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STABILIZATION OF THE WAVE EQUATION WITH BOUNDARY OR INTERNAL DISTRIBUTED DELAY

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Abstract. We consider the wave equation in a bounded region with a smooth boundary with distributed delay on the boundary or into the domain. In both cases, under suitable assumptions, we prove the exponential stability of the solution. These results are obtained by introducing suitable energies and by proving some observability inequalities. For an internal distributed delay, we further show some instability results.

1. Introduction

We study the wave equation subject to Dirichlet boundary conditions on one part of the boundary and dissipative boundary conditions of delay type on the remainder of the boundary. More precisely, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a smooth boundary Γ. We assume that Γ is divided into two closed and disjoint parts Γ_0 and Γ_1; i.e., $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Moreover, we assume that the measure of Γ_0 is positive.

In this domain Ω, we consider the initial-boundary-value problem

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad \Omega \times (0, +\infty), \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty), \tag{1.2}$$

$$\frac{\partial u}{\partial \nu}(t) + \int_{\tau_1}^{\tau_2} \mu(s) u_t(t-s)\,ds + \mu_0 u_t(t) = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \tag{1.3}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{1.4}$$

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$$u_t(x,-t) = f_0(x,-t) \quad \text{in} \quad \Gamma_1 \times (0,\tau_2), \tag{1.5}$$

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative. Moreover, τ_1 and τ_2 are two real numbers with $0 \le \tau_1 < \tau_2$, μ_0 is a positive constant, $\mu : [\tau_1, \tau_2] \to \mathbb{R}$ is an L^∞ function, $\mu \ge 0$ almost everywhere, and the initial data (u_0, u_1, f_0) belong to a suitable space (see below).

Let us denote by $\langle u, v\rangle$ or, equivalently, by $u \cdot v$ the Euclidean inner product between two vectors $u, v \in \mathbb{R}^n$.

We assume that there exists a scalar function $v \in C^2(\overline{\Omega})$ such that

(i) v is strictly convex in $\overline{\Omega}$; that is, there exists a constant $\alpha > 0$ such that

$$\langle D^2(v)(x)\xi, \xi\rangle \ge 2\alpha|\xi|^2, \quad \forall\, x \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^n, \tag{1.6}$$

where $D^2(v)$ denotes the Hessian matrix of v; and

(ii) the vector field $H := \nabla v$ satisfies

$$H(x) \cdot \nu(x) \le 0 \quad \forall x \in \Gamma_0. \tag{1.7}$$

For a discussion about these standard assumptions see [18], where some observability estimates for second-order hyperbolic operators are given.

The above problem can be regarded as a problem with a memory acting only on the time interval $(t-\tau_2, t-\tau_1)$. Indeed, by a change of variable, we see that

$$\int_{\tau_1}^{\tau_2} \mu(s) u_t(t-s)ds = \int_{t-\tau_2}^{t-\tau_1} \mu(t-s) u_t(s)ds.$$

It is well known that, if $\mu \equiv 0$, that is, in absence of delay, the energy of problem (1.1) – (1.4) is exponentially decaying to zero. See for instance Chen [3, 4], Lagnese [15, 16], Lasiecka and Triggiani [17], Komornik and Zuazua [14], Komornik [12, 13]. On the contrary, in the presence of a delay concentrated at a time τ, if the boundary condition (1.3) is replaced by

$$\frac{\partial u}{\partial \nu}(t) + \mu_1 u_t(t-\tau) + \mu_0 u_t(t) = 0 \quad \text{on} \quad \Gamma_1 \times (0,+\infty),$$

some instability phenomena occur. For instance, if $\mu_0 = 0$ there is an instability result of Datko, Lagnese and Polis in one space dimension [7]. If $\mu_0 > \mu_1$ exponential stability is proved by Xu, Yung and Li [25] in the one-dimensional case and by the authors in general space dimensions [21]. In the case $\mu_0 \le \mu_1$ instability examples are also given [25, 21]. As mentioned before our boundary condition, (1.3) can be compared to a boundary condition with memory, the difference being that, for such a boundary condition with memory, $\tau_1 = 0$ and $\tau_2 = t$ and then τ_2 depends on the time

t, which is excluded here. For such systems, the exponential or polynomial decay of the energy is proved in [1, 2, 10, 11, 22, 23, 24], by combining the multiplier method with the use of suitable Lyapounov functional or integral inequalities. Here, under the assumption

$$\mu_0 > \int_{\tau_1}^{\tau_2} \mu(s)ds, \tag{1.8}$$

we will prove an exponential stability result for problem (1.1) – (1.5).

We can consider also the problem with internal feedback:

$$u_{tt} - \Delta u + \mu_0 u_t + \int_{\tau_1}^{\tau_2} a(x)\mu(s)u_t(t-s)ds = 0 \quad \text{in} \quad \Omega \times (0, +\infty), \tag{1.9}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty), \tag{1.10}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \tag{1.11}$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega, \tag{1.12}$$

$$u_t(x, -t) = f_0(x, -t) \quad \text{in} \quad \Omega \times (0, \tau_2), \tag{1.13}$$

where $a \in L^\infty(\Omega)$ is a function such that

$$a(x) \geq 0, \quad \text{a. e. in } \Omega,$$

and

$$a(x) > a_0 > 0, \quad \text{a. e. in } \omega,$$

where $\omega \subset \overline{\Omega}$ is an open neighborhood of Γ_1.

Exponential stability results for the above problem (1.9) – (1.13) in the case $\mu \equiv 0$, that is, without delay, have been obtained by several authors. See for instance Zuazua [26] and Liu [20]. On the contrary, the presence of delay concentrated at a time τ may destabilize the above system. More precisely, if instead of equation (1.9) we consider

$$u_{tt} - \Delta u + \mu_0 u_t + \mu_1 u_t(t - \tau)ds = 0 \quad \text{in} \quad \Omega \times (0, +\infty),$$

the above system is exponentially stable in the case $\mu_0 > \mu_1$ and there are instability examples in the case $\mu_0 \leq \mu_1$ [21]. See also Datko [5, 6] for instability results in one space dimension in the case $\mu_0 = 0$.

