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Stabilization of a Class of Fractional Order Systems With Both Uncertainty and Disturbance

RUNLONG PENG, CUIMEI JIANG, AND RONGWEI GUO
School of Mathematics and Statistics, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, China

Corresponding author: Rongwei Guo (rwguo@qlu.edu.cn)

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ABSTRACT This paper investigates the stabilization problem of fractional order systems with both model uncertainty and external disturbance. By combining the linear feedback control method, the dynamic feedback control method, and the uncertainty and disturbance estimator (UDE)-based control method, respectively, two new UDE-based control methods are developed. Using these methods, the fractional order systems can be stabilized by three steps. In the first step, the linear feedback and dynamic feedback controllers are designed to stabilize the nominal fractional order systems. The second step is to design a UDE-based fractional order controller to estimate the model uncertainty and external disturbance. In the third step, the two controllers are combined into a new controller to realize the stabilization of those fractional order systems. Finally, a numerical example is given to verify the correctness and validity of the proposed methods.

INDEX TERMS Fractional order system, stabilization, linear feedback, dynamic feedback, UDE.

I. INTRODUCTION

Although fractional calculus has a history of more than 300 years, its development is slow due to its lack of practical application background. Nevertheless, fractional order systems (FOSs) can also perform well in some practical problems, and many systems in reality have fractional order dynamic behavior, so the research on FOSs develops rapidly. FOSs play an important role in many aspects, such as signal processing, image processing, automatic control, robotics, and so on [1]—[6]. For FOSs, there are many different kinds of control problems, such as stabilization, synchronization, anti-synchronization, co-existence of synchronization and anti-synchronization, projective synchronization, etc [7]–[19]. Among these control problems, stabilization is the both basic and important problem to be solved for FOSs. Only when the stabilization problem is solved can the other types of control problems be settled. Therefore, it is very important to study the stabilization problem of FOSs.

So far, the stabilization problem of FOSs has been attracted attention of many authors in different scientific fields, such as chemical reactors [20], electrical circuits [21] and microelectro-mechanical systems [22]. For example, stabilization of generalized fractional order chaotic systems [23] is studied via designing the feedback controllers. Control of fractional order Coullet system [24] has been presented via active control. In 2016, based on feedback control, Zheng and Ji have investigated the stabilization of a fractional-order chaotic system without external disturbance and model uncertainty [25]. The stabilization results are proposed for a class of fractional order chaotic systems with unknown parameters by adaptive backstepping technique in [26]. Sliding mode control is also used to realize chaos control of fractional order systems with uncertainty and disturbance in [27]. In [28], based on generalized T-S fuzzy model and adaptive adjustment mechanism, authors obtain a simple but efficient method to control fractional order chaotic systems. Fuzzy control method and LMIs are used to control a class of fractional order uncertain chaotic systems in [29]. Further development on this topic, please refer to Refs.[30]–[34].

However, the designed controllers by some existing methods are too complicated to be used in applications. In order to design a simple and physically controller to stabilize the system, the linear feedback control method is usually used in applications because of its simple structure and good effect. Furthermore, the dynamic feedback control method is given

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in [35-36] is a both effective and simple method for the integer order systems. Therefore we shall generalize the dynamic feedback control method for the integer order systems to solve the stabilization of FOSs.

It should be pointed out that the model uncertainty and external disturbance are not considered carefully in dealing with the control problems of FOSs. However, the model uncertainty and the external disturbance are inevitable in applications. Therefore, it is difficult to implement some existing control methods in practical applications. Luckily, for nonlinear systems, the UDE-based control method has made great progress in solving robot control problems and their applications in the engineering field [37]–[39]. For the study of integer order problems with both model uncertainty and external perturbation, the UED-based control method has been presented in [40]–[43]. Therefore, it is both important and urgent problem to extend the UDE-based control method to FOSs.

The main contribution of this paper is to design a simple and physically controller to stabilize FOSs. The UDE-based linear feedback control method and the UDE-based the dynamic feedback control method are proposed to realize the stabilization problem of FOSs with model uncertainty and external disturbance. It is mainly divided into three steps. In the first step, the nominal system is stabilized by using linear feedback control method and the dynamic feedback control method, respectively, thus the linear feedback controller and the dynamic feedback controller are designed. In the second step, a fractional-order UDE-based controller which can estimate model uncertainty and external disturbance is designed. The third step is to combine the two controllers to realize the stabilization of FOSs. Finally, an example is given to verify the correctness and effectiveness of the proposed method.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. PRELIMINARIES

The derivative of Fractional-order Caputo is defined as

$$D_t^\alpha f(x) = \frac{D^\alpha f(x)}{dt^\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha-n)}\displaystyle\int_\alpha^t \frac{f^{(n)}(\varsigma)}{(t-\varsigma)^{\alpha+1-n}}d\varsigma, & n-1<\alpha<n \\ \dfrac{d^n f(t)}{dt^n}, & \alpha=n \end{cases}$$

where $n=[\alpha]$, $\Gamma(.)$ said function with the following expression

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt$$

The Laplace transform of the Caputo derivative

$$D_t^\alpha f(t) = D_t^{-(n-\alpha)}h(t), \quad h(t)=f^{(n)}(t) \tag{1}$$

where $n-1<\alpha\leq n$.

Taking the Laplace transform of both sides of equation (1), it results in

$$\ell\{D_t^\alpha f(t); s\} = \ell D_t^{-(n-\alpha)}h(t) = s^{-(n-\alpha)}G(s) \tag{2}$$

where

$$\begin{aligned} G(s) &= \ell\{h(t); s\} = \ell\{f^{(n)}(t); s\} \\ &= s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \\ &= s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0), \end{aligned}$$

i.e.,

$$\ell\{D_t^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1<\alpha\leq n.$$

Consider the following FOS

$$D_t^\alpha x(t) = f(x) \tag{3}$$

where $x\in\mathbb{R}^n$ is the state, $f(x)\in\mathbb{R}^n$ is a continuous function vector.

