Stabilization of Continuous-Time Switched Linear Positive Systems

Annalisa Zappavigna, Patrizio Colaneri, Jose' C. Geromel, Richard Middleton

***Abstract*— In this paper we tackle a few problems related to linear positive switched systems. First, we provide a result on state-feedback stabilization of autonomous linear positive switched systems through piecewise linear co-positive Lyapunov functions. This is accompanied by a side result on the existence of a switching law guaranteeing an upper bound to the optimal L_1 cost. Then, the induced L_1 guaranteed cost cost is tackled, through constrained piecewise linear co-positive Lyapunov functions. The optimal L_1 cost control is finally studied via Hamiltonian function analysis.**

I. Introduction

Switched systems [1] belong to the class of hybrid systems [2], i.e. in which there is an interaction between discrete and continuous dynamics, and, in particular, they have continuous dynamics with "isolated" discrete switching events. Some simple examples of switched systems include cars with manual transmission, where changing the gear modifies the continuous behavior of the vehicle; 4WD vehicles where the torque applied to the rear axle can switch between two different levels [3]; chemical processes with variations of the concentrations and so on. Stability of these kind of systems is not a trivial issue [4], [5] and two main problems arise [2], [6]: the first one is to find conditions that guarantee asymptotic stability of a switched system for arbitrary switching signals; the second one occurs if a switched system is not asymptotically stable for arbitrary switching, since in this case it is interesting to identify those switching signals for which it is asymptotically stable.

Positive systems [7], [8], instead, have the peculiar property that any nonnegative input and nonnegative initial state generate a nonnegative state trajectory and output for all times. Positivity of the variables often emerges as the immediate consequence of the nature of the phenomenon itself, such as any variable representing any possible type of resource measured by a quantity like time, money and goods, buffer size and queues, data packets flowing in a network, water and air flows, populations, concentration of any substance, electric charge and light intensity levels. Moreover, probabilities are also positive quantities, and therefore other examples of positive systems include hidden Markov models and phase-type distributions models. Other example include net of interconnected water tanks, industrial processes with chemical reactors, heat exchangers, populations, compartmental systems, pollution models, economical stochastic models and so on. Stabilization problems for positive systems have been studied recently, [9], [10], and can be particularly problematic due to the presence of the positivity constraint on the input variable.

Switched positive systems, [11], are frequently encountered in many application fields. Examples of switched positive systems are straightforward: environmental systems such as rivers whose water flows is regulated by dams, networks of tanks regulated by valves opening, chemical plants where reagents concentration can be varied by means of additional inputs, air conditioning systems. Also many applications in Communications networks involve algorithms that lead to extremely complex positive systems, typically involving significant nonlinearity, abrupt parameter switching and state resets, such as networks employing TCP and other congestion control applications, synchronization problems and wireless power control applications [12]. Despite their ubiquitous nature, a few basic problems, concerning stabilization and performances, still deserve a deeper study. This paper aims at extending results on the stabilization for a continuous-time switched linear system [13], [14] to continuous-time switched linear positive systems of the general form

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0, \tag{1}$$

defined for all $t \geq 0$, where $x(t) \in \mathbb{R}^n_+$ is the state

Zappavigna and Colaneri are with Politecnico di Milano - Dipartimento di Elettronica e Informazione, Via Ponzio 34/5, 20133, Milano, Italy. email: zappavigna-colaneri@elet.polimi.it
Geromel is with UNICAMP - FEE, San Paulo, Brasil. email: geromel@dsce.fee.unicamp.br
Middleton is with NUIM, University of Maynooth, Kilcock Rd, Maynooth, Ireland. email: richard.middleton@nuim.ie

Preprint submitted to 2010 American Control Conference.
Received September 21, 2009.

