19) in the form

$$w=-(-\pi\Delta)^{-1}_{\hat\Omega}\sum_{j=1}^K p_j\varepsilon_j \tag{6.24}$$

To find coefficients $(p_1,\dots,p_K)\equiv\vec p$ we substitute (6.24) into (6.23) taking into account (6.19), (6.13). As a result we get

$$\vec z - A\vec p = \vec g\big(Lz-(\vec p,(-\pi\Delta)^{-1}_{\hat\Omega}\vec\varepsilon)-(\vec e,\vec z-A\vec p)\big), \tag{6.25}$$

where L is the extension operator from (6.19), $\vec z=(z_1,\dots,z_K)$, $A=\|a_{jk}\|$ and

$$z_j=\int\limits_G (Lz(x),\varepsilon_j(x))\,dx,\quad a_{jk}=\int\limits_G ((-\pi\Delta)^{-1}_{\hat\Omega}\varepsilon_k(x),\varepsilon_j(x))\,dx,$$

$$\vec g(u)=(g_1(u),\dots,g_K(u)),$$

$$\vec\varepsilon=(\varepsilon_1(x),\dots,\varepsilon_K(x)),\quad \vec e=(e_1(x),\dots,e_K(x)),\quad (\vec c,\vec e)=\sum_{j=1}^K c_je_j.$$

We showed in *Step 4* of Theorem's 5.1 proof that matrix $A=\|a_{jk}\|$ is positive defined and therefore it is invertible.

Applying to both parts of (6.25) the matrix $-A^{-1}$ we get the equality

$$\vec p = G_z(\vec p) \tag{6.26}$$

where the map $G_z:\mathbb R^K\to\mathbb R^K$ is defined by the relation

$$G_z(\vec p)=A^{-1}z-A^{-1}\vec g(Lz-(\vec e,\vec z)+(\vec p,(-\pi\Delta)^{-1}_{\hat\Omega}\vec\varepsilon)+(\vec e,A\vec p)). \tag{6.27}$$

In virtue of Theorem 6.2 the map $A^{-1}\vec g:\mathbb R^K\to\mathbb R^K$ belongs to the class $C^{1+1/2}$ and $A^{-1}\vec g(0)=0$, $A^{-1}\vec g'(0)=0$. Therefore for sufficiently small $\|\vec p_1\|_{\mathbb R^K}$, $\|\vec p_2\|_{\mathbb R^K}$, $\|z\|_{V_0^1(G)}$ we derive from (6.27) that

$$\begin{aligned}\|G_z(\vec p_1)-G_z(\vec p_2)\| &\leqslant \sup_{\beta\in[0,1]}\|A^{-1}\vec g'(\Gamma z-(\vec e,\vec z)+\\ &+(\beta\vec p_1+(1-\beta)\vec p_2,(-\pi\Delta)^{-1}_{\hat\Omega}\vec\varepsilon)-(\vec e,A[\beta\vec p_1+(1-\beta\vec p_2)]))\|\cdot\|\vec p_1-\vec p_2\|\leqslant\\ &\leqslant\gamma(z,\vec p_1,\vec p_2)\|\vec p_1-\vec p_2\|,\ \text{where } \gamma(z,p_1,p_2)\leqslant\gamma_1(\|z\|^{1/2}_{V_0^1(G)}+\|p_1\|^{1/2}_{\mathbb R^K}+\|p_2\|^{1/2}_{\mathbb R^K}),\end{aligned}$$

and $\gamma_1>0$ is a constant. Therefore the map G_z is a contraction one. Hence by contraction mapping principle ([13]) equation (6.26) has a unique solution $\vec p=(p_1,\dots,p_K)$ if $\|z\|_{V_0^1(G)}$ is sufficiently small. For these $\|z\|_{V_0^1(G)}$ the operator Ext_σ defined in (6.20), (6.19), (6.24) is the desired extension operator. $\square$

6.3. **Theorem on stabilization.** We set

$$V^2(\Omega,\Gamma_0) = V^2(\Omega)\cap V^1(\Omega,\Gamma_0) \tag{6.28}$$

where $V^1(\Omega,\Gamma_0)$ is space (5.18).

Proposition 6.1. *Let $f \in (L_2(\Omega))^3$ and a pair $(\hat{v}(x), \nabla\hat{p}(x))$ belongs to $V^2(\Omega,\Gamma)\times (L_2(\Omega))^3$ and satisfies equations* (2.7), (2.8). *Then there exist an extension $g(x) \in (L_2(G))^2$ of $f(x)$ from Ω to G and an extension $(a(x), \nabla q(x)) in (V^2(G)\cap V_0^1(G))\times (L_2(G))^2$ of $(\hat{v}(x), \nabla\hat{p}(x))$ from Ω to G such that the pair $(a(x), \nabla q(x))$ is a solution of* (2.19), (2.20).

The proof of this simple assertion is absolutely the same as in two-dimension case (see Proposition 5.1 in [7]).

We now are in position to formulate the main result of this paper.

Theorem 6.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary $\partial\Omega$ and $\partial\Omega = \Gamma_0\cup\Gamma$, where Γ, Γ_0 are closed unintersectig surfaces, and $\Gamma \neq \emptyset$. Suppose that an extension $G \subset \mathbb{R}^3$ of Ω satisfies (5.10), (5.11). Let $f(x) \in (L_2(\Omega))^3$ and $(\nabla\hat{v}(x), \nabla\hat{p}(x)) \in V^2(\Omega,\Gamma_0)\times(L_2(\Omega))^3$ satisfy* (2.7), (2.8). *Then for an arbitrary $\sigma > 0$ there exists a sufficiently small $\varepsilon_1 > 0$ such that for each $v_0 \in V^1(\Omega,\Gamma_0)$ satisfying*

$$\|\hat{v} - v_0\|_{V^1(\Omega)} < \varepsilon_1 \tag{6.29}$$

there exists a feedback boundary control $u(t,x)$, $(t,x) \in \Sigma \equiv \mathbb{R}_+\times\Gamma$ which stabilizes Navier—Stokes boundary value problem (2.3)–(2.6) *with the rate* (2.9)*, i.e. the solution v of* (2.3)–(2.6) *satisfies* (2.9).

Proof. Using Proposition 6.1 we extend $\hat{v}(x)$ to $a(x) \in V^1(G)$, and $f(x)$ to $g(x) \in (L_2(G))^2$. As a result we get boundary value problem (2.16)–(2.18) (with certain w_0) and steady-state solution $(a(x), \nabla q(x))$ of this problem. We can suppose that $\sigma > 0$ satisfies (3.20): otherwise we increase σ a little bit and get (3.20). In virtue of Theorem 6.2 in a neighborhood of a there exists a manifold M_- which is invariant with respect to resolving semigroup $S(t, w_0)$ of problem (6.2), (6.3), and for each $w_0 \in M_-$ inequality (6.16) holds. Let ε_1 be so small that it satisfies condition of Theorem 6.3. Then we apply extension operator Ext_σ constructed in Theorem 6.3 to initial condition v_0 of problem (2.3)–(2.6) and take $w_0 = \mathrm{Ext}v_0$ as initial condition for problem (2.16)–(2.18) or for equation (6.2) (that is equivalent). Then since $w_0 \in M_-$, $S(t, w_0) \in M_-$ for each $t \geqslant 0$, and estimate (6.16) holds. We define solution (v, u) of stabilized problem (2.3)–(2.6) by formula (2.22) where $w(t, x) = S(t, w_0)$ is the solution of (6.2), (6.3). Then (2.9) follows from (2.22), (6.16) $\square$

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