Definition 1: Consider the following controlled FOS

$$D_t^\alpha x(t) = f(x) + Bu \tag{4}$$

where $x\in\mathbb{R}^n$ is the state, $f(x)\in\mathbb{R}^n$ is the function vector, $B\in\mathbb{R}^{n\times r}$, $r\geq 1$, and u is the designed controller. If $\lim_{t\to\infty}\|x(t)\|=0$, the system (4) is called to be stabilized by the above controller u.

It is well known that there are many methods to design the controller u. For simplicity, we adopt the linear feedback control method and the dynamic feedback control method to realize the stabilization problem of FOS.

Lemma 1: Consider the system (4). If $(f(x), B)$ is stabilized, then the designed linear feedback controller is given as

$$u = Kx \tag{5}$$

where $K\in\mathbb{R}^{r\times n}$ is a constant matrix.

For the sequel use, three properties of the fractional calculus are stated in the next.

Property 1: Fractional calculus defined by Caputo is a linear algorithm, i.e.,

$$D_t^\alpha(\lambda x+\mu y) = \lambda D_t^\alpha x + \mu D_t^\alpha y$$

Property 2: For FOS(1), $f(x)$ with respect to x satisfies the Lipschitz's condition:

$$\|f(x)-f(y)\|_\infty \leq l\|x-y\|_\infty$$

where $\|.\|_\infty$ is an ∞-norm, l is a positive real number. In particular, when $x=0, f(x)=0$. i.e.,

$$\|f(x)\|_\infty \leq l\|x\|_\infty$$

Property 3: Let $x \in \mathbb{R}$ be a continuous differentiable function, and for any continuous time $t \geq t_0$, i.e.,

$$\frac{1}{2}D_t^\alpha x^2 \leq x D_t^\alpha x, \quad 0 < \alpha < 1$$

Lemma 2: [44]–[45] (Mittag-leffler stability) Consider the system (3) defined by Caputo, let $x = 0$ be the equilibrium point of system (3), and $D \subset \mathbb{R}^n$ be the region containing the origin. Let $V(x) : [0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function, and the local Lipschitz condition is satisfied with respect to x:

$$\beta_1(\|x\|) \leq V(x) \leq \beta_2(\|x\|)$$
$$D_t^\alpha V(x) \leq -\beta_3(\|x\|)$$

where $t \geq 0$, $x \in D$, $\alpha \in (0, 1)$, $\beta_1, \beta_2, \beta_3$ is κ functions, then the fractional order nonlinear system (3) is asymptotically stable at the equilibrium point $x = 0$.

Next, the UDE-based control method is introduced.

Lemma 3: [37]–[39] Consider the following chaotic system with model uncertainty and disturbance

$$\dot{x}(t) = f(x) + BU + \Delta f(x) + d(t) \tag{6}$$

where $x \in \mathbb{R}^n$ is the state, $B \in \mathbb{R}^{n\times r}$, $r \geq 1$, $(f(x), B)$ is controllable, $\Delta f(x)$ is model uncertainty, $d(t)$ is the external disturbance. If a filter $g_f(t)$ satisfies the following conditions:

$$\tilde{u}_d = \hat{u}_d - u_d \tag{7}$$

where $\hat{u}_d = (\dot{x} - F(x) - Bu_{ude}) * g_f(t)$ and $u_d = \Delta f(x) + d(t)$, then UDE-based controller U as follows

$$U = U_s + U_{ude} \tag{8}$$

here $U_s = Kx$, and

$$U_{ude} = B^+\left\{\ell^{-1}[\frac{G_f(s)}{1 - G_f(s)}] * F(x)\right\} - B^+\left\{\ell^{-1}[\frac{sG_f(s)}{1 - G_f(s)}] * x(t)\right\} \tag{9}$$

$B^+ = (B^TB)^{-1}B^T$, $F(x) = f(x) + Bu_s$, $G_f(s) = \ell[g_f(t)]$, ℓ^{-1} represents the inverse Laplace transform, $*$ represents the convolution.

B. PROBLEM FORMULATION

Consider the following FOS

$$D_t^\alpha x(t) = f(x) + Bu + \Delta f(x) + d(t) \tag{10}$$

where $0 < \alpha < 1$, $x \in \mathbb{R}^n$ is the state, $B \in \mathbb{R}^{n\times r}$, $r \geq 1$, $(f(x), B)$ is controllable, $\Delta f(x)$ is model uncertainty, $d(t)$ is the external disturbance.

The main goal of this paper is to design a controller u to stabilize the system (10), i.e.,

$$\lim_{t\to\infty} \|x(t)\| = 0$$

III. MAIN RESULTS

By extending the dynamical feedback control method to the fractional order systems, some new results are obtained.

Theorem 1: Consider system (4). If $(f(x), B)$ is stabilized, then the designed dynamic feedback controller u is proposed as follows

$$u = K(t)x \tag{11}$$

where $K(t) = k(t)B^T$, $D_t^\alpha k(t) \leq -\gamma \sum_{i=1}^n x_i^2$, and

$$\dot{k}(t) = -\gamma x^T x = -\gamma \|x(t)\|^2 = -\gamma \sum_{i=1}^n x_i^2 \tag{12}$$

where $\gamma > 0$ is a constant.

Proof: Substituting the controller u in Eq. (11) into the system (4), we call the system (4) and the system (12) to be the auxiliary system, and introduce the following non-negative function:

$$V(t) = \frac{1}{2}\sum_{i=1}^n x_i^2 + \frac{1}{2\gamma}(k + L)^2$$

where L is a larger constant, i.e., $nl \leq L$.

