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Stabilization of a system modeling temperature and porosity fields in a Kelvin–Voigt-type mixture

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Abstract In this paper, we investigate the asymptotic behavior of solutions to the initial boundary value problem for the interaction between the temperature field and the porosity fields in a homogeneous and isotropic mixture from the linear theory of porous Kelvin–Voigt materials. Our main result is to establish conditions which insure the analyticity and the exponential stability of the corresponding semigroup. We show that under certain conditions for the coefficients we obtain a lack of exponential stability. A numerical scheme is given.

1 Introduction

This article is concerned with a special case of a linear theory for the interaction between the temperature field and the porosity fields in a homogeneous and isotropic mixture from the linear theory of porous Kelvin–Voigt materials. The theory of porous mixtures has been investigated by several authors (see, for instance, [6–8,10] and the references therein). Iesan and Quintanilla [7] considered binary mixtures where the individual components are modeled as porous Kelvin–Voigt materials, and the volume fraction of each constituent was considered as an independent kinematical quantity. The authors assumed that the constituents have a common temperature and that every thermodynamical process that takes place in the mixture satisfies the Clausius–Duhem inequality. At the end, they presented as an application the interaction between the temperature field θ and the porosity fields u and w in a homogeneous and isotropic mixture. We restrict ourselves to the interaction between the temperature field and the porosity fields u and w in a homogeneous and isotropic mixture. Under the same assumptions of Ieşan and Quintanilla [7], we have a system of three equations given by:

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$$\begin{aligned}
&\rho_1^0\,\kappa_1\,u_{tt} - a_{11}\,u_{xx} - a_{12}\,w_{xx} - b_{11}\,u_{xxt} - b_{12}\,w_{xxt} + a_{13}\,u + a_{14}\,w + b_{13}\,u_t + b_{14}\,w_t)\\
&\quad - k_1\,\theta_{xx} - \beta_1\,\theta = 0 \quad \text{in } (0,\ L)\times(0,\ \infty),\\
&\rho_2^0\,\kappa_1\,w_{tt} - a_{12}\,u_{xx} - a_{22}\,w_{xx} - b_{21}\,u_{xxt} - b_{22}\,w_{xxt} - a_{14}\,u + a_{24}\,w + b_{23}\,u_t + b_{24}\,w_t\\
&\quad -k_2\,\theta_{xx} - \beta_2\,\theta = 0 \quad \text{in } (0,\ L)\times(0,\ \infty),\\
&c\,\theta_t - \kappa\,\theta_{xx} + k_1\,u_{xxt} + k_2\,w_{xxt} + \beta_1\,u_t + \beta_2\,w_t = 0 \quad \text{in } (0,\ L)\times(0,\ \infty).
\end{aligned} \tag{1.1}$$

The function $u = u(x,\ t)$ (and $w = w(x,\ t)$) represents the fraction field of a constituent and $\theta = \theta(x,\ t)$ the difference of temperature between the actual state and a reference temperature.

The 24 different parameters of the system (1.1), that is, $\rho_i^0, \kappa_i, k_i, \beta_i, a_{ij}, b_{ij}$, with $i = 1, 2,\ j = 1, \ldots, 4$, $a_{12} = a_{21}, a_{23} = a_{14}$, and c, κ, represent some constitutive coefficients (see [7, Eq. (87)] for more details). However, under assumptions of symmetry, we can make some simplifications. First, mass densities multiplied each one by the respective constitutive coefficients $\rho_1^0\kappa_1$ and $\rho_2^0\kappa_2$ can be summarized to two single parameters ρ_1 and ρ_2, respectively. Second, we assume the symmetry of $B = (b_{ij})$, thus $b_{12} = b_{21}$. Third, we assume a well- balanced exchange coupling between the two fields of porosity, that is, assuming the following symmetry relations simplifying the constitutive coefficients $\zeta^{(1)} = \zeta^{(2)} = -\zeta^{(3)}$ of the model in [7], it can be summarized in only one parameter $\alpha = a_{13} = a_{24} = -a_{14}$. Finally, applying the well balance, this time to the rates of the field of porosity, we can make the following simplification $C^{(1)} = -C^{(2)} = -D^{(1)} = D^{(2)}$ of the model in [7] which we can summarize in the parameters $\alpha_1 = b_{13} = -b_{14} = -b_{23} = b_{24}$. In this case, our equations which govern the fields u, w and θ in the absence of body loads are given by the system

$$\begin{aligned}
&\rho_1\,u_{tt} - a_{11}\,u_{xx} - a_{12}\,w_{xx} - b_{11}\,u_{xxt} - b_{12}\ + w_{xxt}\alpha\,(u-w) + \alpha_1\,(u_t - w_t)\\
&\quad -k_1\,\theta_{xx} - \beta_1\,\theta = 0 \quad \text{in } (0,\ L)\times(0,\ \infty),\\
&\rho_2\,w_{tt} - a_{12}\,u_{xx} - a_{22}\,w_{xx} - b_{12}\,u_{xxt} - b_{22}\,w_{xxt} - \ \alpha\,(u-w) - \alpha_1\,(u_t - w_t)\\
&\quad -k_2\,\theta_{xx} - \beta_2\,\theta = 0 \quad \text{in } (0,\ L)\times(0,\ \infty),\\
&c\,\theta_t - \kappa\,\theta_{xx} + k_1\,u_{xxt} + k_2\,w_{xxt} + \beta_1\,u_t + \beta_2\,w_t = 0 \quad \text{in } (0,\ L)\times(0,\ \infty).
\end{aligned} \tag{1.2}$$

We study the system (1.2) with the following initial conditions:

$$u(x,\ 0) = u_0,\quad u_t(x,\ 0) = u_1,\quad w(x,\ 0) = w_0,\quad w_t(x,\ 0) = w_1,\quad \theta(x,\ 0) = \theta_0 \quad \text{in } (0,\ L) \tag{1.3}$$

and the Dirichlet boundary conditions:

$$u(0,\ t) = u(L,\ t) = 0,\quad w(0,\ t) = w(L,\ t) = 0,\quad \theta(0,\ t) = \theta(L,\ t) = 0 \quad t \in (0,\ \infty). \tag{1.4}$$

We assume that $\rho_1, \rho_2, c, \kappa, \alpha$ and α_1 are positive constants. Since coupling is considered, we consider $\left(\beta_1^2 + \beta_2^2\right)\left(k_1^2 + k_2^2\right) \neq 0$, but the signs of β_i or k_i do not matter in the analysis. The matrix $A = (a_{ij})$ is symmetric and positive definite and $B = (b_{ij}) \neq 0$ is symmetric and non-negative definite, that is,

$$a_{11} > 0,\quad a_{11}\,a_{22} - a_{12}^2 > 0,\quad b_{11} \geq 0,\quad b_{11}\,b_{22} - b_{12}^2 \geq 0.$$

These simplifications and savings in the use of 8 parameters, could be questionable from a standpoint of thermomechanical model. However, our purpose in this work is to investigate the stability of the solutions of the system (1.2)–(1.4). In this sense, it is a simplification without loss of generality. That is, the same results presented here and proved for the system (1.2) could be also obtained for the system (1.1).

The asymptotic behavior as $t \to \infty$ of solutions to the equations of linear thermoelasticity has been studied by many authors. We refer to the book of Liu and Zheng [9] for a general survey on these topics. However, we recall that very few contributions have been performed to study the time behavior of the solutions of non-classical elastic theories. In this direction, we mention the works [1–3, 10, 13] and [14]. In [13], the author treats the theory of elastic mixtures and proves the exponential decay of solutions of the equations of motion of a mixture of two linear isotropic one-dimensional elastic materials when the diffusive force is a function which depends on the point and can be localized. The paper [14] deals with the theory of mixtures. The author states the linear equations of the thermomechanical deformations and studies several suitable conditions to guarantee the exponential stability of solutions. On the other hand, the exponential stability for the case of the thermoelastic mixtures has been studied in [1] and [10]. In [10], the authors prove (generically) the asymptotic stability. In [1], the authors establish conditions to the exponential stability and to the lack of stability of the semigroup. In [2], the authors investigate the asymptotic behavior of solutions of an initial boundary value problem for one-dimensional mixtures. Finally, in [3], the authors investigate the analyticity of the semigroup associated with the initial boundary value problem treated in [2].

We note that we cannot expect that this system always decays in an exponential way. For instance, in the case $\beta_1+\beta_2=0$, $k_1+k_2=0$, $\rho_2\,(a_{11}+a_{12})=\rho_1\,(a_{12}+a_{22})$ and $b_{11}+b_{12}=b_{12}+b_{22}=0$, we can obtain solutions of the form $u=w$ and $\theta=0$. These solutions are undamped and do not decay to zero. These are very particular cases, but we will see that there are some other cases where the solutions decay, but the decay is not so fast to be controlled by an exponential. Our main result is to obtain conditions for the coefficients of the system (1.2) to ensure the exponential stability as well as the analyticity of the semigroup associated with (1.2)–(1.4). We want to emphasize that we follow the same line of reasoning adopted in the papers [1,2] and [14].

This paper is organized as follows. Section 2 outlines briefly the notation, and the well-posedness of the system is established. In Sect. 3, we show the exponential stability of the corresponding semigroup provided that certain conditions are guaranteed. In Sect. 4, we deal with the analyticity of the semigroup. In Sect. 5, we show some conditions where we have the lack of exponential stability of the semigroup. Finally, throughout this paper, C is a generic constant, not necessarily the same at each occasion (it will change from line to line), which depends in an increasing way on the indicated quantities.

2 The existence of the global solution

In this Section, we use the semigroup approach to show the well-posedness of system (1.2)–(1.4). In the standard $L^2(0,\,L)$ space, the scalar product and norm are denoted by

$$\langle \varphi,\ \psi\rangle_{L^2(0,\,L)}=\int_0^L \varphi\,\overline{\psi}\,dx,\quad \|\varphi\|^2_{L^2(0,\,L)}=\int_0^L |\varphi|^2\,dx.$$

We have the Poincaré inequality

$$\|\varphi\|^2_{L^2(0,\,L)}\le C_P\,\|\varphi_x\|^2_{L^2(0,\,L)}\quad \forall\,\varphi\in\ H^1_0(0,\ L)$$

where C_P is the Poincaré constant.

We introduce the face space $\mathcal{H}=H^1_0(0,\ L)\times H^1_0(0,\ L)\times L^2(0,\ L)\times L^2(0,\ L)\times L^2(0,\ L)$ equipped with the inner product given by

$$\langle (u_1,\ w_1,\ v_1,\ \eta_1,\ \theta_1),\ (u_2,\ w_2,\ v_2,\ \eta_2,\ \theta_2)\rangle_{\mathcal{H}}=\int_0^L (a_{11}\,u_{1x}\,\overline{u}_{2x}+a_{22}\,w_{1x}\,\overline{w}_{2x})\,dx+c\int_0^L \theta_1\,\overline{\theta}_2\,dx$$
$$+\int_0^L a_{12}\,(u_{1x}\,\overline{w}_{2x}+w_{1x}\,\overline{u}_{2x})\,dx+\alpha\int_0^L (u_1-w_1)\,(\overline{u_2-w_2})\,dx+\rho_1\int_0^L v_1\,\overline{v}_2\,dx+\rho_2\int_0^L \eta_1\,\overline{\eta}_2\,dx.$$

and the norm induced $\|\cdot\|_{\mathcal{H}}$. We can easily show that the norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to the usual norm in $\mathcal{H}$.

We also consider the linear operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$

$$\mathcal{A}\begin{pmatrix} u\\ w\\ v\\ \eta\\ \theta\end{pmatrix}=\begin{pmatrix} v\\ \eta\\ \frac{1}{\rho_1}\,(a_{11}\,u+a_{12}\,w+b_{11}\,v+b_{12}\,\eta+k_1\,\theta)_{xx}-\frac{\alpha}{\rho_1}\,(u-w)-\frac{\alpha_1}{\rho_2}\,(v-\eta)+\frac{\beta_1}{\rho_1}\,\theta\\ \frac{1}{\rho_2}\,(a_{12}\,u+a_{22}\,w+b_{12}\,v+b_{22}\,\eta+k_2\,\theta)_{xx}+\frac{\alpha}{\rho_1}\,(u-w)+\frac{\alpha_1}{\rho_2}\,(v-\eta)+\frac{\beta_2}{\rho_2}\,\theta\\ \frac{1}{c}\,(\kappa\,\theta-k_1\,v-k_2\,\eta)_{xx}-\frac{\beta_1}{c}\,v-\frac{\beta_2}{c}\,\eta\end{pmatrix}$$

whose domain $\mathcal{D}(\mathcal{A})$ is the subspace of $\mathcal{H}$ consisting of vectors $(u,\ v,\ w,\ \eta,\ \theta)$ such that

$$\begin{aligned}
&v,\ \eta,\ \theta\in H^1_0(0,\ L),\\
&\kappa\,\theta-k_1\,v-k_2\,\eta\in H^2(0,\ L),\\
&a_{11}\,u+a_{12}\,w+b_{11}\,v+b_{12}\,\eta+k_1\,\theta\in H^2(0,\ L),\\
&a_{12}\,u+a_{22}\,w+b_{12}\,v+b_{22}\,\eta+k_2\,\theta\in H^2(0,\ L).
\end{aligned}$$

Taking $u_t = v$ and $w_t = \eta$, (1.2)–(1.4) can be reduced to the following abstract initial value problem for a first-order evolution equation:

$$\frac{d}{dt}U(t) = \mathcal{A}U(t), \quad U(0) = U_0, \quad \forall\, t > 0$$

with $U(t) = (u,\, w,\, u_t,\, w_t,\, \theta)^T$ and $U_0 = (u_0,\, w_0,\, u_1,\, w_1,\, \theta_0)^T$.

First, we show that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions on the space $\mathcal{H}$.

