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Stabilization of nonlinear systems with state and control constraints using Lyapunov-based predictive control ☆

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Abstract

This work considers the problem of stabilization of nonlinear systems subject to state and control constraints, for cases where the state constraints need to be enforced at all times (hard constraints) and where they can be relaxed for some time (soft constraints). We propose a Lyapunov-based predictive control design that guarantees stabilization and state and input constraint satisfaction for all times from an explicitly characterized set of initial conditions. An auxiliary Lyapunov-based analytical bounded control design is used to characterize the stability region of the predictive controller and also provide a feasible initial guess to the optimization problem in the predictive controller formulation. For the case when the state constraints are soft, we propose a switched predictive control strategy that reduces the time during which state constraints are violated, driving the states into the state and input constraints feasibility region of the Lyapunov-based predictive controller. We demonstrate the application of the Lyapunov-based predictive controller designs through a chemical process example.

Keywords: State constraints; Soft constraints; Input constraints; Model predictive control; Bounded Lyapunov-based control; Stability region; Feasibility region

1. Introduction

Control systems are often subject to constraints on their manipulated inputs and state variables. Input constraints arise as a manifestation of the physical limitations inherent in the capacity of control actuators (e.g., bounds on the magnitude of valve opening), and are enforced at all time (hard constraints). State constraints, on the other hand, arise either due to the necessity to keep the state variables within acceptable ranges, to avoid, for example, runaway reactions (in which case they need to be enforced at all times, and treated as hard constraints) or due to the desire to maintain them within desirable bounds dictated by performance considerations (in which case they may be relaxed, and treated as soft constraints). Constraints automatically impose limitations on our ability to steer the dynamics of the closed-loop system at will, and can cause severe deterioration in the nominal closed-loop performance and may even lead to closed-loop instability if not explicitly taken into account at the stage of controller design.

Currently, model predictive control (MPC), also known as receding horizon control (RHC), is one of the few control methods for handling state and input constraints within an optimal control setting and has been the subject of numerous research studies that have investigated the stability properties of MPC (e.g., see [1,20] for extensive surveys of various MPC formulations). In these MPC formulations the stability guarantees are typically based on an assumption of initial feasibility of the optimization problem, and the set of initial conditions, starting from where a given MPC formulation is guaranteed to be feasible, is not explicitly characterized. Attention has also been focused on the problem of state constraints satisfaction [12,31,13,26,6,32,3,27,28], and has typically been analyzed within the soft constraints framework, i.e., with the understanding that state constraints may be relaxed. In the minimum time approach, the state constraints are relaxed upto some time, and set as hard constraints thereafter. In other approaches, they are typically relaxed for all times, and only incorporated in the

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objective function as appropriate penalties on state constraint violation ('softening' of state constraints). In either approach, the problem of providing explicitly the set of initial conditions starting from where stabilization can be achieved and state and input constraints are guaranteed to be feasible has not been addressed.

The desire to implement control approaches that allow for an explicit characterization of their stability properties has motivated significant work on the design of stabilizing control laws, using Lyapunov techniques, that provide explicitly-defined regions of attraction for the closed-loop system; the reader may refer to [17] for a survey of results in this area. In [7,8], a class of Lyapunov-based bounded robust nonlinear controllers (inspired by the results on bounded control originally presented in [19]) that enforce robust stability from an explicitly characterized set of initial conditions, was developed (see [23,11] for utilization of the above controllers within the hybrid predictive control framework, and for stabilization of switched, nonlinear systems, respectively; see also [5] for further details and other applications). Despite their well-characterized stability and constraint-handling properties, those Lyapunov-based controllers are not guaranteed to be optimal with respect to an arbitrary performance criterion, and do not allow for incorporation of performance considerations in the design.

In a recent work, [22], we proposed a Lyapunov-based model predictive control formulation that provides guaranteed stability from an explicitly characterized set of initial conditions in the presence of input constraints. In this work, we propose a Lyapunov-based model predictive control design for stabilization of nonlinear systems with both state and input constraints. The design of the Lyapunov-based MPC uses a bounded controller, with its associated region of stability, as an auxiliary controller that is used to analyze the stability properties of the Lyapunov-based MPC. The proposed Lyapunov-based MPC is shown to possess an explicitly characterized set of initial conditions, starting from where it is guaranteed to be feasible, and hence stabilizing, while enforcing the state and input constraints at all times. For the case when the state constraints are soft, we propose a switched predictive control strategy that reduces the time for which state constraints are violated, driving the states into the state and input constraints feasibility region of the Lyapunov-based predictive controller. We demonstrate the application of the Lyapunov-based predictive controller designs through a chemical process example.

The rest of the paper is organized as follows: in Section 2, we describe the class of systems considered, and briefly review the bounded controller design. In Section 3, we present a Lyapunov-based predictive controller that guarantees stabilization from an explicitly characterized set of initial conditions, while enforcing the state and input constraints at all times. In Section 4, we present another predictive control formulation, and a switching strategy that reduces the time during which the state constraints are violated when state constraints can be treated as soft constraints. Finally, in Section 5, we present simulation results to demonstrate the application of our results.

2. Preliminaries

In this work, we consider the problem of stabilization of continuous-time nonlinear systems with state and input constraints, with the following state-space description:

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \tag{1}$$

$$u \in U, \tag{2}$$

$$x \in X, \tag{3}$$

where $x = [x_1 \cdots x_n]' \in \mathbb{R}^n$ denotes the vector of state variables, $u = [u^1 \cdots u^m]' \in \mathbb{R}^m$ denotes the vector of manipulated inputs, $U \subseteq \mathbb{R}^m$, $X \subseteq \mathbb{R}^n$ denote the constraints on the manipulated inputs and the state variables, respectively, $f(\cdot)$ is a sufficiently smooth $n \times 1$ nonlinear vector function, and $G(\cdot)$ is a sufficiently smooth $n \times m$ nonlinear matrix function. Without loss of generality, it is assumed that the origin is the equilibrium point of the unforced system (i.e. $f(0) = 0$). Throughout the paper, the notation $\|\cdot\|$ will be used to denote the standard Euclidean norm of a vector, while the notation $\|\cdot\|_Q$ refers to the weighted norm, defined by $\|x\|_Q^2 = x'Qx$ for all $x \in \mathbb{R}^n$, where Q is a positive-definite symmetric matrix and x' denotes the transpose of x. The notation $L_f V$ denotes the standard Lie derivative of a scalar function $V(\cdot)$ with respect to the vector function $f(\cdot)$. In order to provide the necessary background for our results in Sections 3 and 4, we will briefly review in the remainder of this section the design procedure for, and the stability properties of, a bounded control design, which will be used to characterize the feasibility region of the Lyapunov-based MPC formulation in Section 3. Throughout the manuscript, we assume that for any $u \in \mathbf{U}$ the solution of the system of Eq. (1) exists and is continuous for all t, and we focus on the state feedback problem where measurements of $x(t)$ are assumed to be available for all t.