variable vector, $\sigma(t)$ is the switching signal, $x_0 \in \mathbb{R}^n_+$ is the initial condition and A_i belongs to the set $\{A_1, \ldots, A_N\}$. In order to be a positive system A_i has to be Metzler, i.e. $a_{ij} \geq 0,\ \forall (i,j),\ i \neq j$. Moreover, we present also a stabilizing switching rule for input-output switched positive systems and the calculation of an upper bound of their induced L_1 norm. The idea that leads to the stabilization is the same used for autonomous systems and it will be presented in detail in the corresponding sections. Finally, we study the stabilization of an autonomous linear switched positive system through optimal control [15], [16], where the continuous control is absent and only the switching signal must be determined [17]. In particular, the sequence of active subsystems may be arbitrary, or it may be subject to constraints given as a pre-specified sequence with arbitrary length or as an arbitrary sequence with pre-specified length.

Throughout this paper, the notation used is standard for positive systems. Capital letters denote matrices, small letters denote vectors. For matrices or vectors, ($'$) indicates transpose. For matrices X or vectors x, the notation X or $x \succ 0$ ($\succeq 0$) indicates that X, or x, has all elements positive (non-negative). The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively.

The paper is organized as follows. Section II is dedicated to state-switching control where the aim is to design a function $u(\cdot)$ such that the system (1) is globally asymptotically stable with the switching rule $\sigma(t) = u(x(t)) \succeq 0,\ \forall t \geq 0$. Section III aims at finding a switching rule ensuring stability and an upper bound of the induced L_1 norm for the switched positive system, while Section IV deals with the switched optimal control for positive systems. Section V concludes the paper.

II. State-switching control

Given a system (1), it is assumed that the state vector $x(t)$ is available for feedback for all $t \geq 0$. Therefore our goal is to determine the function $u(\cdot) : \mathbb{R}^n_+ \to \{1, \ldots, N\}$ such that

$$\sigma(t) = u(x(t)) \tag{2}$$

makes the equilibrium point $x = 0$ of system (1) asymptotically stable. Note that we make no assumption on the stability of the elements of the set $\{A_1, \ldots, A_N\}$. Define the simplex

$$\Lambda := \left\{ \lambda \in \mathbb{R}^N : \sum_{i=1}^{N} \lambda_i = 1,\ \lambda_i \geq 0 \right\} \tag{3}$$

which allows us to introduce the following piecewise co-positive Lyapunov function [18]:

$$v(x) := \min_{i=1,\ldots,N} \alpha_i' x = \min_{\lambda \in \Lambda} \left(\sum_{i=1}^{N} \lambda_i \alpha_i' x \right). \tag{4}$$

The Lyapunov function in (4) is not differentiable everywhere. In particular, let us define the set $I(x) = \{i : v(x) = \alpha_i' x\}$, $v(x)$ fails to be differentiable on $x \in \mathbb{R}^n_+$ such that $I(x)$ is composed of more than one element, that is in the conjunction points of the individual Lyapunov functions $\alpha_i' x$.

Now we will denote by $\mathcal{M}$ the subclass of Metzler matrices consisting of all matrices $\Pi \in \mathbb{R}^{N \times N}$ with elements π_{ji}, such that

$$\pi_{ji} \geq 0\ \forall j \neq i, \quad \sum_{j=1}^{N} \pi_{ji} = 0\ \forall j. \tag{5}$$

As a consequence, any $\Pi \in \mathcal{M}$ has an eigenvalue at zero since $c'\Pi = 0$, where $c' = [1 \cdots 1]$.

Theorem 1 *Consider the linear positive switched system (1) and assume that there exists a set of positive vectors $\{\alpha_1, \ldots, \alpha_N\}$, $\alpha_i \in \mathbb{R}^n_+$, and $\Pi \in \mathcal{M}$, satisfying the coupled co-positive Lyapunov inequalities:*

$$A_i' \alpha_i + \sum_{j=1}^{N} \pi_{ji} \alpha_j \prec 0 \qquad i = 1, \ldots, N. \tag{6}$$