Proposition 2.1 *The operator $\mathcal{A}$ generates a C_0-semigroup $S_{\mathcal{A}}(t)$ of contractions on the space $\mathcal{H}$.*

Proof We will show that $\mathcal{A}$ is a dissipative operator and 0 belongs to the resolvent set of $\mathcal{A}$, denoted by $\rho(\mathcal{A})$. Then, our conclusion will follow using the well-known Lumer–Phillips theorem (see [11]). We observe that if $U = (u,\, w,\, v,\, \eta,\, \theta) \in \mathcal{D}(\mathcal{A})$, then

$$\begin{aligned}
\langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} = {}& a_{11}\int_0^L v_x\,\overline{u}_x\,dx + a_{12}\int_0^L v_x\,\overline{w}_x\,dx + \alpha\int_0^L v\,\overline{u}\,dx - \alpha\int_0^L v\,\overline{w}\,dx \\
&+ a_{12}\int_0^L \eta_x\,\overline{u}_x\,dx + a_{22}\int_0^L \eta_x\,\overline{w}_x\,dx - \alpha\int_0^L \eta\,\overline{u}\,dx + \alpha\int_0^L \eta\,\overline{w}\,dx \\
&- \int_0^L (a_{11}\,u + a_{12}\,w + b_{11}\,v + b_{12}\,\eta + k_1\,\theta)_x\,\overline{v}_x\,dx - \alpha\int_0^L (u - w)\,\overline{v}\,dx \\
&- \alpha_1\int_0^L (v - \eta)\,\overline{v}\,dx + \beta_1\int_0^L \theta\,\overline{v}\,dx + \alpha\int_0^L (u - w)\,\overline{\eta}\,dx + \alpha_1\int_0^L (v - \eta)\,\overline{\eta}\,dx \\
&- \int_0^L (a_{12}\,u + a_{22}\,w + b_{12}\,v + b_{22}\,\eta + k_2\,\theta)_x\,\overline{\eta}_x\,dx + \beta_2\int_0^L \theta\,\overline{\eta}\,dx \\
&- \beta_1\int_0^L v\,\overline{\theta}\,dx - \beta_2\int_0^L \eta\,\overline{\theta}\,dx - \int_0^L (\kappa\,\theta - k_1\,v - k_2\,\eta)_x\,\overline{\theta}_x\,dx.
\end{aligned}$$

Taking the real part of the above equality, we obtain

$$\begin{aligned}
Re\,\langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} = {}& -b_{11}\,\|v_x\|^2_{L^2(0,\,L)} - b_{22}\,\|\eta_x\|^2_{L^2(0,\,L)} \\
&-2\,b_{12}\,Re\int_0^L v_x\,\overline{\eta}_x\,dx - \alpha_1\,\|v - \eta\|^2_{L^2(0,\,L)} - \kappa\,\|\theta_x\|^2_{L^2(0,\,L)}.
\end{aligned}$$

Case I The matrix B is positive definite. In this case, we get

$$\begin{aligned}
Re\,\langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} \le {}& -\kappa\,\|\theta_x\|^2_{L^2(0,\,L)} - \frac{\det B}{2\,b_{22}}\,\|v_x\|^2_{L^2(0,\,L)} - \frac{\det B}{2\,b_{11}}\,\|\eta_x\|^2_{L^2(0,\,L)} \\
&- \alpha_1\,\|v - \eta\|^2_{L^2(0,\,L)} \le 0.
\end{aligned} \tag{2.1}$$

Case II The matrix B is non-negative definite.

(a) $b_{11} > 0$ implies $b_{22} = b_{12}^2/b_{11}$. Then,

$$\begin{aligned}
Re\,\langle \mathcal{A}U,\, U\rangle_{\mathcal{H}} = {}& -\kappa\,\|\theta_x\|^2_{L^2(0,\,L)} - \frac{1}{b_{11}}\,\|b_{11}\,v_x + b_{12}\,\eta_x\|^2_{L^2(0,\,L)} \\
&- \alpha_1\,\|v - \eta\|^2_{L^2(0,\,L)} \le 0.
\end{aligned} \tag{2.2}$$

(b) $b_{11} = 0$ implies $b_{12} = 0$. Then,

$$Re\,\langle \mathcal{A}U,\ U\rangle_{\mathcal{H}} = -\kappa\, \|\theta_x\|^2_{L^2(0,\,L)} - b_{22}\, \|\eta_x\|^2_{L^2(0,\,L)} - \alpha_1\, \|v-\eta\|^2_{L^2(0,\,L)} \le 0. \tag{2.3}$$

Therefore, it results in the three cases that $\mathcal{A}$ is dissipative.

On the other hand, we have that $0 \in \rho(\mathcal{A})$. In fact, given $F = (f,\ g,\ h,\ p,\ q) \in \mathcal{H}$, we must show that there exists a unique $U = (u,\ w,\ v,\ \eta,\ \theta)$ in $\mathcal{D}(\mathcal{A})$ such that

$$\begin{aligned}
&v = f, \quad \eta = g \quad \text{in} \quad H^1_0(0,\ L),\\
&(a_{11}\, u + a_{12}\, w + b_{11}\, v + b_{12}\, \eta + k_1\, \theta)_{xx} - \alpha\,(u-w) - \alpha_1\,(v-\eta) + \beta_1\,\theta = \rho_1\, h \quad \text{in } L^2(0,\ L),\\
&(a_{12}\, u + a_{22}\, w + b_{12}\, v + b_{22}\, \eta + k_2\, \theta)_{xx} + \alpha\,(u-w) + \alpha_1\,(v-\eta) + \beta_2\,\theta = \rho_2\, p \quad \text{in } L^2(0,\ L),\\
&(\kappa\,\theta - k_1\, v - k_2\eta)_{xx} - \beta_1\, v - \beta_2\,\eta = c\, q \quad \text{in } L^2(0,\ L).
\end{aligned}$$

Consider $v = f$ and $\eta = g$. It is known that there is a unique $\theta \in H^1_0(0,\ L)$ satisfying

$$(\kappa\,\theta - k_1\, v - k_2\,\eta)_{xx} = c\, q + \beta_1\, f + \beta_2\, g \in L^2(0,\ L).$$

It follows using the Lax–Milgram theorem that there exists a unique vector function $(u,\ w) \in H^1_0(0,\ L) \times H^1_0(0,\ L)$ such that

$$\begin{aligned}
&(a_{11}\, u + a_{12}\, w + b_{11}\, v + b_{12}\, \eta + k_1\, \theta)_{xx} - \alpha\,(u-w) = \rho_1\, h + \alpha_1\,(v-\eta) - \beta_1\,\theta \in L^2(0,\ L).\\
&(a_{12}\, u + a_{22}\, w + b_{12}\, v + b_{22}\, \eta + k_2\, \theta)_{xx} + \alpha\,(u-w) = \rho_2\, p - \alpha_1\,(v-\eta) - \beta_2\,\theta \in L^2(0,\ L).
\end{aligned}$$

Moreover, it is easy to show that $\|U\|_{\mathcal{H}} \le C\, \|F\|_{\mathcal{H}}$ for a positive constant C. Therefore, we conclude that $0 \in \rho\,(\mathcal{A})$.

Theorem 2.2 *For any $U_0 \in \mathcal{H}$, there exists a unique solution $U(t) = (u,\ w,\ u_t,\ w_t,\ \theta)$ of (1.2)–(1.4) satisfying*

$$\begin{aligned}
u,\ w &\in C([0,\ \infty[:\, H^1_0(0,\ L)) \cap C^1([0,\ \infty[:\, L^2(0,\ L)),\\
\theta &\in C([0,\ \infty[:\, L^2(0,\ L)) \cap L^2(]0,\ \infty[:\, H^1_0(0,\ L)).
\end{aligned}$$

If $U_0 \in \mathcal{D}(\mathcal{A})$, then

$$\begin{aligned}
&u,\ w \in C^1([0,\ \infty[:\ H^1_0(0,\ L)) \cap C^2([0,\ \infty[:\ L^2(0,\ L)),\\
&a_{11}\, u + a_{12}\, w + b_{11}\, u_t + b_{12}\, w_t + k_1\, \theta \in C([0,\ \infty[:\ H^2(0,\ L)),\\
&a_{12}\, u + a_{22}\, w + b_{12}\, u_t + b_{22}\, w_t + k_2\, \theta \in C([0,\ \infty[:\ H^2(0,\ L)),\\
&\theta \in C([0,\ \infty[:\, H^1_0(0,\ L)) \cap C^1([0,\ \infty[:\ L^2(0,\ L)).\\
&\kappa\,\theta - k_1\, u_t - k_2\, w_t \in C([0,\ \infty[:\ H^2(0,\ L)).
\end{aligned}$$

3 Exponential stability

Our main tool is the following theorem established in Gearhart [4] (see also Huang [5] and Prüss [12]).

Theorem 3.1 *Let $S(t)$ be a C_0-semigroup of contractions of linear operators on the Hilbert space $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. Then, $S(t)$ is exponentially stable if and only if*

(i) $i\,\mathbb{R} \subset \rho(\mathcal{A})$;
(ii) $\limsup_{|\lambda|\to\infty} \left\|(i\,\lambda\, I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty$

where $\mathcal{L}(\mathcal{H})$ denotes the space of continuous linear functions in $\mathcal{H}$.

Our starting point to show the exponential stability is the following Lemma:

Lemma 3.2 *Suppose that only one of the items holds*

(a) *B is positive definite;*

(b) *B is non-negative definite and*
  (b.1) $b_{12} \neq -b_{11}$ *or* $b_{12} \neq -b_{22}$;
  (b.2) $b_{12} = -b_{11} = -b_{22}$, $\beta_1 = -\beta_2$ *and* $k_1 \neq -k_2$;
  (b.3) $b_{12} = -b_{11} = -b_{22}$, $\rho_2\,(a_{11} + a_{12}) \neq \rho_1\,(a_{12} + a_{22})$;
  (b.4) $b_{12} = -b_{11} = -b_{22}$, $\beta_1 = \varrho\,k_1$, $\beta_2 = \varrho\,k_2$, $\varrho \neq 0$ *and* $\varrho < \frac{1}{C_P}$, *and* $k_1 \neq -k_2$.

Then, $i\,\mathbb{R} \subset \rho\,(\mathcal{A})$.

Proof We show this result by a contradiction argument. Following the arguments given in Liu and Zheng [9], the proof consists of the following steps:

Step 1 Since $0 \in \rho(\mathcal{A})$, for any real number λ with $\|\lambda\,\mathcal{A}^{-1}\| < 1$, the linear bounded operator $(i\,\lambda\,\mathcal{A}^{-1} - I)$ is invertible; therefore, $i\,\lambda\,I - \mathcal{A} = \mathcal{A}\,(i\,\lambda\,\mathcal{A}^{-1} - I)$ is invertible and its inverse belongs to $\mathcal{L}(\mathcal{H})$, that is, $i\,\lambda \in \rho(\mathcal{A})$. Moreover, $\|(i\,\lambda\,I - \mathcal{A})^{-1}\|$ is a continuous function of λ in the interval $\left(-\,\|\mathcal{A}^{-1}\|^{-1},\ \|\mathcal{A}^{-1}\|^{-1}\right)$.

Step 2 If $\sup\left\{\|(i\,\lambda\,I - \mathcal{A})^{-1}\| :\ |\lambda| < \|\mathcal{A}^{-1}\|^{-1}\right\} = M < \infty$, then for $|\lambda_0| < \|\mathcal{A}^{-1}\|^{-1}$ and $\lambda \in \mathbb{R}$ such that $|\lambda - \lambda_0| < M^{-1}$, we have $\|(\lambda - \lambda_0)\,(i\,\lambda_0\,I - \mathcal{A})^{-1}\| < 1$. Therefore, the operator

$$i\,\lambda\,I - \mathcal{A} = (i\,\lambda_0\,I - \mathcal{A})\,(I + i\,(\lambda - \lambda_0)\,(i\,\lambda_0\,I - \mathcal{A})^{-1})$$

is invertible with its inverse in $\mathcal{L}(\mathcal{H})$, that is, $i\,\lambda \in \rho(\mathcal{A})$. Since λ_0 is arbitrary, we can conclude that $\left\{i\,\lambda :\ |\lambda| < \|\mathcal{A}^{-1}\|^{-1} + M^{-1}\right\} \subset \rho(\mathcal{A})$ and the function $\|(i\,\lambda\,I - \mathcal{A})^{-1}\|$ is continuous in the interval $\left(-\,\|\mathcal{A}^{-1}\|^{-1} - M^{-1},\ \|\mathcal{A}^{-1}\|^{-1} + M^{-1}\right)$.

Step 3 Thus, it follows by item (ii) that if $i\,\mathbb{R} \subset \rho(A)$ is not true, then there exists $\omega \in \mathbb{R}$ with $\|\mathcal{A}^{-1}\|^{-1} \leq |\omega|$ such that $\{i\,\lambda :\ |\lambda| < |\omega|\} \subset \rho(\mathcal{A})$ and $\sup\left\{\|(i\,\lambda\,I - \mathcal{A})^{-1}\| :\ |\lambda| < |\omega|\right\} = \infty$. Therefore, there exists a sequence of real numbers $(\lambda_\nu)_{\nu\in\mathbb{N}}$ with $\lambda_\nu \to \omega$ as $\nu \to \infty$ and $|\lambda_\nu| < |\omega|$, for all $\nu \in \mathbb{N}$, and sequences of vector functions $U_\nu = (u_\nu,\ w_\nu,\ v_\nu,\ \eta_\nu,\ \theta_\nu) \in \mathcal{D}(\mathcal{A})$, $F_\nu = (f_\nu,\ g_\nu,\ h_\nu,\ p_\nu,\ q_\nu) \in \mathcal{H}$, such that $(i\,\lambda_\nu\,I - \mathcal{A})U_\nu = F_\nu$ and $\|U_\nu\|_{\mathcal{H}} = 1$, for all $\nu \in \mathbb{N}$, and $F_\nu \to 0$ in $\mathcal{H}$ when $\nu \to \infty$, that is,

$$\begin{aligned} &i\,\lambda_\nu\,u_\nu - v_\nu = f_\nu \longrightarrow 0 \quad \text{in } H_0^1(0,\ L),\\ &i\,\lambda_\nu\,w_\nu - \eta_\nu = g_\nu \longrightarrow 0 \quad \text{in } H_0^1(0,\ L), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &i\,\lambda_\nu\,\rho_1\,v_\nu - (a_{11}\,u_\nu + a_{12}\,w_\nu + b_{11}\,v_\nu + b_{12}\,\eta_\nu + k_1\,\theta_\nu)_{xx} + \alpha\,(u_\nu - w_\nu)\\ &\quad + \alpha_1\,(v_\nu - \eta_\nu) - \beta_1\,\theta_\nu = \rho_1\,h_\nu \longrightarrow 0 \quad \text{in } L^2(0,\ L), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &i\,\lambda_\nu\,\rho_2\,\eta_\nu - (a_{12}\,u_\nu + a_{22}\,w_\nu + b_{12}\,v_\nu + b_{22}\,\eta_\nu + k_2\,\theta_\nu)_{xx} - \alpha\,(u_\nu - w_\nu)\\ &\quad - \alpha_1\,(v_\nu - \eta_\nu) - \beta_2\,\theta_\nu = \rho_2\,p_\nu \longrightarrow 0 \quad \text{in } L^2(0,\ L), \end{aligned} \tag{3.3}$$

$$i\,\lambda_\nu\,c\,\theta_\nu + \beta_1\,v_\nu + \beta_2\,\eta_\nu - (\kappa\,\theta_\nu - k_1\,v_\nu - k_2\,\eta_\nu)_{xx} = c\,q_\nu \longrightarrow 0 \quad \text{in } L^2(0,\ L). \tag{3.4}$$

We observe that $Re\ \langle(i\,\lambda_\nu\,I - \mathcal{A})U_\nu,\ U_\nu\rangle_{\mathcal{H}} \to 0$ as $\nu \to \infty$.