2.1. Bounded Lyapunov-based control

Consider the system of Eq. (1) for which a control Lyapunov function, V, exists. Using the results in [19] (see also [7]), the following bounded control law can be constructed:

$$u(x) = \begin{Bmatrix} -k(x)(L_G V)'(x), & \|(L_G V)'(x)\| \neq 0 \\ 0, & \|(L_G V)'(x)\| = 0 \end{Bmatrix} := b(x), \tag{4}$$

where

$$k(x) = \frac{L_f^* V(x) + \sqrt{(L_f^* V(x))^2 + (u^{\max}\|(L_G V)'(x)\|)^4}}{\|(L_G V)'(x)\|^2 \left[1 + \sqrt{1 + (u^{\max}\|(L_G V)'(x)\|)^2}\right]}. \tag{5}$$

$L_G V(x) = [L_{g^1} V \cdots L_{g^m} V]$ is a row vector, where g^i is the ith column of G, $L_f^* V = L_f V + \rho V$ and $\rho > 0$, and $u^{\max}$ is a real positive number such that $\|u\| \leqslant u^{\max}$ implies $u \in U$. For the above controller, one can show, using a standard Lyapunov argument, that whenever the closed-loop state, x, evolves within the region described by the set

$$\Phi_{x,u} = \{x \in X : L_f^* V(x) \leqslant u^{\max} \|(L_G V)'(x)\|\}, \tag{6}$$

then the controller satisfies the state and input constraints, and the time-derivative of the Lyapunov function is negative-definite. Therefore, starting from any initial state in the set $\Phi_{x,u}$, asymptotic stability of the constrained closed-loop system can be guaranteed, provided that the state trajectory of the closed-loop system remains within the state-space region described by the set $\Phi_{x,u}$. To ensure this, we need to construct an invariant subset (preferably the largest) of $\Phi_{x,u}$. One way to construct such a subset is using the level sets of V, i.e.,

$$\Omega_{x,u} = \{x \in \mathbb{R}^n : V(x) \leqslant c_{x,u}^{\max}\}, \tag{7}$$

where $c_{x,u}^{\max} > 0$ is the largest number for which $\Omega_{x,u} \subseteq \Phi_{x,u}$. $\Omega_{x,u}$ then provides an estimate of the stability region, starting from where the origin of the constrained closed-loop system, under the control law of Eqs. (4) and (5), is guaranteed to be asymptotically stable and state and input constraints are satisfied for all time.

The bounded controller of Eqs. (4) and (5) possesses a robustness property with respect to measurement errors, that preserves closed-loop stability when the control action is implemented in a discrete (sample and hold) fashion with a sufficiently small hold time (Δ). Specifically, the control law ensures that, for all initial conditions in $\Omega_{x,u}$, the closed-loop state remains in $\Omega_{x,u}$ and eventually converges to some neighborhood of the origin (we will refer to this neighborhood as Ω^b) whose size depends on Δ. This robustness property, was formalized in [22] for the problem of stabilization under input constraints, and carries over to the case of input and state constraints. This property will be exploited in the Lyapunov-based predictive controller design of Section 3 and is formalized in Proposition 1 below (the proof of the proposition is similar to that of Proposition 1 in [22], and is omitted for brevity). For further results on the analysis and control of sampled-data nonlinear systems, the reader may refer to [14,24,16,30].

Proposition 1. *Consider the constrained system of Eq.* (1), *under the bounded control law of Eqs.* (4) *and* (5) *with* $\rho > 0$ *and let* $\Omega_{x,u}$ *be the stability region estimate under continuous implementation of the bounded controller. Let* $u(t) = u(j\Delta)$ *for all* $j\Delta \leqslant t < (j+1)\Delta$ *and* $u(j\Delta) = b(x(j\Delta))$, $j = 0, \ldots, \infty$. *Then, given any positive real number d, there exist positive real numbers* Δ^*, δ' *and* ε^* *such that if* $\Delta \in (0, \Delta^*]$ *and* $x(0) := x_0 \in \Omega_{x,u}$, *then* $x(t) \in \Omega_{x,u} \subseteq X$ *and* $\limsup_{t\to\infty} \|x(t)\| \leqslant d$. *Also, if* $V(x_0) \leqslant \delta'$ *then* $V(x(\tau)) \leqslant \delta' \, \forall \tau \in [0, \Delta)$ *and if* $\delta' < V(x_0) \leqslant c_{x,u}^{\max}$, *then* $\dot V(x(\tau)) \leqslant -\varepsilon^* \, \forall \tau \in [0, \Delta)$.

Remark 1. Control Lyapunov function (CLF)-based stabilization of nonlinear systems has been studied extensively in the nonlinear control literature (e.g., see [2,19,29]). The construction of constrained CLFs (i.e., CLFs that take the constraints into account) remains a difficult problem (especially for nonlinear systems) that is the subject of ongoing research. For several classes of nonlinear systems that arise commonly in the modeling of engineering systems, systematic and computationally feasible methods are available for constructing unconstrained CLFs (CLFs for the unconstrained system) by exploiting the system structure. Examples include the use of quadratic functions for feedback linearizable systems and the use of backstepping techniques to construct CLFs for systems in strict feedback form. In this work, the bounded controllers in Eqs. (4) and (5) are designed using unconstrained CLFs, which are also used to explicitly characterize the associated regions of stability via Eqs. (7) and (6). While the resulting estimates do not necessarily capture the entire domain of attraction (this remains an open problem even for linear systems), we will use them throughout the paper for a concrete illustration of the basic ideas of the results. It is possible to obtain substantially improved estimates by using, for example, a combination of several CLFs (see, for example, [9,23]).

3. Lyapunov-based model predictive control

Consider model predictive control of the system of Eq. (1) with hard state and input constraints. We present here a Lyapunov–based MPC formulation (see Remark 3 for a discussion on this formulation and its relationship to other Lyapunov-based formulations) that guarantees feasibility of the optimization problem subject to hard constraints on the state and input, and hence constrained stabilization of the closed-loop system from an explicitly characterized set of initial conditions. For this MPC design, the control action at state x and time t is obtained by solving, online, a finite horizon optimal control problem of the form:

$$P(x,t) : \min \quad \{J(x,t,u(\cdot)) | u(\cdot) \in S, \; x \in X\}, \tag{8}$$

$$\text{s.t.} \quad \dot x = f(x) + G(x)u, \tag{9}$$

$$\dot V(x(\tau)) \leqslant -\varepsilon^* \quad \forall \tau \in [t, t+\Delta) \quad \text{if } V(x(t)) > \delta', \tag{10}$$

$$V(x(\tau)) \leqslant \delta' \quad \forall \tau \in [t, t+\Delta) \; \textit{if} \; V(x(t)) \leqslant \delta', \tag{11}$$

where $S = S(t, T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into U, and T is the specified horizon. Eq. (9) is the nonlinear model describing the time evolution of the state x, V is the Lyapunov function used in the bounded controller design and δ', ε^* are defined in Proposition 1. A control $u(\cdot)$ in S is characterized by the sequence $\{u[j]\}$ where $u[j] := u(j\Delta)$ and satisfies $u(t) = u[j]$ for all $t \in [j\Delta, (j+1)\Delta)$. The performance index is given by

$$J(x,t,u(\cdot)) = \int_t^{t+T} [\|x^u(s;x,t)\|_Q^2 + \|u(s)\|_R^2]\, \mathrm{d}s, \tag{12}$$

where Q and R are positive semi-definite, and strictly positive definite, symmetric matrices, respectively, and $x^u(s;x,t)$ denotes the solution of Eq. (1), due to control u, with initial state x at time t. The minimizing control $u^0(\cdot) \in S$ is then applied to the plant over the interval $[j\Delta, (j+1)\Delta)$ and the procedure is repeated indefinitely. This defines an implicit model predictive control law

$$M(x) := \arg\min(J(x, t, u(\cdot))) := u_1. \quad (13)$$

Closed-loop stability and state and input constraint feasibility properties of the closed-loop system under the Lyapunov-based predictive controller are inherited from the bounded controller under discrete implementation and are formalized in Proposition 2 below.