In this paper, assuming

$$\mu_0 > \|a\|_\infty \int_{\tau_1}^{\tau_2} \mu(s)ds, \tag{1.14}$$

we will prove the exponential decay of the energy of problem (1.9) – (1.13).

The paper is organized as follows. In Section 2 we will prove the well posedness of problems (1.1) – (1.5) and (1.9) – (1.13) using semigroup theory. In Section 3 we will give the exponential stability of the problem with boundary feedback while in Section 4 we will deal with the problem with internal feedback. Finally, in Section 5 we will give an instability example if our assumption (1.14) is not verified.

2. WELL POSEDNESS

In this section we will prove that systems (1.1) – (1.5) and (1.9) – (1.13) are well posed using semigroup theory.

We start by considering the problem with boundary feedback (1.1) – (1.5). Let us set

$$z(x,\rho,t,s) = u_t(x,t-\rho s), \quad x \in \Gamma_1, \quad \rho \in (0,1), \quad s \in (\tau_1,\tau_2), \quad t > 0. \tag{2.1}$$

Then, problem (1.1) – (1.5) is equivalent to

$$u_{tt}(x,t) - \Delta u(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty), \tag{2.2}$$

$$s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0 \quad \text{in} \quad \Gamma_1 \times (0,1) \times (0,+\infty) \times (\tau_1,\tau_2), \tag{2.3}$$

$$u(x,t) = 0 \quad \text{on} \quad \Gamma_0 \times (0,+\infty), \tag{2.4}$$

$$\frac{\partial u}{\partial \nu}(x,t) = -\mu_0 u_t(x,t) - \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\,ds \text{ on } \Gamma_1 \times (0,+\infty), \tag{2.5}$$

$$z(x,0,t,s) = u_t(x,t) \quad \text{on} \quad \Gamma_1 \times (0,\infty) \times (\tau_1,\tau_2), \tag{2.6}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{2.7}$$

$$z(x,\rho,0,s) = f_0(x,\rho,s) \quad \text{in} \quad \Gamma_1 \times (0,1) \times (0,\tau_2). \tag{2.8}$$

If we set $U := (u, u_t, z)^T$, then $U' := (u_t, u_{tt}, z_t)^T = \left(u_t, \Delta u, -s^{-1} z_\rho\right)^T$. Therefore, problem (2.2) – (2.8) can be rewritten as

$$\begin{cases} U' = \mathcal{A} U \\ U(0) = (u_0, u_1, f_0)^T \end{cases} \tag{2.9}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u \\ -s^{-1} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}) := \Big\{ \ (u,v,z)^T \in$$

$$\big(E(\Delta, L^2(\Omega)) \cap H^1_{\Gamma_0}(\Omega)\big) \times H^1_{\Gamma_0}(\Omega) \times L^2(\Gamma_1 \times (\tau_1, \tau_2); H^1(0,1)) \ : \\ \frac{\partial u}{\partial \nu}(x) = -\mu_0 v(x) - \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s)ds \text{ on } \Gamma_1; \\ v(x) = z(x,0,s) \text{ on } \Gamma_1 \ \Big\}, \tag{2.10}$$

where, as usual, $H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}$, and $E(\Delta, L^2(\Omega)) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$. Recall that for a function $u \in E(\Delta, L^2(\Omega))$, the normal derivative $\frac{\partial u}{\partial \nu}$ belongs to $H^{-1/2}(\Gamma_1)$ and the next Green's formula is valid (see Section 1.5 of [9]):

$$\int_\Omega \nabla u \nabla w dx = -\int_\Omega \Delta u w dx + \langle \frac{\partial u}{\partial \nu}; w\rangle_{\Gamma_1}, \quad \forall w \in H^1_{\Gamma_0}(\Omega), \tag{2.11}$$

where $\langle \cdot ; \cdot \rangle_{\Gamma_1}$ means the duality pairing between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$.

Note further that, for $(u,v,z)^T \in \mathcal{D}(\mathcal{A})$, $\frac{\partial u}{\partial \nu}$ belongs to $L^2(\Gamma_1)$, since $z(\cdot,1,s)$ is in $L^2(\Gamma_1)$. Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1 \times (0,1) \times (\tau_1, \tau_2)). \tag{2.12}$$

We will show that $\mathcal{A}$ generates a C_0 semigroup on $\mathcal{H}$, under the assumption

$$\mu_0 \geq \int_{\tau_1}^{\tau_2} \mu(s) ds. \tag{2.13}$$

Let us define on the Hilbert space $\mathcal{H}$ the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} := \int_\Omega \{\nabla u(x) \nabla \tilde{u}(x) + v(x)\tilde{v}(x)\} dx \\ + \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} s\mu(s) \int_0^1 z(x,\rho,s)\tilde{z}(x,\rho,s) d\rho ds d\Gamma. \tag{2.14}$$

Theorem 2.1. *Assume that* (2.13) *holds. Then, for any initial datum $U_0 \in \mathcal{H}$ there exists a unique solution $U \in C([0,+\infty), \mathcal{H})$ of problem* (2.9). *Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then $U \in C([0,+\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0,+\infty), \mathcal{H})$.*

Proof. Let $U = (u,v,z)^T \in \mathcal{D}(\mathcal{A})$. Then

$$(\mathcal{A}U, U) = \left\langle \begin{pmatrix} v \\ \Delta u \\ -s^{-1} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_\Omega \{\nabla v(x) \nabla u(x) + v(x) \Delta u(x)\} dx \\ - \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} \mu(s) \int_0^1 z_\rho(x,\rho,s) z(x,\rho,s) d\rho ds d\Gamma.$$

Then, by Green's formula,

$$(\mathcal{A}U, U) = \int_{\Gamma_1} \frac{\partial u}{\partial \nu}(x) v(x) d\Gamma - \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} \mu(s) \int_0^1 z_\rho(x, \rho, s) z(x, \rho, s) d\rho ds d\Gamma. \tag{2.15}$$