(a) If $\det B > 0$, it follows from (2.1) that

$$\|\theta_{\nu x}\|^2_{L^2(0,\,L)} + \|v_{\nu x}\|^2_{L^2(0,\,L)} + \|\eta_{\nu x}\|^2_{L^2(0,\,L)} \longrightarrow 0 \quad \text{as } \nu \to \infty.$$

From (3.1) we obtain

$$\|u_{\nu x}\|^2_{L^2(0,\,L)} + \|w_{\nu x}\|^2_{L^2(0,\,L)} \longrightarrow 0 \quad \text{as } \nu \longrightarrow \infty.$$

Thus, $\lim\limits_{\nu\to\infty} \|U_\nu\|_{\mathcal{H}} = 0$ and we have a contradiction.

(b)–(b.1) If $\det B = 0$ and $b_{11} > 0$, then it follows from (2.2) that

$$\|\theta_{\nu x}\|^2_{L^2(0,\,L)} + \|v_\nu - \eta_\nu\|^2_{L^2(0,\,L)} + \|(b_{11}\,v_\nu + b_{12}\,\eta_\nu)_x\|^2_{L^2(0,\,L)} \longrightarrow 0 \quad \text{as } \nu \to \infty. \tag{3.5}$$

Thus, $\theta_\nu \to 0$ and $b_{11}\,v_\nu + b_{12}\,\eta_\nu \to 0$ in $H_0^1(0,\ L)$. Since $(u_\nu)_{\nu\in\mathbb{N}}$ and $(w_\nu)_{\nu\in\mathbb{N}}$ are bounded sequences in $H_0^1(0,\ L)$, there exist subsequences, still denoted by $(u_\nu)_{\nu\in\mathbb{N}}$, $(w_\nu)_{\nu\in\mathbb{N}}$, such that $u_\nu \to u$ and $w_\nu \to w$

in $L^2(0,\ L)$. From (3.1), we have that $v_\nu \to v$, $\eta_\nu \to \eta$, $v_\nu - \eta_\nu \to v - \eta$, $b_{11}\, v_\nu + b_{12}\, \eta_\nu \to b_{11}\, v + b_{12}\, \eta$ in $L^2(0,\ L)$. It results from (3.5) that

$$v = \eta \quad \text{and} \quad b_{11}\, v + b_{12}\, \eta = 0. \tag{3.6}$$

Since $b_{12} \neq -b_{11}$, we have that $v = \eta = 0$. It follows from (3.1) that $u = w = 0$. On the other hand, we can conclude from (3.2), (3.3) that

$$(a_{11}\, u_\nu + a_{12}\, w_\nu + b_{11}\, v_\nu + b_{12}\, w_\nu)_{\nu\in\mathbb{N}} \quad \text{and} \quad (a_{12}\, u_\nu + a_{22}\, w_\nu + b_{12}\, v_\nu + b_{22}\, w_\nu)_{\nu\in\mathbb{N}} \tag{3.7}$$

converge to zero in $H_0^1(0,\ L)$. Therefore, from (3.5), we obtain that

$$a_{11}\, u_\nu + a_{12}\, w_\nu \to 0 \quad \text{and} \quad a_{12}\, u_\nu + a_{22}\, w_\nu \to 0 \quad \text{in } H_0^1(0,\ L).$$

Since (a_{ij}) is positive definite, it follows that $u_\nu \to 0$, $w_\nu \to 0$ in $H_0^1(0,\ L)$. Thus, we have a contradiction and the result follows. If $b_{11} = 0$ then $b_{22} > 0$, and we can use similar manipulations.

(b)–(b.2) Suppose that $b_{12} = -\,b_{11} = -\,b_{22}$ and $\beta_1 = -\,\beta_2$. Multiplying (3.4) by $\kappa_1\, v_\nu + \kappa_2\, \eta_\nu$, we have

$$\begin{aligned}&\langle i\, \lambda_\nu\, c\, \theta_\nu,\ k_1\, v_\nu + k_2\, \eta_\nu\rangle_{L^2(0,\,L)} + \langle \beta_1\, v_\nu + \beta_2\, \eta_\nu,\, k_1\, v_\nu + k_2\, \eta_\nu\rangle_{L^2(0,\,L)}\\ &\quad + \kappa\, \langle \theta_{\nu x},\ (k_1\, v_\nu + k_2\, \eta_\nu)_x\rangle_{L^2(0,\,L)} - \|(k_1\, v_\nu + k_2\, \eta_\nu)_x\|^2_{L^2(0,\,L)} = \langle c\, q_\nu,\ k_1\, v_\nu + k_2\, \eta_\nu\rangle_{L^2(0,\,L)}.\end{aligned}$$

Using the Cauchy–Schwartz inequality

$$\begin{aligned}\frac{1}{2}\, \|(k_1\, v_\nu + k_2\, \eta_\nu)_x\|^2_{L^2(0,\,L)} \leq\ & \|c\, q_\nu\|_{L^2(0,\,L)} \|k_1\, v_\nu\\ &+ k_2\, \eta_\nu\|_{L^2(0,\,L)} + |\lambda_\nu| \|c\, \theta_\nu\|_{L^2(0,\,L)} \|k_1\, v_\nu + k_2\, \eta_\nu\|_{L^2(0,\,L)}\\ &+ |\beta_1|\, \|v_\nu - \eta_\nu\|_{L^2(0,\,L)} \|k_1\, v_\nu + k_2\, \eta_\nu\|_{L^2(0,\,L)} + \kappa^2\, \|\theta_{\nu x}\|^2_{L^2(0,\,L)},\end{aligned}$$

since the sequence $(k_1\, v_\nu + k_2\, \eta_\nu)_{\nu\in\mathbb{N}}$ is bounded in $L^2(0,\ L)$, and using (3.5), we conclude that

$$k_1\, v_\nu + k_2\, \eta_\nu \longrightarrow 0 \quad \text{in } H_0^1(0,\ L).$$

Hence, $k_1\, v + k_2\, \eta = 0$. Since $k_1 \neq -k_2$, we have from (3.6) that $v = \eta = 0$. We can use similar arguments to show that $u = w = 0$ and $\lim_{\nu\to\infty} \|U_\nu\|_{\mathcal{H}} = 0$.

(b)–(b.3) Similarly, we get that the sequences given in (3.7) converge in $H_0^1(0,\ L)$. Using again (3.5), we obtain that $(a_{11}\, u_\nu + a_{12}\, w_\nu)_{\nu\in\mathbb{N}}$ and $(a_{12}\, u_\nu + a_{22}\, w_\nu)_{\nu\in\mathbb{N}}$ converge in $H_0^1(0,\ L)$. We can conclude that $u_\nu \to u$, $w_\nu \to w$, $v_\nu \to v$ and $\eta_\nu \to \eta$ in $H_0^1(0,\ L)$. From (3.6), we have $v = \eta$, and from (3.1) $u = w$. Therefore, from (3.2), (3.3) we have

$$-\,\omega^2\, u - \frac{a_{11} + a_{12}}{\rho_1}\, u_{xx} = 0 \quad \text{and} \quad -\,\omega^2\, u - \frac{a_{12} + a_{22}}{\rho_2}\, u_{xx} = 0.$$

Since $(a_{11} + a_{12})/\rho_1 \neq (a_{12} + a_{22})/\rho_2$, we conclude that $u = 0$ and hence $w = v = \eta = 0$. The result follows.

(b)–(b.4). From (3.4), we have

$$\begin{aligned}&\langle i\, \lambda_\nu\, c\, \theta_\nu,\ k_1\, v_\nu + k_2\, \eta_\nu\rangle_{L^2(0,\,L)} + \varrho\, \|k_1\, v_\nu + k_2\, \eta_\nu\|^2_{L^2(0,\,L)} + \kappa\, \langle \theta_{\nu x},\ (k_1\, v_\nu + k_2\, \eta_\nu)_x\rangle_{L^2(0,\,L)}\\ &\quad - \|(k_1\, v_\nu + k_2\, \eta_\nu)_x\|^2_{L^2(0,\,L)} = \langle c\, q_\nu,\, k_1\, v_\nu + k_2\, \eta_\nu\rangle_{L^2(0,\,L)}.\end{aligned}$$

If $\varrho < 0$, then

$$\begin{aligned}&(-\,\varrho)\, \|k_1\, v_\nu + k_2\, \eta_\nu\|^2_{L^2(0,\,L)} + \frac{1}{2}\, \|(k_1\, v_\nu + k_2\, \eta_\nu)_x\|^2_{L^2(0,\,L)}\\ &\quad \leq \|c\, q_\nu\|_{L^2(0,\,L)} \|k_1\, v_\nu + k_2\, \eta_\nu\|_{L^2(0,\,L)} + |\lambda_\nu|\, \|c\, \theta_\nu\|_{L^2(0,\,L)}\, \|k_1\, v_\nu\\ &\qquad + k_2\, \eta_\nu\|_{L^2(0,\,L)} + \kappa^2\, \|\theta_{\nu x}\|^2_{L^2(0,\,L)}.\end{aligned}$$

If $0 < \varrho < \frac{1}{C_P}$, then

$$(-\varrho\, C_P + 1)\, \|(k_1\, v_\nu + k_2\, \eta_\nu)_x\|^2_{L^2(0,\,L)} \leq \|c\, q_\nu\|_{L^2(0,\,L)} \|k_1\, v_\nu + k_2\, \eta_\nu\|_{L^2(0,\,L)} + |\lambda_\nu|\, \|c\, \theta_\nu\|_{L^2(0,\,L)}\, \|k_1\, v_\nu + k_2\, \eta_\nu\|_{L^2(0,\,L)} + \kappa\, \|\theta_{\nu x}\, \|_{L^2(0,\,L)}\, \|(k_1\, v_\nu + k_2\, \eta_\nu)_x\|_{L^2(0,\,L)}.$$

Therefore, $k_1\, v_\nu + k_2\, \eta_\nu \to 0$ in $H_0^1(0,\,L)$ and $k_1\, v + k_2\, \eta = 0$. Since $v = \eta$ and $k_1 \neq -k_2$, we have that $v = \eta = 0$. We can use similar arguments to show that $\lim_{\nu\to\infty} \|U_\nu\|_{\mathcal{H}} = 0$. In the next theorem, we will consider only the case in which the matrix B is non-negative definite. However, in the next Section, we will prove that when B is positive definite, the semigroup $S_{\mathcal{A}}(t)$ is analytic and therefore it is exponentially stable.

Theorem 3.3 *Suppose that B is non-negative definite and at least one of the following items occurs:*

(a) $b_{12} \neq -b_{11}$ *or* $b_{12} \neq -b_{22}$;
(b) $b_{12} = -b_{11} = -b_{22}$
  (b.1) $\beta_1 = -\beta_2$ *and* $k_1 \neq -k_2$;
  (b.2) $(\beta_1,\, \beta_2) = \varrho\,(k_1,\, k_2)$, $\varrho \neq 0$ *and* $\varrho < \frac{1}{C_P}$, *and* $k_1 \neq -k_2$;
  (b.3) $\rho_2\,(a_{11} + a_{12}) \neq \rho_1\,(a_{12} + a_{22})$.