Proposition 2. *Consider the constrained system of Eq.* (1) *under the MPC law of Eqs.* (8)–(13) *with* $\Delta \leqslant \Delta^*$ *where* Δ^* *was defined in Proposition* 1. *Then, given any* $x_0 \in \Omega_{x,u}$, *where* $\Omega_{x,u}$ *was defined in Eq.* (7), *the optimization problem of Eqs.* (8)–(13) *is feasible for all times*, $x(t) \in \Omega_{x,u} \subseteq X$ *for all* $t \geqslant 0$ *and* $\limsup_{t\to\infty} \|x(t)\| \leqslant d$.

Proof. The proof of this proposition is divided in to three parts. In the first part, we show that for all $x_0 \in \Omega_{x,u}$, the predictive control design of Eqs. (8)–(13) is feasible. We then show that $\Omega_{x,u}$ is invariant under the predictive control algorithm of Eqs. (8)–(13), and that the state and input constraints are satisfied for all times. Finally, we prove practical stability for the closed-loop system.

Part 1: Consider some $x_0 \in \Omega_{x,u}$ under the predictive controller of Eqs. (8)–(13), with a prediction horizon $T = N\Delta$, where Δ is the hold time and $1 \leqslant N < \infty$ is the number of the prediction steps. The initial condition can either be such that $V(x_0) \leqslant \delta'$ or $\delta' < V(x_0) \leqslant c_{x,u}^{\max}$.

Case 1: If $\delta' < V(x_0) \leqslant c_{x,u}^{\max}$, the control input trajectory under the bounded controller of Eqs. (4) and (5) provides a feasible solution to the constraint of Eq. (10) (see Proposition 1). A feasible initial guess for the optimization problem of Eqs. (8)–(13) therefore exists, and, in particular, is given by $u(j\Delta) = b(j\Delta)$, $j = 1, \ldots, N$. Note that if $u = b(\cdot)$ for $t = [0, \Delta]$, and $\Delta \in (0, \Delta^*]$, then $\dot{V} \leqslant -\varepsilon^*$ and $b(\cdot) \in U$ (since $b(\cdot)$ is computed using the bounded controller of Eqs. (4) and (5)). Also, under discrete implementation of the bounded controller of Eqs. (4) and (5), for any $x_0 \in \Omega_{x,u}$, $x(t) \in \Omega_{x,u}\, \forall t \geqslant 0$, therefore, the constraint $x(t) \in X$ is also satisfied by this control trajectory (note that $\Omega_{x,u} \subseteq \Phi_{x,u} \subseteq X$).

Case 2: If $V(x_0) \leqslant \delta'$, once again we infer from Proposition 1 that the control input trajectory provided by the bounded controller of Eqs. (4) and (5) provides a feasible initial guess, given by $u(j\Delta) = b(x(j\Delta))$, $j = 1, \ldots, N$ (recall from Proposition 1, that under the bounded controller of Eqs. (4) and (5), if $V(x_0) \leqslant \delta'$ then $V(x(t)) \leqslant \delta'\ \forall\, t \geqslant 0$). This shows that for all $x_0 \in \Omega_{x,u}$, the Lyapunov-based predictive controller of Eqs. (8)–(13) is feasible for all times.

Part 2: As shown in Part 1, for any $\delta' < V(x_0) \leqslant c_{x,u}^{\max}$, the constraint of Eq. (10) in the optimization problem is feasible. Upon implementation, therefore, the value of the Lyapunov function decreases, and since $\Omega_{x,u}$ is a level set of V, the closed-loop state trajectory cannot escape $\Omega_{x,u}$. On the other hand, if $V(x_0) \leqslant \delta'$, feasibility of the constraint of Eq. (11) guarantees that the closed-loop state trajectory evolves such that $V(x(t)) \leqslant \delta'\, \forall t \geqslant 0$. In both cases, $\Omega_{x,u}$ continues to be an invariant region under the Lyapunov-based predictive controller of Eqs. (8)–(13). Also, since $\Omega_{x,u} \subseteq \Phi_{x,u} \subseteq X$, we have that $x(t) \in X$ for all $t \geqslant 0$.

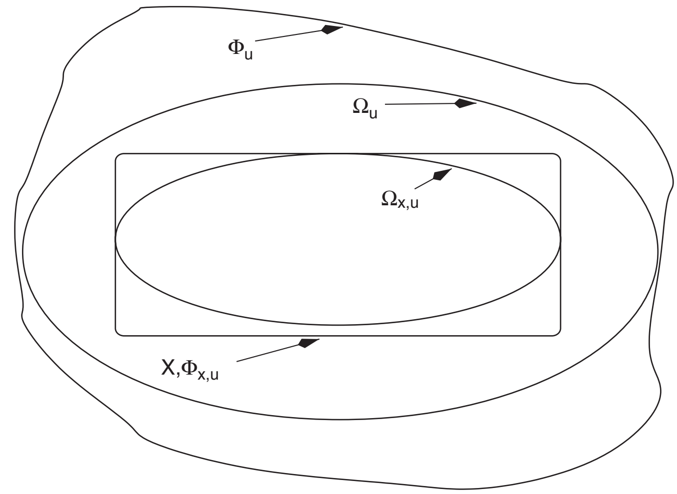

Fig. 1. A schematic representing the input (ϕ_u, Ω_u) and state and input ($\phi_{x,u}$, $\Omega_{x,u}$) constrained stability regions, together with the set (X) describing the state constraints.