Differentiating V along the trajectory of the auxiliary system and applying Property 1, Property 2 and Property 3, we get

$$\begin{aligned}
D_t^\alpha V(t) &= \frac{1}{2}\sum_{i=1}^n D_t^\alpha x_i^2 + \frac{1}{2\gamma}D_t^\alpha (k + L)^2 \\
&\leq \sum_{i=1}^n x_i D_t^\alpha x_i + \frac{k + L}{\gamma} D_t^\alpha (k + L) \\
&= \sum_{i=1}^n x_i D_t^\alpha x_i + \frac{k + L}{\gamma} D_t^\alpha k \\
&= \sum_{i=1}^n x_i (f(x_i) + kx_i) + \frac{k + L}{\gamma} D_t^\alpha k \\
&= \sum_{i=1}^n x_i f(x_i) + \sum_{i=1}^n kx_i^2 + \frac{k + L}{\gamma} D_t^\alpha k \\
&\leq nl \sum_{i=1}^n x_i^2 + \sum_{i=1}^n kx_i^2 + \frac{k + L}{\gamma} D_t^\alpha k \\
&\leq nl \sum_{i=1}^n x_i^2 + \sum_{i=1}^n kx_i^2 - (k + L)\sum_{i=1}^n x_i^2 \\
&= nl \sum_{i=1}^n x_i^2 - L\sum_{i=1}^n x_i^2 \\
&= (nl - L)\sum_{i=1}^n x_i^2 \leq 0 \\
&= (nl - L)V_1(t) \leq 0,
\end{aligned}$$

where $V_1(t) = \sum_{i=1}^n x_i^2$.

Integrating the integer order of the above inequality, it results in

$$\begin{aligned}(nl-L)&\int_{t_0}^{t} V_1(q)dq\\ &\geq \int_{t_0}^{t} D_q^{\alpha} V(\varphi)dq\\ &= \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\int_{t_0}^{q}\frac{V'(\varphi)}{(q-\varphi)^{\alpha}}d\varphi dq\\ &= \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\int_{\varphi}^{t}\frac{V'(\varphi)}{(q-\varphi)^{\alpha}}dqd\varphi\\ &= \frac{1}{\Gamma(2-\alpha)}\int_{t_0}^{t} V'(\varphi)(t-\varphi)^{1-\alpha}d\varphi\\ &= -\frac{V(t_0)(t-t_0)^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t} V(\varphi)(t-\varphi)^{-\alpha}d\varphi\\ &\geq -\frac{V(t_0)(t-t_0)^{1-\alpha}}{\Gamma(2-\alpha)}.\end{aligned}$$

Then

$$\int_{t_0}^{t} V_1(q)dq \leq \frac{V(t_0)(t-t_0)^{1-\alpha}}{(nl-L)\Gamma(2-\alpha)}$$

Taking the limit of both sides of the inequality

$$\lim_{t\to\infty}\int_{t_0}^{t} V_1(q)dq \leq \lim_{t\to\infty}\frac{V(t_0)(t-t_0)^{1-\alpha}}{(nl-L)\Gamma(2-\alpha)}$$

then we obtain

$$\begin{aligned}\lim_{t\to\infty}\frac{\int_{t_0}^{t} V_1(q)dq}{(t-t_0)^{1-\alpha}} &\leq \lim_{t\to\infty}\frac{V(t_0)}{(nl-L)\Gamma(2-\alpha)}\\ &= \frac{V(t_0)}{(nl-L)\Gamma(2-\alpha)}.\end{aligned}$$

There are two cases for $\lim_{t\to\infty}\int_{t_0}^{t} V_1(q)dq$.

Case 1: If $\lim_{t\to\infty}\int_{t_0}^{t} V_1(q)dq < +\infty$, we get

$$\lim_{t\to\infty} V_1(t) = \lim_{t\to\infty}\|x_i\|^2 = 0.$$

Case 2: If $\lim_{t\to\infty}\int_{t_0}^{t} V_1(q)dq = +\infty$, using the L'Hospital principle, it results in

$$\begin{aligned}\lim_{t\to\infty}\frac{\int_{t_0}^{t} V_1(q)dq}{(t-t_0)^{1-\alpha}} &= \lim_{t\to\infty} V_1(t)(t-t_0)^{\alpha}\\ &\leq \lim_{t\to\infty}\frac{V(t_0)}{(nl-L)\Gamma(2-\alpha)}.\end{aligned}$$

The integral of order α in Eq. (10) is performed below

$$V(t)-V(t_0) \leq \frac{nl-L}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{V(q)}{(t-q)^{1-\alpha}}dq \leq 0,$$

that is

$$V_1(t) \leq V(t) \leq V(t_0),$$

this means that $V_1(t)$ must be bounded.
So there is a $T > 0$ there is:

$$V_1(t) \leq \frac{V(t_0)}{(nl-L)\Gamma(2-\alpha)},$$

So for all of the $t \geq T$, we have:

$$\lim_{t\to\infty} V_1(t) = \lim_{t\to\infty}\|x\|^2 = 0.$$

According to Lemma 2, it can be known that system (4) is asymptotically stable at the equilibrium point $x = 0$ by the controller u. □

Theorem 2: Considering the FOS (10). If $(f(x), B)$ is stabilized and a filter $g_f(t)$ satisfies the following condition:

$$\tilde{u}_d = \hat{u}_d - u_d, \tag{13}$$

where $\hat{u}_d = (D_t^{\alpha}x - F(x) - Bu_{ude}) * g_f(t)$, and $u_d = \Delta f(x) + d(t)$, then UDE-based controller u as follows

$$u = u_s + u_{ude} \tag{14}$$

where $u_s = Kx$, and K is given in Eq. (5) or Eq. (11), and u_{ude} is presented as follows

$$\begin{aligned}u_{ude} &= B^{+}\left\{\ell^{-1}[\frac{G_f(s)}{1-G_f(s)}] * F(x)\right\}\\ &\quad -B^{+}\left\{\ell^{-1}[\frac{s^{\alpha}G_f(s)}{1-G_f(s)}] * x(t)\right\},\ 0<\alpha\leq 1\end{aligned} \tag{15}$$

Proof: Substituting the controller u in Eq. (14) into the system (10), we get

$$D_t^{\alpha}x(t) = F(x) + u_d + Bu_{ude} \tag{16}$$

where $F(x) = f(x,t) + Bu_s$, $u_d = \Delta f(x,t) + d(t)$, and u_{ude} is to be designed.

