

STABILIZATION OF SECOND ORDER EVOLUTION EQUATIONS BY A CLASS OF UNBOUNDED FEEDBACKS

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Abstract. In this paper we consider second order evolution equations with unbounded feedbacks. Under a regularity assumption we show that observability properties for the undamped problem imply decay estimates for the damped problem. We consider both uniform and non uniform decay properties.

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1. INTRODUCTION

In recent years an important literature was devoted to the controllability and stabilizability of second order infinite dimensional systems coming from elasticity (see for instance Lions [16] and references therein). According to the classical principle of Russell (see [23]) if a system is uniformly stabilizable by using colocated actuators and sensors then it is exactly controllable by using the same actuators (*i.e.*, the same input operator). As far as we know the converse of this assertion was not proved in a general framework. The only result available in the literature supposes that the input operator is bounded (see Haraux [9]) in the energy space or they are based on non local feed-backs (see for instance Komornik [12] and references therein). In the applications for PDE's systems this situation leads to non-local feedbacks given, in particular, by Riccati type operators. However in many PDE systems the exponential stabilizability with colocated actuators and sensors was proved by direct methods (see Lagnese [14], Komornik and Zuazua [13]) by using multiplier techniques. The aim of this paper is to give a class of unbounded input operators for which exact controllability implies uniform stabilizability by colocated actuators and sensors.

More precisely, let X be a complex Hilbert space with norm and inner product denoted respectively by $\|\cdot\|_X$ and $\langle \cdot, \cdot \rangle_X$. Let A be a linear unbounded self-adjoint and strictly positive operator in X . Let $\mathcal{D}(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$. Denote by $(\mathcal{D}(A^{\frac{1}{2}}))'$ the dual space of $\mathcal{D}(A^{\frac{1}{2}})$ obtained by means of the inner product in X .

Further, let U be a complex Hilbert space (which will be identified to its dual space) with norm and inner product respectively denoted by $\|\cdot\|_U$ and $\langle \cdot, \cdot \rangle_U$ and let $B \in \mathcal{L}(U, (\mathcal{D}(A^{\frac{1}{2}}))')$.

Most of the linear control problems coming from elasticity can be written as

$$\begin{cases} x''(t) + Ax(t) + Bu(t) = 0, \\ x(0) = z^0, x'(0) = z^1, \end{cases} \quad (1.1)$$

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where $x : [0, T] \rightarrow X$ is the state of the system, $u \in L^2(0, T; U)$ is the input function and we denoted the differentiation with respect to time by “ \prime ”.

We define the energy of $x(t)$ at instant t by

$$E(x(t)) = \frac{1}{2} \left\{ \|x'(t)\|_X^2 + \|A^{\frac{1}{2}}x(t)\|_X^2 \right\}.$$

Simple formal calculations give

$$E(x(0)) - E(x(t)) = -\langle Bu(t), x'(t) \rangle_{\mathcal{D}(A^{\frac{1}{2}}), (\mathcal{D}(A^{\frac{1}{2}}))'}, \quad \forall t \geq 0. \tag{1.2}$$

This is why, in many problems, coming in particular from elasticity, the input u is given in the feedback form $u(t) = B^*x'(t)$, which obviously gives a nonincreasing energy and which corresponds to colocated actuators and sensors. The aim of this paper is to give sufficient conditions making the corresponding closed loop system

$$x''(t) + Ax(t) + BB^*x'(t) = 0, \tag{1.3}$$

$$x(0) = x^0, \quad x'(0) = x^1, \tag{1.4}$$

uniformly stable in the energy space $\mathcal{D}(A^{\frac{1}{2}}) \times X$. In the case of non uniform stability we give sufficient conditions for weaker decay properties.

In order to obtain the characterization of decay properties of the damped problem *via* observability inequalities for the conservative problem we will use the assumption below. This assumption is less restrictive than the boundedness of B which was the basic hypothesis in [9].

(H)

If $\beta > 0$ is fixed and $C_\beta = \left\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda = \beta \right\}$, the function

$$\lambda \in \mathbb{C}_+ = \left\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0 \right\} \rightarrow H(\lambda) = \lambda B^*(\lambda^2 I + A)^{-1} B \in \mathcal{L}(U) \tag{1.5}$$

is bounded on C_β .

An equivalent statement of (H) is given at Section 3. Under this alternative form this assumption can be verified for PDE systems (such as the systems in the examples below), by proving results called (in the PDE community) “hidden regularity results”.

The main novelties brought in by this paper are the following:

- (a) we give a sufficient and necessary condition for the exponential stability of all finite energy solutions of (1.3, 1.4) by using only the undamped problem (*i.e.* corresponding to $B = 0$ in (1.3));
- (b) in the case of non exponential stability in the energy space we give an explicit decay rate for all initial data lying in a more regular space.

Our approach has common points with the result obtained in [9] for feedbacks which are bounded in the energy space. The main difference is that we replace the assumption of boundedness of B by the assumption (H). Moreover our methods are related to those proposed in [28] for a general class of first order systems (see the coming paper [26] for a description of the connections between our results and those in [28]).

The paper is organized as follows. In the second section we give precise statements of the main results. Some regularity results implied by (H) are given in Section 3. Section 4 contains the proof of the main results. The last section is devoted to some applications.

2. STATEMENT OF THE MAIN RESULTS

Let $x(t)$ be a solution of (1.3, 1.4). Simple formal calculations show that a sufficiently smooth solution of (1.3, 1.4) satisfies the energy estimate

$$E(x(0)) - E(x(t)) = \int_0^t \|B^*x'(s)\|_U^2 ds, \quad \forall t \geq 0. \tag{2.1}$$

In particular (2.1) implies that

$$E(x(t)) \leq E(x(0)), \quad \forall t \geq 0.$$

Estimate above suggests that the natural well-posedness space for (1.3, 1.4) is $V \times X$ where $V = \mathcal{D}(A^{\frac{1}{2}})$ and $\|x\|_V = \|A^{\frac{1}{2}}x\|_X, \forall x \in V$.

The existence and uniqueness of finite energy solutions of (1.3, 1.4) can be obtained by standard semigroup methods. This why the results below are given without proofs.

Proposition 2.1. *Suppose that $(x^0, x^1) \in V \times X$. Then the problem (1.3, 1.4) admits a unique solution*

$$x(t) \in C(0, T; V) \cap C^1(0, T; X)$$

such that $B^*x(\cdot) \in H^1(0, T; U)$ and

$$\|(B^*x)'(\cdot)\|_{L^2(0, T; U)}^2 \leq C\|(x^0, x^1)\|_{V \times X}^2, \tag{2.2}$$

where the constant $C > 0$ is independent of (x^0, x^1) . Moreover $x(t)$ satisfies the energy estimate (2.1).

Let us now consider the problem

$$\phi''(t) + A\phi(t) = 0, \tag{2.3}$$

$$\phi(0) = x^0, \phi'(0) = x^1. \tag{2.4}$$

It is well known that (2.3, 2.4) is well-posed in $\mathcal{D}(A) \times V$ and in $V \times X$.

The following theorem is a direct generalisation of the result in [9].

Theorem 2.2. *Assume that the hypothesis (H) is verified. Then, the system described by (1.3, 1.4) is exponentially stable in $V \times X$ if and only if there exists $T > 0, C > 0$ such that*

$$\int_0^T \|(B^*\phi)'(t)\|_U^2 dt \geq C\|(x^0, x^1)\|_{V \times X}^2, \quad \forall (x^0, x^1) \in \mathcal{D}(A) \times V. \tag{2.5}$$

Remark 2.3. Assumption (H) is not necessary for the implication: uniform exponential stability \Rightarrow (2.5). The latter follows from (indirectly) Russell's principle [23].

The statement of our second main result requires some notations.

Consider the unbounded linear operator

$$\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \rightarrow V \times X, \quad \mathcal{A}_d = \begin{pmatrix} 0 & I \\ -A & -BB^* \end{pmatrix}, \tag{2.6}$$

where

$$\mathcal{D}(\mathcal{A}_d) = \left\{ (u, v) \in V \times X, Au + BB^*v \in X, v \in V \right\}.$$

Let X_1, X_2, Y_1, Y_2 be four Banach spaces such that

$$V \times X \subset X_1 \times X_2, \mathcal{D}(\mathcal{A}_d) \subset Y_1 \times Y_2 \subset V \times X,$$

$$\forall z \in \mathcal{D}(\mathcal{A}_d), \|z\|_{\mathcal{D}(\mathcal{A}_d)} \sim \|z\|_{Y_1 \times Y_2}$$

and

$$[Y_1 \times Y_2, X_1 \times X_2]_\theta = V \times X, \tag{2.7}$$

for a fixed real number $\theta \in]0, 1[$, where $[\cdot, \cdot]_\theta$ denotes the interpolation space (see for instance Triebel [24]).

Let $\mathcal{G} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that \mathcal{G} is continuous, invertible, increasing on \mathbb{R}_+ and suppose that the function $x \rightarrow \frac{1}{x^{1-\theta}} \mathcal{G}(x)$ increasing on $(0, 1)$.

In the case of non exponential decay in the energy space we give explicit decay estimates valid for regular initial data, as stated in the result below:

Theorem 2.4. *Assume that assumption (H) is verified and that the function \mathcal{G} satisfies assumptions above. Then the following assertions hold true:*

1. *If for all $(x^0, x^1) \in \mathcal{D}(A) \times V$ we have*

$$\int_0^T \|(B^* \phi)'(t)\|_U^2 dt \geq C \|(x^0, x^1)\|_{V \times X}^2 \mathcal{G} \left(\frac{\|(x^0, x^1)\|_{X_1 \times X_2}^2}{\|(x^0, x^1)\|_{V \times X}^2} \right), \tag{2.8}$$

for some constant $C > 0$ then there exists a constant $C_1 > 0$ such that for all $t > 0$ and for all $(x^0, x^1) \in \mathcal{D}(\mathcal{A}_d)$ we have

$$E(x(t)) \leq C_1 \left[\mathcal{G}^{-1} \left(\frac{1}{1+t} \right) \right]^{\frac{\theta}{1-\theta}} \|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^2. \tag{2.9}$$

2. *If for all $(x^0, x^1) \in \mathcal{D}(A) \times V$ we have*

$$\int_0^T \|(B^* \phi)'(t)\|_U^2 dt \geq C \|(x^0, x^1)\|_{X_1 \times X_2}^2, \tag{2.10}$$

for some constant $C > 0$ then there exists a constant $C_2 > 0$ such that for all $t > 0$ and for all $(x^0, x^1) \in \mathcal{D}(\mathcal{A}_d)$ we have

$$E(x(t)) \leq \frac{C_2}{(1+t)^{\frac{\theta}{1-\theta}}} \|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^2. \tag{2.11}$$

Remark 2.5. 1. Estimates similar to (2.11) were first given by Russell [22] in the case of bounded feedback controls. Russell’s method cannot be directly extended to unbounded feedbacks.