Part 3: Finally, consider an initial condition, x_0, such that $\delta' < V(x_0) \leqslant c_{x,u}^{\max}$. Since the optimization problem continues to be feasible for all $t \geqslant 0$, we have that $V(x(t+\Delta)) < V(x(t))$ for all $\delta' < V(x(t)) \leqslant c_{x,u}^{\max}$. All trajectories originating in $\Omega_{x,u}$, therefore, converge to the set defined by $\Omega^t := \{x \in \mathbb{R}^n : V(x) \leqslant \delta'\}$. For $V(x_0) \leqslant \delta'$, the feasibility of the optimization problem of Eqs. (8)–(13) implies $V(x(t)) \leqslant \delta'\, \forall t \geqslant 0$. Therefore, for all $x_0 \in \Omega_{x,u}$, $\limsup_{t\to\infty} V(x(t)) \leqslant \delta'$. Note that since $V(\cdot)$ is a continuous function of the state, one can find a finite, positive real number, δ', such that $V(x) \leqslant \delta'$ implies $\|x\| \leqslant d$. For such a choice of δ', we therefore have that $\limsup_{t\to\infty} \|x(t)\| \leqslant d$.This completes the proof of Proposition 2. □

Remark 2. The set $\Phi_{x,u}$ defined by Eq. (6) can be understood as the intersection of the sets $\Phi_u = \{x \in \mathbb{R}^n : L_f^* V(x) \leqslant u^{\max} \|(L_G V)'(x)\|\}$, which defines the set where $\dot{V} < 0$ and $\|u\| \leqslant u^{\max}$ (the set defined by $\Omega_u = \{x \in \mathbb{R}^n : V(x) \leqslant c_u^{\max}\}$ where $c_u^{\max} > 0$ is the largest number for which $\Omega_u \subseteq \Phi_u$, therefore, is the region of guaranteed feasibility for the Lyapunov-based predictive controller in the absence of state constraints, see [22]), and the set X describing the state constraints. It can be argued that the set $\Phi_{x,u}$ is a sufficiently nonconservative estimate of the set of initial conditions within which the level set, $\Omega_{x,u}$, can be constructed (see Fig. 1). To this end, consider the following cases: if $X \subset \Phi_u$ (as in Fig. 1), the closed-loop system cannot be initialized outside of X, because in that case the state constraints are violated at the outset (see Section 4 for how this can be handled in the case that the state constraints can be treated as soft constraints). On the other hand, if $\Phi_u \subset X$, the closed-loop system state cannot be initialized from outside of Φ_u, since outside of this set, negative definiteness of $\dot{V}$ is not guaranteed under input constraints (this points to the fact that there may be initial conditions that satisfy the state constraints, but starting from where it may not be possible to stabilize the system due to the presence of input constraints). Also, it is possible to generate a larger estimate of Φ_u, by using, for example, a family of Lyapunov functions [9], to cover larger portions of the set X. Note also that, in general, merely initializing the closed-loop system inside of X is not sufficient to guarantee subsequent satisfaction of the state constraints because the closed-loop state trajectory, even if it eventually stabilizes, may do so in a way that it goes out of X, and violates the state constraints before stabilizing (this necessitates the construction of the invariant set $\Omega_{x,u}$).

Remark 3. Note that the predictive controller formulation of Eqs. (8)–(13) requires that the value of the Lyapunov function decrease during the first step only. Practical stability of the closed-loop system is achieved since, due to the receding nature of controller implementation, only the first move of the set of calculated moves is implemented and the problem is resolved at the next time step. If the optimization problem is initially feasible, and continues to be feasible, then every control move that is implemented, enforces a decay in the value of the Lyapunov function, leading to stability (see Remark 4 for how initial and subsequent feasibility is guaranteed for an explicitly characterized set of initial conditions). Lyapunov-based predictive control approaches (see, for example, [18,25]) typically incorporate a similar Lyapunov function decay constraint, albeit requiring the constraint of Eq. (10) to hold at the *end* of the prediction horizon as opposed to only the first time step. An input trajectory that only requires the value of the Lyapunov function value to decrease at the end of the horizon may involve the state trajectory going out of the level set (and therefore, possibly out of the state constraint satisfaction region, violating the state constraints), and motivates using a constraint that requires the Lyapunov function to decrease during the first time step (this also facilitates the explicit characterization of the feasibility region).

Remark 4. For $0 < \Delta \leqslant \Delta^*$, the constraint of Eq. (10), is guaranteed to be satisfied (the control action computed by the bounded controller design provides a feasible initial guess to the optimization problem). Note that this is so because the constraint requires the Lyapunov function value to decay, not at the *end* of the prediction horizon (as is customarily done in Lyapunov-based MPC approaches), but only during the first time step. Furthermore, since the state is initialized in $\Omega_{x,u}$, which is a level set of V, the closed-loop system evolves so as to stay within $\Omega_{x,u}$, thereby guaranteeing feasibility at future times. Since the level set $\Omega_{x,u}$ is completely contained in the set defining the constraints on the states, and the state trajectory under the predictive controller continues to evolve within this set, the state constraints are satisfied at all times.

Remark 5. For linear systems, the work in [26] uses the minimum time approach, where the smallest time, $T_{\min}$, beyond which the state constraints can be satisfied on an infinite horizon is identified, and the state constraints are relaxed upto that time. The need to satisfy the constraints after $T_{\min}$ may, however, result in large violations of the state constraints for times prior to $T_{\min}$. Note also that even if the state constraints are relaxed upto a time, initial conditions starting from where the closed-loop system can be stabilized are not explicitly characterized. Furthermore, while possible for linear systems, the computation of $T_{\min}$ is a more difficult task for nonlinear systems. Even if it is computable, the state constraints are not enforced at all times, and hence not strictly treated as hard constraints. In contrast, the Lyapunov-based MPC of Eqs. (8)–(13) guarantees feasibility and stabilization from an explicitly characterized set of initial conditions while enforcing the state and input constraints at all times.

Remark 6. Note that the constraints of Eqs. (10)–(11) incorporate stability considerations in the control design, which override the performance considerations specified by the objective function of Eq. (12) (which means that the implemented control action may not be the one dictated by the performance objective of Eq. (12) alone). In contrast to analytic bounded control methods, however, the predictive control design allows for specification of performance objective in the control design that ultimately determines the choice of control action out of the allowable (feasible) set of control moves determined by the constraints on the manipulated input and the states and the stability requirements.

Remark 7. Together with the computational difficulties of solving a nonlinear optimization problem at each time step, one of the key challenges that impact on the practical implementation of nonlinear MPC (NMPC) is the inherent difficulty of characterizing, a priori, the set of initial conditions starting from where a given NMPC controller is guaranteed to stabilize the closed-loop system, or for a given set of initial conditions, to identify the value of the prediction horizon for which the optimization problem will be feasible subject to state and input constraints. The Lyapunov-based predictive controller formulation guarantees initial and subsequent feasibility of the optimization problem irrespective of the choice of the prediction horizon and also provides, at the same time, an explicit characterization of a set of initial conditions starting from where stability is guaranteed. In addition, the optimization problem in the predictive controller is initialized with a feasible initial guess, which substantially reduces the computational burden. Note also that any other Lyapunov-based analytic control design can be used as the auxiliary controller, and the choice is not limited to the bounded controller used in this paper. The Lyapunov-based analytic control design should, however, provide an explicit characterization of the state and input constrained stability region, and be robust with respect to discrete implementation.