Integrating by parts in ρ, we have

$$\int_0^1 z_\rho(x, \rho, s) z(x, \rho, s) d\rho = - \int_0^1 z_\rho(x, \rho, s) z(x, \rho, s) d\rho + z^2(x, 1, s) - z^2(x, 0, s);$$

that is,

$$\begin{aligned} &\int_{\Gamma_1} \int_{\tau_1}^{\tau_2} \mu(s) \int_0^1 z_\rho(x, \rho, s) z(x, \rho, s) d\rho ds d\Gamma \\ &\quad = \frac{1}{2} \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} \mu(s) \{z^2(x, 1, s) - z^2(x, 0, s)\} ds d\Gamma. \end{aligned} \tag{2.16}$$

Therefore, from (2.15) and (2.16),

$$\begin{aligned} (\mathcal{A}U, U) &= - \int_{\Gamma_1} v(x) \left[-\mu_0 v(x) - \int_{\tau_1}^{\tau_2} \mu(s) z(x, 1, s) \right] d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} \mu(s) [z^2(x, 1, s) - z^2(x, 0, s)] ds d\Gamma \\ &= -\mu_0 \int_{\Gamma_1} v^2(x) d\Gamma - \int_{\Gamma_1} v(x) \left(\int_{\tau_1}^{\tau_2} \mu(s) z(x, 1, s) ds \right) d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} \mu(s) z^2(x, 1, s) ds d\Gamma + \frac{1}{2} \int_{\tau_1}^{\tau_2} \mu(s) ds \int_{\Gamma_1} v^2(x) d\Gamma. \end{aligned}$$

Now, using Cauchy-Schwarz's inequality, we can estimate

$$\begin{aligned} &\left| \int_{\Gamma_1} v(x) \left(\int_{\tau_1}^{\tau_2} \mu(s) z(x, 1, s) ds \right) d\Gamma \right| \\ &\qquad \le \frac{1}{2} \int_{\Gamma_1} v^2(x) \left(\int_{\tau_1}^{\tau_2} \mu(s) ds \right) d\Gamma + \frac{1}{2} \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} \mu(s) z^2(x, 1, s) ds d\Gamma. \end{aligned}$$

Therefore, from the assumption (2.13),

$$(\mathcal{A}U, U) \le \left(-\mu_0 + \int_{\tau_1}^{\tau_2} \mu(s) ds \right) \int_{\Gamma_1} v^2(x) d\Gamma \le 0; \tag{2.17}$$

that is, the operator $\mathcal{A}$ is dissipative.

Now, we will show that $\lambda I - \mathcal{A}$ is surjective for a fixed $\lambda > 0$. Given $(f, g, h)^T \in \mathcal{H}$, we seek $U = (u, v, z)^T \in \mathcal{D}(\mathcal{A})$ a solution of

$$(\lambda I - \mathcal{A}) \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix};$$

that is, satisfying

$$\begin{cases} \lambda u - v = f \\ \lambda v - \Delta u = g \\ \lambda z + s^{-1} z_\rho = h. \end{cases} \tag{2.18}$$

Suppose that we have found u with the right regularity. Then, we set

$$v := \lambda u - f \tag{2.19}$$

and we can determine z. Indeed, by (2.10),

$$z(x, 0, s) = v(x), \quad \text{for} \quad x \in \Gamma_1, \quad s \in (\tau_1, \tau_2), \tag{2.20}$$

and, from (2.18),

$$\lambda z(x, \rho, s) + s^{-1} z_\rho(x, \rho, s) = h(x, \rho, s), \quad \text{for} \quad x \in \Gamma_1, \ \rho \in (0, 1), \ s \in (\tau_1, \tau_2). \tag{2.21}$$

Then, by (2.20) and (2.21), we obtain

$$z(x, \rho, s) = e^{-\lambda \rho s} v(x) + s e^{-\lambda \rho s} \int_0^\rho h(x, \sigma, s) e^{\lambda \sigma s} d\sigma.$$

So, from (2.19), on $\Gamma_1 \times (0, 1) \times (\tau_1, \tau_2)$,

$$z(x, \rho, s) = \lambda u(x) e^{-\lambda \rho s} - f(x) e^{-\lambda \rho s} + s e^{-\lambda \rho s} \int_0^\rho h(x, \sigma, s) e^{\lambda \sigma s} d\sigma, \tag{2.22}$$

and, in particular,

$$z(x, 1, s) = \lambda u(x) e^{-\lambda s} + z_0(x, s), \quad x \in \Gamma_1, \ s \in (\tau_1, \tau_2), \tag{2.23}$$

with $z_0 \in L^2(\Gamma_1 \times (\tau_1, \tau_2))$ defined by

$$z_0(x, s) = -f(x) e^{-\lambda s} + s e^{-\lambda s} \int_0^1 h(x, \sigma, s) e^{\lambda \sigma s} d\sigma, \quad x \in \Gamma_1, \ s \in (\tau_1, \tau_2). \tag{2.24}$$

By (2.19) and (2.18), the function u satisfies

$$\lambda(\lambda u - f) - \Delta u = g;$$

that is,

$$\lambda^2 u - \Delta u = g + \lambda f. \tag{2.25}$$

Problem (2.25) can be reformulated as

$$\int_\Omega (\lambda^2 u - \Delta u) w dx = \int_\Omega (g + \lambda f) w dx, \quad \forall w \in H^1_{\Gamma_0}(\Omega). \tag{2.26}$$