Then, it is easy to obtain

$$u_d = D_t^{\alpha}x(t) - F(x) - Bu_{ude}$$

According to the idea of UDE-control method in [37-39], the u_d is estimated by

$$\hat{u}_d = u_d * g_f(t) = (D_t^{\alpha}x(t) - F(x) - Bu_{ude}) * g_f(t),$$

Noting the condition (13), if $Bu_{ude} = -\hat{u}_d$, then the conclusion of this theorem is obtained, i.e.,

$$\begin{aligned}Bu_{ude} &= -\hat{u}_d\\ &= -u_d * g_f(t)\\ &= -(D_t^{\alpha}x(t) - F(x) - Bu_{ude}) * g_f(t)\end{aligned} \tag{17}$$

Taking the Laplace transform of both sides of this equation (17), we get

$$BU_{ude}(s) = -s^{\alpha}X(s)G_f(s) + F(s)G_f(s) + BU_{ude}(s)G_f(s)$$

then

$$BU_{ude}(s) - BU_{ude}(s)G_f(s) = -s^{\alpha}X(s)G_f(s) + F(s)G_f(s)$$

thus

$$U_{ude}(s) = B^{+}\frac{1}{1-G_f(s)}\{-s^{\alpha}X(s)G_f(s) + F(s)G_f(s)\}$$

i.e.,

$$U_{ude}(s) = B^{+}\left\{[\frac{G_f(s)}{1-G_f(s)}]F(s) - [\frac{s^{\alpha}G_f(s)}{1-G_f(s)}]X(s)\right\}$$

Thus, $u_{ude} = \ell^{-1}[U_{ude}]$ is obtained. □

In conclusion, three steps are required to accomplish the stabilization of the fractional order system with both model uncertainty and external disturbance

(I) The controller u_s is designed for the nominal FOSs by using Lemma 1 and Theorem 1;

(II) The controller u_{ude} is obtained by choosing an appropriate filter to estimate the model uncertainty and external disturbance;

(III) The controller u is proposed by setting $u = u_s + u_{ude}$.

IV. ILLUSTRATIVE EXAMPLE WITH NUMERICAL SIMULATION

In this section, the fractional order hyper-Chen system is used to verify the correctness and effectiveness of the proposed method.

Example 1: The fractional order hyper-Chen system with both disturbance and uncertainty is described as follows

$$D_t^\alpha x = f(x) + Bu + \Delta f(x) + d(t) \tag{18}$$

where $x \in \mathbb{R}^4$ is the state, and

$$f(x) = \begin{pmatrix} 37(x_2 - x_1) \\ -9x_1 - x_1x_3 + 26x_2 \\ -3x_3 + x_1x_2 + x_1x_3 - x_4 \\ -8x_4 + x_2x_3 - x_1x_3 \end{pmatrix} \tag{19}$$

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Delta f(x) = \begin{pmatrix} 0 \\ 0.03x_1x_2 \\ 0 \\ 0 \end{pmatrix} \tag{20}$$

$$d(t) = \begin{pmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \\ d_4(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 500 \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

or

$$d(t) = \begin{pmatrix} d_1(t) \\ d_2(t) \\ d_3(t) \\ d_4(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 20\sin(2t) \\ 0 \\ 0 \end{pmatrix} \tag{22}$$

$$u = u_s + u_{ude} \tag{23}$$

is the controller to be designed, $u_s \in \mathbb{R}$ and $u_{ude} \in \mathbb{R}$.

The controlled nominal fractional order hyper-Chen system is given as

$$D_t^\alpha x = f(x) + Bu_s \tag{24}$$

where $f(x)$ is given in (19), and B is presented in (20)

According to the results in Section III, we should firstly design the controller u_s to stabilize the system (24). There are two methods, i.e., the linear feedback control method and the dynamic feedback control method, are to be used in the next.

Consider the uncontrolled nominal fractional order hyper-Chen system (24), i.e., $u_s = 0$. It is easy to obtain that if $x_2 = 0$, then the following subsystem

$$\begin{aligned} D_t^\alpha x_1 &= -37x_1 \\ D_t^\alpha x_3 &= -3x_3 + x_1x_3 - x_4 \\ D_t^\alpha x_4 &= -8x_4 \end{aligned} \tag{25}$$

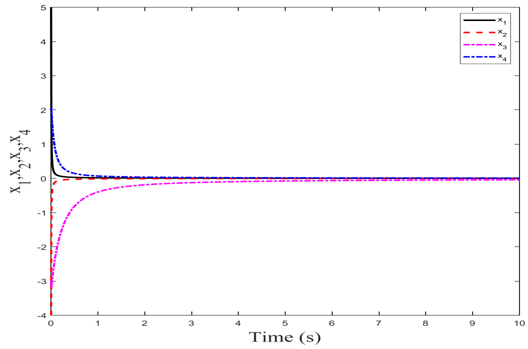

FIGURE 1. The system (24) with $\alpha = 0.8$ is stabilized.

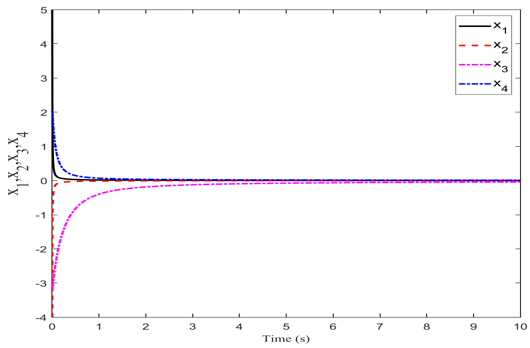

FIGURE 2. The system (24) with $\alpha = 0.95$ is stabilized.

is globally asymptotically stable, which implies that $(f(x), B)$ is stabilized.