4. Handling soft state constraints via controller switching

Consider now the nonlinear system of Eq. (1) where the state constraints represent desired bounds on the values of the state variables. In this case, the state constraints can be treated as soft constraints, allowing their violation for some period of time. It is important, nevertheless, to implement control action that reduces the time for which constraints are violated. We propose in this section a control design that uses two predictive formulations and switches between them. First we design a predictive controller that, while respecting input constraints, drives the state trajectory into the feasible region of the predictive controller formulation of Proposition 2, in a way that reduces the time for which the state constraints are violated, and then implement the predictive controller of Proposition 2 to achieve stabilization together with state and input constraint satisfaction for the rest of the time. To this end, we first cast the system of Eq. (1) as a switched system of the form:

$$\dot{x} = f(x) + G(x)u_{i(t)} \quad i \in \{1, 2\}, \tag{14}$$

where $i : [0, \infty) \to \{1, 2\}$ is the switching signal which is assumed to be a piecewise continuous (from the right) function of time, implying that only a finite number of switches between the two controllers is allowed on any finite-time interval. The index, $i(t)$, represents a discrete state that indexes the control input, u, with the understanding that $i(t) = 1$ if and only if $u_i(x(t)) = u_1$ (i.e., the Lyapunov-based MPC formulation of Eqs. (8)–(13) is used) and $i(t) = 2$ if and only if $u_i(x(t)) = u_2$ (i.e., an MPC formulation designed to reduce the time of state constraint violation, is used). Theorem 1 below presents both the control law, u_2, and the switching law.

Theorem 1. *Consider the switched nonlinear system of Eq.* (14), *for which there exists a control Lyapunov function V, and for a given pair of positive real numbers* (d, ρ), Δ *is chosen such that* $\Delta \leqslant \Delta^*$, *where* Δ^* *was defined in Proposition* 1. *Given any initial condition* $x_0 \in \Omega_u$, *where* Ω_u *was defined in Remark* 2, *let* T_b *be the time it takes for the bounded controller of Eqs.* (4) *and* (5), *under discrete implementation with a discretization step* Δ, *to achieve* $x(T_b) \in \Omega_{x,u}$. *Consider the following optimization problem*:

$$u = \arg\min(J) := u_2, \tag{15}$$

$$J = qV(x(t + \Delta)) + \int_t^{t+T} [\|u(s)\|_R^2]\,\mathrm{d}s, \tag{16}$$

where $q > 0$, $R > 0$, T *is the prediction horizon given by* $T = T_b - t$, *subject to the following constraints*:

$$\dot{x} = f(x) + G(x)u, \tag{17}$$

$$u \in U, \tag{18}$$

$$V(x(t + k\Delta)) \leqslant V(x(t+(k-1)\Delta)), \quad k = 1, \ldots, T/\Delta, \tag{19}$$

$$x(t + T) \in \Omega_{x,u}. \tag{20}$$

Let T_{switch} *be the earliest time such that* $x(T_{\mathrm{switch}}) \in \Omega_{x,u}$, *where* $\Omega_{x,u}$ *was defined in Eq.* (7), *under the controller of Eqs.* (15)–(20). *Then, the following switching law*:

$$i(t) = \left\{ \begin{array}{ll} 2, & 0 \leqslant t \leqslant T_{\mathrm{switch}} \\ 1, & t > T_{\mathrm{switch}} \end{array} \right\} \tag{21}$$

ensures, for the closed-loop system, that $x(T_b) \in \Omega_{x,u}$, $T_{\mathrm{switch}} \leqslant T_b$, $x(t) \in \Omega_{x,u} \subseteq X \; \forall t > T_{\mathrm{switch}}$ *and* $\limsup_{t\to\infty} \|x(t)\| \leqslant d$.

Proof. The proof of the theorem proceeds as follows: we first show that the optimization problem of Eqs. (15)–(20) is feasible for all $0 \leqslant t \leqslant T_{\mathrm{switch}}$, $x(T_{\mathrm{switch}}) \in \Omega_{x,u}$ and that $T_{\mathrm{switch}} \leqslant T_b$. Then, we use the result of Proposition 2 to show that for $t > T_{\mathrm{switch}}$, the controller of Eqs. (8)–(13) ensures that $x(t) \in \Omega_{x,u} \subseteq X \; \forall \; t > T_{\mathrm{switch}}$ and $\limsup_{t\to\infty} \|x(t)\| \leqslant d$.

Case 1: Consider $x_0 \in \Omega_u \backslash \Omega_{x,u}$. Using the result of Proposition 1 in [22], we have that under discrete implementation of the bounded controller of Eqs. (4) and (5) with $\Delta \leqslant \Delta^*$, the state trajectory, starting from x_0, evolves such that $x(t) \in \Omega_u$ and $V(x(t + \Delta)) \leqslant V(x(t))$ for all $t \geqslant 0$. From the definition of T_b, we have that $x(T_b) \in \Omega_{x,u}$ under discrete implementation of the bounded controller. The optimization problem of Eqs. (15)–(20) is guaranteed to be initially feasible, since a feasible initial guess can always be obtained using the control input trajectory under the bounded controller and is given by

$$u(k\Delta) = b(x(k\Delta)), \quad k = 1, \ldots, T/\Delta.$$

Subsequently, the tail of the solution at the first time step:

$$u(k\Delta), \; k = 2, \ldots, T/\Delta,$$

is a feasible initial guess for the constraints in the optimization problem at the next time step (at the next time step, the horizon reduces from $T = T_b$ to $T = T_b - \Delta$). Note that under the implementation of the solution of the control trajectory at the first time step, we get

$$V(x(t + k\Delta)) \leqslant V(x(t + (k - 1)\Delta)), \quad k = 1, \ldots, T/\Delta.$$

Under the implementation of the tail, therefore, we have that

$$V(x(t + k\Delta)) \leqslant V(x(t + (k - 1)\Delta)), \quad k = 2, \ldots, T/\Delta$$

and also $x(T_b) \in \Omega_{x,u}$, which is the constraint that the optimization problem needs to enforce at the next time step. The optimization problem of Eqs. (15)–(20), therefore, is guaranteed to be initially and successively feasible, and hence $x(T_b) \in \Omega_{x,u}$.

By definition of T_{switch}, if the state trajectory enters $\Omega_{x,u}$ before T_b, then T_{switch} is set to that value, hence $x(T_{\mathrm{switch}}) \in \Omega_{x,u}$ where $T_{\mathrm{switch}} \leqslant T_b$. From Proposition 2, we get that for all $x(T_{\mathrm{switch}}) \in \Omega_{x,u}$, $x(t) \in \Omega_{x,u} \subseteq X \; \forall t \geqslant T_{\mathrm{switch}}$ and $\limsup_{t\to\infty} \|x(t)\| \leqslant d$.

Case 2: For any initial condition $x_0 \in \Omega_{x,u} \subseteq \Omega_u$ we have that $T_b = T_{\mathrm{switch}} = 0$ (since $x_0 \in \Omega_{x,u}$), and the switching law of Eq. (21) dictates that the controller of Eqs. (8)–(13) is implemented for all times. Since $x_0 \in \Omega_{x,u}$, from Proposition 2, we get that $x(t) \in \Omega_{x,u} \subseteq X \; \forall t \geqslant 0$ and $\limsup_{t\to\infty} \|x(t)\| \leqslant d$. This completes the proof of Theorem 1. □

Remark 8. The implementation of the predictive controller of Theorem 1 is described algorithmically below:

(1) Given the system model of Eq. (1), the constraints on the input and states, and a control Lyapunov function, V, design

the bounded controller of Eqs. (4) and (5) and compute the stability region estimate under the input constraints, Ω_u, and that under the state and input constraints, $\Omega_{x,u}$.