Integrating by parts,

$$\begin{aligned}
\int_\Omega (\lambda^2 u - \Delta u) w dx &= \int_\Omega (\lambda^2 u w + \nabla u \nabla w) dx - \int_{\Gamma_1} \frac{\partial u}{\partial \nu} w d\Gamma \\
&= \int_\Omega (\lambda^2 u w + \nabla u \nabla w) dx + \int_{\Gamma_1} \left(\mu_0 v w + w \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s) ds \right) d\Gamma \\
&= \int_\Omega (\lambda^2 u w + \nabla u \nabla w) dx \\
&\quad + \int_{\Gamma_1} \left\{ \mu_0 (\lambda u - f) w + w \int_{\tau_1}^{\tau_2} \mu(s) (\lambda u(x) e^{-\lambda s} + z_0(x,s) \right\} ds d\Gamma,
\end{aligned}$$

where we have used (2.19) and (2.23). Therefore, (2.26) can be rewritten as

$$\begin{aligned}
\int_\Omega (\lambda^2 u w + \nabla u \nabla w) dx &+ \int_{\Gamma_1} (\mu_0 \lambda u w d\Gamma + \int_{\Gamma_1} \lambda u w \int_{\tau_1}^{\tau_2} \mu(s) e^{-\lambda s} ds d\Gamma \\
&= \int_\Omega (g + \lambda f) w dx + \mu_0 \int_{\Gamma_1} f w d\Gamma \\
&\quad - \int_{\Gamma_1} w \int_{\tau_1}^{\tau_2} \mu(s) z_0(x,s) ds d\Gamma, \quad \forall w \in H^1_{\Gamma_0}(\Omega).
\end{aligned} \tag{2.27}$$

As the left-hand side of (2.27) is coercive on $H^1_{\Gamma_0}(\Omega)$, the Lax-Milgram lemma guarantees the existence and uniqueness of a solution $u \in H^1_{\Gamma_0}(\Omega)$ of (2.27). If we consider $w \in \mathcal{D}(\Omega)$ in (2.27), we have that u solves in $\mathcal{D}'(\Omega)$

$$\lambda^2 u - \Delta u = g + \lambda f, \tag{2.28}$$

and thus $u \in E(\Delta, L^2(\Omega))$.

Using Green's formula (2.11) in (2.27) and using (2.28), we obtain

$$\frac{\partial u}{\partial \nu} + \left(\mu_0 + \int_{\tau_1}^{\tau_2} \mu(s) e^{-\lambda s} ds \right) \lambda u = \mu_0 f - \int_{\tau_1}^{\tau_2} \mu(s) z_0(x,s) ds \quad \text{on } \Gamma_1. \tag{2.29}$$

Therefore, from (2.29),

$$\frac{\partial u}{\partial \nu} = -\mu_0 v - \int_{\tau_1}^{\tau_2} \mu(s) z(\cdot, 1, s) ds \quad \text{on} \quad \Gamma_1,$$

where we have used (2.19) and (2.23). So, we have found that $(u,v,z)^T \in \mathcal{D}(\mathcal{A})$ which verifies (2.18). Hence, the well posedness result follows from the Hille–Yosida theorem. $\square$

Now, we consider the problem with internal feedback (1.9) – (1.13). Let us define

$$z(x,\rho,t,s) = u_t(x,t-\rho s), \quad x \in \Omega,\ \rho \in (0,1),\ s \in (\tau_1,\tau_2),\ t>0. \tag{2.30}$$

Then, problem (1.9) – (1.13) can be rewritten as

$$u_{tt} - \Delta u + a(x)\left[\mu_0 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)ds\right] = 0 \quad \text{in} \quad \Omega \times (0,+\infty), \tag{2.31}$$

$$s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0 \ \text{ in } \Omega \times (0,1) \times (0,+\infty) \times (\tau_1,\tau_2), \tag{2.32}$$

$$u(x,t) = 0 \quad \text{on} \quad \Gamma_0 \times (0,+\infty), \tag{2.33}$$

$$\frac{\partial u}{\partial \nu}(x,t) = 0 \quad \text{on} \quad \Gamma_1 \times (0,+\infty), \tag{2.34}$$

$$z(x,0,t,s) = u_t(x,t) \quad \text{on} \quad \Omega \times (0,+\infty) \times (\tau_1,\tau_2), \tag{2.35}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{2.36}$$

$$z(x,\rho,0,s) = g_0(x,\rho,s) \quad \text{in} \quad \Omega \times (0,1) \times (0,\tau_2). \tag{2.37}$$

If we define $U := (u, u_t, z)^T$, then

$$U' := (u_t, u_{tt}, z_t)^T = \left(u_t, \Delta u - a(\mu_0 u_t + \int_{\tau_1}^{\tau_2} \mu(s) z(\cdot,1,\cdot,s)ds), -s^{-1} z_\rho\right)^T.$$

Therefore, problem (2.31) – (2.37) can be rewritten as

$$\begin{cases} U' = \hat{\mathcal{A}} U \\ U(0) = (u_0, u_1, g_0)^T, \end{cases} \tag{2.38}$$

where the operator $\hat{\mathcal{A}}$ is defined by

$$\hat{\mathcal{A}} \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u - a\mu_0 v - a\int_{\tau_1}^{\tau_2} \mu(s) z(\cdot,1,s)ds \\ -s^{-1} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\hat{\mathcal{A}}) := \Big\{ (u,v,z)^T \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times H^1(\Omega) \times L^2(\Omega \times (\tau_1,\tau_2); H^1(0,1)) \ : \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1;\ v(x) = z(x,0,s) \text{ in } \Omega \Big\}. \tag{2.39}$$

Denote by $\hat{\mathcal{H}}$ the Hilbert space

$$\hat{\mathcal{H}} := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)), \tag{2.40}$$

equipped with the inner product

$$\begin{aligned}\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle_{\hat{\mathcal{H}}} &:= \int_\Omega \{\nabla u(x) \nabla \tilde{u}(x) + v(x)\tilde{v}(x)\} dx \\ &+ \int_\Omega a(x) \int_{\tau_1}^{\tau_2} \int_0^1 \mu(s) z(x,\rho,s)\tilde{z}(x,\rho,s) d\rho ds dx.\end{aligned} \tag{2.41}$$

Under the assumption

$$\mu_0 \geq \|a\|_\infty \int_{\tau_1}^{\tau_2} \mu(s) ds, \tag{2.42}$$

arguing analogously to the case of boundary feedback, we can show that the operator $\hat{\mathcal{A}}$ generates a C_0 semigroup on $\hat{\mathcal{H}}$. This gives the following well-posedness result.