Then, $S_{\mathcal{A}}(t)$ is exponentially stable, that is, there exist two positive constants $M > 1$ and μ such that

$$\|S_{\mathcal{A}}(t)\|_{\mathcal{L}(\mathcal{H})} \leq M\, e^{-\mu\, t} \quad \text{for every } t \geq 0.$$

Proof From Theorem 3.1 and Lemma 3.2, it is sufficient to prove that (ii) is true. Given $\lambda \in \mathbb{R}$ and $F = (f,\, g,\, h,\, p,\, q) \in \mathcal{H}$, let $U = (u,\, w,\, v,\, \eta,\, \theta) \in \mathcal{D}(\mathcal{A})$ be the solution of $(i\,\lambda\, I - \mathcal{A})U = F$, that is,

$$i\,\lambda\, u - v = f \quad \text{in} \quad H_0^1(0,\, L), \tag{3.8}$$

$$i\,\lambda\, w - \eta = g \quad \text{in} \quad H_0^1(0,\, L), \tag{3.9}$$

$$i\,\lambda\, \rho_1\, v - (a_{11}\, u + a_{12}\, w + b_{11}\, v + b_{12}\, \eta + k_1\theta)_{xx} + \alpha\,(u - w) + \alpha_1\,(v - \eta) - \beta_1\,\theta = \rho_1\, h \quad \text{in } L^2(0,\, L), \tag{3.10}$$

$$i\,\lambda\, \rho_2\, \eta - (a_{12}\, u + a_{22}\, w + b_{12}\, v + b_{22}\, \eta + k_2\,\theta)_{xx} - \alpha\,(u - w) - \alpha_1\,(v - \eta) - \beta_2\,\theta = \rho_2\, p \quad \text{in } L^2(0,\, L), \tag{3.11}$$

$$i\,\lambda\, c\,\theta + \beta_1\, v + \beta_2\, \eta - (\kappa\theta - k_1\, v - k_2\, \eta)_{xx} = c\, q \quad \text{in } L^2(0,\, L). \tag{3.12}$$

If $\det B = 0$ and $b_{11} > 0$ (the other case is similarly analyzed), it follows from (2.2) that

$$Re\ \langle (i\,\lambda\, I - \mathcal{A})U,\, U\rangle_{\mathcal{H}} \;=\; \kappa\, \|\theta_x\|^2_{L^2(0,\,L)} + \frac{1}{b_{11}}\, \|b_{11}\, v_x + b_{12}\, \eta_x\|^2_{L^2(0,\,L)} + \alpha_1\, \|v - \eta\|^2_{L^2(0,\,L)}. \tag{3.13}$$

On the other hand, taking the real part of the $\mathcal{H}$-inner product between $(i\,\lambda\, I - \mathcal{A})U = F$ and U, we have

$$Re\ \langle (i\,\lambda\, I - \mathcal{A})U,\, U\rangle_{\mathcal{H}} = \ Re\langle F,\, U\rangle_{\mathcal{H}}. \tag{3.14}$$

Replacing (3.13) into the left-hand side of (3.14), then it follows that there exists a positive constant such that

$$\|\theta_x\|^2_{L^2(0,\,L)} + \|(b_{11}\, v + b_{12}\, \eta)_x\|^2_{L^2(0,\,L)} + \|v - \eta\|^2_{L^2(0,\,L)} \leq C\, \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \tag{3.15}$$

Taking the inner product of (3.10) with u and (3.11) with w, using (3.8), (3.9), we obtain

$$a_{11}\int_0^L |u_x|^2\,dx + a_{12}\int_0^L w_x\,\overline{u}_x\,dx + \int_0^L (b_{11}\,v + b_{12}\,\eta)_x\,\overline{u}_x\,dx + k_1\int_0^L \theta_x\,\overline{u}_x\,dx$$
$$+\alpha\int_0^L (u-w)\,\overline{u}\,dx + \alpha_1\int_0^L (v-\eta)\,\overline{u}\,dx - \beta_1\int_0^L \theta\,\overline{u}\,dx = \rho_1\int_0^L h\,\overline{u}\,dx + \rho_1\int_0^L v\,(\overline{v+f})\,dx,$$
$$a_{22}\int_0^L |w_x|^2\,dx + a_{12}\int_0^L u_x\,\overline{w}_x\,dx + \int_0^L (b_{12}\,v + b_{22}\,\eta)_x\,\overline{w}_x\,dx + k_2\int_0^L \theta_x\,\overline{w}_x\,dx$$
$$-\alpha\int_0^L (u-w)\,\overline{w}\,dx - \alpha_1\int_0^L (v-\eta)\,\overline{w}\,dx - \beta_2\int_0^L \theta\,\overline{w}\,dx = \rho_2\int_0^L p\,\overline{w}\,dx + \rho_2\int_0^L \eta\,(\overline{\eta+g})\,dx.$$

Adding these equalities, using the Young and Cauchy–Schwartz inequalities and performing straightforward calculations, we obtain

$$\begin{aligned}
&\frac{\det A}{2\,a_{22}}\int_0^L |u_x|^2\,dx + \frac{\det A}{2\,a_{11}}\int_0^L |w_x|^2\,dx\\
&\le \rho_1\int_0^L |v|^2\,dx + \rho_2\int_0^L |\eta|^2\,dx + \alpha_1\int_0^L |v-\eta|\,|u-w|\,dx\\
&\quad + C\left(\int_0^L |\theta_x|^2\,dx\right)^{1/2}\left[\left(\int_0^L |u_x|^2\,dx\right)^{1/2} + \left(\int_0^L |w_x|^2\,dx\right)^{1/2}\right]\\
&\quad + \int_0^L |(b_{11}\,v + b_{12}\,\eta)_x|\,|u_x|\,dx + \rho_1\int_0^L |v|\,|f|\,dx + \rho_1\int_0^L |h|\,|u|\,dx\\
&\quad + \int_0^L |(b_{12}\,v + b_{22}\,\eta)_x|\,|w_x|\,dx + \rho_2\int_0^L |\eta|\,|g|\,dx + \rho_2\int_0^L |p|\,|w|\,dx.
\end{aligned}\tag{3.16}$$

(a) From (3.15) and using that $b_{12} \neq -b_{11}$, we have

$$\|v\|^2_{L^2(0,\,L)} + \|\eta\|^2_{L^2(0,\,L)} \le C\,\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}}.\tag{3.17}$$

It follows from (3.15), (3.16) and (3.17) that

$$\|u_x\|^2_{L^2(0,\,L)} + \|w_x\|^2_{L^2(0,\,L)} \le C\,\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}}.\tag{3.18}$$

Therefore, from (3.15), (3.17) and (3.18), we conclude that

$$\|(i\,\lambda\,I - \mathcal{A})^{-1}F\|_{\mathcal{H}} \le C\,\|F\|_{\mathcal{H}}\quad \forall\,\lambda \in \mathbb{R}.\tag{3.19}$$

(b)–(b.1) Taking the inner product in $L^2(0,\,L)$ of (3.12) by $k_1\,v + k_2\,\eta$ and using the Gauss Theorem we obtain

$$\int_0^L |(k_1\,v + k_2\,\eta)_x|^2\,dx = i\,\lambda\,c\int_0^L \theta\,(\overline{k_1\,v + k_2\,\eta})\,dx + \kappa\int_0^L \theta_x\,(\overline{k_1\,v + k_2\,\eta})_x\,dx$$
$$+\int_0^L (\beta_1\,v + \beta_2\,\eta)\,(\overline{k_1\,v + k_2\,\eta})\,dx - c\int_0^L q\,(\overline{k_1\,v + k_2\,\eta})\,dx.$$

Using the Young and Cauchy–Schwartz inequalities, there exists a positive constant C such that

$$\int_0^L |(k_1\, v + k_2\, \eta)_x|^2\, dx \le C \left| \int_0^L \theta\, \overline{\lambda\, i\, (k_1\, v + k_2\, \eta)}\, dx \right| + C \int_0^L |\theta_x|^2\, dx + C \int_0^L |\beta_1\, v + \beta_2\, \eta|^2\, dx + C\, \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}.$$

From (3.15), we obtain that

$$\int_0^L |(k_1\, v + k_2\, \eta)_x|^2\, dx \le C \left| \int_0^L \theta\, \overline{\lambda\, i\, (k_1\, v + k_2\, \eta)}\, dx \right| + C\, \|F\|_{\mathcal{H}}\, \|U\|_{\mathcal{H}}. \tag{3.20}$$

On the other hand, multiplying the Eqs. (3.10) by k_1/ρ_1, (3.11) by k_2/ρ_2 and adding the result, we obtain

$$\begin{aligned} i\, \lambda\, (k_1\, v + k_2\, \eta) = &-\alpha \left(\frac{k_1}{\rho_1} - \frac{k_2}{\rho_2}\right)(u - w) - \alpha_1 \left(\frac{k_1}{\rho_1} - \frac{k_2}{\rho_2}\right)(v - \eta) + \left(\frac{k_1 \beta_1}{\rho_1} + \frac{k_2\, \beta_2}{\rho_2}\right)\theta \\ &+ k_1\, h + k_2\, p + \frac{k_1}{\rho_1}\, (a_{11}\, u + a_{12}\, w + b_{11}\, v + b_{12}\, \eta + k_1\, \theta)_{xx} \\ &+ \frac{k_2}{\rho_2}\, (a_{12}\, u + a_{22}\, w + b_{12}\, v + b_{22}\, \eta + k_2\, \theta)_{xx} \quad \text{in } L^2(0,\ L). \end{aligned} \tag{3.21}$$

Taking the inner product of θ with $i\, \lambda\, (k_1\, v + k_2\, \eta)$ in $L^2(0,\ L)$, using that $b_{11} = b_{22} = -b_{12}$, (3.21) and the Gauss Theorem, it follows that

$$\begin{aligned} \int_0^L \theta\, \overline{i\, \lambda\, (k_1\, v + k_2\, \eta)}\, dx = &- \int_0^L \left[\left(\frac{k_1}{\rho_1}\, a_{11} + \frac{k_2}{\rho_2}\, a_{12}\right) \theta_x\, \overline{u}_x + \left(\frac{k_1}{\rho_1}\, a_{12} + \frac{k_2}{\rho_2}\, a_{22}\right) \theta_x\, \overline{w}_x \right] dx \\ &- b_{11} \left(\frac{k_1}{\rho_1} - \frac{k_2}{\rho_2}\right) \int_0^L \theta_x\, (\overline{v}_x - \overline{\eta}_x)\, dx - \int_0^L \theta\, (\overline{k_1\, h + k_2\, p})\, dx \\ &+ \left(\frac{k_1\, \beta_1}{\rho_1} + \frac{k_2\, \beta_2}{\rho_2}\right) \int_0^L |\theta|^2\, dx - \left(\frac{k_1^2}{\rho_1} + \frac{k_2^2}{\rho_2}\right) \int_0^L |\theta_x|^2\, dx \\ &- \alpha \left(\frac{k_1}{\rho_1} - \frac{k_2}{\rho_2}\right) \int_0^L \theta\, (\overline{u - w})\, dx - \alpha_1 \left(\frac{k_1}{\rho_1} - \frac{k_2}{\rho_2}\right) \int_0^L \theta\, (\overline{v - \eta})\, dx. \end{aligned}$$

Then, from (3.15), we have

$$\left| \int_0^L \theta\, \overline{\lambda\, i\, (k_1\, v + k_2\, \eta)}\, dx \right| \le C\, \|\theta_x\|_{L^2(0,\, L)} \left(\|u_x\|_{L^2(0,\, L)} + \|w_x\|_{L^2(0,\, L)}\right) + C\, \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \tag{3.22}$$

Substituting (3.22) in (3.20) yields

$$\|(k_1\, v + k_2\, \eta)_x\|^2_{L^2(0,\, L)} \le C\, \|\theta_x\|_{L^2(0,\, L)} \left(\|u_x\|_{L^2(0,\, L)} + \|w_x\|_{L^2(0,\, L)}\right) + C\, \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}.$$

Using the Poincaré inequality, we obtain

$$\|k_1\, v + k_2\, \eta\|^2_{L^2(0,\, L)} \le C\, \|\theta_x\|_{L^2(0,\, L)} \left(\|u_x\|_{L^2(0,\, L)} + \|w_x\|_{L^2(0,\, L)}\right) + C\, \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}.$$

Since $k_1 \neq -k_2$ and from (3.15), we conclude that

$$\|v\|^2_{L^2(0,\,L)} + \|\eta\|^2_{L^2(0,\,L)} \leq C\,\|\theta_x\|_{L^2(0,\,L)}\left(\|u_x\|_{L^2(0,\,L)} + \|w_x\|_{L^2(0,\,L)}\right) + C\,\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{3.23}$$

Substituting (3.23) in (3.16) we obtain (3.18). Using (3.18) in (3.23), we get (3.17). Therefore, (3.19) holds.

(b)–(b.2) Taking the inner product in $L^2(0,\ L)$ of (3.12) with $k_1\,v + k_2\,\eta$, using the Gauss Theorem and the fact that $(\beta_1,\ \beta_2) = \varrho\,(k_1,\ k_2)$, we obtain

$$\begin{aligned}(-\varrho)\int_0^L |k_1\,v + k_2\,\eta|^2\,dx + \int_0^L |(k_1\,v + k_2\,\eta)_x|^2\,dx \\ = i\,\lambda\,c\int_0^L \theta\,(\overline{k_1\,v + k_2\,\eta})\,dx + \kappa\int_0^L \theta_x\,(\overline{k_1\,v + k_2\,\eta})_x\,dx - c\int_0^L q\,(\overline{k_1\,v + k_2\,\eta})\,dx.\end{aligned}$$

Therefore if $\varrho < 0$,

$$\begin{aligned}(-\varrho)\int_0^L |k_1\,v + k_2\,\eta|^2\,dx + \frac{1}{2}\int_0^L |(k_1\,v + k_2\,\eta)_x|^2\,dx \\ \leq C\left|\int_0^L \theta\,\overline{\lambda\,i\,(\beta_1\,v + \beta_2\,\eta)}\,dx\right| + C\,\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.\end{aligned}$$

If $0 < \varrho < \frac{1}{C_P}$, then

$$\begin{aligned}(-\varrho\,C_P + 1)\int_0^L |(k_1\,v + k_2\,\eta)_x|^2\,dx \\ \leq C\left|\int_0^L \theta\,\overline{\lambda\,i\,(\beta_1\,v + \beta_2\,\eta)}\,dx\right| + C\,\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.\end{aligned}$$

From (3.22), we conclude that

$$\int_0^L |(k_1\,v + k_2\,\eta)_x|^2\,dx \leq\ C\,\|\theta_x\|_{L^2(0,\,L)}\left(\|u_x\|_{L^2(0,\,L)} + \|w_x\|_{L^2(0,\,L)}\right) + C\,\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}.$$

Since $k_1 \neq -k_2$, we obtain

$$\|v\|^2_{L^2(0,\,L)} + \|\eta\|^2_{L^2(0,\,L)} \leq C\,\|\theta_x\|_{L^2(0,\,L)}\left(\|u_x\|_{L^2(0,\,L)} + \|w_x\|_{L^2(0,\,L)}\right) + C\,\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}.$$

Similarly, we obtain (3.19).