2. If $\theta \in (0, \frac{1}{2})$ then the identity function satisfies the assumptions on \mathcal{G} in Theorem 2.4. In this case (2.11) is a consequence of (2.9). However if $\theta \in (\frac{1}{2}, 1)$ then the identity function does not satisfy the assumptions on \mathcal{G} in the Theorem 2.4. In this second case (2.11) is not a consequence of (2.9).

3. SOME REGULARITY RESULTS

Consider the evolution problem

$$y''(t) + Ay(t) = Bv(t), \tag{3.1}$$

$$y(0) = y'(0) = 0. \tag{3.2}$$

A natural question is the regularity of y when $v \in L^2(0, T; U)$. By applying standard energy estimates we can easily check that $y \in C(0, T; X) \cap C^1(0, T; V)$. However if B satisfies a certain admissibility condition then y is more regular. More precisely the following result, which is a version of the general transposition method (see, for instance, Lions and Magenes [17]), holds true.

Lemma 3.1. *Suppose that $v \in L^2(0, T; U)$ and that the solutions ϕ of (2.3, 2.4) are such that $B^*\phi(\cdot) \in H^1(0, T; U)$ and there exists a constant $C > 0$ such that*

$$\|(B^*\phi)'(\cdot)\|_{L^2(0, T; U)} \leq C\|(x^0, x^1)\|_{V \times X}, \quad \forall (x^0, x^1) \in V \times X. \tag{3.3}$$

Then the problem (3.1, 3.2) admits a unique solution having the regularity

$$y \in C(0, T; V) \cap C^1(0, T; X). \tag{3.4}$$

Proof. Let

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times V$$

and denote by $[\mathcal{D}(\mathcal{A})]'$ the dual space of $\mathcal{D}(\mathcal{A})$ with respect to the pivot space $V \times X$.

If we put $Z = \begin{pmatrix} y \\ y' \end{pmatrix}$ it is clear that (3.1, 3.2) can be written as

$$Z' + \mathcal{A}Z(t) = \mathcal{B}v(t), \quad Z(0) = 0,$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix} : V \times X \rightarrow [\mathcal{D}(\mathcal{A})]',$$

$$\mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} : U \rightarrow [\mathcal{D}(\mathcal{A})]'$$

It well known that \mathcal{A} is a skew adjoint operator so it generates a group of isometries in $[\mathcal{D}(\mathcal{A})]'$, denoted by $\mathcal{S}(t)$.

After simple calculations we get that the operator $\mathcal{B}^* : \mathcal{D}(\mathcal{A}) \rightarrow U$ is given by

$$\mathcal{B}^* \begin{pmatrix} u \\ v \end{pmatrix} = B^*v, \quad \forall (u, v) \in \mathcal{D}(\mathcal{A}).$$

This implies that

$$\mathcal{B}^*\mathcal{S}^*(t) \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = B^*\phi', \quad \forall (x^0, x^1) \in \mathcal{D}(\mathcal{A}),$$

with ϕ satisfying (2.3, 2.4). From the inequality above and (3.3) we deduce that there exists a constant $C > 0$ such that

$$\int_0^T \left\| \mathcal{B}^* \mathcal{S}^*(t) \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \right\|_U^2 dt \leq C \|(x^0, x^1)\|_{V \times X}^2, \quad \forall (x^0, x^1) \in \mathcal{D}(\mathcal{A}).$$

According to Theorem 3.1 in [6] (p. 187) the inequality above implies the interior regularity (3.5).

Proposition 3.2. *Suppose that $v \in L^2(0, T; U)$ and that the problem (3.1, 3.2) admits a unique solution having the regularity*

$$y \in C(0, T; V) \cap C^1(0, T; X). \tag{3.5}$$

*Then hypothesis (H) holds if and only if $B^*y(\cdot) \in H^1(0, T; U)$ and there exists a constant $C > 0$ such that*

$$\|(B^*y)'(\cdot)\|_{L^2(0, T; U)} \leq C \|v\|_{L^2(0, T; U)}, \quad \forall v \in L^2(0, T; U). \tag{3.6}$$

Proof. As equation (3.1) is time reversible, after extending v by zero for $t \in \mathbb{R} \setminus [0, T]$, we can solve (3.1, 3.2), for $t \in \mathbb{R}$. By this way, we obtain a function, denoted also by y , such that

$$\begin{aligned} y &\in C(\mathbb{R}; V) \cap C^1(\mathbb{R}; X) \cap L^2(\mathbb{R}; V), \\ y(t) &= 0, \quad \forall t \leq 0, \end{aligned} \tag{3.7}$$

and y satisfies (3.1, 3.2) for all $t \in \mathbb{R}$.

Let $\widehat{y}(\lambda)$, where $\lambda = \gamma + i\eta$, $\gamma > 0$ and $\eta \in \mathbb{R}$, be the Laplace (with respect to t) transform of y . Since y satisfies (3.7), estimate (3.6) is equivalent to the fact that the function $t \rightarrow e^{-\gamma t} B^*y(t)$ belongs to $H^1(\mathbb{R}; U)$ and that there exists a constant $M_1 > 0$ such that

$$\|e^{-\gamma \cdot} B^*y(\cdot)\|_{H^1(\mathbb{R}; U)}^2 \leq M_1 \|v(\cdot)\|_{L^2(\mathbb{R}; U)}^2.$$

Equivalently, by the Parseval identity (see for instance Doetsch [8], p. 212), it suffices to prove that the function

$$\eta \rightarrow (\gamma + i\eta) B^* \widehat{y}(\gamma + i\eta)$$

belongs to $L^2(\mathbb{R}_\eta; U)$, for some $\gamma > 0$, and that there exists a constant $M_2 > 0$ such that

$$\|(\gamma + i\eta) B^* \widehat{y}(\gamma + i\eta)\|_{L^2(\mathbb{R}_\eta; U)}^2 \leq M_2 \int_{-\infty}^{+\infty} \|\widehat{v}(\gamma + i\eta)\|_U^2 d\eta. \tag{3.8}$$

It can be easily checked that \widehat{y} satisfies:

$$\lambda^2 \widehat{y}(\lambda) + A \widehat{y}(\lambda) = B \widehat{v}(\lambda), \quad \forall Re \lambda > 0. \tag{3.9}$$

Relation above implies, for $Re \lambda > 0$ that

$$\lambda B^* \widehat{y}(\lambda) = H(\lambda) \widehat{v}(\lambda), \quad \forall Re \lambda > 0, \tag{3.10}$$

where $H(\lambda)$ is defined in (1.5). Assumption (H) implies the existence of a constant $M_2 > 0$ such that (3.8) holds true. This ends the proof of the fact that assumption (H) implies that (3.6) holds for all finite energy solution of (3.1, 3.2).

Suppose now that (3.6) holds true. By using the time reversibility and the invariance with respect to translations (in time) of (3.1) we obtain that (3.1, 3.2) is well posed for all input $v \in L^2(\mathbb{R}, U)$, v compactly supported. More precisely, we have

$$\int_{\text{supp}(v)} \|(B^*y)'(t)\|_U^2 dt \leq C \int_{\text{supp}(v)} \|v(t)\|_U^2 dt,$$

for all compactly supported $v \in L^2(\mathbb{R}, U)$, with the same constant as in (3.6).

Using (3.10) it follows that

$$\|H(\gamma + i\eta)\widehat{v}(\gamma + i\eta)\|_{L^2(\mathbb{R}_\eta, U)}^2 \leq C \|\widehat{v}(\lambda)\|_{L^2(\mathbb{R}_\eta, U)}^2, \tag{3.11}$$

for all compactly supported $v \in L^2(\mathbb{R}, U)$.

By density it follows that (3.11) holds for all $v \in L^2(\mathbb{R}, U)$. We have thus proved that (3.6) implies that (H) holds true.

Proposition 3.3. *Suppose that hypothesis (H) is satisfied. Then for $(x^0, x^1) \in V \times X$ we have that $B^*\phi(\cdot) \in H^1(0, T; U)$ and there exist $C, T > 0$ such that the solution $\phi(t)$ of (2.3, 2.4) satisfies (3.3). In the other words assumption (H) implies (3.3).*

Proof. Suppose that hypothesis (H) is satisfied. Let $x(t) \in C(0, T; V) \cap C^1(0, T; X)$ be the unique solution of (1.3, 1.4). By Proposition 2.1 we know that $B^*x \in H^1(0, T; U)$ and that (2.1) holds true. Let ϕ be the solution of (2.3, 2.4). We clearly have $\psi = x - \phi \in C(0, T; V) \cap C^1(0, T; X)$ and ψ satisfies

$$\begin{cases} \psi''(t) + A\psi(t) = BB^*x'(t), \text{ in } C(0, T; V'), \\ \psi(0) = \psi'(0) = 0. \end{cases}$$

By applying now Proposition 3.2 with $v = B^*x' \in L^2(0, T; U)$ we obtain that

$$\int_0^T \|(B^*\psi)'(t)\|_U^2 dt \leq C \int_0^T \|(B^*x)'(t)\|_U^2 dt. \tag{3.12}$$

Since $B^*\phi = B^*x - B^*\psi$ relations (2.1) and (3.12) imply the conclusion of the proposition.