Theorem 2.2. *Assume* (2.42). *Then, for any initial datum $U_0 \in \hat{\mathcal{H}}$, there exists a unique solution $U \in C([0,+\infty), \hat{\mathcal{H}})$ of problem* (2.38). *Moreover, if $U_0 \in \mathcal{D}(\hat{\mathcal{A}})$, then $U \in C([0,+\infty), \mathcal{D}(\hat{\mathcal{A}})) \cap C^1([0,+\infty), \hat{\mathcal{H}})$.*

3. BOUNDARY STABILITY ESTIMATE

In this section we will prove exponential stability of problem (1.1) – (1.5) under the assumption (1.8).

First of all, note that assumption (1.8) implies that there exists a positive constant c_0 such that

$$\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s) ds - \frac{c_0}{2}(\tau_2 - \tau_1) > 0. \tag{3.1}$$

Define the energy of a solution of problem (1.1) – (1.5) as

$$\begin{aligned}E(t) &:= \frac{1}{2} \int_\Omega \{u_t^2(x,t) + |\nabla u(x,t)|^2\} dx \\ &+ \frac{1}{2} \int_{\Gamma_1} \int_{\tau_1}^{\tau_2} s[\mu(s) + c_0] \int_0^1 u_t^2(x, t - \rho s) d\rho ds d\Gamma.\end{aligned} \tag{3.2}$$

We can prove that the energy is decreasing. More precisely, we have the following result.

Proposition 3.1. *There exists a positive constant C such that for any regular solution of problem* (1.1) – (1.5) *we have*

$$E'(t) \leq -C\left\{\int_{\Gamma_1} u_t^2(x,t)d\Gamma + \int_{\Gamma_1}\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)dsd\Gamma\right\}. \tag{3.3}$$

Proof. Differentiating (3.2) we obtain

$$E'(t) = \int_\Omega \{u_t u_{tt} + \nabla u \nabla u_t\}dx + \int_{\Gamma_1}\int_{\tau_1}^{\tau_2} s[\mu(s)+c_0]\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)d\rho dsd\Gamma.$$

Applying Green's formula, we have

$$E'(t) = \int_{\Gamma_1} u_t \frac{\partial u}{\partial \nu}d\Gamma + \int_{\Gamma_1}\int_{\tau_1}^{\tau_2} s[\mu(s)+c_0]\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)d\rho dsd\Gamma. \tag{3.4}$$

Now, observe that

$$-su_t(x,t-\rho s) = u_\rho(x,t-\rho s), \tag{3.5}$$

and

$$s^2 u_{tt}(x,t-\rho s) = u_{\rho\rho}(x,t-\rho s). \tag{3.6}$$

Therefore,

$$\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)d\rho = -\int_0^1 s^{-3}u_\rho(x,t-\rho s)u_{\rho\rho}(x,t-\rho s)d\rho, \tag{3.7}$$

from which follows, integrating by parts in ρ,

$$\begin{aligned}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} s[\mu(s)+c_0]\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)d\rho dsd\Gamma \\ = \frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}[\mu(s)+c_0][u_t^2(x,t)-u_t^2(x,t-s)]dsd\Gamma.\end{aligned} \tag{3.8}$$

Using (3.4), (3.8) and the boundary condition (1.3) on Γ_1, we have

$$\begin{aligned}E'(t) = &-\mu_0\int_{\Gamma_1} u_t^2(x,t)d\Gamma - \int_{\Gamma_1} u_t(x,t)\left\{\int_{\tau_1}^{\tau_2}\mu(s)u_t(x,t-s)ds\right\}d\Gamma \\ &-\frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}[\mu(s)+c_0]u_t^2(x,t-s)d\Gamma + \frac{1}{2}\int_{\Gamma_1} u_t^2(x,t)\int_{\tau_1}^{\tau_2}[\mu(s)+c_0]dsd\Gamma.\end{aligned} \tag{3.9}$$

Now, from Cauchy-Schwarz's inequality,

$$\begin{aligned}\left|\int_{\Gamma_1} u_t(t)\int_{\tau_1}^{\tau_2}\mu(s)u_t(t-s)dsd\Gamma\right| &\le \int_{\Gamma_1}|u_t(t)|\int_{\tau_1}^{\tau_2}\mu(s)|u_t(t-s)|dsd\Gamma\\ &\le \int_{\Gamma_1}|u_t(t)|\left(\int_{\tau_1}^{\tau_2}\mu(s)ds\right)^{\frac{1}{2}}\left(\int_{\tau_1}^{\tau_2}\mu(s)u_t^2(t-s)ds\right)^{\frac{1}{2}}d\Gamma \qquad (3.10)\\ &\le \frac{1}{2}\int_{\Gamma_1}u_t^2(t)\left(\int_{\tau_1}^{\tau_2}\mu(s)ds\right)d\Gamma+\frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}\mu(s)u_t^2(t-s)dsd\Gamma.\end{aligned}$$

So, from (3.9) and (3.10) we obtain

$$\begin{aligned}E'(t)\le \left(-\mu_0+\int_{\tau_1}^{\tau_2}\mu(s)ds+\frac{c_0}{2}(\tau_2-\tau_1)\right)\int_{\Gamma_1}u_t^2(x,t)d\Gamma\\ -\frac{c_0}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2}u_t^2(x,t-s)dsd\Gamma,\end{aligned}$$

which, recalling (3.1), proves the proposition. □

Now, we can prove a boundary observability estimate for problem (1.1) – (1.5).

Proposition 3.2. *There is a time $T^0>0$ such that for all times $T>T^0$ there exists a positive constant C_0 (depending on T) for which*

$$E(0)\le C_0\int_0^T\int_{\Gamma_1}\left\{u_t^2(x,t)+\int_{\tau_1}^{\tau_2}u_t^2(x,t-s)ds\right\}d\Gamma dt, \qquad (3.11)$$

for any regular solution u of problem (1.1) – (1.5).