(b)–(b.3) From (3.15) we have

$$\|\theta_x\|^2_{L^2(0,\,L)} + \|v_x - \eta_x\|^2_{L^2(0,\,L)} + \|v - \eta\|^2_{L^2(0,\,L)} \leq C\,\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}, \tag{3.24}$$

for a positive constant C. Hence, from (3.24) we obtain

$$\|u_x - w_x\|^2_{L^2(0,\,L)} \leq C\,\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}} \quad |\lambda| > 1. \tag{3.25}$$

Taking the inner product of

$$i\,\lambda\,(v-\eta)-\left[\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w+\left(\frac{b_{11}}{\rho_1}-\frac{b_{12}}{\rho_2}\right)v+\left(\frac{b_{12}}{\rho_1}-\frac{b_{22}}{\rho_2}\right)\eta\right.$$
$$\left.+\left(\frac{k_1}{\rho_1}-\frac{k_2}{\rho_2}\right)\theta\right]_{xx}+\left(\frac{\alpha}{\rho_1}-\frac{\alpha}{\rho_2}\right)(u-v)+\left(\frac{\alpha_1}{\rho_1}-\frac{\alpha_1}{\rho_2}\right)(v-\eta)-\left(\frac{\beta_1}{\rho_1}-\frac{\beta_2}{\rho_2}\right)\theta=h-p$$

with $\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w$ and using that $i\,\lambda\,(u+w)=(v+\eta)+(f+g)$, we get

$$\int_0^L\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u_x+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w_x\right|^2dx$$
$$\leq\; b_{11}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\int_0^L\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u_x+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w_x\right|\,|\,v_x-\eta_x|\,dx$$
$$+\,\alpha\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\int_0^L|u-w|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w\right|dx$$
$$+\,\alpha_1\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\int_0^L|v-\eta|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w\right|dx$$
$$+\int_0^L|v-\eta|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)v+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)\eta\right|dx$$
$$+\int_0^L|v-\eta|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)f+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)g\right|dx$$
$$+\left|\frac{k_1}{\rho_1}-\frac{k_2}{\rho_2}\right|\int_0^L|\theta_x|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u_x+\left(\frac{a_{12}}{\rho_1}+\frac{a_{22}}{\rho_2}\right)w_x\right|\,dx$$
$$+\left|\frac{\beta_1}{\rho_1}-\frac{\beta_2}{\rho_2}\right|\int_0^L|\theta|\left|\left(\frac{a_{11}}{\rho_1}-\frac{a_{12}}{\rho_2}\right)u+\left(\frac{a_{12}}{\rho_1}+\frac{a_{22}}{\rho_2}\right)w\right|dx$$
$$+\int_0^L|h-p|\left|\left(\frac{a_{11}}{\rho_1}+\frac{a_{12}}{\rho_2}\right)u+\left(\frac{a_{12}}{\rho_1}-\frac{a_{22}}{\rho_2}\right)w\right|dx.$$

Using the Cauchy–Schwartz and Young inequalities, we obtain

$$\|(\rho_2a_{11}-\rho_1a_{12})\,u_x+(\rho_2\,a_{12}-\rho_1\,a_{22})\,w_x\|^2_{L^2(0,\,L)}$$
$$\leq C\left(\|v-\eta\|_{L^2(0,\,L)}\,\|U\|_{\mathcal{H}}+\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}\right)\quad|\lambda|>1,\;\;C>0.$$

By hypothesis, from (3.25) and the last estimate, we have

$$\|u_x\|^2_{L^2(0,\,L)}+\|w_x\|^2_{L^2(0,\,L)}\leq C\left(\|v-\eta\|_{L^2(0,\,L)}\,\|U\|_{\mathcal{H}}+\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}\right)\quad|\lambda|>1,\;\;C>0. \tag{3.26}$$

Now, taking the inner product in $L^2(0,\ L)$ of (3.10) with u, (3.11) with w and using (3.8), (3.9), we obtain

$$\begin{aligned}\int_0^L |v|^2\,dx &= \frac{a_{11}}{\rho_1}\int_0^L |u_x|^2\,dx + \frac{\alpha}{\rho_1}\int_0^L |u|^2\,dx + \frac{a_{12}}{\rho_1}\int_0^L w_x\,\overline{u}_x\,dx - \frac{\alpha}{\rho_1}\int_0^L w\,\overline{u}\,dx \\ &\quad - \frac{\beta_1}{\rho_1}\int_0^L \theta\,\overline{u}\,dx - \int_0^L v\,\overline{f}\,dx - \int_0^L h\,\overline{u}\,dx + \frac{1}{\rho_1}\int_0^L (b_{11}\,v_x + b_{12}\,\eta_x)\overline{u}_x\,dx \\ &\quad + \frac{\alpha_1}{\rho_1}\int_0^L v\,\overline{u}\,dx - \frac{\alpha_1}{\rho_1}\int_0^L \eta\,\overline{u}\,dx + k_1\int_0^L \theta_x\,\overline{u}_x\,dx \end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\int_0^L |\eta|^2\,dx &= \frac{a_{12}}{\rho_2}\int_0^L u_x\,\overline{w}_x\,dx - \frac{\alpha}{\rho_2}\int_0^L u\,\overline{w}\,dx + \frac{a_{22}}{\rho_2}\int_0^L |w_x|^2\,dx + \frac{\alpha}{\rho_2}\int_0^L |w|^2\,dx \\ &\quad - \frac{\beta_2}{\rho_2}\int_0^L \theta\,\overline{w}\,dx - \int_0^L \eta\,\overline{g}\,dx - \int_0^L p\,\overline{w}\,dx + \frac{1}{\rho_2}\int_0^L (b_{12}\,v_x + b_{22}\,\eta_x)\,\overline{w}_x\,dx \\ &\quad - \frac{\alpha_1}{\rho_1}\int_0^L v\,\overline{w}\,dx + \frac{\alpha_1}{\rho_1}\int_0^L \eta\,\overline{w}\,dx + k_2\int_0^L \theta_x\,\overline{u}_x\,dx. \end{aligned} \tag{3.28}$$

Combining (3.24), (3.26), (3.27) and (3.28) yields

$$\|v\|^2_{L^2(0,\,L)} + \|\eta\|^2_{L^2(0,\,L)} \le C\left(\|\,v - \,\eta\|_{L^2(0,\,L)}\,\|U\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}\right) \quad |\lambda| > 1,\ \ C > 0.$$

The last estimate together with (3.16) implies

$$\|u_x\|^2_{L^2(0,\,L)} + \|w_x\|^2_{L^2(0,\,L)} \le C\left(\|\,v - \eta\|_{L^2(0,\,L)}\,\|U\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}\right).$$

Therefore, there exists a positive constant C such that

$$\|U\|^2_{\mathcal{H}} \le C\left(\|\,v - \eta\|_{L^2(0,\,L)}\,\|U\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}\right) \quad |\lambda| > 1.$$

From (3.24)

$$\|(i\,\lambda\,I - A)^{-1}F\|_{\mathcal{H}} \le C\,\|F\|_{\mathcal{H}} \quad |\lambda| > 1.$$

The result follows.

4 Analyticity

In this Section, we will show that the semigroup $S_{\mathcal{A}}(t)$ is analytic. In order to show this, our main tool will be the following theorem whose proof can be found in Liu and Zheng [9].

Theorem 4.1 *Let $S(t)$ be a C_0-semigroup of contractions of linear operators in a Hilbert space $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. Suppose that $i\,\mathbb{R} \subset \rho(\mathcal{A})$. Then, $S(t)$ is analytic if and only if*

$$\limsup_{|\lambda|\to\infty}\ \|\lambda\,(i\,\lambda\,I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{4.1}$$

From Lemma 3.2, we have that the imaginary axis is contained in $\rho(\mathcal{A})$. Thus, in order to prove that the semigroup $S_{\mathcal{A}}(t)$ is analytic it remains to show (4.1). With this aim, in the next theorems of this Section, we will show that there is a positive constant C, independent on λ, such that

$$|\lambda|\,\|(i\,\lambda\, I-\mathcal{A})^{-1}\|_{\mathcal{H}}\le C \quad \forall\,\lambda\in\mathbb{R}.$$

Theorem 4.2 *Suppose that B is positive definite. Then, the semigroup $S_{\mathcal{A}}(t)$ is analytic.*

Proof Given $\lambda\in\mathbb{R}$ and $F=(f,\,g,\,h,\,p,\,q)\in\mathcal{H}$, let $U=(u,\,w,\,v,\,\eta,\,\theta)\in\mathcal{D}(\mathcal{A})$ be the solution of $(i\,\lambda\, I-\mathcal{A})U=F$, that is,

$$i\,\lambda\, u-v=f \quad \text{in } H_0^1(0,\,L), \tag{4.2}$$

$$i\,\lambda\, w-\eta=g \quad \text{in } H_0^1(0,\,L), \tag{4.3}$$

$$\begin{aligned} &i\,\lambda\,\rho_1\, v-(a_{11}\,u+a_{12}\,w+b_{11}\,v+b_{12}\,\eta+k_1\,\theta)_{xx}\\ &\quad+\alpha\,(u-w)+\alpha_1\,(v-\eta)-\beta_1\,\theta=\rho_1\, h \quad \text{in } L^2(0,\,L), \end{aligned} \tag{4.4}$$

$$\begin{aligned} &i\,\lambda\,\rho_2\,\eta-(a_{12}\,u+a_{22}\,w+b_{12}\,v+b_{22}\,\eta+k_2\,\theta)_{xx}\\ &\quad-\alpha\,(u-w)-\alpha_1\,(v-\eta)-\beta_2\,\theta=\rho_2\, p \quad \text{in } L^2(0,\,L), \end{aligned} \tag{4.5}$$

$$i\,\lambda\, c\,\theta+\beta_1\, v+\beta_2\,\eta-(\kappa\theta-k_1\,v-k_2\,\eta)_{xx}=c\,q \quad \text{in } L^2(0,\,L). \tag{4.6}$$

It results from

$$Re\,\langle(i\,\lambda\, I-\mathcal{A})U,\,U\rangle_{\mathcal{H}}=Re\,\langle F,\,U\rangle_{\mathcal{H}} \tag{4.7}$$

and (2.1) that

$$\|\theta_x\|^2_{L^2(0,\,L)}+\|v_x\|^2_{L^2(0,\,L)}+\|\eta_x\|^2_{L^2(0,\,L)}+\|v-\eta\|_{L^2(0,\,L)}\le C\,\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \tag{4.8}$$

for a positive constant C. From (3.16) and Young's inequality, there exists a positive constant C such that

$$\begin{aligned} \|u_x\|^2_{L^2(0,\,L)}+\|w_x\|^2_{L^2(0,\,L)}+\|u-w\|^2_{L^2(0,\,L)}&\le C\,\|(b_{11}\,v+b_{12}\,\eta)_x\|^2_{L^2(0,\,L)}+C\,\|(b_{12}\,v+b_{22}\,\eta)_x\|^2_{L^2(0,\,L)}\\ &\quad+\|v\|^2_{L^2(0,\,L)}+\|\eta\|^2_{L^2(0,\,L)}+C\,\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+C\,\|v-\eta\|^2_{L^2(0,\,L)}. \end{aligned}$$

From (4.8), we get

$$\|U\|_{\mathcal{H}}\le C\,\|F\|_{\mathcal{H}} \tag{4.9}$$

for a positive constant C. On the other hand, since $\mathrm{Im}\langle(i\,\lambda\, I-\mathcal{A})U,\,U\rangle_{\mathcal{H}}=\mathrm{Im}\langle F,\,U\rangle_{\mathcal{H}}$, we have

$$\lambda\,\|U\|^2_{\mathcal{H}}\le|\,\mathrm{Im}\langle\mathcal{A}U,\,U\rangle_{\mathcal{H}}|+\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{4.10}$$

with

$$\begin{aligned} \mathrm{Im}\langle\mathcal{A}U,\,U\rangle_{\mathcal{H}}=\;&2\,i\,a_{11}\,\mathrm{Im}\int_0^L v_x\,\overline{u}_x\,dx+2\,i\,a_{12}\,\mathrm{Im}\int_0^L v_x\,\overline{w}_x\,dx+2\,i\,\alpha\,\mathrm{Im}\int_0^L(v-\eta)\,(\overline{u-w})\,dx\\ &+2\,i\,a_{12}\,\mathrm{Im}\int_0^L\eta_x\,\overline{u}_x\,dx+2\,i\,a_{22}\,\mathrm{Im}\int_0^L\eta_x\,\overline{w}_x\,dx-2\,i\,\beta_1\,\mathrm{Im}\int_0^L\theta\,\overline{v}\,dx\\ &-2\,i\,\beta_2\,\mathrm{Im}\int_0^L\theta\,\overline{\eta}\,dx+2\,i\,k_2\,\mathrm{Im}\int_0^L\theta_x\,\overline{\eta}_x\,dx+2\,i\,k_1\,\mathrm{Im}\int_0^L\theta_x\,\overline{v}_x\,dx. \end{aligned} \tag{4.11}$$

It follows from (4.8), (4.9), (4.10) and (4.11) that there exists a positive constant C such that

$$\lambda\,\|U\|_{\mathcal{H}}\le C\,\|F\|_{\mathcal{H}}\quad\forall\,\lambda\in\mathbb{R}\quad\Longleftrightarrow\quad|\lambda|\,\|(i\,\lambda\, I-A)^{-1}\|_{\mathcal{L}(\mathcal{H})}\le C\quad\forall\,\lambda\in\mathbb{R}. \tag{4.12}$$

□

Theorem 4.3 *Suppose that B is non-negative definite and only one of the below items occurs:*

(a) $(b_{12} \neq -b_{11}$ *or* $b_{12} \neq -b_{22})$ *and* $(b_{11}k_2 \neq b_{12}\, k_1$ *or* $b_{12}\, k_2 \neq b_{22}\, k_1)$;
(b) $b_{12} = -\,b_{11} = -\,b_{22}$
  (b.1) $\beta_1 = -\,\beta_2$ *and* $k_1 \neq -\,k_2$;
  (b.2) $(\beta_1,\, \beta_2) = \varrho\,(k_1,\, k_2),\ \varrho \neq 0$ *and* $\varrho < \frac{1}{C_P},\, k_1 \neq -\,k_2$;
  (b.3) $\rho_2\,(a_{11} + a_{12}) \neq \rho_1\,(a_{12} + a_{22})$ *and* $k_2 \neq -\,k_1$.

Then, the semigroup $S_{\mathcal{A}}(t)$ is analytic.