Corollary 3.4. *Suppose that assumption (H) is satisfied. Then, for all $v \in L^2(0, T; U)$, problem (3.1, 3.2) admits a unique solution y satisfying (3.5) and (3.6).*

Proof. Suppose that assumption (H) is satisfied. Then Proposition 3.3 and Lemma 3.1 imply that problem (3.1, 3.2) admits a unique solution y satisfying (3.5). Finally Proposition 3.2 implies that y satisfies (3.6). \square

4. PROOF OF THE MAIN RESULTS

Let $x(t) \in C(0, T; V) \cap C^1(0, T; X)$ be the solution of (1.3, 1.4). Then $x(t)$ can be written as

$$x(t) = \phi(t) + \psi(t), \tag{4.1}$$

where $\phi(t)$ satisfies (2.3, 2.4) and $\psi(t)$ satisfies

$$\psi''(t) + A\psi(t) = -BB^*x'(t), \tag{4.2}$$

$$\psi(0) = \psi'(0) = 0. \tag{4.3}$$

The main ingredient of the proof of Theorem 2.2 and of the proof of Theorem 2.4 is the following result:

Lemma 4.1. *Let $(x^0, x^1) \in V \times X$ and suppose that (H) is verified. Then the solution $x(t)$ of (1.3, 1.4) and the solution $\phi(t)$ of (2.3, 2.4) satisfy*

$$C_1 \int_0^T \|(B^* \phi)'(t)\|_U^2 dt \leq \int_0^T \|(B^* x)'(t)\|_U^2 dt \leq 4 \int_0^T \|(B^* \phi)'(t)\|_U^2 dt, \tag{4.4}$$

where $C_1 > 0$ is a constant independent of (x^0, x^1) .

Remark 4.2. By Proposition 2.1, $(B^* x)'(\cdot) \in L^2(0, T; U)$. So, equation (4.2) makes sense. The result above shows that the L^2 norm of $\|(B^* x)'(\cdot)\|_U$ is equivalent to the L^2 norm of $\|(B^* \phi)'(\cdot)\|_U$ (notice that $\|(B^* \phi)'(\cdot)\|_U \in L^2(0, T)$ by Prop. 3.3).

Proof of Lemma 4.1. We prove (4.4) for $x(t)$ satisfying (1.3, 1.4) and $\phi(t)$ solution of (2.3, 2.4).

Relation (4.1) implies that

$$\int_0^T \|(B^* \phi)'(\cdot)\|_U^2 dt \leq 2 \left\{ \int_0^T \|(B^* x)'(\cdot)\|_U^2 dt + \int_0^T \|(B^* \psi)'(\cdot)\|_U^2 dt \right\}.$$

Estimate above combined with inequality (3.6) in Proposition 3.2 implies the existence of a constant $C_1 > 0$, independent of (x^0, x^1) , such that

$$C_1 \int_0^T \|(B^* \phi)'(\cdot)\|_U^2 dt \leq \int_0^T \|(B^* x)'(\cdot)\|_U^2 dt. \tag{4.5}$$

On the other hand, according to Remark 4.2 and to relation (4.1) we have that

$$\|(B^* \phi)'(\cdot)\|_U \in L^2(0, T).$$

This means that (4.2) can be rewritten as

$$\psi''(t) + A\psi(t) + B(B^* \psi)'(t) = -B(B^* \phi)'(t). \tag{4.6}$$

We denote now by $w(t)$ the extension of $(B^* \phi)'$ obtained by defining $w(t) = 0, t \in \mathbb{R} \setminus [0, T]$. We still denote by $\psi(t)$ the solution of

$$\begin{cases} \psi''(t) + A\psi(t) + B(B^* \psi)' = -Bw(t), & t \in \mathbb{R}, \\ \psi(0) = \psi'(0) = 0. \end{cases} \tag{4.7}$$

We clearly have $\psi(t) = 0$ for $t \in \mathbb{R} \setminus [0, T]$.

Taking the Laplace transform we get

$$\lambda^2 \widehat{\psi}(\lambda) + A\widehat{\psi}(\lambda) + \lambda B B^* \widehat{\psi}(\lambda) = -B\widehat{w}(\lambda), \forall \lambda = \gamma + i\eta, \gamma > 0.$$

The equality above holds in $(\mathcal{D}(A^{\frac{1}{2}}))'$.

By applying $\bar{\lambda} \widehat{\psi} \in \mathcal{D}(A^{\frac{1}{2}})$ to the equality above, we get

$$\lambda |\lambda|^2 \|\widehat{\psi}(\lambda)\|_X^2 + \bar{\lambda} \|A^{\frac{1}{2}} \widehat{\psi}(\lambda)\|_X^2 + \|\lambda B^* \widehat{\psi}(\lambda)\|_U^2 = -\langle w(\lambda), \bar{\lambda} B^* \widehat{\psi}(\lambda) \rangle_U.$$

Taking the real part of each term, we get

$$\int_{\mathbb{R}_\eta} \|\lambda B^* \widehat{\psi}(\lambda)\|_U^2 d\eta \leq \frac{1}{2} \int_{\mathbb{R}_\eta} \|\widehat{w}(\lambda)\|_U^2 d\eta + \frac{1}{2} \int_{\mathbb{R}_\eta} \|\lambda B^* \widehat{\psi}(\lambda)\|_U^2 d\eta.$$

Parseval identity implies

$$\|(B^*\psi)'(t)\|_{L^2(0,T;U)}^2 \leq \|(B^*\phi)'(t)\|_{L^2(0,T;U)}^2. \tag{4.8}$$

Relation (4.1) and inequality above imply that

$$\|(B^*x)'(t)\|_{L^2(0,T;U)}^2 \leq 4 \|(B^*\phi)'(t)\|_{L^2(0,T;U)}^2. \tag{4.9}$$

Inequalities (4.5) and (4.9) obviously yield the conclusion (4.4).

We can now prove the first main result.

Proof of Theorem 2.2. All finite energy solutions of (1.3, 1.4) satisfy the estimate

$$E(x(t)) \leq Me^{-\omega t}E(x(0)), \quad \forall t \geq 0, \tag{4.10}$$

where $M, \omega > 0$ are constants independent of (x^0, x^1) , if and only if there exist a time $T > 0$ and a constant $C > 0$ (depending on T) such that

$$E(x(0)) - E(x(T)) \geq CE(x(0)), \quad \forall (x^0, x^1) \in V \times X.$$

By (2.1) relation above is equivalent to the inequality

$$\int_0^T \|(B^*x)'(s)\|_U^2 ds \geq CE(x(0)), \quad \forall (x^0, x^1) \in V \times X.$$

From Lemma 4.1 it follows that the system (1.3, 1.4) is exponentially stable if and only if

$$\int_0^T \|(B^*\phi)'(s)\|_U^2 ds \geq CE(x(0)), \quad \forall (x^0, x^1) \in \mathcal{D}(A) \times V$$

holds true. By density it follows that (1.3, 1.4) is exponentially stable if and only if (2.5) holds true. This ends up the proof of Theorem 2.2.

Remark 4.3. By analyzing the proof above we notice that the proof of the inequality (4.9) does not require assumption (H). More precisely, the inequality (4.8) can be also obtained in the following direct manner:

Let $(x^0, x^1) \in V \times X$ such that $B^*\phi \in H^1(0, T; U)$. Then, by formally multiplying (4.7) by ψ' , it follow that the function $\psi = x - \phi$ satisfies

$$\|(\psi(t), \psi'(t))\|_{V \times X}^2 + \int_0^t \left(\|B^*\psi'(s)\|_U^2 + \|B^*x'(s)\|_U^2 \right) ds = \int_0^t \|B^*\phi'(s)\|_U^2 ds.$$

This implies that (4.9) holds.

This means that the result mentioned in Remark 2.3 can be also established by a direct method. □

Before giving the proof of Theorem 2.4, we need a technical lemma. This lemma extends a result in Jaffard *et al.* [11]. For a proof we refer to Ammari *et al.* [1].

Lemma 4.4. *Let (\mathcal{E}_k) be a sequence of positive real numbers satisfying*

$$\mathcal{E}_{k+1} \leq \mathcal{E}_k - C\mathcal{E}_{k+1}^{2+\alpha}, \quad \forall k \geq 0, \tag{4.11}$$

where $C > 0$ and $\alpha > -1$ are constants. Then there exists a positive constant M such that

$$\mathcal{E}_k \leq \frac{M}{(k+1)^{\frac{1}{1+\alpha}}}, \quad \forall k \geq 0. \tag{4.12}$$

□

Proof of Theorem 2.4. By density (2.8) implies that for all $(x^0, x^1) \in V \times X$ we have

$$\int_0^T \|(B^* \phi)'(t)\|_{\bar{U}}^2 dt \geq C \|(x^0, x^1)\|_{V \times X}^2 \mathcal{G} \left(\frac{\|(x^0, x^1)\|_{X_1 \times X_2}^2}{\|(x^0, x^1)\|_{V \times X}^2} \right).$$

By applying Lemma 4.1 we obtain that the solution $x(t)$ of (1.3, 1.4) satisfies the following inequality

$$\int_0^T \|(B^* x)'(t)\|_{\bar{U}}^2 dt \geq C \|(x^0, x^1)\|_{V \times X}^2 \mathcal{G} \left(\frac{\|(x^0, x^1)\|_{X_1 \times X_2}^2}{\|(x^0, x^1)\|_{V \times X}^2} \right), \quad \forall (x^0, x^1) \in V \times X.$$

Relation above and (2.1) imply the existence of a constant $K > 0$ such that

$$\|(x(T), x'(T))\|_{V \times X}^2 \leq \|(x^0, x^1)\|_{V \times X}^2 - K \|(x^0, x^1)\|_{V \times X}^2 \mathcal{G} \left(\frac{\|(x^0, x^1)\|_{X_1 \times X_2}^2}{\|(x^0, x^1)\|_{V \times X}^2} \right), \quad \forall (x^0, x^1) \in \mathcal{D}(\mathcal{A}_d). \tag{4.13}$$