Proof. From Proposition 6.3 of [18], for T greater than a sufficiently large time $\hat{T}$, and any $\varepsilon>0$, we have

$$\mathcal{E}(0)\le c\int_0^T\int_{\Gamma_N}\left\{\left(\frac{\partial u}{\partial\nu}\right)^2+u_t^2\right\}d\Gamma dt+c\|u\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}, \qquad (3.12)$$

for a suitable constant c (depending on T), where $\mathcal{E}(\cdot)$ denotes the standard energy for the wave equation; that is,

$$\mathcal{E}(t):=\frac{1}{2}\int_\Omega\{u_t^2(x,t)+|\nabla u(x,t)|^2\}dx. \qquad (3.13)$$

Estimate (3.12) is obtained by Carleman estimates under the assumption that there exists a function v of class C^2 satisfying (1.6) and (1.7). The function v is needed to construct a suitable weight function for Carleman estimates. From the boundary condition (1.3) it follows that

$$\left|\frac{\partial u}{\partial \nu}\right| \leq \frac{\mu_0}{2} u_t^2(t) + \frac{1}{2}\left|\int_{\tau_1}^{\tau_2} \mu(s) u_t(t-s)ds\right|^2, \tag{3.14}$$

and from Cauchy-Schwarz's inequality,

$$\left|\int_{\tau_1}^{\tau_2} \mu(s) u_t(t-s)ds\right|^2 \leq \int_{\tau_1}^{\tau_2} \mu(s)ds \left(\int_{\tau_1}^{\tau_2} \mu(s) u_t^2(t-s)ds\right). \tag{3.15}$$

Then, by (3.12), (3.14) and (3.15), we have

$$\mathcal{E}(0) \leq c\int_0^T \int_{\Gamma_1} \left\{u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)ds\right\} d\Gamma dt + c\|u\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}, \tag{3.16}$$

for a suitable positive constant c. Now, note that

$$E(t) = \mathcal{E}(t) + E_B(t),$$

where

$$E_B(t) := \frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} s[\mu(s)+c_0]\int_0^1 u_t^2(x,t-\rho s)d\rho ds d\Gamma. \tag{3.17}$$

In particular,

$$E_B(0) = \frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} s[\mu(s)+c_0]\int_0^1 u_t^2(x,-\rho s)d\rho ds d\Gamma. \tag{3.18}$$

By a change of variable in (3.18) we obtain, for $T \geq \tau_2$,

$$\begin{aligned} E_B(0) &= \frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} [\mu(s)+c_0]\int_0^s u_t^2(x,t-s)dt ds d\Gamma \\ &\leq \frac{1}{2}\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} [\mu(s)+c_0]\int_0^T u_t^2(x,t-s)dt ds d\Gamma \\ &\leq c\int_0^T\int_{\Gamma_1}\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)ds d\Gamma dt. \end{aligned} \tag{3.19}$$

Denote by $T^0 := \max\{\tau_2, \hat{T}\}$. Then, from (3.16) and (3.19), for any $T > T^0$ we have

$$\begin{aligned} E(0) &= \mathcal{E}(0) + E_B(0) \\ &\leq c\int_0^T\int_{\Gamma_1}\left\{u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)ds\right\} d\Gamma dt \\ &\quad + c\|u\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}, \end{aligned} \tag{3.20}$$

for a suitable positive constant c depending on T.

Now, in order to obtain estimate (3.11) we need to absorb the lower order term $\|u\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}$. This can be done by applying a compactness-uniqueness argument analogously to Proposition 3.2 of [21]. We give the details for the reader's convenience.

We prove (3.11) arguing by contradiction. Suppose that (3.11) is not true. Therefore, there is a sequence $\{u_n\}_n$ of solutions of problem (1.1)−(1.5) such that

$$E^n(0) > n\int_0^T\int_{\Gamma_1}\Big\{u_{nt}^2(x,t)+\int_{\tau_1}^{\tau_2}u_{nt}^2(x,t-s)ds\Big\}d\Gamma dt, \tag{3.21}$$

where we have denoted by $E^n(\cdot)$ the energy $E(\cdot)$ related to the solution u_n.

From (3.20) we have

$$\begin{aligned}E^n(0)\le c\Big\{\int_0^T\int_{\Gamma_1}\Big[u_{nt}^2(x,t)+\int_{\tau_1}^{\tau_2}u_{nt}^2(x,t-s)ds\Big]d\Gamma dt\\ +\|u_n\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}\Big\}.\end{aligned} \tag{3.22}$$

Then, from (3.21) and (3.22) we can deduce that

$$\begin{aligned}&n\int_0^T\int_{\Gamma_1}\Big[u_{nt}^2(x,t)+\int_{\tau_1}^{\tau_2}u_{nt}^2(x,t-s)ds\Big]d\Gamma dt\\ &<c\Big\{\int_0^T\int_{\Gamma_1}\Big[u_{nt}^2(x,t)+\int_{\tau_1}^{\tau_2}u_{nt}^2(x,t-s)ds\Big]d\Gamma dt+c\|u_n\|_{H^{\frac12+\varepsilon}(\Omega\times(0,T))}\Big\};\end{aligned}$$

that is,

$$\begin{aligned}(n-c)\int_0^T\int_{\Gamma_1}\Big\{u_{nt}^2(x,t)+\int_{\tau_1}^{\tau_2}u_{nt}^2(x,t-s)ds\Big\}d\Gamma dt\\ <c\|u_n\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}.\end{aligned} \tag{3.23}$$

Renormalizing, we obtain a sequence $\{w_n\}_n$ of solutions of problem (1.1) – (1.5) with

$$\|w_n\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))}=1, \tag{3.24}$$

and

$$\int_0^T\int_{\Gamma_1}\Big\{w_{nt}^2(x,t)+\int_{\tau_1}^{\tau_2}w_{nt}^2(x,t-s)ds\Big\}d\Gamma dt<\frac{c}{n-c}. \tag{3.25}$$

From (3.24), (3.25) and (3.22) it follows that the sequence $\{w_n\}_n$ is bounded in $H^1(\Omega\times(0,T))$. Since $H^1(\Omega\times(0,T))$ is compactly embedded in $H^{1/2+\varepsilon}(\Omega\times$