Then, the state-switching control (2) *with*

$$u(x(t)) = \arg \min_{i=1,\ldots,N} \alpha_i' x(t) \tag{7}$$

makes the equilibrium solution $x = 0$ of the system (1) *globally asymptotically stable.*

Proof: Since the Lyapunov function (4) is not differentiable for all $t \geq 0$, we need to deal with the Dini derivative [19]:

$$D^+ v(x(t)) = \limsup_{h \to 0^+} \frac{v(x(t+h)) - v(x(t))}{h} \tag{8}$$

Assume, in accordance with (7), that at an arbitrary $t \geq 0$, the state-switching control is given by $\sigma(t) = u(x(t)) = i$ for some $i \in I(x(t))$. Therefore, remembering also that (5) is valid for $\Pi \in \mathcal{M}$ and that $\alpha_j' x(t) \geq \alpha_i' x(t)$ for all

$j = 1, \dots, N$, we have

$$\begin{aligned} D^+v(x(t)) &= \limsup_{h\to 0^+} \frac{v(x(t)+hA_ix(t)) - v(x(t))}{h} \\ &= \min_{k\in I(x(t))} \alpha_k' A_i x(t) \\ &\le \alpha_i' A_i x(t) \le -\sum_{j=1}^{N} \pi_{ji}\alpha_j' x(t) \\ &= -\pi_{ii}\alpha_i' x(t) - \sum_{j\ne i} \pi_{ji}\alpha_j' x(t) \\ &\le -\pi_{ii}\alpha_i' x(t) - \sum_{j\ne i} \pi_{ji}\alpha_i' x(t) \\ &= -\sum_{j=1}^{N} \pi_{ji}\alpha_i' x(t) = 0, \end{aligned} \tag{9}$$

which proves the proposed theorem since the Lyapunov function $v(x(t))$ defined in (4) is radially unbounded. ■

Remark 1 *Theorem 3 does not require the set $\{A_1, \dots, A_N\}$ be exclusively composed of Hurwitz matrices. With $\Pi \in \mathcal{M}$ a necessary condition for the Lyapunov Metzler inequalities to be feasible with respect to $\{\alpha_1, \dots, \alpha_N\}$ is that matrices $A_i + \pi_{ii}I$ have to be asymptotically stable for all $i = 1, \dots, N$. Since $\pi_{ii} \le 0$, this condition does not imply the asymptotic stability of A_i.*

Now, let us introduce a guaranteed cost associated with the proposed state-switching control law (7).

Theorem 2 *Consider the linear positive switched system (1) and let $q \in \mathbb{R}_+^n$ be given. Assume that there exists a set of positive vectors $\{\alpha_1, \dots, \alpha_N\}$, $\alpha_i \in \mathbb{R}_+^n$ and $\Pi \in \mathcal{M}$, satisfying the coupled co-positive Lyapunov inequalities:*

$$A_i'\alpha_i + \sum_{j=1}^{N} \pi_{ji}\alpha_j + q \prec 0 \qquad i = 1, \dots, N. \tag{10}$$

The state-switching control (2) *with* $u(x(t))$ *given by* (7) *makes the equilibrium solution* $x = 0$ *of the system* (1) *globally asymptotically stable and*

$$\int_0^\infty q'x(t)dt \le \min_{i=1,\dots,N} \alpha_i' x_0 \tag{11}$$

Proof: If (10) holds, then (6) holds too, so we can say that the equilibrium point $x = 0$ for system (1) is globally asymptotically stable under that control law (7).