Proof Given $\lambda \in \mathbb{R}$ and $F = (f,\, g,\, h,\, p,\, q) \in \mathcal{H}$, let $U = (u,\, w,\, v,\, \eta,\, \theta) \in \mathcal{D}(\mathcal{A})$ be the solution of (4.2)–(4.6) and suppose $b_{11} > 0$. It results from (2.2), (4.7) that

$$\|\theta_x\|^2_{L^2(0,\,L)} + \|v - \eta\|^2_{L^2(0,\,L)} + \|(b_{11}\, v + b_{12}\, \eta)_x\|^2_{L^2(0,\,L)} \leq C\, \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}. \tag{4.13}$$

We prove in Theorem 3.3 that there exists a positive constant C such that

$$\|u_x\|^2_{L^2(0,\,L)} + \|w_x\|^2_{L^2(0,\,L)} + \|v\|^2_{L^2(0,\,L)} + \|\eta\|^2_{L^2(0,\,L)} \leq C\, \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \tag{4.14}$$

Taking the inner product of (4.6) with $k_1\, v + k_2\, \eta$, we obtain

$$\int_0^L i\,\lambda\, c\,\theta\, \overline{(k_1\, v + k_2\, \eta)}\, dx + \int_0^L (\beta_1\, v + \beta_2\, \eta)\, \overline{(k_1\, v + k_2\, \eta)}\, dx - \int_0^L (\kappa\,\theta - k_1\, v - k_2\, \eta)_{xx}\, \overline{(k_1\, v + k_2\, \eta)}\, dx$$
$$= c \int_0^L q\, \overline{(k_1\, v + k_2\, \eta)}\, dx.$$

Using the Gauss theorem we obtain

$$\begin{aligned} &-\langle c\,\theta,\, i\,\lambda\,(k_1\, v + k_2\, \eta)\rangle_{L^2(0,\,L)} + \langle \beta_1\, v + \beta_2\, \eta,\ k_1\, v + k_2\, \eta\rangle_{L^2(0,\,L)} \\ &\quad + \kappa\, \langle \theta_x,\ (k_1\, v + k_2\, \eta)_x\rangle_{L^2(0,\,L)} - \|(k_1\, v + k_2\, \eta)_x\|^2_{L^2(0,\,L)} = c\, \langle q,\ k_1\, v + k_2\, \eta\rangle_{L^2(0,\,L)}. \end{aligned}$$

Using the Young and Cauchy–Schwartz inequalities, we get

$$\|(k_1\, v + k_2\, \eta)_x\|^2_{L^2(0,\,L)} \leq C\, \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + C\, \|\beta_1\, v + \beta_2\, \eta\|^2_{L^2(0,\,L)} + C\, |\langle \theta,\ i\, k_1\, \lambda\, v + i\, k_2\, \lambda\, \eta\rangle_{L^2(0,\,L)}|.$$

From (4.14), we have

$$\|(k_1\, v + k_2\, \eta)_x\|^2_{L^2(0,\,L)} \leq C\, \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + C\, |\langle \theta,\ i\, k_1\, \lambda\, v + i\, k_2\, \lambda \eta\rangle_{L^2(0,\,L)}|. \tag{4.15}$$

From (4.4), (4.5) and performing straightforward calculations, we obtain

$$\begin{aligned} \langle \theta,\ i\, k_1\, \lambda\, v + i\, k_2\, \lambda\, \eta\rangle_{L^2(0,\,L)} = &-\frac{k_1 a_{11}}{\rho_1}\, \langle \theta_x,\ u_x\rangle_{L^2(0,\,L)} - \frac{k_1 a_{12}}{\rho_1}\, \langle \theta_x,\ w_x\rangle_{L^2(0,\,L)} \\ &-\frac{k_1}{\rho_1}\, \langle \theta_x,\, (b_{11}\, v + b_{12}\, \eta)_x\rangle_{L^2(0,\,L)} - \frac{k_2\, a_{12}}{\rho_2}\, \langle \theta_x, u_x\rangle - \frac{k_2 a_{22}}{\rho_2}\, \langle \theta_x,\, w_x\rangle_{L^2(0,\,L)} \\ &-\frac{k_2}{\rho_2}\, \langle \theta_x,\ (b_{12}\, v + b_{22}\, \eta)_x\rangle_{L^2(0,\,L)} - \left(\frac{k_1^2}{\rho_1} + \frac{k_2^2}{\rho_2}\right) \|\theta_x\|^2_{L^2(0,\,L)} \\ &-\alpha\left(\frac{k_1}{\rho_1} - \frac{k_2}{\rho_2}\right) \langle \theta,\ u - w\rangle_{L^2(0,\,L)} \\ &-\alpha_1\left(\frac{k_1}{\rho_1} - \frac{k_2}{\rho_2}\right) \langle \theta,\ v - \eta\rangle_{L^2(0,\,L)} + \left(\frac{k_1 \beta_1}{\rho_1} + \frac{k_2\, \beta_2}{\rho_2}\right) \|\theta\|^2_{L^2(0,\,L)} \\ &+ \langle \theta,\ k_1\, h + k_2\, p\rangle_{L^2(0,\,L)}. \end{aligned}$$

Then from (4.13), (4.14), we get

$$|\langle \theta,\ i\,k_1\,\lambda\,v + i\,k_2\,\lambda\,\eta\rangle_{L^2(0,\,L)}| \leq C\,\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}},$$

for a positive constant C. The last estimate together with (4.15) implies

$$\|(k_1\,v + k_2\,\eta)_x\|^2_{L^2(0,\,L)} \leq C\,\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{4.16}$$

(a) Since $b_{11}\,k_1 \neq b_{12}\,k_2$, it results from (4.13), (4.16) that

$$\|v_x\|^2_{L^2(0,\,L)} + \|\eta_x\|^2_{L^2(0,\,L)} \leq C\,\|U\|_{\mathcal{H}}\,\|F\|_{\mathcal{H}}, \tag{4.17}$$

for a positive constant C. We conclude from (4.11), (4.13), (4.14) and (4.17) that

$$|\mathrm{Im}\langle \mathcal{A}U,\ U\rangle_{\mathcal{H}}| \leq C\,\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

From (4.10), we have

$$\lambda\,\|U\|^2_{\mathcal{H}} \leq C\,\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Following the same reasoning as above, we obtain (4.12). The proof of the item (b) is similar. □

5 About the lack of exponential stability

In this Section, we will show that there are cases where the lack of exponential stability of the semigroup occurs. However, we observe that the case

$$b_{11} = b_{22} = -\,b_{12}, \quad \rho_2\,(a_{11}+a_{12}) = (a_{22}+a_{12})\,\rho_1, \quad \beta_1 \neq -\beta_2 \quad \text{and} \quad k_1 \neq -\,k_2$$

is not studied in this work.

To show the lack of exponential stability, we will show that the condition (b) of Theorem 4.1 is not true. To do this, it is sufficient to show the existence of sequences $F_\nu \in \mathcal{H}$, $\xi_\nu \in \mathbb{R}$ such that $(F_\nu)_{\nu\in\mathbb{N}}$ is bounded, $|\xi_\nu| \to \infty$ and $\|(i\,\xi_\nu\,I - \mathcal{A})^{-1}F_\nu\| \to \infty$ as $\nu \to \infty$.

We denote by $\varphi_\nu \in H^1_0(0,\ L) \cap H^2(0,\ L)$ and $\lambda_\nu \in \mathbb{R}$ the sequences of eigenvectors and eigenvalues, respectively, of the operator $-\,\partial^2_x$, that is,

$$-\,\varphi_{\nu xx} = \lambda_\nu\varphi_\nu \quad \text{in } (0,\ L)$$

with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots,\ \lambda_\nu \to \infty$ as $\nu \to \infty$ and such that the sequence $(\varphi_\nu)_{\nu\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,\ L)$.

Theorem 5.1 *Suppose that $b_{11} = b_{22} = -\,b_{12}$ and $\rho_2\,(a_{11}+a_{12}) = (a_{22}+a_{12})\,\rho_1$. Moreover, assume that only one of the following items is true:*

(a) $\beta_1 = -\,\beta_2$ *and* $k_1 = -\,k_2$*;*
(b) $\beta_1 \neq -\,\beta_2$ *and* $k_1 = -\,k_2$.

Then, $S_{\mathcal{A}}(t)$ is not exponentially stable.

Proof First of all, we observe that $b_{11} \neq 0$ and $k_1 \neq 0$.

For each $\nu \in \mathbb{N}$, we take $F_\nu = (0,\ 0,\ a\,\rho_1^{-1}\,\varphi_\nu,\ b\,\rho_2^{-1}\,\varphi_\nu,\ 0) \in \mathcal{H}$, with $a,\ b \in \mathbb{R}$, and we denote by $U_\nu = (u_\nu,\ w_\nu,\ v_\nu,\ \eta_\nu,\ \theta_\nu)$ the solution of the resolvent equation

$$(i\,\lambda\,I - \mathcal{A})U_\nu = F_\nu, \quad \lambda \in \mathbb{R}. \tag{5.1}$$

For each $\nu \in \mathbb{N}$, the solutions of (5.1) are of the form $u_\nu = A_\nu\, \varphi_\nu$, $w_\nu = B_\nu\, \varphi_\nu$ and $\theta_\nu = C_\nu\, \varphi_\nu$. Thus, we get the system

$$v_\nu = i\,\lambda\, u_\nu, \quad \eta_\nu = i\,\lambda\, w_\nu, \tag{5.2}$$

$$\begin{aligned} &-\rho_1\,\lambda^2\, A_\nu + \lambda_\nu\,(a_{11} + i\,\lambda\, b_{11})\, A_\nu + \lambda_\nu\,(a_{12} - i\,\lambda\, b_{11})\, B_\nu + \lambda_\nu\, k_1\, C_\nu \\ &\quad + \alpha\,(A_\nu - B_\nu) + i\,\alpha_1\,\lambda\,(A_\nu - B_\nu) - \beta_1\, C_\nu = a, \qquad (5.3)\\ &\quad - \rho_2\,\lambda^2\, B_\nu + \lambda_\nu\,(a_{12} - i\,\lambda\, b_{11})\, A_\nu + \lambda_\nu\,(a_{22} + i\,\lambda\, b_{11})\, B_\nu + \lambda_\nu\, k_2\, C_\nu \\ &\quad - \alpha\,(A_\nu - B_\nu) - i\alpha_1\,\lambda\,(A_\nu - B_\nu) - \beta_2\, C_\nu = b, \qquad (5.4)\\ &(i\,c\,\lambda + \kappa\,\lambda_\nu)\, C_\nu + i\,\lambda\,(\beta_1 - k_1\,\lambda_\nu)\, A_\nu + i\,\lambda\,(\beta_2 - k_2\,\lambda_\nu)\, B_\nu = 0. \qquad (5.5) \end{aligned}$$

Adding (5.3) with (5.4), we get

$$\left(-\,\lambda^2 + \frac{(a_{11} + a_{12})\,\lambda_\nu}{\rho_1}\right)(\rho_1\, A_\nu + \rho_2\, B_\nu) + (\lambda_\nu\,(k_1 + k_2) - (\beta_1 + \beta_1))\, C_\nu = a + b. \tag{5.6}$$

(a) Substituting $a = b = 1$, $\beta_1 = -\,\beta_2$ and $k_1 = -k_2$ in (5.6), we obtain

$$\left(-\,\lambda^2 + \frac{(a_{11} + a_{12})\,\lambda_\nu}{\rho_1}\right)(\rho_1\, A_\nu + \rho_2\, B_\nu) = 2. \tag{5.7}$$

Taking $\lambda = \xi_\nu = \sqrt{\frac{a_{11}+a_{12}}{\rho_1}\lambda_\nu - 1}$ in (5.1), it results from (5.7) that

$$\rho_1\, A_\nu + \rho_2\, B_\nu = 2,$$

i.e.,

$$A_\nu = \tau_1 - \tau_2\, B_\nu \quad \text{with} \quad \tau_1 = \frac{2}{\rho_1} \quad \text{and} \quad \tau_2 = \frac{\rho_2}{\rho_1}. \tag{5.8}$$

Substituting (5.8) into (5.5), we get

$$C_\nu = -\,\frac{i\,\tau_1\,\xi_\nu\,(\beta_1 - k_1\,\lambda_\nu)}{\kappa\,\lambda_\nu + i\,c\,\xi_\nu} + \frac{i\,(1 + \tau_2)\,\xi_\nu\,(\beta_1 - k_1\,\lambda_\nu)}{\kappa\,\lambda_\nu + i\,c\,\xi_\nu}\, B_\nu. \tag{5.9}$$

Now, substituting (5.8), (5.9) into (5.3), we obtain

$$B_\nu = \frac{P_\nu + i\, Q_\nu}{R_\nu + i\, S_\nu}$$

where

$$\begin{aligned} P_\nu &= (-\,1 + \alpha\,\tau_1 + \rho_1\,\tau_1)\,\kappa\,\lambda_\nu - \kappa\, a_{12}\,\tau_1\,\lambda_\nu^2 - c\,\alpha_1\,\tau_1\,\xi_\nu^2 - c\, b_{11}\,\tau_1\,\xi_\nu^2\,\lambda_\nu,\\ Q_\nu &= (-\,c + (\alpha + c)\,\tau_1\,\rho_1 + \tau_1\,\beta_1^2)\,\xi_\nu + (\kappa\,\alpha_1\,\tau_1 - c\,\tau_1\, a_{12} - 2\,\beta_1\, k_1\,\tau_1)\,\xi_\nu\,\lambda_\nu\\ &\quad + (\kappa\, b_{11}\,\tau_1 + \tau_1\, k_1^2)\,\xi_\nu\,\lambda_\nu^2,\\ R_\nu &= (\tau_2\,\rho_1 + \alpha\,(1 + \tau_2))\,\kappa\,\lambda_\nu - \kappa(1 + \tau_2)\, a_{12}\,\lambda_\nu^2 - c\,\alpha_1\,(1 + \tau_2)\,\xi_\nu^2 - c\,(1 + \tau_2)\, b_{11}\,\xi_\nu^2\,\lambda_\nu,\\ S_\nu &= (c\,\tau_2\,\rho_1 + (1 + \tau_2)\,(c\,\alpha + \beta_1^2)\,\xi_\nu + (1 + \tau_2)\,(\kappa\,\alpha_1 - c\, a_{12} - 2\,\beta_1\, k_1)\,\xi_\nu\,\lambda_\nu\\ &\quad + (1 + \tau_2)\,(\kappa\, b_{11} + k_1^2)\,\xi_\nu\,\lambda_\nu^2. \end{aligned}$$

Since

$$\lim_{\nu\to\infty} |B_\nu| = \frac{\tau_1}{1 + \tau_2}$$

it follows

$$\lim_{\nu\to\infty} \|\eta_\nu\| = \lim_{\nu\to\infty} \xi_\nu\, |B_\nu| = \infty.$$

Therefore,

$$\lim_{\nu\to\infty} \|U_\nu\|_{\mathcal{H}} = \infty.$$

The proof follows.