(2) Given the size of the ball that the state is required to converge to, d, compute Δ^* for the predictive controller of Proposition 2 and choose $\Delta \in (0, \Delta^*]$ for the purpose of MPC implementation.

(3) If $x_0 \in \Omega_{x,u}$, proceed to step 6, else continue.

(4) For $x_0 \in \Omega_u$, compute the time T_b taken by the bounded controller under discrete implementation, with a discretization time Δ, to drive the state trajectory inside $\Omega_{x,u}$.

(5) Implement the controller of Eqs. (15)–(20) until the time (T_{switch}) that the state trajectory enters $\Omega_{x,u}$ (this is guaranteed to happen, by T_b at the latest), at which time, switch to the predictive controller of Proposition 2 (Eqs. (8)–(13)).

(6) Implement the predictive controller of Proposition 2 (Eqs. (8)–(13)) to achieve stabilization under input and state constraints.

Remark 9. For linear systems, the problem of state constraints satisfaction is typically handled by relaxing the constraints, while appropriately penalizing the state constraint violation within the objective function [32], or solving a multi-objective problem [28] that minimizes both the duration and size of state constraint violation. While these approaches do away with any potential infeasibility due to the state constraints, they do not limit the time for which the state constraints are violated. The use of the bounded controller, to obtain an estimate of the time within which the state trajectory can be driven inside $\Omega_{x,u}$ allows the use of this value in the constraint of Eq. (20) and guarantees state constraint satisfaction by that time. Furthermore, since the objective function minimizes the Lyapunov function value itself at the next time instance, and the target set, $\Omega_{x,u}$, is a level set, it is likely that the resulting control action will drive the trajectory inside the feasible region faster, and result in a smaller time (compared to implementing the bounded controller itself) for which the state constraints are violated (see the simulation example for a demonstration).

Remark 10. The problem of implementing MPC with guaranteed stability regions was recently addressed for linear systems under state [10] and output [21] feedback control, and for input constrained nonlinear systems with [23] and without [9] uncertainty, by means of a hybrid predictive control structure that embeds the implementation of MPC within the stability region of a Lyapunov-based bounded controller and uses the bounded controller as a fall-back component that can be switched to in the event of infeasibility or instability of the predictive controller. In this work, the switching takes place between two different predictive control formulations (the controller of Eqs. (15)–(20) and the controller of Eqs. (8)–(13)). Unlike the hybrid predictive control structure, switching between the two controllers here is not to provide a fall-back mechanism in the event of infeasibility (the Lyapunov-based predictive controller of Eqs. (8)–(13) is guaranteed to be feasible from an explicitly characterized set of initial conditions), but rather to use the controller of Eqs. (15)–(20) to guide the system trajectory into the state and input constrained stability region of the Lyapunov-based predictive control design of Eqs. (8)–(13). The bounded controller design is used in this work, not as a fall-back, but for the purpose of providing an estimate of the feasibility region (and, therefore, the stability region) for the Lyapunov-based predictive controller, and feasible initial guesses for the control moves (the decision variables in the optimization problem).

Remark 11. Note that while the closed-loop system is formulated as a switched system (see Eq. (14)), the switching law restricts the number of switches between the two controllers to at most one (the switch takes place only if $x_0 \in \Omega_u \backslash \Omega_{x,u}$). The closed-loop system, therefore, does not need to satisfy the multiple Lyapunov function stability criteria [4,15], which would have to be satisfied if there were back and forth switchings between the controllers over the infinite time interval. The switching law of Eq. (21) also avoids any chattering by allowing only a finite number of switches (in this case, one) over any finite time interval.

5. Application to a chemical process example

Consider a continuous stirred tank reactor where an irreversible, first-order exothermic reaction of the form $A \xrightarrow{k} B$ takes place. The mathematical model for the process takes the form:

$$\dot{C}_A = \frac{F}{V}(C_{A0} - C_A) - k_0 e^{(-E/RT_R)} C_A,$$
$$\dot{T}_R = \frac{F}{V}(T_{A0} - T_R) + \frac{(-\Delta H)}{\rho c_p} k_0 e^{(-E/RT_R)} C_A + \frac{Q_\sigma}{\rho c_p V}, \tag{22}$$

where C_A denotes the concentration of the species A, T_R denotes the temperature of the reactor, Q_σ is the heat removed from the reactor, with σ indexing the control law, V is the volume of the reactor, k_0, E, ΔH are the pre-exponential constant, the activation energy, and the enthalpy of the reaction and c_p and ρ, are the heat capacity and fluid density in the reactor. The values of all process parameters can be found in Table 1. The control objective is to stabilize

Table 1
Process parameters and steady-state values

$V = 0.1\,\mathrm{m}^3$
$R = 8.314\,\mathrm{KJ/Kmol\,K}$
$C_{A0_s} = 1.0\,\mathrm{Kmol/m}^3$
$T_{A0_s} = 310.0\,\mathrm{K}$
$Q_s = 0.0\,\mathrm{KJ/min}$
$\Delta H = -4.78 \times 10^4\,\mathrm{KJ/Kmol}$
$k_0 = 72 \times 10^9\,\mathrm{min}^{-1}$
$E = 8.314 \times 10^4\,\mathrm{KJ/Kmol}$
$c_p = 0.239\,\mathrm{KJ/Kg\,K}$
$\rho = 1000.0\,\mathrm{Kg/m}^3$
$F = 100 \times 10^{-3}\,\mathrm{m}^3/\mathrm{min}$
$T_{Rs} = 395.33\,\mathrm{K}$
$C_{As} = 0.57\,\mathrm{Kmol/m}^3$

the reactor at the unstable equilibrium point $(C_A^s, T_R^s) = (0.57\,\text{Kmol/m}^3, 395.3\,\text{K})$, while keeping the state variables between $C_A^{\min} = 0.41\,\text{Kmol/m}^3 \leqslant C_A \leqslant 0.73\,\text{Kmol/m}^3 = C_A^{\max}$ and $T_R^{\min} = 392.3\,\text{K} \leqslant T_R \leqslant 398.3\,\text{K} = T_R^{\max}$ using the rate of heat input, Q, and change in inlet concentration of species A, $\Delta C_A = C_{A0} - C_{A0_s}$ as manipulated inputs with constraints: $|Q| \leqslant 0.0167\,\text{KJ/min}$ and $|\Delta C_{A0}| \leqslant 1\,\text{Kmol/m}^3$. We construct a bounded controller of the form of Eq. (4) using $V(x) = x'Px$ where $x = (C_A - C_A^s,\ T_R - T_R^s)$,

$$P = \begin{bmatrix} 9.35 & 0.41 \\ 0.41 & 0.02 \end{bmatrix},$$

where the matrix P was computed using the linearized system. The computation of the stability region estimates, under input constraints Ω_u and under state and input constraints $\Omega_{x,u}$, however, was done using the nonlinear system dynamics, and are shown in Fig. 2. The parameters in the objective function of Eq. (12) are chosen as $Q = qI$, with $q = 1.0$, and $R = rI$, with $r = 1.0$ and those in the objective function of Eq. (16) are chosen as $q = 10.0$ and $R = rI$, with $r = 0.01$. The constrained nonlinear optimization problem is solved using the MATLAB subroutine fmincon, and the set of ODEs is integrated using the MATLAB solver ODE45.