Moreover

$$v(x) = \min_{i=1,\dots,N} (\alpha_i' x) \tag{12}$$

then

$$\begin{aligned} D^+(v(x)) &= \min_{k\in I(x(t))} \alpha_k' A_i x \le \alpha_i' A_i x \\ &\le -\pi_{ii}\alpha_i' x - \sum_{j\ne i} \pi_{ji}\alpha_i' x - q'x = -q'x \end{aligned} \tag{13}$$

Hence

$$D^+(v(x)) \le -q'x(t) \tag{14}$$

which, after integration, gives

$$\begin{aligned} v(x(t)) - v(0) &= \int_0^t D^+v(x(\tau))d\tau \\ &\le -\int_0^t q'x(\tau)d\tau. \end{aligned} \tag{15}$$

Due to asymptotic stability $v(x(t))$ goes to zero as t goes to infinity, therefore

$$\int_0^t q'x(\tau)d\tau \le v(0) = \min_{i=1,\dots,N} \alpha_i' x_0. \tag{16}$$

This concludes the proof. ■

Notice that (10) is not linear in the unknowns variables π_{ij}, α_i. Therefore, we need an alternative reformulation in order to allow for an efficient numerical search. In particular, the idea is to obtain a simpler, even if more conservative, stability condition that can be expressed by means of LMIs.

The next corollary shows that, working with a subclass of $\mathcal{M}$-matrices, characterized by having the same diagonal elements, this goal is accomplished.

Corollary 1 *Let $q \in \mathbb{R}_+^n$ be given. Assume that there exists a set of positive vectors $\{\alpha_1, \dots, \alpha_N\}$, $\alpha_i \in \mathbb{R}_+^n$ and a scalar $\gamma > 0$ satisfying the modified coupled co-positive Lyapunov inequalities:*

$$A_i'\alpha_i + \gamma(\alpha_j - \alpha_i) + q \prec 0 \qquad i \ne j = 1, \dots, N. \tag{17}$$

The state-switching control (2) *with* $u(x(t))$ *given by* (7) *makes the equilibrium solution* $x = 0$ *of the system* (1) *globally asymptotically stable and*

$$\int_0^\infty q'x(t)dt \le \min_{i=1,\dots,N} \alpha_i' x_0 \tag{18}$$

Proof: The matrix $\Pi \in \mathcal{M}$ has been chosen such that $\pi_{ii} = -\gamma$, therefore

$$\gamma^{-1} \sum_{j\ne i} \pi_{ji} = 1 \qquad \forall i = 1, \dots, N \tag{19}$$

Since $\pi_{ji} \geq 0 \; \forall j \neq i, \; j, i = 1, \ldots, N$ we can multiply (17) by π_{ji}, summing up $\forall j \neq i, \; j, i = 1, \ldots, N$ and finally multiplying the result by $\gamma^{-1} > 0$, so obtaining

$$A_i' \alpha_i + q \prec - \sum_{j=1}^{N} \pi_{ji} \alpha_j \qquad \forall i = 1, \ldots, N \tag{20}$$

Hence the upper bound (11) of Theorem 2 holds. ∎

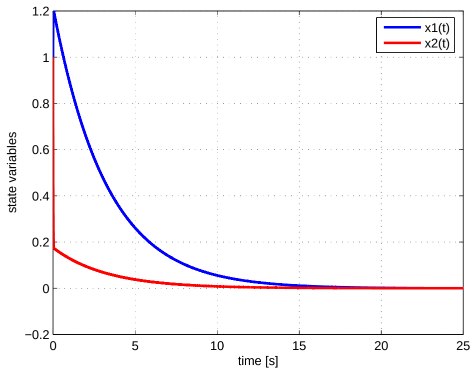

Fig. 1. Time simulation of the state switching control

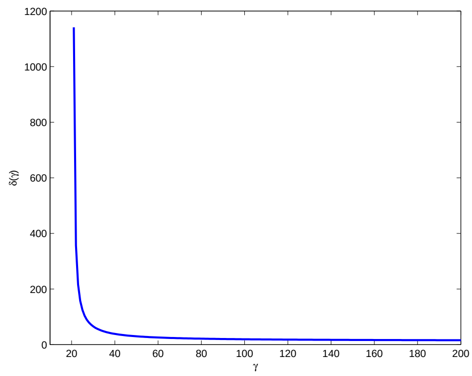

Fig. 2. Guaranteed cost as a function of γ.