(b) Substituting $a = -\beta_1$, $b = -\beta_2$ and $k_1 = -k_2$ in (5.6), we obtain

$$\left(-\lambda^2 + \frac{a_{11}+a_{12}}{\rho_1}\lambda_\nu\right)(\rho_1 A_\nu + \rho_2 B_\nu) - (\beta_1+\beta_2)\, C_\nu = -(\beta_1+\beta_2). \tag{5.10}$$

Taking $\lambda = \xi_\nu = \sqrt{\frac{a_{11}+a_{12}}{\rho_1}\lambda_\nu}$ in (5.1), it follows from (5.10) that

$$C_\nu = 1. \tag{5.11}$$

Substituting (5.11) into (5.5), we get

$$A_\nu = \frac{-c\,\xi_\nu + i\,\kappa\,\lambda_\nu}{\xi_\nu\,(\beta_1 - k_1\,\lambda_\nu)} - \frac{\beta_2 + k_1\,\lambda_\nu}{\beta_1 - k_1\,\lambda_\nu} B_\nu. \tag{5.12}$$

We obtain from (5.3)

$$A_\nu - B_\nu = \frac{-\lambda_\nu\, k_1}{\alpha - a_{12}\,\lambda_\nu + i\,\xi_\nu\,(\alpha_1 + b_{11}\,\lambda_\nu)}. \tag{5.13}$$

Substituting (5.12) into (5.13) and performing straightforward calculations, we obtain

$$B_\nu = \frac{\beta_1 - k_1\,\lambda_\nu}{(\beta_1+\beta_2)}\left[\frac{-c\,\xi_\nu + i\,\kappa\,\lambda_\nu}{\xi_\nu\,(\beta_1 - k_1\,\lambda_\nu)} + \frac{k_1\,\lambda_\nu\,(\alpha - a_{12}\,\lambda_\nu - i\,\xi_\nu\,(\alpha_1 + b_{11}\,\lambda_\nu))}{(\alpha - a_{12}\,\lambda_\nu)^2 + \xi_\nu^2\,(\alpha_1 + b_{11}\,\lambda_\nu)^2}\right].$$

Therefore,

$$\operatorname{Im}\, B_\nu = \frac{1}{(\beta_1+\beta_2)}\left[\kappa\,\frac{\lambda_\nu}{\xi_\nu} - \frac{k_1\,\xi_\nu\,\lambda_\nu\,(\alpha_1 + b_{11}\,\lambda_\nu)\,(\beta_1 - k_1\,\lambda_\nu)}{(\alpha - a_{12}\,\lambda_\nu)^2 + \xi_\nu^2\,(\alpha_1 + b_{11}\,\lambda_\nu)^2}\right],$$

i.e.,

$$\begin{aligned}\operatorname{Im}\, B_\nu = {} & \frac{1}{(\beta_1+\beta_2)} \\ & \times \frac{\lambda_\nu}{\xi_\nu}\left[\kappa - \frac{\alpha_1\,\beta_1\,k_1\,\xi_\nu + b_{11}\,\beta_1\,k_1\,\xi_\nu\,\lambda_\nu - \alpha_1\,k_1^2\,\xi_\nu\,\lambda_\nu - k_1^2\,b_{11}\,\xi_\nu\,\lambda_\nu^2}{\alpha^2\,\xi_\nu^{-1} - 2\,\alpha\,a_{12}\,\lambda_\nu\,\xi_\nu^{-1} + a_{12}^2\,\lambda_\nu^2\,\xi_\nu^{-1} + \alpha_1^2\,\xi_\nu + 2\,\alpha_1\,b_{11}\,\xi_\nu\,\lambda_\nu + b_{11}^2\,\xi_\nu\,\lambda_\nu^2}\right].\end{aligned}$$

Thus,

$$\lim_{\nu\to\infty} \operatorname{Im}\, B_\nu = \frac{1}{(\beta_1+\beta_2)}\left(\kappa + \frac{k_1^2}{b_{11}}\right)\lim_{\nu\to\infty}\frac{\lambda_\nu}{\xi_\nu} = \infty.$$

Hence, $\lim_{\nu\to\infty} |B_\nu| = \infty$. Therefore,

$$\lim_{\nu\to\infty} \|\eta_\nu\|^2_{L^2(0,\,L)} = \lim_{\nu\to\infty}\int_0^L |\gamma_\nu\, B_\nu\, w_\nu|^2\, dx = \lim_{\nu\to\infty} |\gamma_\nu\, B_\nu| = \infty.$$

Consequently, $\lim_{\nu\to\infty} \|U_\nu\|_{\mathcal{H}} = \infty$. The proof follows and the theorem is complete.

6 Numerical examples

The following numerical examples show the asymptotic behavior of the solution of (1.2) due to the exponential stability when the conditions (a) or (b) of Lemma 3.2 are verified, and the lack of exponential stability, when they are not verified.

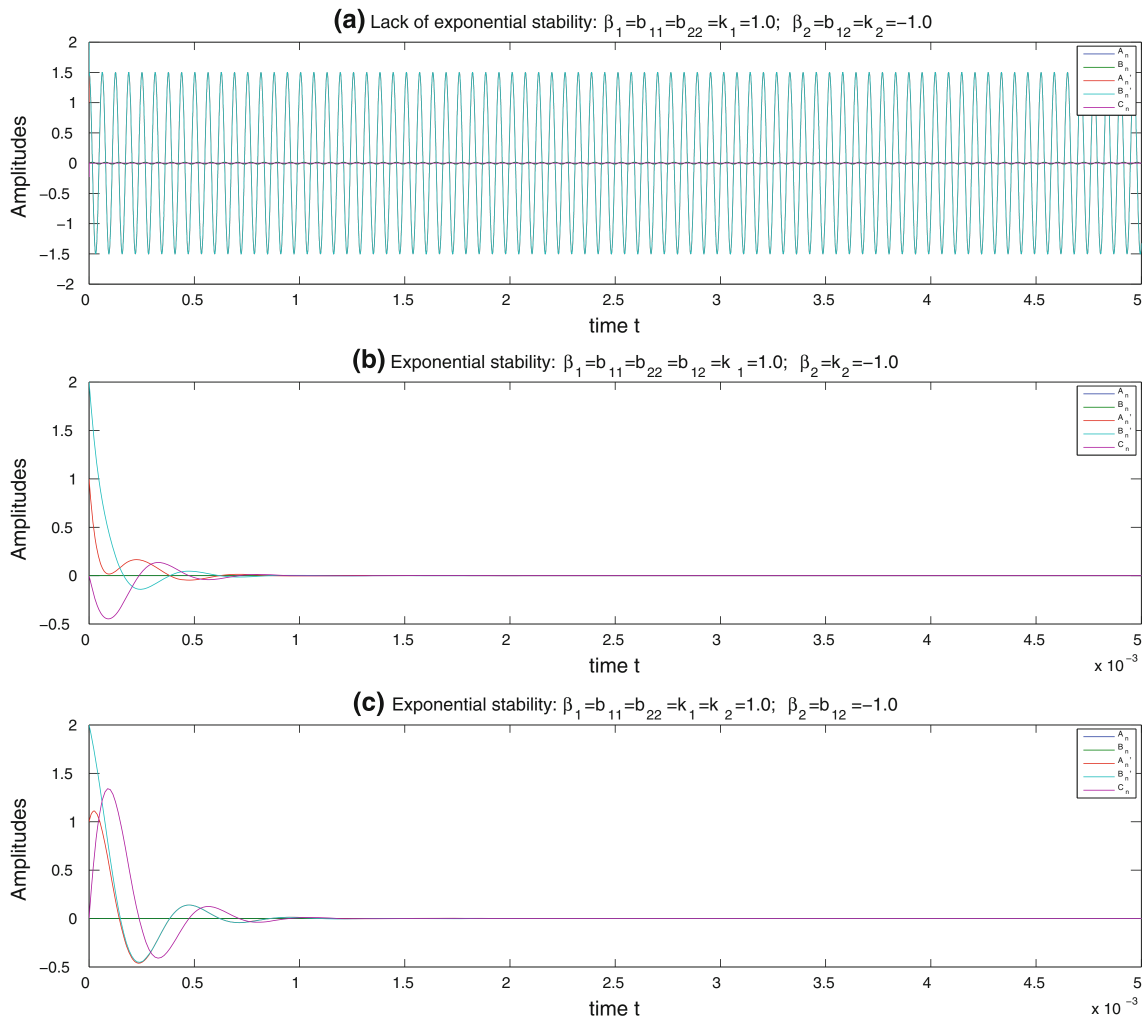

Fig. 1 Example I. Different evolution of the amplitudes: **a** lack of exponential stability ($\beta_1 = b_{11} = b_{22} = k_1 = 1.0$, $\beta_2 = b_{12} = k_2 = -1.0$); **b** exponential stability ($\beta_1 = b_{11} = b_{22} = b_{12} = k_1 = 1.0$, $\beta_2 = k_2 = -1.0$); **c** exponential stability ($\beta_1 = b_{11} = b_{22} = k_1 = k_2 = 1.0$, $\beta_2 = b_{12} = -1.0$)

6.1 Example I. Amplitudes for sample sinusoidal initial condition

We consider here a similar example as in the above Section. That is, we choose $L = \pi$, and we suppose that the solutions are of the form $u_n = A_n(t)\, \sin(nx)$, $w_n = B_n(t)\, \sin(nx)$, $v_n = A_n'(t)\, \sin(nx)$, $\eta_n = B_n'(t)\, \sin(nx)$, and $\theta_n = C_n(t)\, \sin(nx)$. In this case, the amplitudes $(A_n,\ B_n,\ C_n)$ verify the following system of ODEs:

$$\begin{aligned}
\rho_1\, A_n'' &= -n^2 \left(a_{11}\, A_n + a_{12}\, B_n + b_{11}\, A_n' + b_{12}\, B_n' + k_1\, \theta\right) - \alpha\, (A_n - B_n) - \alpha_1\, (A_n' - B_n') + \beta_1\, C_n, \\
\rho_2\, B_n'' &= -n^2 \left(a_{12}\, A_n + a_{22}\, B_n + b_{12}\, A_n' + b_{22}\, B_n'\right) + \alpha\, (A_n - B_n) + \alpha_1\, (A_n' - B_n') + \beta_2\, C_n, \\
c\, C_n' &= -n^2 \left(\kappa\, C_n - k_1\, A_n' - k_2\, B_n'\right) - \beta_1\, A_n' - \beta_2\, B_n'.
\end{aligned} \tag{6.14}$$

Thus, we consider the system (6.14) with the parameter values $a_{11} = a_{22} = 1.0$, $a_{12} = 0.0$, and $\rho_1 = \rho_2 = \alpha = \alpha_1 = c = \kappa = 1.0$.

Figure 1 represents the evolution in time of the three amplitudes A_n, B_n, and C_n, and the derivatives A_n' and B_n' (which are the amplitudes of v and η, respectively), for $n = 100$. For the numerical simulation, we use the Runge–Kutta–Fehlberg method RKF45, with the standard solver `ode45()` of Matlab. The case (a) is a simulation for $0 \le t \le 5.0$, and the cases (b) and (c) are simulations for $0 \le t \le 0.001$.

The first Fig. 1(a) represents an example with lack of exponential stability when the hypotheses of Theorem 5.1 are verified:

$$b_{11} = b_{22} = -\, b_{12}, \quad \rho_2\,(a_{11} + a_{12}) = (a_{22} + a_{12})\,\rho_1, \quad k_1 = -\, k_2.$$

The second Fig. 1(b) represents an exponential stability example when the following hypothesis of Theorem 3.3 is verified: B is non-negative definite and

$$b_{12} \neq -\, b_{11} \quad \text{or} \quad b_{12} \neq -\, b_{22}.$$

More precisely, we take $b_{12} = b_{11} = b_{12} = 1$.

The third Fig. 1(c) represents again an exponentially stable case when the following hypothesis of Theorem 3.3 is verified:

$$b_{12} = -\, b_{11} = -\, b_{22}, \quad \beta_1 = -\, \beta_2, \quad k_1 \neq -\, k_2.$$

More precisely, we take $\beta_1 = b_{11} = b_{22} = k_1 = k_2 = 1.0, \quad \beta_2 = b_{12} = -1.0$.

We observe clearly in the Figs. 1(b) and (c) that the 5 amplitudes $A_n(t)$, $B_n(t)$, $A'_n(t)$, $B'_n(t)$ and $C_n(t)$ tend to zero very fast. However, both (b) and (c) are exponentially stable.

6.2 Example II. Asymptotic behavior for a small initial condition

Here, we compute numerically the solution of the system (2.2), with $L = 1.0, T = 2.0$, and the initial condition:

$$v(x,\, 0) = \begin{cases} 0 & \text{if } 0.0 \le x \le 0.4, \\ 10\,(x - 0.4) & \text{if } 0.4 \le x \le 0.5, \\ 10\,(0.6 - x) & \text{if } 0.5 \le x \le 0.6, \\ 0 & \text{if } 0.6 \le x \le 1.0, \end{cases} \quad \eta(x,\, 0) = \begin{cases} 0 & \text{if } 0.0 \le x \le 0.4, \\ 20\,(x - 0.4) & \text{if } 0.4 \le x \le 0.5, \\ 20\,(0.6 - x) & \text{if } 0.5 \le x \le 0.6, \\ 0 & \text{if } 0.6 \le x \le 1.0, \end{cases} \tag{6.15}$$

and $u(x,\, 0) = w(x,\, 0) = \theta(x,\, 0) = 0.0$. We remark that the initial condition defined in (6.15) has two peaks of height 1 and 2, respectively, and support in $(0.4;\, 0.6)$. Additionally, we consider the same parameter values of Example I, $a_{11} = a_{22} = 1.0$, $a_{12} = 0.0$, and $\rho_1 = \rho_2 = \alpha = \alpha_1 = c = \kappa = 1.0$.