Case 1: according to Lemma 1, the controller u_s is designed as

$$u_s = \begin{pmatrix} 0 & -100 & 0 & 0 \end{pmatrix} x = -100\, x_2. \tag{26}$$

In the following, numerical simulation is carried out with the initial conditions: $x(0) = [5, -4, -3, 2]^T$. Figure 1 and Figure 2 show that the fractional order hyper-Chen system (24) is stabilized by the above controller u_s when $\alpha = 0.8$ and $\alpha = 0.95$, respectively.

Case 2: according to Theorem 1, another controller u_s is designed as

$$u_s = k(t)B^T x = k(t)\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} x = k(t)x_2. \tag{27}$$

and the dynamic gain $k(t)$ is updated by (12).

Next, numerical simulation is carried out with the initial conditions: $x(0) = [5, -4, -3, 2]^T$, $k(0) = -1$. Figure 3 and Figure 5 show that the fractional order hyper-Chen system (24) is stabilized by the controller u_s when $\alpha = 0.8$ and $\alpha = 0.95$, respectively. Figure 4 and Figure 6 show that the feedback gain $k(t)$ converge to negative constants, respectively.

The second step is to design the controller u_{ude} for the fractional order Chen-hyper system (18).

According to Theorem 2, the controller u_{ude} is designed. Thus, the controller $u = u_s + u_{ude}$ in (18) is obtained.

It should be pointed out that the controller u_{ude} is varied with the order α for the fractional order problem.

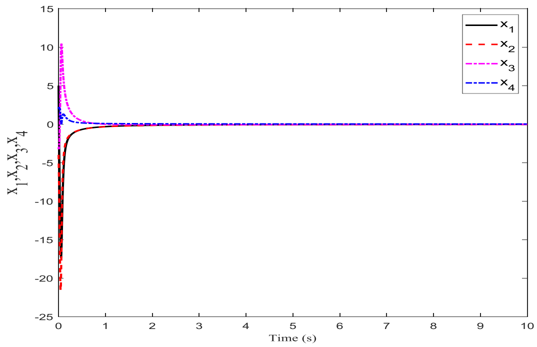

FIGURE 3. The system (24) with $\alpha = 0.8$ is stabilized.

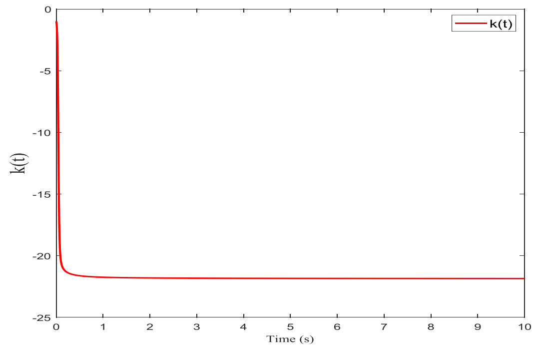

FIGURE 4. $k(t)$ converges to a constant.

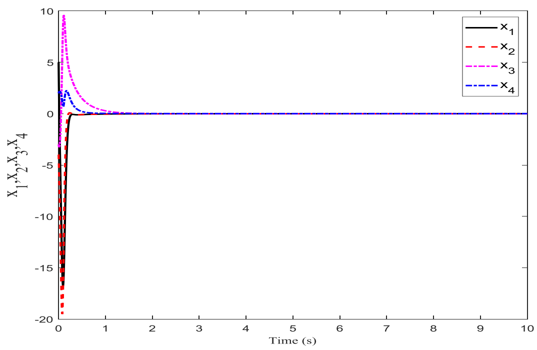

FIGURE 5. The system (24) with $\alpha = 0.95$ is stabilized.

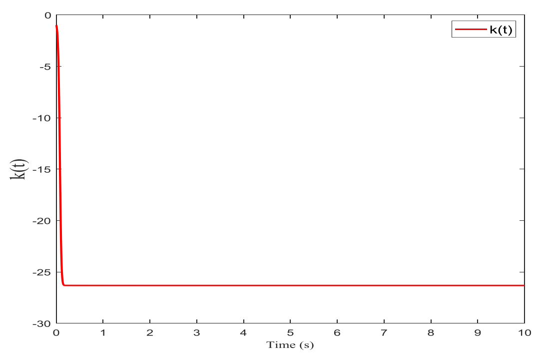

FIGURE 6. $k(t)$ converges to a constant.

For example, when $\alpha = 0.8$, the controller u_{ude} is as follows:

$$u_{ude} = B^{+}\left\{\ell^{-1}[\frac{G_f(s)}{1-G_f(s)}] * F(x)\right\} -B^{+}\left\{\ell^{-1}[\frac{s^{0.8}G_f(s)}{1-G_f(s)}] * x(t)\right\}$$

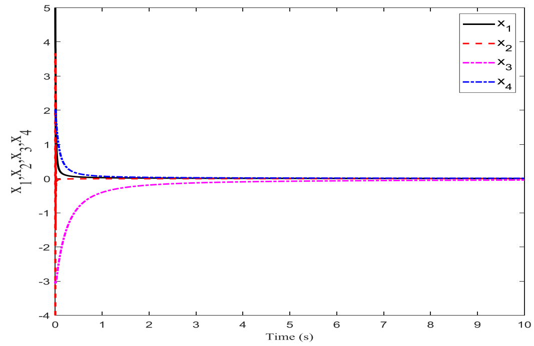

FIGURE 7. The system (18) with $\alpha = 0.8$ is stabilized.

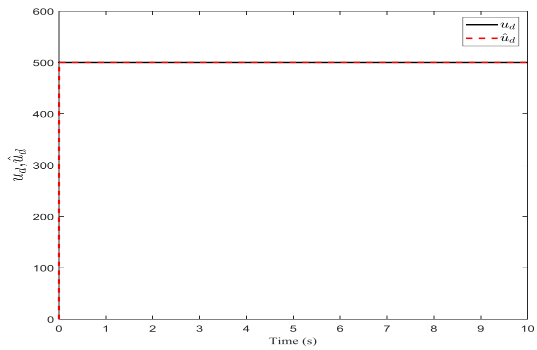

FIGURE 8. $\hat{u}_d$ tends to u_d.