Example 1 *Let us consider the example with $N = 2$ and matrices*

$$A_1 = \begin{bmatrix} -5 & 30 \\ 2 & -10 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 18 \\ 10 & -80 \end{bmatrix},$$

which are both positive and unstable. Considering $q = [1 \; 2]'$ and the initial condition $x_0 = [1 \;\; 1]'$, the problem of theorem 2 has been solved fixing the Π matrix elements and minimizing the function $\delta(\alpha) = \alpha_1' x_0 + \alpha_2' x_0$, that is an upper bound of the objective function $\min_i \alpha_i' x_0$, fulfilling (10). *This procedure enables us to determine the minimum $\delta^\star(\alpha) = 19.158$, corresponding to $\pi_{11} = -91$, $\pi_{22} = -100$ $\alpha_1 = [5.6795 \; 4.4984]'$ and $\alpha_2 = [5.8817 \; 3.0984]'$. Figure 1 shows the trajectories of the state variable $x(t) \in \mathbb{R}^2$ versus time for the switching system controlled by the state switching rule $\sigma(t) = u(x(t))$ given by* (7)*. As it can be seen, the proposed control strategy is effective in stabilizing the system under consideration. If we consider the problem formulation* (17)*, following the same rationale explained above, we can minimize the objective function that is now $\delta(\gamma)$. Figure 2 shows the behavior of the function $\delta(\gamma)$, which presents an horizontal asymptote for increasing values of γ. The minimum $\delta^\star(\gamma) = 13.06$ corresponds to $\alpha_1 = [4.2286 \; 2.3046]'$ and $\alpha_2 = [4.2294 \; 2.2993]'$. The state variables behavior is the same shown in Figure 1.*

Interestingly, a lower bound to the performance can be obtained following the same rationale of Theorem 2.

Theorem 3 *Assume that there exists a set of positive vectors $\{\alpha_1, \ldots, \alpha_N\}$, $\alpha_i \in \mathbb{R}_+^n$, and $\Pi \in \mathcal{M}$, satisfying the coupled co-positive Lyapunov inequalities:*

$$A_i' \alpha_j + \sum_{k=1}^{N} \pi_{ki} \alpha_k + q \succeq 0 \qquad i, j = 1, \ldots, N. \tag{21}$$

Then

$$\inf_\sigma \int_0^\infty q' x(t) dt \; \geq \max_{i=1,\ldots,N} \alpha_i' x(0)$$

Proof: Here we take

$$v(x) = \max_i \alpha_i' x$$

Then,

$$\begin{aligned} D^+ v(x(t)) &= \limsup_{h \to 0^+} \frac{v(x(t) + h A_i x(t)) - v(x(t))}{h} \\ &= \max_{k \in I(x(t))} \alpha_k' A_i x(t) \\ &\geq \alpha_j' A_i x(t) \\ &\geq - \sum_{k=1}^{N} \pi_{ki} \alpha_k' x(t) - q' x(t) \\ &= - \pi_{ii} \alpha_i' x(t) - \sum_{k \neq i} \pi_{ki} \alpha_k' x(t) - q' x(t) \\ &\geq - \pi_{ii} \alpha_i' x(t) - \sum_{k \neq i} \pi_{ki} \alpha_i' x(t) - q' x(t) \\ &= \; - q' x(t) \end{aligned} \tag{22}$$

Preprint submitted to 2010 American Control Conference.
Received September 21, 2009.

which proves the proposed theorem, since the optimal trajectory is such that $V(x(t)) \to 0$ as $t \to \infty$. ■

III. GUARANTEED INDUCED L_1 NORM

The computation of an induced norm for input-output switched system is quite difficult. Here we look at a switching rule that ensure stability and an upper bound of the induced L_1 norm for the switched positive system

$$\dot{x} = A_\sigma x + B_\sigma w \quad (23)$$

$$z = C_\sigma x + D_\sigma w \quad (24)$$

Since the system is positive, matrix A_i is Metzler and matrices B_i, C_i, D_i are positive.