By using (2.7) (see again [24]), we obtain for fixed $\theta \in (0, 1)$

$$\frac{\|(x^0, x^1)\|_{X_1 \times X_2}^2}{\|(x^0, x^1)\|_{V \times X}^2} \geq \frac{\|(x^0, x^1)\|_{X_1 \times X_2}^{\frac{2-2\theta}{\theta}}}{\|(x^0, x^1)\|_{V \times X}^{\frac{2-2\theta}{\theta}}}, \quad \forall (x^0, x^1) \in \mathcal{D}(\mathcal{A}_d). \tag{4.14}$$

By using (4.14) combined with the fact that the function $t \rightarrow \|(x(t), x'(t))\|_{V \times X}^2$ is nonincreasing, the function \mathcal{G} is increasing and relation (4.22) we obtain the existence of a constant $K_1 > 0$ such that

$$\|(x(T), x'(T))\|_{V \times X}^2 \leq \|(x^0, x^1)\|_{V \times X}^2 - K_1 \|(x^0, x^1)\|_{V \times X}^2 \mathcal{G} \left(\frac{\|(x(T), x'(T))\|_{V \times X}^{\frac{2-2\theta}{\theta}}}{\|(x^0, x^1)\|_{Y_1 \times Y_2}^{\frac{2-2\theta}{\theta}}} \right). \tag{4.15}$$

Estimate (4.15) remains valid in successive intervals $[kT, (k+1)T]$, so, we have

$$\begin{aligned} & \|(x((k+1)T), x'((k+1)T))\|_{V \times X}^2 \leq \|(x(kT), x'(kT))\|_{V \times X}^2 \\ & - K_1 \|(x(kT), x'(kT))\|_{V \times X}^2 \mathcal{G} \left(\frac{\|(x((k+1)T), x'((k+1)T))\|_{V \times X}^{\frac{2-2\theta}{\theta}}}{\|(x(kT), x'(kT))\|_{Y_1 \times Y_2}^{\frac{2-2\theta}{\theta}}} \right). \end{aligned}$$

Since \mathcal{A}_d generates a semigroup of contractions in $\mathcal{D}(\mathcal{A}_d)$ and the graph norm on $\mathcal{D}(\mathcal{A}_d)$ is equivalent to $\|\cdot\|_{Y_1 \times Y_2}$, relations above imply the existence of a constant $K_2 > 0$ such that

$$\|(x((k+1)T), x'((k+1)T))\|_{V \times X}^2 \leq \|(x(kT), x'(kT))\|_{V \times X}^2$$

$$\begin{aligned}
 & -K_2 \|(x(kT), x'(kT))\|_{V \times X}^2 \mathcal{G} \left(\frac{\|(x((k+1)T), x'((k+1)T))\|_{V \times X}^{\frac{2-2\theta}{\theta}}}{\|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^{\frac{2-2\theta}{\theta}}} \right), \\
 & \forall (x^0, x^1) \in \mathcal{D}(\mathcal{A}_d).
 \end{aligned} \tag{4.16}$$

If we adopt now the notation

$$\mathcal{E}_k = \mathcal{G} \left(\frac{\|(x(kT), x'(kT))\|_{V \times X}^{\frac{2-2\theta}{\theta}}}{\|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^{\frac{2-2\theta}{\theta}}} \right), \tag{4.17}$$

the inequalities (4.23) implies

$$\frac{\|(x((k+1)T), x'((k+1)T))\|_{V \times X}^2}{\|(x(kT), x'(kT))\|_{V \times X}^2} \frac{\mathcal{E}_k}{\mathcal{E}_{k+1}} \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_k \mathcal{E}_{k+1}. \tag{4.18}$$

Since, the function $t \rightarrow \|(x(t), x'(t))\|_{V \times X}^2$ is nonincreasing and the function \mathcal{G} is increasing, relation (4.18) implies

$$\frac{\|(x((k+1)T), x'((k+1)T))\|_{V \times X}^2}{\|(x(kT), x'(kT))\|_{V \times X}^2} \frac{\mathcal{E}_k}{\mathcal{E}_{k+1}} \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^2. \tag{4.19}$$

According to (4.24), relation (4.19) gives,

$$\begin{aligned}
 & \frac{\frac{1}{\left[\frac{\|(x(kT), x'(kT))\|_{V \times X}^{\frac{2-2\theta}{\theta}}}{\|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^{\frac{2-2\theta}{\theta}}} \right]^{\frac{\theta}{1-\theta}}} \mathcal{G} \left(\frac{\|(x(kT), x'(kT))\|_{V \times X}^{\frac{2-2\theta}{\theta}}}{\|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^{\frac{2-2\theta}{\theta}}} \right)}{\frac{1}{\left[\frac{\|(x((k+1)T), x'((k+1)T))\|_{V \times X}^{\frac{2-2\theta}{\theta}}}{\|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^{\frac{2-2\theta}{\theta}}} \right]^{\frac{\theta}{1-\theta}}} \mathcal{G} \left(\frac{\|(x((k+1)T), x'((k+1)T))\|_{V \times X}^{\frac{2-2\theta}{\theta}}}{\|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^{\frac{2-2\theta}{\theta}}} \right)}} \mathcal{E}_{k+1} \\
 & \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^2.
 \end{aligned} \tag{4.20}$$

Relation (4.20) combined with that the function $x \rightarrow \frac{1}{x^{1-\theta}} \mathcal{G}(x)$ is increasing in $(0, 1)$, gives

$$\mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^2, \quad \forall k \geq 0. \tag{4.21}$$

By applying Lemma 4.4 and using relation (4.24) we obtain the existence of a constant $M > 0$ such that

$$\|(x(kT), x'(kT))\|_{V \times X}^2 \leq \left[\mathcal{G}^{-1} \left(\frac{M}{k+1} \right) \right]^{\frac{\theta}{1-\theta}} \|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^2, \quad \forall k \geq 0,$$

which obviously implies (2.9).

Proof of second assertion of Theorem 2.4. By density (2.10) implies that for all $(x^0, x^1) \in V \times X$ we have

$$\int_0^T \|(B^* \phi)'(t)\|_{\mathcal{U}}^2 dt \geq C \|(x^0, x^1)\|_{X_1 \times X_2}^2.$$

Then, Lemma 4.1 combined with (2.7) and (2.1) imply the existence of a constant $K > 0$ such that

$$\begin{aligned} \|(x(T), x'(T))\|_{V \times X}^2 &\leq \|(x^0, x^1)\|_{V \times X}^2 - K \frac{\|(x^0, x^1)\|_{X_1 \times X_2}^{\frac{2}{\theta}}}{\|(x^0, x^1)\|_{V \times X}^{\frac{2-2\theta}{\theta}}}, \\ &\forall (x^0, x^1) \in \mathcal{D}(\mathcal{A}_d). \end{aligned} \tag{4.22}$$

Following the same steps as in the proof of the first assertion of Theorem 2.4 we obtain the existence of a constant $C > 0$ such that for all $k \geq 0$ we have

$$\begin{aligned} \|(x((k+1)T), x'((k+1)T))\|_{V \times X}^2 &\leq \|(x(kT), x'(kT))\|_{V \times X}^2 \\ -C \frac{\|(x((k+1)T), x'((k+1)T))\|_{V \times X}^{\frac{2}{\theta}}}{\|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^{\frac{2-2\theta}{\theta}}}, &\quad \forall (x^0, x^1) \in \mathcal{D}(\mathcal{A}_d). \end{aligned} \tag{4.23}$$

If we adopt the notation

$$\mathcal{H}_k = \frac{\|(x(kT), x'(kT))\|_{V \times X}^2}{\|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^2}, \tag{4.24}$$

relation (4.23) gives

$$\mathcal{H}_{k+1} \leq \mathcal{H}_k - C\mathcal{H}_{k+1}^{\frac{1}{\theta}}, \quad \forall k \geq 0. \tag{4.25}$$

By applying Lemma 4.4 and using relation (4.25) we obtain the existence of a constant $M > 0$ such that

$$\|(x(kT), x'(kT))\|_{V \times X}^2 \leq \frac{M \|(x^0, x^1)\|_{\mathcal{D}(\mathcal{A}_d)}^2}{(k+1)^{\frac{\theta}{1-\theta}}}, \quad \forall k \geq 0,$$

which obviously implies (2.10). □

5. SOME APPLICATIONS

Now, we give some applications of Theorem 2.2 and Theorem 2.4. Some of them are new and some were obtained by different methods in previous literature.

5.1. First example: Stabilization of the string

We consider the following initial and boundary problem:

$$(I) \begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t}(\xi, t) \delta_\xi = 0, & (x, t) \in (0, 1) \times (0, +\infty), \\ u(0, t) = u(1, t) = 0, & t \in (0, +\infty), \\ u(x, 0) = u^0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u^1(x), & x \in (0, 1), \end{cases}$$

where $\xi \in (0, 1)$ and δ_ξ is the Dirac mass concentrated in the point $\xi \in (0, 1)$.