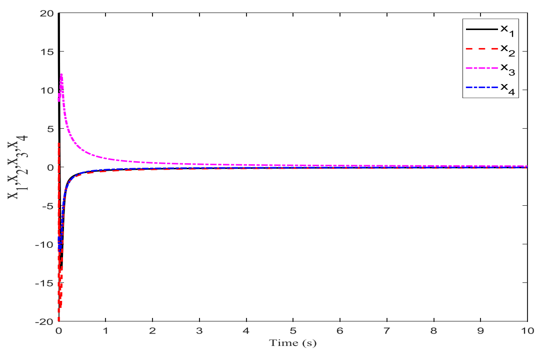

FIGURE 9. The system (18) with $\alpha = 0.8$ is stabilized.

For the system (18) with $d(t)$ is given in (21) and u_s is presented in (26), numerical simulation is carried out with the initial conditions: $x(0) = [5, -4, -3, 2]^T$. Figure 7 shows that the system (18) is stabilized by the controller u, Figure 8 shows that $\hat{u}_d$ tends to u_d.

For the system (18) with $d(t)$ is given in (21) and u_s is presented in (27), numerical simulation is carried out with the initial conditions: $x(0) = [5, -4, -3, 2]^T$, $k(0) = -1$. Figure 9 shows that the system (18) is stabilized by the controller u, Figure 10 shows that $\hat{u}_d$ tends to u_d, Figure 11 shows that the feedback gain $k(t)$ converges to a negative constant.

For the system (18) with $d(t)$ is given in Eq. (22) and u_s is presented in (26), numerical simulation is carried out with the initial conditions: $x(0) = [5, -4, -3, 2]^T$, Figure 12 shows

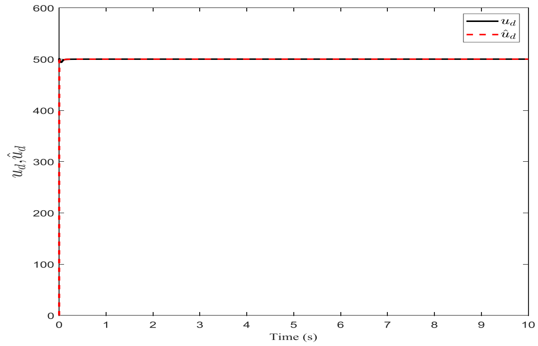

FIGURE 10. $\hat{u}_d$ **tends to** u_d**.**

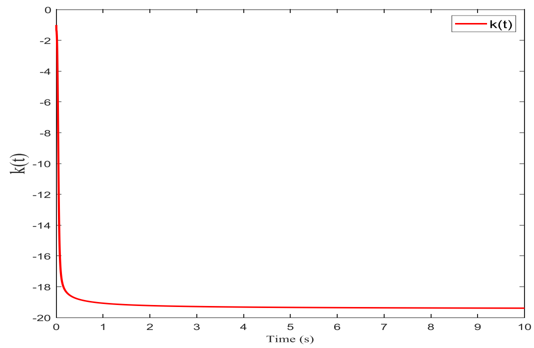

FIGURE 11. $k(t)$ **tends to a constant.**

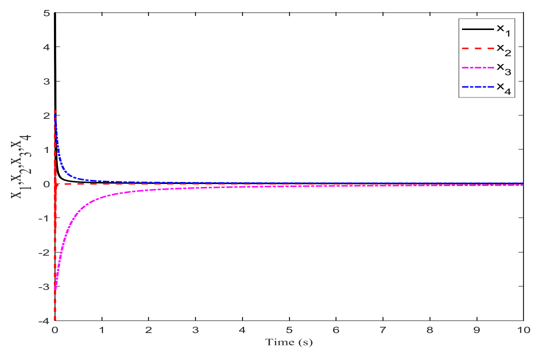

FIGURE 12. **The system (18) with** $\alpha = 0.8$ **is stabilized.**

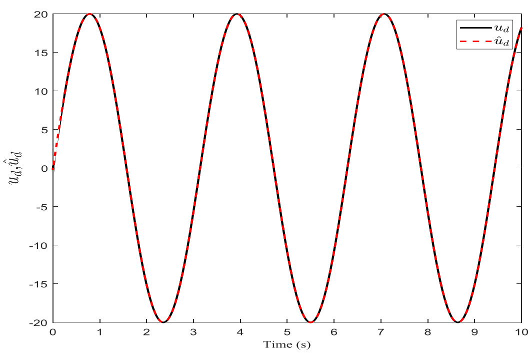

FIGURE 13. $\hat{u}_d$ **tends to** u_d**.**

that the system (18) is stabilized by the controller u, Figure 13 shows that $\hat{u}_d$ tends to u_d.