$(0,T))$, there exists a subsequence, still denoted by $\{w_n\}_n$ for simplicity of notation, such that $w_n \to w$ strongly in $H^{1/2+\varepsilon}(\Omega \times (0,T))$. Thus, from (3.24),

$$\|w\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))} = 1. \tag{3.26}$$

Moreover, by (3.25),

$$\int_0^T \int_{\Gamma_1} \left\{ w_t^2(x,t) + \int_{\tau_1}^{\tau_2} w_t^2(x,t-s)ds \right\} d\Gamma dt = 0.$$

Therefore, we have that

$$w_t = 0 \quad \text{on} \quad \Gamma_1 \times (0,T)$$

and

$$\frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \times (0,T).$$

Now, put $v := w_t$. Then, v solves in a distributional sense

$$v'' - \Delta v = 0 \quad \text{in} \quad \Omega \times (0,T),$$

with

$$v = 0 \quad \text{on} \;\; \Gamma \times (0,T), \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \;\; \Gamma_1 \times (0,T).$$

Therefore, from Holmgren's uniqueness theorem (see [19], Chapter I, Theorem 8.2, page 92) we have $v \equiv 0$. This implies that w is constant in time; that is, $w(x,t) = w(x)$. Then, w satisfies

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_0 \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma_1 \end{cases}$$

and so $w \equiv 0$. This is in contradiction with (3.24). Then, the observability inequality (3.11) is proved. □

From (3.11) easily follows the stability estimate.

Theorem 3.3. *Let the assumption* (1.8) *be satisfied. Then, there exist positive constant* γ_1, γ_2 *such that, for any solution of problem* (1.1) – (1.5),

$$E(t) \le \gamma_1 E(0) e^{-\gamma_2 t}, \quad \forall t \ge 0. \tag{3.27}$$

Proof. From (3.3), we have

$$E(T) - E(0) \le -C \int_0^T \int_{\Gamma_1} \left\{ u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)ds \right\} d\Gamma dt. \tag{3.28}$$

By (3.28) and the observability estimate (3.11), we obtain

$$\begin{aligned} E(T) \le E(0) &\le C_0 \int_0^T \int_{\Gamma_1} \left\{ u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)ds \right\} d\Gamma dt \\ &\le C_0 C^{-1}(E(0) - E(T)), \end{aligned}$$

then $E(T) \le \tilde{C}E(0)$, with $\tilde{C} < 1$. This easily implies the stability estimate (3.27), since our system (1.1) – (1.5) is invariant by translation and the energy E is decreasing. □

4. Internal stability estimate

In this section we will prove an exponential stability estimate for problem (1.9) – (1.13) under the assumption (1.14).

Note that assumption (1.14) implies that there exists a positive constant c_1 such that

$$\mu_0 - \|a\|_\infty \Big[\int_{\tau_1}^{\tau_2} \mu(s)ds - \frac{c_1}{2}(\tau_2 - \tau_1) \Big] > 0. \tag{4.1}$$

We define the energy of a solution of problem (1.9)–(1.13) by

$$\begin{aligned} E_0(t) : &= \frac{1}{2} \int_\Omega \left\{ u_t^2 + |\nabla u|^2 \right\} dx \\ &+ \frac{1}{2} \int_\Omega a(x) \int_{\tau_1}^{\tau_2} s[\mu(s) + c_1] \int_0^1 u_t^2(x, t - \rho s) d\rho ds dx, \end{aligned} \tag{4.2}$$

where c_1 is a constant verifying (4.1).

We can prove that the energy is decreasing and that a suitable estimate holds on the derivative of the energy.

Proposition 4.1. *There exists a positive constant C such that, for any regular solution of the problem* (1.9) – (1.13), *we have*

$$E_0'(t) \le -C \int_\Omega a(x) \left\{ u_t^2(x,t) + \Big(\int_{\tau_1}^{\tau_2} u_t^2(x, t-s)ds \Big) \right\} dx. \tag{4.3}$$

Proof. Differentiating (4.2) and using Green's formula, we have

$$\begin{aligned} E_0'(t) &= \int_\Omega u_t(u_{tt} - \Delta u)dx \\ &+ \frac{d}{dt} \frac{1}{2} \int_\Omega a \int_{\tau_1}^{\tau_2} s[\mu(s) + c_1] \int_0^1 u_t^2(x, t - \rho s) d\rho ds dx \end{aligned} \tag{4.4}$$

$$= \int_\Omega u_t\Big(- \mu_0 u_t - a \int_{\tau_1}^{\tau_2} \mu(s) u_t(t-s)ds\Big)dx$$
$$+\frac{d}{dt}\frac{1}{2}\int_\Omega a \int_{\tau_1}^{\tau_2} s[\mu(s)+c_1]\int_0^1 u_t^2(x,t-\rho s)d\rho ds dx$$
$$= -\mu_0 \int_\Omega u_t^2 dx - \int_\Omega a u_t \int_{\tau_1}^{\tau_2} \mu(s)u_t(t-s)ds$$
$$+\frac{d}{dt}\frac{1}{2}\int_\Omega a \int_{\tau_1}^{\tau_2} s[\mu(s)+c_1]\int_0^1 u_t^2(x,t-\rho s)d\rho ds dx.$$

Now, observe that

$$\frac{d}{dt}\frac{1}{2}\int_\Omega a \int_{\tau_1}^{\tau_2} s[\mu(s)+c_1]\int_0^1 u_t^2(x,t-\rho s)d\rho ds dx \tag{4.5}$$
$$= \int_\Omega a \int_{\tau_1}^{\tau_2} s[\mu(s)+c_1]\int_0^1 u_t(x,t-\rho s)u_{tt}(x,t-\rho s)d\rho ds dx.$$

Integrating by parts we obtain

$$\int_0^1 u_\rho(x,t-\rho s)u_{\rho\rho}(x,t-\rho s)d\rho = \frac{1}{2}s^2[u_t^2(x,t-\rho s)]_0^1 = \frac{s^2}{2}[u_t^2(t-s)-u_t^2(t)]. \tag{4.6}$$