In this case, we have:

$$X = L^2(0, 1), U = \mathbb{R}, V = H_0^1(0, 1),$$

and

$$A = -\frac{d^2}{dx^2}, \mathcal{D}(A) = H^2(0, 1) \cap H_0^1(0, 1), Bk = k \delta_\xi, \forall k \in \mathbb{R}. \tag{5.1}$$

Then, \mathcal{A}_d is given by

$$\mathcal{A}_d \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{d^2 u}{dx^2} - v(\xi) \delta_\xi \end{pmatrix},$$

$$\forall (u, v) \in \mathcal{D}(\mathcal{A}_d) = \left\{ (u, v) \in [H_0^1(0, 1) \cap H^2(0, \xi) \cap H^2(\xi, 1)] \times H_0^1(0, 1), \frac{du}{dx}(\xi^+) - \frac{du}{dx}(\xi^-) = v(\xi) \right\}.$$

Denote by Q the set of all rational numbers. Let us also denote by \mathcal{S} the set of all numbers $\rho \in (0, 1)$ such that $\rho \notin Q$ and if $[0, a_1, \dots, a_n, \dots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. Let us notice that \mathcal{S} is obviously uncountable and, by classical results on diophantine approximation (*cf.* [7], p. 120), its Lebesgue measure is equal to zero. In particular, by Euler–Lagrange theorem (see Lang [15], p. 57) \mathcal{S} contains all $\xi \in (0, 1)$ such that ξ is an irrational quadratic number (*i.e.* satisfying a second degree equation with rational coefficients). According to a classical result (see for instance Tucsnak [25] and the references therein) if $\xi \in \mathcal{S}$ then there exists a constant $C_\xi > 0$ such that

$$|\sin(n\pi\xi)| \geq \frac{C_\xi}{n}, \quad \forall n \geq 1. \tag{5.2}$$

Stability results for (I) are then an immediate consequence of Theorem 2.2 and Theorem 2.4. We have the following result:

Theorem 5.1. 1. For any $\xi \in (0, 1)$ the system described by (I) is not exponentially stable in $V \times L^2(0, 1)$.

2. For all $\xi \in \mathcal{S}$ and for all $t \geq 0$ we have

$$\left\| \left(u(t), \frac{\partial u}{\partial t}(t) \right) \right\|_{V \times L^2(0,1)}^2 \leq \frac{C_\xi}{t+1} \|(u^0, u^1)\|_{\mathcal{D}(\mathcal{A}_d)}^2, \tag{5.3}$$

$$\forall (u^0, u^1) \in \mathcal{D}(\mathcal{A}_d),$$

where $C_\xi > 0$ is a constant depending only on ξ .

3. If $\epsilon > 0$ then, for almost all $\xi \in (0, 1)$ and for all $t \geq 0$ we have

$$\left\| \left(u(t), \frac{\partial u}{\partial t}(t) \right) \right\|_{V \times L^2(0,1)}^2 \leq \frac{C_{\xi, \epsilon}}{(t+1)^{\frac{1}{1+\epsilon}}} \|(u^0, u^1)\|_{\mathcal{D}(\mathcal{A}_d)}^2,$$

$$\forall (u^0, u^1) \in \mathcal{D}(\mathcal{A}_d), \tag{5.4}$$

where $C_{\xi, \epsilon} > 0$ is a constant depending only on ξ and ϵ .

Remark 5.2. The first assertion of the theorem above was proved by a different method in Bamberger *et al.* [3].

In this case the problem (2.3, 2.4) becomes

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0, (0, 1) \times (0, +\infty), \tag{5.5}$$

$$\phi(0, t) = \phi(1, t) = 0, (0, +\infty), \tag{5.6}$$

$$\phi(x, 0) = u^0(x), \frac{\partial \phi}{\partial t}(x, 0) = u^1(x), (0, 1). \tag{5.7}$$

Lemma 5.3. *The operators A and B defined by (5.1) satisfy assumption (H).*

Proof. Let $k \in \mathbb{R}$. It can be easily checked that $v = (\lambda^2 + A)^{-1} Bk$ satisfies:

$$\lambda^2 v(x) - \frac{d^2 v}{dx^2}(x) = 0, x \in (0, \xi) \cup (\xi, 1), Re\lambda > 0, \tag{5.8}$$

$$v(0) = v(1) = 0, \tag{5.9}$$

$$[v]_{\xi} = 0, \left[\frac{dv}{dx} \right]_{\xi} = k, \tag{5.10}$$

where we denote by $[g]$ the jump of the function g at the point ξ .

The solutions of (5.8, 5.9) have the form

$$v(x) = \begin{cases} A \operatorname{sh}(\lambda x), & x \in (0, \xi), \\ B \operatorname{sh}[\lambda(x - 1)], & x \in (\xi, 1), \end{cases}$$

where A, B are constants.

Consequently, the solutions of (5.8–5.10) have the following form

$$v(x) = \begin{cases} \frac{1}{\lambda} \frac{\operatorname{sh}[\lambda(\xi - 1)] \operatorname{sh}(\lambda x)}{\operatorname{sh}(\lambda)} k, & x \in (0, \xi), \\ \frac{1}{\lambda} \frac{\operatorname{sh}(\lambda \xi) \operatorname{sh}[\lambda(x - 1)]}{\operatorname{sh}(\lambda)} k, & x \in (\xi, 1). \end{cases}$$

Then, the function $H(\lambda) = \lambda B^*(\lambda^2 + A)^{-1} B$ associated to problem (I) is given by the following expression

$$H(\lambda) = \frac{\operatorname{sh}(\lambda \xi) \operatorname{sh}[\lambda(\xi - 1)]}{\operatorname{sh}(\lambda)}, Re\lambda > 0.$$

We easily check that

$$\sup_{\lambda \in C_\beta} |H(\lambda)| \leq \frac{\text{ch}(\beta\xi) \text{ch}[\beta(\xi - 1)]}{\text{sh}(\beta)}.$$

Thus (H) is satisfied.

The observability inequality concerning the trace at the point $x = \xi$ of the solutions of (5.5–5.7) is given in the proposition below.

Proposition 5.4. *Let $T > 0$ be fixed. Then the following assertions hold true.*

1. *For all $\xi \in \mathcal{S}$ the solution ϕ of (5.5–5.7) satisfies*

$$\int_0^T \left[\frac{\partial \phi}{\partial t}(\xi, t) \right]^2 dt \geq C_\xi \left(\|u^0\|_{L^2(0,1)}^2 + \|u^1\|_{H^{-1}(0,1)}^2 \right),$$

$$\forall (u^0, u^1) \in V \times L^2(0, 1), \tag{5.11}$$

where $C_\xi > 0$ is a constant depending only on ξ .

2. *For all $\epsilon > 0$ and for almost all $\xi \in (0, 1)$ the solution ϕ of (5.5–5.7) satisfies*

$$\int_0^T \left[\frac{\partial \phi}{\partial t}(\xi, t) \right]^2 dt \geq C_{\xi, \epsilon} \left(\|u^0\|_{H^{-\epsilon}(0,1)}^2 + \|u^1\|_{H^{-1-\epsilon}(0,1)}^2 \right),$$

$$\forall (u^0, u^1) \in V \times L^2(0, 1), \tag{5.12}$$

where $C_{\xi, \epsilon} > 0$ is a constant depending only on ξ and ϵ .

3. *The result in assertion 1 is sharp in the sense that, for all $\xi \in (0, 1)$, there exists a sequence $(u_m^0, u_m^1) \subset V \times L^2(0, 1)$ such that the corresponding sequence of solutions (ϕ_m) of (5.5–5.7) with initial data (u_m^0, u_m^1) satisfies*

$$\lim_{m \rightarrow \infty} \frac{\int_0^T \left[\frac{\partial \phi}{\partial t}(\xi, t) \right]^2 dt}{\|u^0\|_{H^\epsilon(0,1)}^2 + \|u^1\|_{H^{-1+\epsilon}(0,1)}^2} = 0. \tag{5.13}$$

Proof. If we put

$$u^0(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \quad u^1(x) = \sum_{n=1}^{\infty} nb_n \sin(n\pi x) \tag{5.14}$$

with

$$(na_n), (nb_n) \subset l^2(\mathbb{R}),$$

then we clearly have

$$\frac{\partial \phi}{\partial t}(\xi, t) = \sum_{n \geq 1} \left(-n\pi a_n \sin(n\pi t) \sin(n\pi \xi) + n\pi b_n \cos(n\pi t) \sin(n\pi \xi) \right). \tag{5.15}$$

From Ingham’s inequality (see Ingham [10]) we obtain, for all $T > 2$, the existence of a constant $C_T > 0$ such that the solution ϕ of (5.5–5.7) satisfies

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(\xi, t) \right|^2 dt \geq C_T \sum_{n \geq 1} n^2 (a_n^2 + b_n^2) |\sin(n\pi\xi)|^2. \tag{5.16}$$

Relations (5.16) and (5.2) imply the existence of a constant $K_{T,\xi} > 0$ such that

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(\xi, t) \right|^2 dt \geq K_{T,\xi} \sum_{n=0}^{\infty} [a_n^2 + b_n^2], \quad \forall \xi \in \mathcal{S},$$

which is exactly (5.11).

In order to prove (5.12) we use a well-known result asserting that, for all $\epsilon > 0$ there exists a set $B_\epsilon \subset (0, 1)$ having the Lebesgue measure equal to 1 and a constant $C > 0$, such that for any, $\rho \in B_\epsilon$,

$$|\sin(n\pi\rho)| \geq \frac{C}{n^{1+\epsilon}}, \quad \forall n \geq 1. \tag{5.17}$$

Let us notice that by Roth’s theorem B_ϵ contains all numbers in $(0, 1)$ having the property that ξ is an algebraic irrational (see Cassals [7] for details). Inequalities (5.16) and (5.17) obviously imply (5.12).

We still have to show the existence of a sequence satisfying (5.13). By using continous fractions (see again [25] and references therein for details) we construct a sequence $(q_m) \subset \mathbb{N}$ such that $q_m \rightarrow \infty$ and

$$|\sin(q_m\pi\xi)| \leq \frac{\pi}{q_m}, \forall m \geq 1. \tag{5.18}$$

Using (5.15) and (5.47) a simple calculation shows that the sequence $(\phi_m^0, \phi_m^1) = (\sin(q_m\pi x), 0)$ satisfies (5.13).

Proof of Theorem 5.1. According to Theorem 2.2, the solutions of (I) satisfy the estimate

$$\left\| \left(u(t), \frac{\partial u}{\partial t}(t) \right) \right\|_{V \times L^2(0,1)}^2 \leq M e^{-\omega t} \|(u^0, u^1)\|_{V \times L^2(0,1)}^2, \quad \forall t \geq 0, \tag{5.19}$$

where $M, \omega > 0$ are constants depending only on ξ , if and only if the solution ϕ of (5.5–5.7) satisfies

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(\xi, s) \right|^2 ds \geq C \|(u^0, u^1)\|_{V \times L^2(0,1)}^2, \quad \forall (u^0, u^1) \in V \times L^2(0, 1).$$

The inequality above clearly contradicts assertion 3 in Proposition 5.4. So assumption (5.38) is false. We end up in this way the proof of the first assertion of theorem.