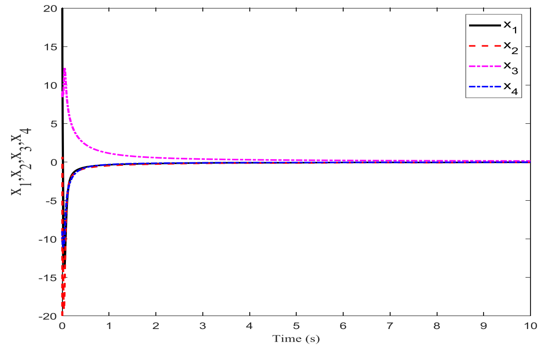

FIGURE 14. **The system (18) with** $\alpha = 0.8$ **is stabilized.**

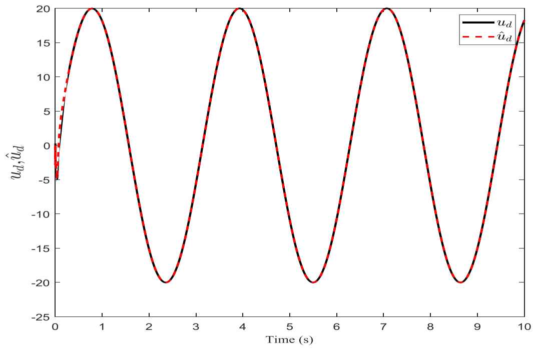

FIGURE 15. $\hat{u}_d$ **tends to** u_d**.**

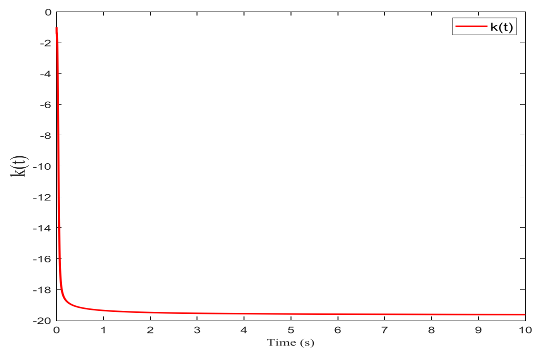

FIGURE 16. $k(t)$ **converges to a constant.**

For the system (18) with $d(t)$ is given in (22) and u_s is presented in (27), numerical simulation is carried out with the initial conditions: $x(0) = [5, -4, -3, 2]^T, k(0) = -1$, Figure 14 shows that the system (21) is stabilized by the controller u, Figure 15 shows that $\hat{u}_d$ tends to u_d, Figure 16 shows that the feedback gain $k(t)$ converges to a negative constant.

When $\alpha = 0.95$, the controller u_{ude} is as follows:

$$u_{ude} = B^{+}\left\{\ell^{-1}[\frac{G_f(s)}{1-G_f(s)}] * F(x)\right\} - B^{+}\left\{\ell^{-1}[\frac{s^{0.95}G_f(s)}{1-G_f(s)}] * x(t)\right\}.$$

For the system (18) with $d(t)$ is given in Eq. (21) and u_s is presented in (26), numerical simulation is carried out with the initial conditions: $x(0) = [5, -4, -3, 2]^T$. Figure 17 shows

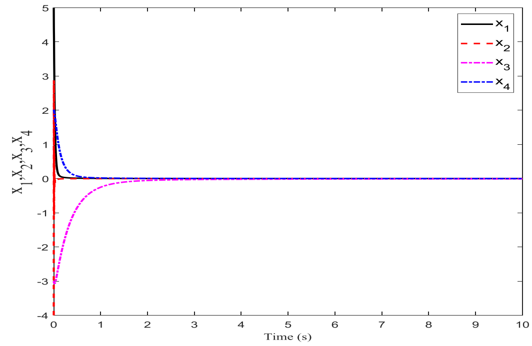

FIGURE 17. The system (18) with $\alpha = 0.95$ is stabilized.

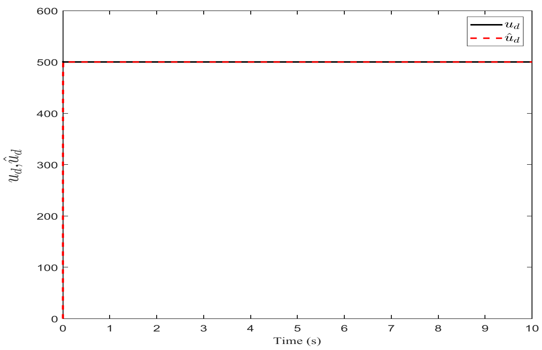

FIGURE 18. $\hat{u}_d$ tends to u_d.

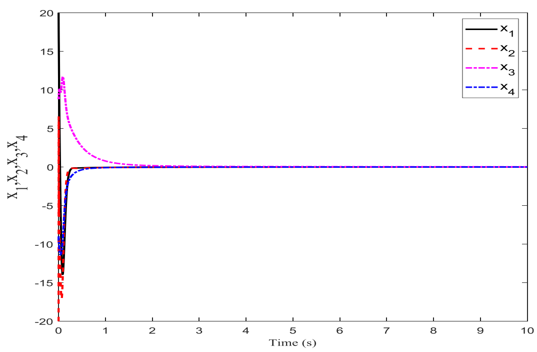

FIGURE 19. The system (18) with $\alpha = 0.95$ is stabilized.

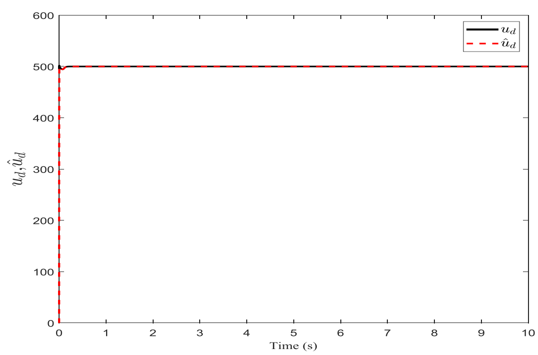

FIGURE 20. $\hat{u}_d$ tends to u_d.

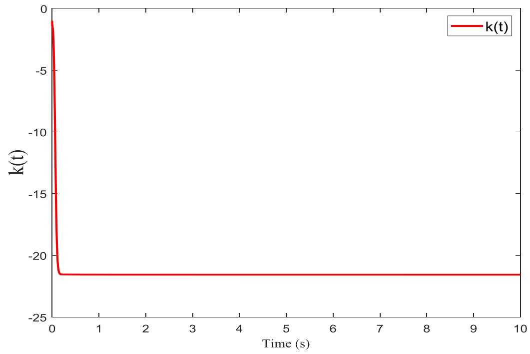

FIGURE 21. $k(t)$ converges to a constant.