We aim at finding a switching rule

$$u(x(t)) = \arg \min_{i=1,\dots,N} \alpha_i' x(t) \quad (25)$$

that makes the equilibrium solution $x = 0$ of the system (23)-(24) globally asymptotically stable and

$$J = \sup_{w \in L_1, w \neq 0} \frac{\int_0^\infty q_z' z(t) dt}{\int_0^\infty q_w' w(t) dt} \leq \gamma \quad (26)$$

where $q_z = [1\,1\,...\,1]'$ and $q_w = [1\,1\,...\,1]'$ of appropriate dimensions.

Theorem 4 *Assume that there exists a set of positive vectors $\{\alpha_1, \dots, \alpha_N\}$, $\alpha_i \in \mathbb{R}_+^n$, and $\Pi \in \mathcal{M}$, satisfying the coupled co-positive inequalities $\forall i = 1, 2, \cdots, N$:*

$$A_i'\alpha_i + \sum_{j=1}^N \pi_{ji}\alpha_j + C_i' q_z \prec 0 \quad (27)$$

$$B_i'\alpha_i + D_i' q_z \prec \gamma q_w \quad (28)$$

The state-switching control (2) *with*

$$u(x(t)) = \arg \min_{i=1,\dots,N} \alpha_i' x(t) \quad (29)$$

makes the equilibrium solution $x = 0$ of the system (1) *globally asymptotically stable and $J \leq \gamma$.*

Proof: We consider again the Lyapunov function (4). Since (27) implies (6), the system is readily seen to be stable under the action given by (29). Moreover, we have

$$\begin{aligned} D^+ v(x(t)) &= \limsup_{h \to 0^+} \frac{v(x(t) + hA_i x(t)) - v(x(t))}{h} \\ &= \min_{k \in I(x(t))} \alpha_k'(A_i x(t) + B_i w) \\ &\leq \alpha_i'(A_i x(t) + B_i w) < \\ &< -\sum_{j=1}^N \pi_{ji}\alpha_j' x(t) - q_z' C_i x(t) + \alpha_i' B_i w(t) \\ &< -\pi_{ii}\alpha_i' x(t) - \sum_{j \neq i} \pi_{ji}\alpha_j' x(t) \\ &\quad + \gamma q_w' w(t) - q_z' z(t) \\ &\leq \gamma q_w' w(t) - q_z' z(t), \end{aligned} \quad (30)$$

so that, for each $w \in L_1$ it follows

$$\int_0^\infty (q_z' z(t) - \gamma q_w w(t)) dt < \min_i \alpha_i' x(0)$$

The result is proved by letting $x(0) \to 0$. ■

Remark 2 *Notice that the L_1 induced norm of a stable positive system (A_i, B_i, C_i, D_i) is $\|G_i(0)\|_1$, where $G_i(s)$ is its transfer function. This fact can be also expressed in terms of linear inequalities as follows: $\|G_i(0)\|_1 < \gamma$ if and only if there exist a positive vector α_i such that*

$$A_i'\alpha_i + C_i' q_z = 0 \quad (31)$$

$$B_i'\alpha_i + D_i' q_z \prec \gamma q_w \quad (32)$$

Therefore, when all systems are stable, the attenuation level $\min_i \|G_i(0)\|_1$ can be attained by choosing the constant switching signal $\sigma(t) = \arg \min_i \|G_i(0)\|_1$. Notice, however that (27), (28) are certainly feasible only for $\gamma > \max_i \|G_i(0)\|$, taking $\Pi = 0 \in \mathcal{M}$.