We first demonstrate the implementation of the Lyapunov-based predictive controller of Proposition 2 (Eqs. (8)–(13)) for

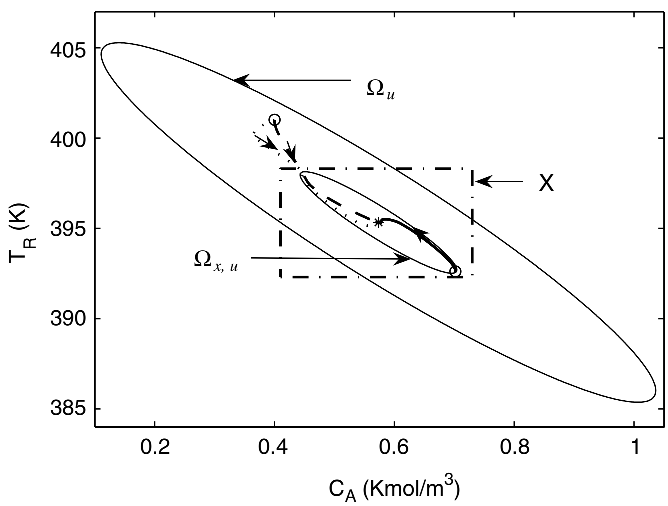

Fig. 2. Closed-loop state trajectory under the predictive controller of Proposition 2 (solid line), under the bounded controller (dashed line) and under the predictive controller of Theorem 1 (dotted line).

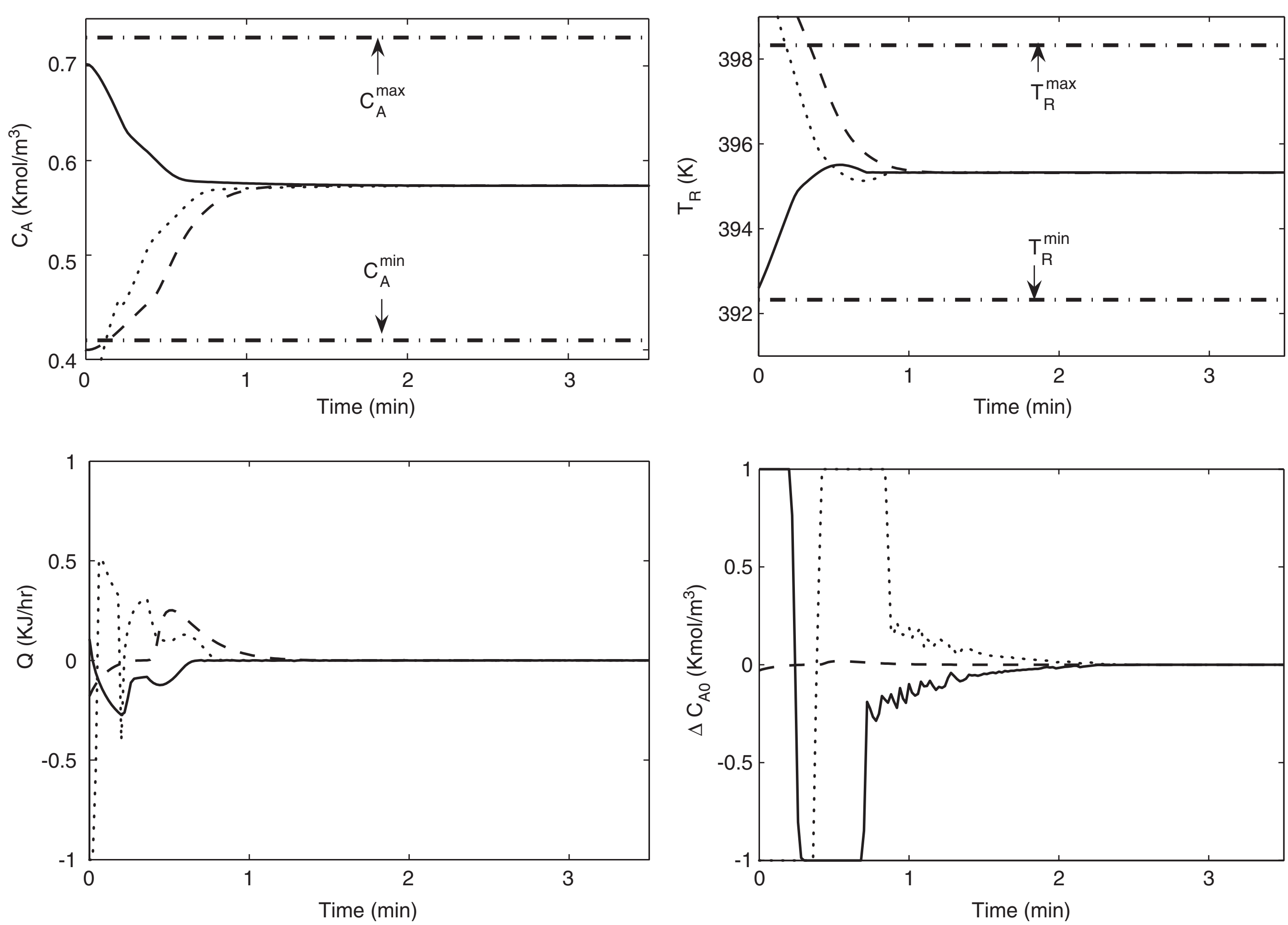

Fig. 3. Closed-loop state (top) and input (bottom) profiles under the predictive controller of Proposition 2 (solid lines), under the bounded controller (dashed lines) and under the predictive controller of Theorem 1 (dotted lines).

the case when the state constraints are hard constraints and need to be satisfied at all times. To this end, we consider an initial condition that belongs to the state and input constrained stability region of the predictive controller, $\Omega_{x,u}$. As shown by the solid line in Fig. 2, starting from the initial condition $(C_A, T_{\mathrm{R}}) = (0.702\,\mathrm{Kmol/m^3}, 392.6\,\mathrm{K})$, successful stabilization of the closed-loop system is achieved, together with state and input constraint satisfaction for all times. The corresponding state and input profiles are shown in Fig. 3.