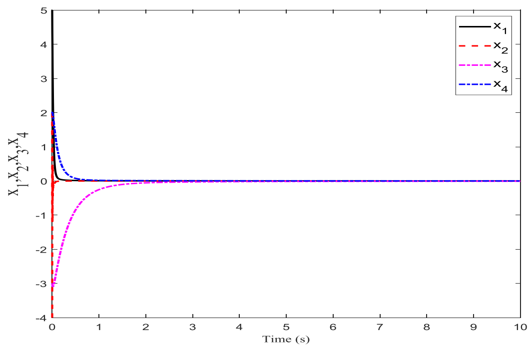

FIGURE 22. The system (18) with $\alpha = 0.95$ is stabilized.

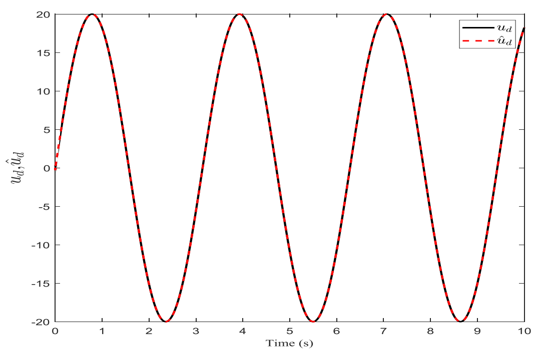

FIGURE 23. $\hat{u}_d$ tends to u_d.

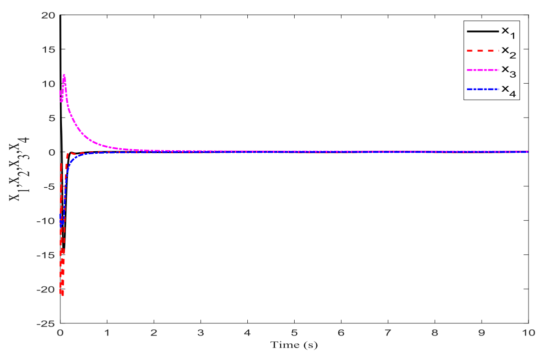

FIGURE 24. The system (18) with $\alpha = 0.95$ is stabilized.

that the system (18) is stabilized by the controller u, Figure 18 shows that $\hat{u}_d$ tends to u_d.

For the system (18) with $d(t)$ is given in Eq. (21) and u_s is presented in (27), numerical simulation is carried out with

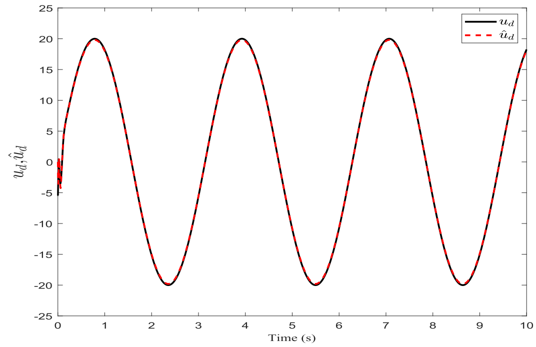

FIGURE 25. $\hat{u}_d$ **tends to** u_d**.**

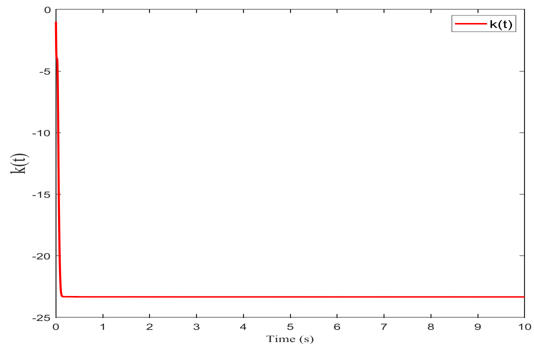

FIGURE 26. $k(t)$ **tends to a constant.**

the initial conditions: $x(0) = [5, -4, -3, 2]^T$, $k(0) = -1$. Figure 19 shows that the system (18) is stabilized by the controller u, Figure 20 shows that $\hat{u}_d$ tends to u_d, Figure 21 shows that the feedback gain $k(t)$ converges to a negative constant.

For the system (18) with $d(t)$ is given in Eq. (22) and u_s is presented in (26), numerical simulation is carried out with the initial conditions: $x(0) = [5, -4, -3, 2]^T$, Figure 22 shows that the system (21) is stabilized by the controller u, Figure 23 shows that $\hat{u}_d$ tends to u_d.

For the system (18) with $d(t)$ is given in Eq. (22) and u_s is presented in Eq. (27), numerical simulation is carried out with the initial conditions: $x(0) = [5, -4, -3, 2]^T$, $k(0) = -1$, Figure 24 shows that the system (18) is stabilized by the controller u, Figure 25 shows that $\hat{u}_d$ tends to u_d, Figure 26 shows that the feedback gain $k(t)$ converges to a negative constant.

V. CONCLUSION

In conclusion, we have studied the stabilization problem of FOSs with both disturbance and uncertainty. The integer order UDE-based controller is extended to the fractional order systems, and then the fractional-order UDE-based controller is obtained. Combining the linear, the dynamic feedback methods, and the UDE-based control method, respectively, two new UDE-based control methods have been obtained. The FOSs with both model uncertainty and external disturbance is stabilized by three steps. In the last section, a numerical example with numerical simulations has been given to verify the correctness and effectiveness of the proposed method.

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RUNLONG PENG was born in Liaocheng, Shandong, China, in 1996. He received the B.S. degree from the Qilu Institute of Technology, in 2019, where he is currently pursuing the M.S. degree. His research interests including nonlinear systems control and chaos.

CUIMEI JIANG received the B.S. degree from Shandong Normal University, the M.S. degree from the Ocean University of China, and the Ph.D. degree from Shandong University. She is currently a Lecturer with the School of Mathematical Sciences, Qilu University of Technology. Her research interests include nonlinear dynamics, chaos control, and chaotic synchronization.

RONGWEI GUO received the B.S. degree from the University of Jinan, China, in 2001, the M.S. degree from Shanghai University, China, in 2004, and the Ph.D. degree from Shandong University, China, in 2011. He is currently a Professor with the School of Mathematics and Statistics, Qilu University of Technology. His research interests include nonlinear systems control and switched systems.