Example 2 *Let us consider the example with $N = 2$ and matrices A_1, A_2 of Example 1. Moreover, $B_1 = B_2 = [1\ 1]'$, $C_1 = C_2 = [1\ 1]$, and $w = |sin(t)e^{-0.5t}|$. Following the same rationale of example 1 we have achieved $\gamma = 10.06$ corresponding to $\pi_{11} = -93$, $\pi_{22} = -100$, $\alpha_1 = [5.0957\ 3.963]'$ and $\alpha_2 = [5.2737\ 2.7346]'$. Figure 3 shows the state variables behavior. As it can be seen, the proposed control strategy is effective in stabilizing the system under consideration.*

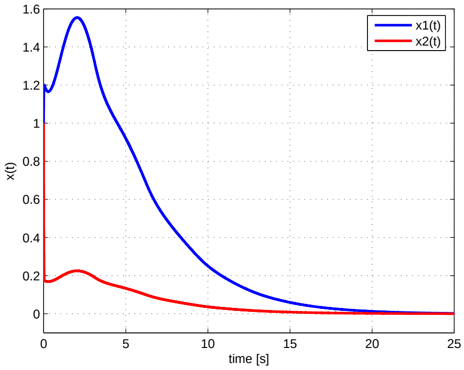

Fig. 3. Time simulation of the state switching control

IV. Optimal Control

In this section, we consider the optimal control problem for an autonomous linear switched positive system on a finite time interval. The cost functional to be minimized over all admissible switching sequences is given by

$$J(x_0, x, \sigma) = \int_0^{t_f} q'x(t)dt + \varsigma' x(t_f) \tag{33}$$

where $x(t)$ is a solution of (1) with the switching signal $\sigma(t)$. Vectors q and ς are assumed to have all positive entries.

The optimal switching signal, the corresponding trajectory and the optimal cost functional will be denoted as $\sigma^o(t, x_0)$, $x^o(t)$ and $J(x_0, x^o, \sigma^o)$ respectively.

The Hamiltonian function relative to system (1) and cost functional (33) is given by

$$H(x, \sigma, p) = q'x + p'A_\sigma x \tag{34}$$

Theorem 5 *Let $\sigma^o(t, x_0) : [0, t_f] \times \mathbb{R}^n_+ \to \mathcal{I} = \{1, \dots, N\}$ be an admissible switching signal relative to x_0 and $x^o(t)$ be the corresponding trajectory. If $\alpha(t)$ is a positive vector solution of the system of differential equations*

$$\dot{x}^o(t) = A_{\sigma^o(t,x_0)} x^o(t) \tag{35}$$

$$-\dot{\alpha}(t) = A'_{\sigma^o(t,x_0)} \alpha(t) + q \tag{36}$$

$$\sigma^o(t, x_0) = \arg\min_{i \in \mathcal{I}} \{\alpha'(t) A_i x^o(t)\} \tag{37}$$

with the boundary condition $x^o(0) = x_0$ and $\alpha(t_f) = \varsigma$ then $\sigma^o(t, x_0)$ is the optimal switching signal relative to x_0 and the value of the optimal cost functional is

$$J(x_0, x^o, \sigma^o) = \alpha'(0)x_0 \tag{38}$$

Proof: The scalar function

$$v(x, t) = \alpha(t)'x \tag{39}$$

is a generalized solution of the HJBE

$$0 = \frac{\partial v}{\partial t}(x, t) + H\left(x(t), \sigma^o(t, x_0), \frac{\partial v}{\partial x}(x, t)'\right) \tag{40}$$

where

$$H(x, \sigma, p) = q'x + p'A_\sigma x \tag{41}$$

In fact

$$\frac{\partial v}{\partial x}(x, t) = \alpha(t)' \tag{42}$$

$$\frac{\partial v}{\partial t}(x, t) = \dot{\alpha}(t)'x \tag{43}$$

so that, for almost all $t \in [0, t_f]$

$$\begin{aligned} &\dot{\alpha}(t)'x(t) + q'x(t) + \alpha(t)'A_{\sigma^o(t,x_0)}x(t) = \\ &= -\alpha(t)'A_{\sigma^o(t,x_0)}x(t) - q'x(t) + \\ &\quad + q'x(t) + \alpha(t)'A_{\sigma^o(t,x_0)}x(t) = \\ &= 0 \end{aligned} \tag{44}$$

Moreover it satisfies the boundary condition

$$v(x(t_f), t_f) = \alpha(t_f)'x(t_f) = \varsigma' x(t_f). \tag{45}$$

This completes the proof. ■

Notice that computation of the optimal control law as discussed in Theorem 5 is quite demanding. This is due to the two point nature of the problem that requires a back integration of (36) with a fixed final condition and a forward integration of the system (35) with a given initial condition.