Consider now the case where the state constraints reflect desirable bounds on the state variables, and can be treated as soft constraints. In this case, the closed-loop state could be initialized in Ω_u, from initial conditions where state constraints are initially violated. Starting from an initial condition that violates the state constraint on the temperature, $(C_A, T_{\mathrm{R}}) = (0.4\,\mathrm{Kmol/m^3}, 401\,\mathrm{K})$, it takes 0.34 min for the state trajectory to enter $\Omega_{x,u}$ under the implementation of the bounded controller (see dashed lines in Figs. 2 and 3). Setting $T_{\mathrm{b}}=0.34$ min, therefore and implementing the predictive controller of Theorem 1, we find that the controller is able to drive the state trajectory inside of $\Omega_{x,u}$ at $t = 0.18$ min, substantially reducing the time for which the soft constraints are violated (see dotted lines in Figs. 2 and 3). After $T_{\mathrm{switch}} = 0.18$ min the predictive controller of Proposition 2 is employed to successfully achieve stabilization in the presence of state and input constraints.

6. Conclusions

In this work, we considered the problem of stabilization of nonlinear systems subject to state and control constraints. We proposed a Lyapunov-based MPC design that guarantees stabilization and state and input constraint satisfaction from an explicitly characterized set of initial conditions. An auxiliary Lyapunov-based analytical bounded control design was used to characterize the stability region of the predictive controller and also provide a feasible initial guess to the optimization problem in the predictive controller formulation. For the case when the state constraints are soft, we proposed a switched predictive control strategy that reduces the time during which state constraints are violated, driving the states into the state and input constraints feasibility region of the Lyapunov-based predictive controller. We demonstrated the application of the Lyapunov-based predictive controller designs through a chemical process example.

References

[1] F. Allgower, H. Chen, Nonlinear model predictive control schemes with guaranteed stability, in: R. Berber, C. Kravaris (Eds.), NATO ASI on Nonlinear Model Based Process Control, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998, pp. 465–494.

[2] Z. Artstein, Stabilization with relaxed controls, Nonlinear Anal. 7 (1983) 1163–1173.

[3] A. Bemporad, M. Morari, Robust model predictive control: a survey, in: A. Garulli, A. Tesi, A. Vicino (Eds.), Robustness in Identification and Control, Lecture Notes in Control and Information Sciences, vol. 245, Springer, Berlin, 1999, pp. 207–266.

[4] M.S. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, IEEE Trans. Automat. Control 43 (1998) 475–482.

[5] P.D. Christofides, N.H. El-Farra, Control of Nonlinear and Hybrid Process Systems: Designs for Uncertainty, Constraints and Time-Delays, Springer, New York, 2005.

[6] N.M.C. de Oliveira, L.T. Biegler, Constraint handling and stability properties of model-predictive control, A.I.Ch.E. J. 40 (1994) 1138–1150.

[7] N.H. El-Farra, P.D. Christofides, Integrating robustness, optimality, and constraints in control of nonlinear processes, Chem. Eng. Sci. 56 (2001) 1841–1868.

[8] N.H. El-Farra, P.D. Christofides, Bounded robust control of constrained multivariabe nonlinear processes, Chem. Eng. Sci. 58 (2003) 3025–3047.

[9] N.H. El-Farra, P. Mhaskar, P.D. Christofides, Hybrid predictive control of nonlinear systems: method and applications to chemical processes, Internat. J. Rob. Nonlinear Control 4 (2004) 199–225.

[10] N.H. El-Farra, P. Mhaskar, P.D. Christofides, Uniting bounded control and MPC for stabilization of constrained linear systems, Automatica 40 (2004) 101–110.

[11] N.H. El-Farra, P. Mhaskar, P.D. Christofides, Output feedback control of switched nonlinear systems using multiple Lyapunov functions, Systems Control Lett. 54 (2005) 1163–1182.

[12] C.E. Garcia, A.M. Morshedi, Quadratic-programming solution of dynamic matrix control (QDMC), Chem. Eng. Commun. 46 (1986) 73–87.

[13] H. Genceli, M. Nikolaou, Robust stability analysis of constrained L_1-norm model predictive control, A.I.Ch.E. J. 39 (1993) 1954–1965.

[14] J.W. Grizzle, P.V. Kokotovic, Feedback linearization of sampled-data systems, IEEE Trans. Automat. Control 33 (1988) 857–859.

[15] J. Hespanha, A.S. Morse, Stability of switched systems with average dwell time, in: Proceedings of 38th IEEE Conference on Decision and Control, Phoenix, AZ, 1999, pp. 2655–2660.

[16] N. Kazantzis, A functional equations approach to nonlinear discrete-time feedback stabilization through pole-placement, Systems Control Lett. 43 (2001) 361–369.

[17] P. Kokotovic, M. Arcak, Constructive nonlinear control: a historical perspective, Automatica 37 (2001) 637–662.

[18] S.L.D. Kothare, M. Morari, Contractive model predictive control for constrained nonlinear systems, IEEE Trans. Automat. Control 45 (2000) 1053–1071.

[19] Y. Lin, E.D. Sontag, A universal formula for stabilization with bounded controls, Systems Control Lett. 16 (1991) 393–397.

[20] D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O.M. Scokaert, Constrained model predictive control: Stability and optimality, Automatica 36 (2000) 789–814.

[21] P. Mhaskar, N.H. El-Farra, P.D. Christofides, Hybrid predictive control of process systems, A.I.Ch.E. J. 50 (2004) 1242–1259.

[22] P. Mhaskar, N.H. El-Farra, P.D. Christofides, Predictive control of switched nonlinear systems with scheduled mode transitions, IEEE Trans. Automat. Contr. 50 (2005) 1670–1680.

[23] P. Mhaskar, N.H. El-Farra, P.D. Christofides, Robust hybrid predictive control of nonlinear systems, Automatica 41 (2005) 209–217.

[24] D. Nesic, A.R. Teel, P.V. Kokotovic, Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations, Systems Control Lett. 38 (1999) 259–270.

[25] J.A. Primbs, V. Nevistic, J.C. Doyle, A receding horizon generalization of pointwise min-norm controllers, IEEE Trans. Automat. Contr. 45 (2000) 898–909.

[26] J.B. Rawlings, K.R. Muske, The stability of constrained receding horizon control, IEEE Trans. Automat. Control 38 (1993) 1512–1516.

[27] P.O.M. Scokaert, D.Q. Mayne, J.B. Rawlings, Suboptimal model predictive control (feasibility implies stability), IEEE Trans. Automat. Control 44 (1999) 648–654.

[28] P.O.M. Scokaert, J.B. Rawlings, Feasibility issues in linear model predictive control, A.I.Ch.E. J. 45 (1999) 1259–1649.

[29] R. Sepulchre, M. Jankovic, P. Kokotovic, Constructive Nonlinear Control, Springer, Berlin, Heidelberg, 1997.

[30] L. Zaccarian, A.R. Teel, D. Nesic, On finite gain L_P stability of nonlinear sampled-data systems, Systems Control Lett. 49 (2003) 201–212.
[31] E. Zafiriou, A.L. Marchal, Stability of SISO quadratic dynamic matrix control with hard output constraints, A.I.Ch.E. J. 37 (1991) 1550–1560.
[32] A. Zheng, M. Morari, Stability of model predictive control with mixed constraints, IEEE Trans. Automat. Control 40 (1995) 1818–1823.