We pass now to the proof of the second assertion of this theorem. Let $\xi \in \mathcal{S}$. By Proposition 5.4, the solution ϕ of (5.5–5.7) satisfies the inequality

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(\xi, t) \right|^2 dt \geq K_1 \left(\|u^0\|_{L^2(0,1)}^2 + \|u^1\|_{H^{-1}(0,1)}^2 \right), \quad \forall (u^0, u^1) \in V \times L^2(0, 1),$$

where $K_1 > 0$ is a constant. The conclusion (5.3) follows now by simply using the Theorem 2.4 (with $X_1 = L^2(0, 1)$, $X_2 = H^{-1}(0, 1)$, $Y_1 \times Y_2 = [H_0^1(0, 1) \cup H^2(0, \xi) \cup H^2(\xi, 1)] \times H_0^1(0, 1)$, $\mathcal{G}(x) = x, \forall x \in \mathbb{R}^+, \theta = \frac{1}{2}$).

Let us now suppose that $\epsilon > 0$ and that ξ belongs to the set B_ϵ . From (5.12), it follows (5.4) by Theorem 2.4 (with $X_1 = H^{-\epsilon}(0, 1)$, $X_2 = H^{-1-\epsilon}(0, 1)$, $Y_1 \times Y_2 = [H_0^1(0, 1) \cup H^2(0, \xi) \cup H^2(\xi, 1)] \times H_0^1(0, 1)$, $\mathcal{G}(x) = x, \forall x \in \mathbb{R}^+$, $\theta = \frac{1}{2+\epsilon}$).

Remark 5.5. The same method can be applied for a string with different boundary conditions (see Ammari *et al.* [2]). □

5.2. Second example: Stabilization of the Bernoulli–Euler beam

We consider the following initial and boundary value problem:

$$\frac{\partial^2 u}{\partial t^2}(x, t) + \frac{\partial^4 u}{\partial x^4}(x, t) + \frac{\partial u}{\partial t}(\xi, t) \delta_\xi = 0, \quad 0 < x < 1, \quad t > 0, \tag{5.20}$$

$$u(0, t) = u(1, t) = \frac{\partial^2 u}{\partial x^2}(0, t) = \frac{\partial^2 u}{\partial x^2}(1, t) = 0, \quad t > 0, \tag{5.21}$$

$$u(x, 0) = u^0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u^1(x), \quad 0 < x < 1. \tag{5.22}$$

Here u denotes the transverse displacement of the beam, δ_ξ is the Dirac mass concentrated at the point $\xi \in (0, 1)$ and we suppose that the length of the beam is equal to 1.

In this case,

$$X = L^2(0, 1), \quad U = \mathbb{R}, \quad V = H^2(0, 1) \cap H_0^1(0, 1),$$

and

$$\begin{aligned} A &= \frac{d^4}{dx^4}, \quad \mathcal{D}(A) = \left\{ u \in H^4(0, 1) \cap H_0^1(0, 1), \frac{d^2 u}{dx^2}(0) = \frac{d^2 u}{dx^2}(1) = 0 \right\}, \\ Bk &= k\delta_\xi, \quad \forall k \in \mathbb{R}. \end{aligned} \tag{5.23}$$

\mathcal{A}_d is given by

$$\mathcal{A}_d \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\frac{d^4 u}{dx^4} - v(\xi) \delta_\xi \end{pmatrix},$$

$$\forall (u, v) \in \mathcal{D}(\mathcal{A}_d) = \left\{ (u, v) \in [H^2(0, 1) \cap H^4(0, \xi) \cap H^4(\xi, 1)] \times H^2(0, 1), \quad u(0) = v(0) \right.$$

$$\left. = u(1) = v(1) = \frac{d^2 u}{dx^2}(0) = \frac{d^2 u}{dx^2}(1) = 0, \frac{d^2 u}{dx^2}(\xi+) = \frac{d^2 u}{dx^2}(\xi-), \frac{d^3 u}{dx^3}(\xi+) - \frac{d^3 u}{dx^3}(\xi-) = -v(\xi) \right\}.$$

The stability results can now be stated as follows.

Theorem 5.6. 1. For any $\xi \in (0, 1)$ the system described by (5.20–5.22) is not exponentially stable in $V \times L^2(0, 1)$.

2. For all ξ in the set \mathcal{S} defined in Section 5.1 and for all $t \geq 0$ we have

$$\left\| \left(u(t), \frac{\partial u}{\partial t}(t) \right) \right\|_{V \times L^2(0,1)}^2 \leq \frac{C_\xi}{(t+1)^2} \|(u^0, u^1)\|_{\mathcal{D}(\mathcal{A}_d)}^2, \quad \forall (u^0, u^1) \in \mathcal{D}(\mathcal{A}_d), \tag{5.24}$$

where $C_\xi > 0$ is a constant depending only on ξ .

3. If $\epsilon > 0$ then, for almost all $\xi \in (0, 1)$ and for all $t \geq 0$ we have

$$\left\| \left(u(t), \frac{\partial u}{\partial t}(t) \right) \right\|_{V \times L^2(0,1)}^2 \leq \frac{C_{\xi,\epsilon}}{(t+1)^{\frac{2}{1+\epsilon}}} \|(u^0, u^1)\|_{\mathcal{D}(\mathcal{A}_d)}^2, \quad \forall (u^0, u^1) \in \mathcal{D}(\mathcal{A}_d), \tag{5.25}$$

where $C_{\xi,\epsilon} > 0$ is a constant depending only on ξ and ϵ .

In this case the problem (2.3, 2.4) becomes

$$\frac{\partial^2 \phi}{\partial t^2}(x, t) + \frac{\partial^4 \phi}{\partial x^4}(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \tag{5.26}$$

$$\phi(0, t) = \phi(1, t) = \frac{\partial^2 \phi}{\partial x^2}(0, t) = \frac{\partial^2 \phi}{\partial x^2}(1, t) = 0, \quad t > 0, \tag{5.27}$$

$$\phi(x, 0) = u^0(x), \quad \frac{\partial \phi}{\partial t}(x, 0) = u^1(x), \quad 0 < x < 1. \tag{5.28}$$

Lemma 5.7. *The operators A and B defined in (5.23) satisfy assumption (H).*

Proof. Let $k \in \mathbb{R}$. It can be easily checked that $v = (\lambda^2 - A)^{-1} Bk$ satisfies:

$$\lambda^2 v(x) + \frac{d^4 v}{dx^4}(x) = 0, \quad x \in (0, \xi) \cup (\xi, 1), \quad Re\lambda > 0, \tag{5.29}$$

$$v(0) = v(1) = \frac{d^2 v}{dx^2}(0) = \frac{d^2 v}{dx^2}(1) = 0, \quad Re\lambda > 0, \tag{5.30}$$

$$[v]_\xi = \left[\frac{dv}{dx} \right]_\xi = \left[\frac{d^2 v}{dx^2} \right]_\xi = 0, \tag{5.31}$$

$$\left[\frac{d^3 v}{dx^3} \right]_\xi = k, \quad Re\lambda > 0, \tag{5.32}$$

where we denote by $[f]_\xi$ the jump of the function f at the point ξ .

Recall that $\lambda = iw^2$ where $w \in \left\{ re^{i\gamma}; r > 0, \gamma \in] - \frac{\pi}{2}, 0[\right\}$. Then, the solutions of (5.29–5.31) have the following form

$$v(x) = \begin{cases} -\frac{1}{2w^3} \left\{ \frac{\sin[w(\xi - 1)] \sin(wx)}{\sin(w)} - \frac{\text{sh}(wx) \text{sh}[w(\xi - 1)]}{\text{sh}(w)} \right\} k, & x \in (0, \xi), \\ -\frac{1}{2w^3} \left\{ \frac{\sin(w\xi) \sin[w(x - 1)]}{\sin(w)} - \frac{\text{sh}(w\xi) \text{sh}[w(x - 1)]}{\text{sh}(w)} \right\} k, & x \in (\xi, 1). \end{cases}$$

Then the function $H(\lambda) = \lambda B^*(\lambda^2 - A)^{-1}B$ is given by

$$H(\lambda = iw^2) = \frac{i}{2w} \left\{ -\frac{\sin(w\xi) \sin[w(\xi - 1)]}{\sin(w)} + \frac{\text{sh}(w\xi) \text{sh}[w(\xi - 1)]}{\text{sh}(w)} \right\}, \forall \text{Re}\lambda > 0.$$

By Lemma 3.4 in [1] we conclude that $H(\lambda)$ is bounded on C_β .

The observability inequalities are given in the proposition below.

Proposition 5.8. *Let $T > 0$ be fixed and $\mathcal{S} \subset [0, 1]$ be the set introduced in Section 5.1. Then*

1. *For all $\xi \in \mathcal{S}$ the solution ϕ of (5.26–5.28) satisfies*

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(\xi, t) \right|^2 dt \geq C_\xi \left(\|u^0\|_{H^1(0,1)}^2 + \|u^1\|_{H^{-1}(0,1)}^2 \right),$$

$$\forall (u^0, u^1) \in V \times L^2(0, 1), \tag{5.33}$$

where $C_\xi > 0$ is a constant depending only on ξ .

2. *For all $\epsilon > 0$ and for almost all $\xi \in (0, 1)$ the solution ϕ of (5.26–5.28) satisfies*

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(\xi, t) \right|^2 dt \geq C_{\xi, \epsilon} \left(\|u^0\|_{H^{1-\epsilon}(0,1)}^2 + \|u^1\|_{H^{-1-\epsilon}(0,1)}^2 \right),$$

$$\forall (u^0, u^1) \in V \times L^2(0, 1), \tag{5.34}$$

where $C_{\xi, \epsilon} > 0$ is a constant depending only on ξ and ϵ .