V. Conclusions

In this paper we have dealt with the stabilization of linear switched positive systems. First of all we have found a stabilizing switching signal for both autonomous and input-output linear switched positive systems. For the latter an upper bound of the induced L_1 norm has been proposed. In both cases, the determination of a guaranteed cost has been addressed. Finally, an optimal control approach has been studied for autonomous linear switching positive systems, without any constraint on the switching signal.

References

[1] D. Liberzon, *Switching in Systems and Control*. Birkhäuser, 2003.

[2] R. A. DeCarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, 2000.

[3] G. Panzani, M. Corno, M. Tanelli, A. Zappavigna, S. Savaresi, A. Fortina, and S. Campo, “Combined performance and stability optimisation via central transfer case active control in four-wheeled vehicles,” in *Submitted to the 48th IEEE Conference on Decision and Control, 2009*.

[4] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, “Stability criteria for switched and hybrid systems,” *SIAM Review*, 2007.

[5] Z. Sun and S. S. Ge, “Analysis and synthesis of switched linear control systems,” *Automatica*, vol. 41, pp. 181–195, 2005.

[6] D. Liberzon and A. S. Morse, “Basic problems in stability and design of switched systems,” *IEEE control systems magazine*, 1999.

[7] S. Rinaldi and L. Farina, *Positive Linear Systems*. Wiley Interscience Series, 2000.

[8] L. Benvenuti and L. Farina, “Eigenvalue regions for positive systems,” *Systems and Control Letters*, vol. 51, pp. 325–330, 2004.

[9] T. Kaczorek, “Stabilization of positive linear systems,” in *Proceedings of the 37th IEEE Conference on Decision and Control, Tampa, Florida, USA*, 1998.

[10] P. D. Leenheer and D. Aeyels, “Stabilization of positive linear systems,” *Systems and Control Letters*, vol. 44, no. 4, pp. 259–271, 2001.

[11] L. Gurvits, R. Shorten, and O. Mason, “On the stability of switched positive linear systems,” *IEEE Transactions on Automatic Control*, 2007.

[12] R. Shorten and O. Mason, “On linear copositive Lyapunov functions and the stability of switched positive linear systems,” *IEEE Transactions on Automatic Control*, 2007.

[13] J. C. Geromel and P. Colaneri, “Stability and stabilization of continuous-time switched linear systems,” *SIAM Journal on Control and Optimization*, vol. 45, no. 5, pp. 1915–1930, 2006.

[14] P. Colaneri, J. C. Jeromel, and P. Bolzern, “Dynamic output feedback control of switched linear systems,” *IEEE Trans. Automatic Control*, vol. 53, no. 3, pp. 720–733, 2008.

[15] M. Athans and P. L. Falb, *Optimal Control: An Introduction to the Theory and Applications*. McGraw-Hill, 1966.

[16] A. Locatelli, *Optimal Control, An Introduction*. Birkäuser, Basel, 2001.

[17] X. Xu and P. J. Antsaklis, “Optimal Control of Switched Autonomous Systems,” in *Proceedings of the 41th IEEE CDC*, 2002.

[18] R. Shorten and O. Mason, “Quadratic and copositive lyapunov functions and the stability of positive switched linear systems,” in *Proceedings of the American Control Conference*, 2007.

[19] K. M. Garg, *Theory of Differentiation: A Unified Theory of Differentiation Via New Derivate Theorems and New Derivatives*. Wiley-Interscience, 1998.