In order to compare these numerical results with those of Example I and the previous Sect. 5, we assume that

$$u(x,\, t) \;=\; \sum_{k=1}^{\infty} A_k(t)\, \sin(k-1)\,\pi\, x, \quad w(x,\, t) \;=\; \sum_{k=1}^{\infty} B_k(t)\, \sin(k-1)\,\pi\, x,$$
$$\theta(x,\, t) = \sum_{k=1}^{\infty} C_k(t)\, \cos(k-1)\,\pi\, x, \tag{6.16}$$

and therefore, we extend the initial conditions (6.15) by odd functions, in the interval $(-\, L,\, 0)$.

On the other hand, if we discretize the space dimension $(-\, L,\, L) = (-1,\, 1)$ in $2\,N - 1$ subintervals $I_j = (j\,\delta\, x,\, (j+1)\,\delta\, x)$, with $\delta\, x = 1/(2\,N)$, and $j = -\, N, \ldots, N - 1$ and we approximate the solution $U(x,\, t)$ of the system (2.2) by piecewise functions equal to $U_j(t)$ in each subinterval, then we can take the Discrete Fourier Transform of the solution:

$$\widetilde{U}_k(t) = \sum_{j=1}^{2\,N} U_j(t)\, e^{-\,\pi\, i\,(k-1)(j-1)/N}, \tag{6.17}$$

and we reconstruct the solution by the Inverse Discrete Fourier Transform:

$$U_j(t) = \frac{1}{2\,N} \sum_{k=1}^{2\,N} \widetilde{U}_k(t)\, e^{\pi\, i\,(k-1)(j-1)/N}. \tag{6.18}$$

We note that if we define $\widetilde{U}_k(t) = (\widetilde{u}_j(t),\ \widetilde{w}_j(t),\ \widetilde{v}_j(t),\ \widetilde{\eta}_j(t),\ \widetilde{\theta}_j(t))^T$, then $A_k(t) = -\operatorname{Im}(\widetilde{u}_j(t)),\ B_k(t) = -\operatorname{Im}(\widetilde{w}_j(t))$, $C_k(t) = -\operatorname{Im}(\widetilde{\theta}_j(t))$ and the following system of ODEs is verified:

$$
\begin{aligned}
\widetilde{u}_k' &= \widetilde{v}_k, \quad \widetilde{w}_k' = \widetilde{\eta}_k,\\
\rho_1 \widetilde{v}_k' &= -\pi^2 (k-1)^2 \left(a_{11} \widetilde{u}_k + a_{12} \widetilde{w}_k + b_{11} \widetilde{v}_k + b_{12} \widetilde{\eta}_k + k_1 \widetilde{\theta}_k\right) - \alpha (\widetilde{u}_k - \widetilde{w}_k) - \alpha_1 (\widetilde{v}_k - \widetilde{\eta}_k) + \beta_1 \widetilde{\theta}_k,\\
\rho_2 \widetilde{\eta}_k' &= -\pi^2 (k-1)^2 \left(a_{12} \widetilde{u}_k + a_{22} \widetilde{w}_k + b_{12} \widetilde{v}_k + b_{22} \widetilde{\eta}_k + k_2 \widetilde{\theta}_k\right) + \alpha (\widetilde{u}_k - \widetilde{w}_k) + \alpha_1 (\widetilde{v}_k - \widetilde{\eta}_k) + \beta_2 \widetilde{\theta}_k,\\
c\, \widetilde{\theta}_k' &= -\pi^2 (k-1)^2 \left(\kappa\, \widetilde{\theta}_k - k_1 \widetilde{v}_k - k_2 \widetilde{\eta}_k\right) - \beta_1 \widetilde{v}_k - \beta_2 \widetilde{\eta}_k.
\end{aligned}
\tag{6.19}
$$

We make simulations for $N = 1024$ using in this case the Stiff solver `ode15s()` of MATLAB to compute each one of the 1024 system of Eqs. (6.19), and we reconstruct the solution by the Inverse Discrete Fourier Transform (6.18).

Figures 2 and 3 represent the evolution of the solutions $(u,\ w,\ v,\ \eta,\ \theta)$, with the same parameters b_{ij} and β_i of Example I, cases (a), (b) and (c). Figure 2 shows the lack of exponential stability with $\beta_1 = b_{11} = b_{22} = k_1 = 1.0$, $\beta_2 = b_{12} = k_2 = -1.0$, and Fig. 3 shows the exponential stability with $\beta_1 = b_{11} = b_{22} = b_{12} = k_1 = 1.0$, $\beta_2 = k_2 = -1.0$. In both figures, $u(x,\ t)$ is graphed at the top left, $w(x,\ t)$ at the top right, $v(x,\ t)$ at the bottom left, $\eta(x,\ t)$ at the bottom right, and $\theta(x,\ t)$ at the center.

Finally, in Fig. 4, it is represented the norm $\mathcal{H}$ of the numerical solution of (2.2) for the 5 first cases of Example I ((a), (b) and (c)). More precisely, we plot the function:

$$
t \mapsto \sqrt{\sum_{j=1}^{N} h \left[\left(\frac{u_j(t) - u_{j-1}(t)}{h} \right)^2 + \left(\frac{w_j(t) - u_{j-1}(t)}{h} \right)^2 + v_j^2(t) + w_j^2(t) + \left(\theta_j(t) - h \sum_{\ell} \theta_\ell(t) \right)^2 \right]}.
$$

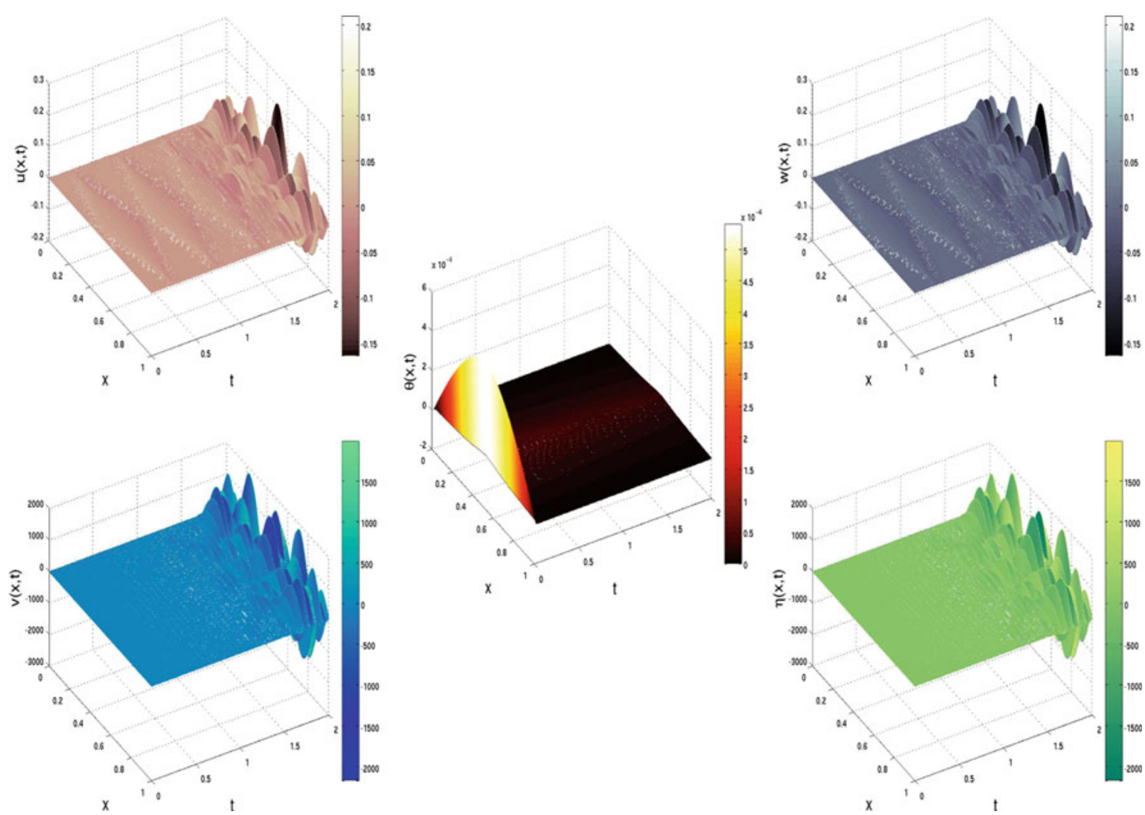

Fig. 2 Example II. Lack of exponential stability. Numerical solutions u, w, v, η, θ. Case $\beta_1 = b_{11} = b_{22} = k_1 = 1.0$, $\beta_2 = b_{12} = k_2 = -1.0$

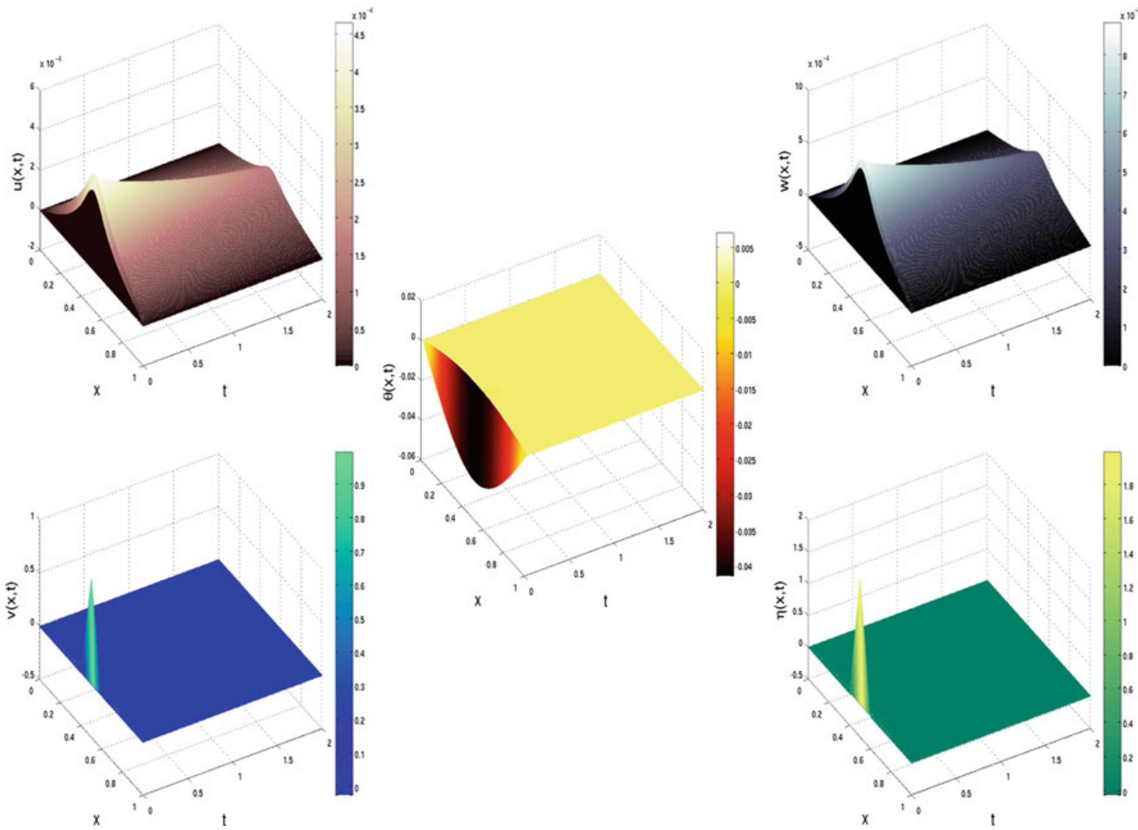

Fig. 3 Example II. Exponential stability. Numerical solutions u, w, v, η, θ. Case $\beta_1 = b_{11} = b_{22} = b_{12} = k_1 = 1.0, \beta_2 = k_2 = -1.0$

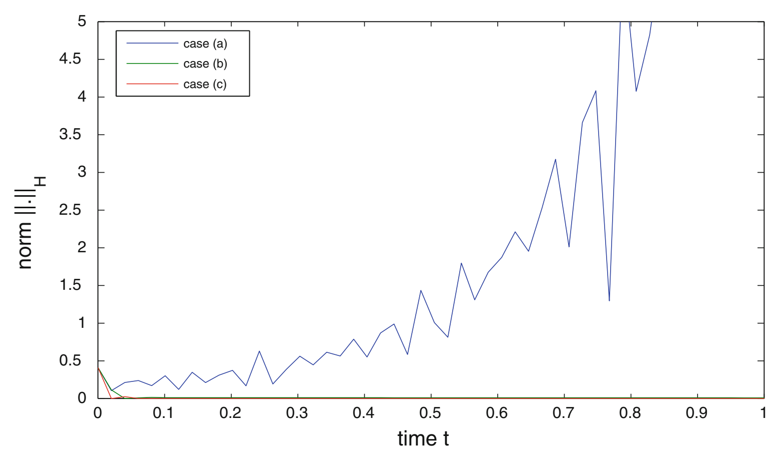

Fig. 4 Example II. Evolution in time of $t \mapsto \|U(\cdot\,,\ t)\|_{\mathcal{H}}$ for: **a** Lack of exponential stability $\beta_1 = b_{11} = b_{22} = k_1 = 1.0,\ \beta_2 = b_{12} = k_2 = -1.0$; **b** Exponential stability $\beta_1 = b_{11} = b_{22} = k_1 = k_2 = 1.0, \beta_2 = b_{12} = -1.0$; **c** Exponential stability $\beta_1 = b_{11} = b_{22} = b_{12} = k_1 = 1.0,\ \ \beta_2 = k_2 = -1.0$

We observe that in general, in the cases of lack of exponential stability, the curves diverge when $t \to \infty$ (case (a)), and the curves tend to zero in the exponentially stable cases ((b) and (c)).

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