3. *The result in assertion 1 is sharp in the sense that, for all $\xi \in (0, 1)$, there exists a sequence $(u_m^0, u_m^1) \subset V \times L^2(0, 1)$ such that the corresponding sequence of solutions (ϕ_m) of (5.26, 5.27) with initial data (u_m^0, u_m^1) satisfies*

$$\lim_{m \rightarrow \infty} \frac{\int_0^T \left| \frac{\partial \phi_m}{\partial t}(\xi, t) \right|^2 dt}{\|u_m^0\|_{H^{1+\epsilon}(0,1)}^2 + \|u_m^1\|_{H^{-1+\epsilon}(0,1)}^2} = 0. \tag{5.35}$$

Proof. Let

$$u^0(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \quad u^1(x) = \sum_{n=1}^{\infty} nb_n \sin(n\pi x),$$

with

$$(n^2 a_n), (n b_n) \in l^2(\mathbb{R}).$$

Which implies that

$$\frac{\partial \phi}{\partial t}(\xi, t) = \sum_{n=1}^{\infty} [-n^2 \pi^2 a_n \sin(n^2 \pi^2 t) \sin(n \pi \xi) + n^2 \pi^2 b_n \cos(n^2 \pi^2 t) \sin(n \pi \xi)]. \tag{5.36}$$

From (5.36) and Ingham’s inequality (see [10]), we obtain that, for all $T > 2$, there exists a constant $C_T > 0$ such that

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(\xi, t) \right|^2 dt \geq C_T \sum_{n=1}^{\infty} [n^4 a_n^2 \sin^2(n \pi \xi) + n^4 b_n^2 \sin^2(n \pi \xi)]. \tag{5.37}$$

Suppose now that ξ belongs to the set \mathcal{S} defined in Section 5.1. Then relations (5.37) and (5.2) imply the existence of a constant $K_{T,\xi} > 0$ such that

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(\xi, t) \right|^2 dt \geq K_{T,\xi} \sum_{n=1}^{\infty} [n^2 a_n^2 + n^2 b_n^2], \quad \forall \xi \in \mathcal{S},$$

which is exactly (5.33).

Assertions 2 and 3 of the proposition can be proved by simply adapting the proof of Proposition 5.4. So we skip the details. □

Proof of Theorem 5.6. 1. According to Theorem 2.2, all finite energy solutions of (5.20–5.22) satisfy the estimate

$$\left\| (u(t), \frac{\partial u}{\partial t}(t)) \right\|_{V \times L^2(0,1)}^2 \leq M e^{-\omega t} \|(u^0, u^1)\|_{V \times L^2(0,1)}^2, \quad \forall t \geq 0, \tag{5.38}$$

where $M, \omega > 0$ are constants depending only on ξ , if and only if the solution ϕ of (5.26–5.28) satisfies

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(\xi, s) \right|^2 ds \geq C \|(u^0, u^1)\|_{V \times L^2(0,1)}^2, \quad \forall (u^0, u^1) \in V \times L^2(0, 1).$$

The inequality above clearly contradicts assertion 3 in Proposition 5.8. So Assumption (5.38) is false. We end up in this way the proof of the first assertion of this theorem.

We pass now to the proof of the second assertion of this theorem. Let $\xi \in \mathcal{S}$. By Proposition 5.8, the solution ϕ of (5.26–5.28) satisfies the inequality

$$\int_0^T \left| \frac{\partial \phi}{\partial t}(\xi, t) \right|^2 dt \geq K_1 \left(\|u^0\|_{H^1(0,1)}^2 + \|u^1\|_{H^{-1}(0,1)}^2 \right), \quad \forall (u^0, u^1) \in V \times L^2(0, 1),$$

where $K_1 > 0$ is a constant. The conclusion (5.24) follows now by simply using the Theorem 2.4 (with $X_1 = H^1(0, 1)$, $X_2 = H^{-1}(0, 1)$, $Y_1 \times Y_2 = [H^2(0, 1) \cap H^4(0, 1) \cap H^4(\xi, 1)] \times H^2(0, 1)$, $\theta = \frac{2}{3}$).

Let us now suppose that $\epsilon > 0$ and that ξ belongs to the set B_ϵ , introduced in the proof of Proposition 5.4. From (5.34), it follows (5.25) by Theorem 2.4 (with $X_1 = H^{1-\epsilon}(0, 1)$, $X_2 = H^{-1-\epsilon}(0, 1)$, $Y_1 \times Y_2 = [H^2(0, 1) \cap H^4(0, 1) \cap H^4(\xi, 1)] \times H^2(0, 1)$, $\theta = \frac{2}{3+\epsilon}$).

Remark 5.9. For different boundary condition there exist points $\xi \in (0, 1)$ for which we have exponential decay (see Rebarber [20] and [1] for characterization of these points). □

5.3. Third example: Dirichlet boundary stabilization of the wave equation in a disk

Let $B_R = \{x \in \mathbb{R}^2, \|x\|_e < R\}$ with boundary $\partial B_R = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$, where Γ_0, Γ_1 are disjoint parts of the boundary relatively open in ∂B_R , $\text{int}(\Gamma_0) \neq \emptyset$. Above $\|\cdot\|_e$ denotes the Euclidean norm in \mathbb{R}^2 . We consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, B_R \times (0, +\infty), \tag{5.39}$$

$$u = \frac{\partial}{\partial \nu}(Gu'), \Gamma_0 \times (0, +\infty), \tag{5.40}$$

$$u = 0, \Gamma_1 \times (0, +\infty), \tag{5.41}$$

$$u(x, 0) = u^0(x), \frac{\partial u}{\partial t}(x, 0) = u^1(x), B_R, \tag{5.42}$$

where ν is the unit normal vector of ∂B_R pointing towards the exterior of B_R and $G = (-\Delta)^{-1} : H^{-1}(B_R) \rightarrow H_0^1(B_R)$.

Denote: $A = -\Delta, \mathcal{D}(A) = H_0^1(B_R)$. We will also denote by A the extension of this operator $A : L^2(B_R) \rightarrow (H^2(B_R) \cap H_0^1(B_R))'$. Notice that $(H^2(B_R) \cap H_0^1(B_R))'$ is the dual of $\mathcal{D}(A^{\frac{1}{2}})$ with respect to the pivot space $H^{-1}(B_R)$. Moreover define

$$B \in \mathcal{L}(L^2(\Gamma_0), (H^2(B_R) \cap H_0^1(B_R))')$$

by $Bv = ADv, \forall v \in L^2(\Gamma_0)$, where $D \in \mathcal{L}(L^2(\Gamma_0), L^2(B_R))$ is the Dirichlet map *i.e.*, $Df = g$ if and only if

$$\begin{cases} \Delta g = 0, B_R, \\ g = f, \Gamma_0, g = 0, \Gamma_1. \end{cases}$$

Problem (5.39–5.42) can be rewritten in $V \times X = L^2(B_R) \times H^{-1}(B_R)$ as follows

$$\begin{cases} u'' + Au + BB^*u' = 0, \\ u(0) = u^0, u'(0) = u^1. \end{cases}$$

Then, according to Proposition 2.1 the problem (5.39–5.42) is well-posed in $L^2(B_R) \times H^{-1}(B_R)$, *i.e.*, for all $(u^0, u^1) \in L^2(B_R) \times H^{-1}(B_R)$, equations (5.39–5.42) admit a unique solution

$$u \in C([0, \infty); L^2(B_R)) \cap C^1([0, \infty); H^{-1}(B_R)).$$

Moreover $u(x, t)$ satisfies the following energy estimate

$$\|(u^0, u^1)\|_{L^2(B_R) \times H^{-1}(B_R)}^2 - \left\| \left(u(t), \frac{\partial u}{\partial t}(t) \right) \right\|_{L^2(B_R) \times H^{-1}(B_R)}^2 = 2 \int_0^t \int_{\Gamma_0} \left| \frac{\partial [G(\frac{\partial u}{\partial t})]}{\partial \nu}(x, s) \right|^2 d\Gamma_0 ds. \tag{5.43}$$

Let ϕ be the solution of the following problem

$$\frac{\partial^2 \phi}{\partial t^2} - \Delta \phi = 0, B_R \times (0, \infty), \tag{5.44}$$

$$\phi = 0, \partial B_R \times (0, \infty), \tag{5.45}$$

$$\phi(x, 0) = \phi^0(x), \frac{\partial \phi}{\partial t}(x, 0) = \phi^1(x), B_R. \tag{5.46}$$

Denote

$$\mathcal{A}_d = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix},$$

with

$$\mathcal{D}(\mathcal{A}_d) = \left\{ (u, v) \in L^2(B_R) \times H^{-1}(B_R) / (v, \Delta u) \in L^2(B_R) \times H^{-1}(B_R), u|_{\Gamma_0} = \frac{\partial[Gv]}{\partial \nu}, u|_{\Gamma_1} = 0 \right\}.$$

If $(u, v) \in \mathcal{D}(\mathcal{A}_d)$ we denote

$$\|(u, v)\|_{\mathcal{D}(\mathcal{A}_d)}^2 = \|(u, v)\|_{L^2(B_R) \times H^{-1}(B_R)}^2 + \|(v, \Delta u)\|_{L^2(B_R) \times H^{-1}(B_R)}^2.$$

The following holds.

Theorem 5.10. *Suppose that there exists $T_0 > 0$ such that the solution ϕ of (5.44–5.46) satisfy for all $T > T_0$*

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma_0 dt \geq C \|(\phi^0, \phi^1)\|_{H_0^1(B_R) \times L^2(B_R)}^2, \tag{5.47}$$

where C is a positive constant which depends only on T_0 .

Then all finite energy solutions $(u(t), u'(t))$ of (5.39–5.42) decay exponentially to zero in $L^2(B_R) \times H^{-1}(B_R)$.

Similar results were proved in a more general geometry and by different methods (see for instance Bardos *et al.* [4] and references therein).

Remark 5.11. Necessary and sufficient condition such that assumption (5.47) is satisfied is given in Bardos *et al.* [5].

Theorem 5.12. *For all $(u^0, u^1) \in \mathcal{D}(\mathcal{A}_d)$ there exists a constant $K > 0$ such that*

$$\left\| \left(u(t), \frac{\partial u}{\partial t}(t) \right) \right\|_{L^2(B_R) \times H^{-1}(B_R)}^2 \leq \frac{K \|(u^0, u^1)\|_{\mathcal{D}(\mathcal{A}_d)}^2}{\ln(1+t)}, \quad \forall t > 0. \tag{5.48}$$

Technicals results

To prove Theorem 5.10 and Theorem 5.12 we need the following technicals results.

Let $\beta > 0$ be a fixed real number and $C_\beta = \{ \lambda \in \mathbb{C} \mid Re\lambda = \beta \}$. Denote by J_l the Bessel functions of integer order of first kind:

$$\begin{cases} \frac{1}{r} \frac{d}{dr} \left(r \frac{dJ_l}{dr}(r) \right) + \left(1 - \frac{l^2}{r^2} \right) J_l(r) = 0, \text{ for } r > 0, \\ J_l \text{ is regular in } r = 0. \end{cases} \tag{5.49}$$

Proposition 5.13. *There exists a constant $c_\beta > 0$ (depends only on β and R), such that*

$$\sup_{\lambda \in C_\beta} \frac{1}{|\lambda|} \left| i\lambda \frac{J'_l(i\lambda R)}{J_l(i\lambda R)} - \frac{l}{R} \right| \leq c_\beta.$$

Proof. The above estimate follows from the uniform asymptotic expansions of $J_l(z)$ (for $|l|$ and $|z|$ big enough). The reader is referred to Olver and Watson for the derivation and the proof of these asymptotic expansions. According to Olver and Watson, there are three regions for the asymptotic expansions of $J_l(z)$: $|z| \gg |l|$, $|l| \ll |z|$ and the transition region where $|z| \sim |l|$. Making use of the formulas given in Olver [18] and Watson [27] (pp. 231 and 249), in each region we compute the asymptotic expansion of $\frac{J'_l(z)}{J_l(z)}$. The following holds.

For $|l| \gg |\text{Im}\lambda|$,

$$\left| i\lambda \frac{J'_l(i\lambda R)}{J_l(i\lambda R)} - \frac{l}{R} \right| \leq C,$$

where C is independent in l and in $\text{Im}\lambda$.

For $|l| \sim |\text{Im}\lambda|$,

$$i\lambda \frac{J'_l(i\lambda R)}{J_l(i\lambda R)} \sim \lambda^{\frac{2}{3}}.$$

For $|l| \ll |\text{Im}\lambda|$,

$$\left| i\lambda \frac{J'_l(i\lambda R)}{J_l(i\lambda R)} - \frac{l}{R} \right| \leq C|\lambda|,$$

where C is independent in l and $\text{Im}\lambda$.

The estimates above end the proof.

Proposition 5.14. *For all $(\phi^0, \phi^1) \in L^2(B_R) \times H^{-1}(B_R)$ and for all $\Gamma_0 \subset \partial B_R$, $\text{int}(\Gamma_0) \neq \emptyset$, there exist $T, C_1 > 0$ such that*

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial}{\partial \nu} \left[G \left(\frac{\partial \phi}{\partial t} \right) \right] \right|^2 d\Gamma_0 dt \geq \|\phi^0, \phi^1\|_{L^2(B_R) \times H^{-1}(B_R)}^2 \exp \left(- 2C_1^2 \frac{\|(\phi^0, \phi^1)\|_{L^2(B_R) \times H^{-1}(B_R)}^2}{\|(\phi^0, \phi^1)\|_{H^{-1}(B_R) \times H^{-2}(B_R)}^2} \right).$$

Proof. Let $\psi = G(\frac{\partial \phi}{\partial t})$, then ψ satisfy (5.44, 5.45) with $(\psi^0 = G(\phi^1), \psi^1 = -\phi_0) \in H_0^1(B_R) \times L^2(B_R)$. Thus, according to Theorem 1 in Robbiano [21], for all $\Gamma_0 \subset \partial B_R$, there exist $T, C_1 > 0$ such that

$$\|(G(\phi_1), -\phi_0)\|_{L^2(B_R) \times H^{-1}(B_R)} \leq \frac{C_1 \|(G(\phi_1), -\phi_0)\|_{H_0^1(B_R) \times L^2(B_R)}}{\left[\ln \left(2 + \frac{\|(G(\phi_1), -\phi_0)\|_{H_0^1(B_R) \times L^2(B_R)}}{\left(\int_0^T \int_{\Gamma_0} \left| \frac{\partial}{\partial \nu} [G(\frac{\partial \phi}{\partial t})] \right|^2 d\Gamma_0 dt \right)^{\frac{1}{2}}} \right) \right]^{\frac{1}{2}}}.$$

Which obviously implies,

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial}{\partial \nu} \left[G \left(\frac{\partial \phi}{\partial t} \right) \right] \right|^2 d\Gamma_0 dt \geq \|\phi^0, \phi^1\|_{L^2(B_R) \times H^{-1}(B_R)}^2 \exp \left(-2C_1^2 \frac{\|(\phi^0, \phi^1)\|_{L^2(B_R) \times H^{-1}(B_R)}^2}{\|(\phi^0, \phi^1)\|_{H^{-1}(B_R) \times H^{-2}(B_R)}^2} \right).$$

□

Proof of Theorem 5.10. By Theorem 2.2 it suffices to verify (H).

Notice first that

$$B^*v = \frac{\partial[G(v)]}{\partial \nu}, \forall v \in L^2(B_R).$$

The function $H(\lambda) \in \mathcal{L}(L^2(\Gamma_0))$ associated to the problem (5.39–5.42) has an explicit representation in terms of Bessell functions. The function $w = (\lambda^2 - \Delta)^{-1} Bg$ satisfies

$$\begin{cases} \lambda^2 w - \Delta w = 0, & B_R, \\ w = g, \Gamma_0, w = 0, & \Gamma_1, \end{cases}$$

where $g \in L^2(\Gamma_0)$.

If we consider the Fourier expansion

$$f = \begin{cases} g, \Gamma_0, \\ 0, \Gamma_1, \end{cases} = \sum_{l \in \mathbb{Z}} f_l e^{il\theta} \in L^2(\partial B_R),$$

a simple calculation using (5.49) shows

$$w(x) = w(r, \theta) = \sum_{l \in \mathbb{Z}} \frac{J_l(i\lambda r)}{J_l(i\lambda R)} f_l e^{il\theta}.$$

Thus,

$$G(w)(x) = G(w)(r, \theta) = \frac{1}{\lambda^2} \sum_{l \in \mathbb{Z}} \left(\frac{J_l(i\lambda r)}{J_l(i\lambda R)} - \left(\frac{r}{R} \right)^l \right) f_l e^{il\theta}.$$

Which obviously implies that

$$H(\lambda)g = \lambda \frac{\partial[G(w)]}{\partial \nu} \Big|_{r=R} = \frac{1}{\lambda} \sum_{l \in \mathbb{Z}} \left(i\lambda \frac{J'_l(i\lambda R)}{J_l(i\lambda R)} - \frac{l}{R} \right) f_l e^{il\theta}, \forall Re\lambda > 0.$$

By Proposition 5.13 we obtain

$$\|H(\lambda)g\|_{L^2(\Gamma_0)}^2 = \frac{1}{|\lambda|^2} \sum_{l \in \mathbb{Z}} \left| i\lambda \frac{J'_l(i\lambda R)}{J_l(i\lambda R)} - \frac{l}{R} \right|^2 |f_l|^2 \leq c_\beta^2 \|f\|_{L^2(\partial B_R)}^2 = c_\beta^2 \|g\|_{L^2(\Gamma_0)}^2,$$

where the positive constant c_β depends only on β and R . From this estimate we conclude that the function $H(\lambda)$ is bounded on C_β , *i.e.* (H) holds true.

Thus, making use of Theorem 2.2, we obtain the exponential stability of all finite energy solutions of (5.39–5.42). □

Proof of Theorem 5.12. According to Proposition 5.14, for all $(u^0, u^1) \in L^2(B_R) \times H^{-1}(B_R)$ and for all $\Gamma_0 \subset \partial B_R$, there exist $T > 0$ and $C_1 > 0$ such that

$$\int_0^T \left\| \frac{\partial [G(\frac{\partial \phi}{\partial t})]}{\partial \nu} \right\|_{L^2(\Gamma_0)}^2 dt \geq \|(u^0, u^1)\|_{L^2(B_R) \times H^{-1}(B_R)}^2 \exp\left(-2C_1^2 \frac{\|(u^0, u^1)\|_{L^2(B_R) \times H^{-1}(B_R)}^2}{\|(u^0, u^1)\|_{H^{-1}(B_R) \times H^{-2}(B_R)}^2}\right).$$

Since (H) is satisfied (see the proof of Th. 5.10), then Lemma 4.1 implies that the solution $u(x, t)$ of (5.39–5.42) satisfies the following inequality

$$\int_0^T \left\| \frac{\partial [G(\frac{\partial u}{\partial t})]}{\partial \nu} \right\|_{L^2(\Gamma_0)}^2 dt \geq C \|(u^0, u^1)\|_{L^2(B_R) \times H^{-1}(B_R)}^2 \exp\left(-2C_1^2 \frac{\|(u^0, u^1)\|_{L^2(B_R) \times H^{-1}(B_R)}^2}{\|(u^0, u^1)\|_{H^{-1}(B_R) \times H^{-2}(B_R)}^2}\right),$$

$$\forall (u^0, u^1) \in L^2(B_R) \times H^{-1}(B_R).$$

Estimate (5.48) follow from Theorem 2.4 (with $X_1 = H^{-1}(B_R), X_2 = H^{-2}(B_R), Y_1 \times Y_2 = H^1(B_R) \times L^2(B_R), \mathcal{G}(x) = \exp(-\frac{2C_1^2}{x}), \forall x > 0, \theta = \frac{1}{2}$). □

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