So, recalling (3.5) and (3.6), from (4.5) and (4.6) we have

$$\frac{d}{dt}\frac{1}{2}\int_\Omega a \int_{\tau_1}^{\tau_2} s[\mu(s)+c_1]\int_0^1 u_t^2(x,t-\rho s)d\rho ds dx \tag{4.7}$$
$$= -\int_\Omega a \int_{\tau_1}^{\tau_2} s[\mu(s)+c_1]\int_0^1 s^{-3} u_\rho(x,t-\rho s)u_{\rho\rho}(x,t-\rho s)d\rho ds dx$$
$$= -\frac{1}{2}\int_\Omega a \int_{\tau_1}^{\tau_2} [\mu(s)+c_1][u_t^2(t-s)-u_t^2(t)]ds dx.$$

From (4.4) and (4.7) we deduce

$$E_0'(t) = -\mu_0\int_\Omega u_t^2 dx - \int_\Omega a u_t \int_{\tau_1}^{\tau_2}\mu(s)u_t(t-s)ds dx \tag{4.8}$$
$$-\frac{1}{2}\int_\Omega a \int_{\tau_1}^{\tau_2}[\mu(s)+c_1]u_t^2(t-s)ds dx + \frac{1}{2}\int_\Omega a u_t^2 \int_{\tau_1}^{\tau_2}[\mu(s)+c_1]ds dx.$$

Now, note that from Cauchy–Schwarz's inequality, we may write

$$\left|\int_\Omega a u_t(t)\int_{\tau_1}^{\tau_2}\mu(s)u_t(t-s)ds dx\right| \le \int_\Omega a|u_t(t)|\int_{\tau_1}^{\tau_2}\mu(s)|u_t(t-s)|ds dx$$

$$\leq \int_\Omega a|u_t| \left(\int_{\tau_1}^{\tau_2} \mu(s)ds\right)^{\frac{1}{2}} \left(\int_{\tau_1}^{\tau_2} \mu(s)u_t^2(t-s)ds\right)^{\frac{1}{2}} dx \tag{4.9}$$
$$\leq \frac{1}{2}\int_\Omega au_t^2(x,t)\left(\int_{\tau_1}^{\tau_2} \mu(s)ds\right)dx + \frac{1}{2}\int_\Omega a\int_{\tau_1}^{\tau_2} \mu(s)u_t^2(t-s)dsdx.$$

From (4.8) and (4.9) it follows that

$$E_0'(t) \leq -\mu_0 \int_\Omega u_t^2 dx + \frac{1}{2}\int_\Omega au_t^2(x,t)\int_{\tau_1}^{\tau_2} \mu(s)dsdx \tag{4.10}$$
$$+\frac{1}{2}\int_\Omega a\int_{\tau_1}^{\tau_2} \mu(s)u_t^2(t-s)dsdx - \frac{1}{2}\int_\Omega a\int_{\tau_1}^{\tau_2}[\mu(s)+c_1]u_t^2(t-s)dsdx$$
$$+\frac{1}{2}\int_\Omega au_t^2(x,t)\int_{\tau_1}^{\tau_2}[\mu(s)+c_1]dsdx,$$

and therefore

$$E_0'(t) \leq -\int_\Omega \left[\mu_0 - a\int_{\tau_1}^{\tau_2}\mu(s)ds - \frac{c_1}{2}(\tau_2-\tau_1)\right]u_t^2(x,t)dx \tag{4.11}$$
$$-\frac{c_1}{2}\int_\Omega a\int_{\tau_1}^{\tau_2} u_t^2(t-s)dsdx.$$

The claim immediately follows from our assumption (1.14) on μ_0 and μ, recalling that c_1 is a positive constant satisfying (4.1). $\square$

To prove the exponential stability result we need a suitable observability estimate.

Proposition 4.2. *There exists a time T^0 such that for all times $T > T^0$ there is a positive constant C_0 (depending on T) for which*

$$E_0(0) \leq C_0 \int_0^T \int_\Omega a(x)\left\{u_t^2(x,t) + \left(\int_{\tau_1}^{\tau_2} u_t^2(x,t-s)ds\right)\right\}dxdt \tag{4.12}$$

for any regular solution u of problem (1.9) – (1.13).

Proof. We can write $E_0(t) = \mathcal{E}(t) + E_I(t)$, where $\mathcal{E}(t)$ is the standard energy for the wave equation defined in (3.13) and

$$E_I(t) := \frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2} s[\mu(s)+c_1]\int_0^1 u_t^2(x,t-\rho s)d\rho dsdx. \tag{4.13}$$

Let w be the solution of the homogeneous problem for the wave equation with mixed Dirichlet–Neumann boundary condition,

$$w_{tt}(x,t) - \Delta w(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty) \tag{4.14}$$

$$w(x,t)=0 \quad \text{on} \quad \Gamma_0 \times (0,+\infty) \tag{4.15}$$

$$\frac{\partial w}{\partial \nu}(x,t)=0 \quad \text{on} \quad \Gamma_1 \times (0,+\infty) \tag{4.16}$$

$$w(x,0)=w_0(x) \quad \text{and} \quad w_t(x,0)=w_1(x) \quad \text{in} \quad \Omega. \tag{4.17}$$

Denote by $\mathcal{E}_w(t)$ the standard energy for the wave equation corresponding to w; that is,

$$\mathcal{E}_w(t)=\frac{1}{2}\int_\Omega \{w_t^2(x,t)+|\nabla w(x,t)|^2\}dx. \tag{4.18}$$

Note that $\mathcal{E}_w(t)$ is constant.

It is well known that for problem (4.14)–(4.17) we have the following observability estimate

$$\mathcal{E}_w(0) \leq C_1 \int_0^T \int_\omega w_t^2(x,t)dxdt, \tag{4.19}$$

for all times $T > \overline{T}$. This easily follows, for instance, from an estimate of [18] and standard arguments with multipliers.

Now, we can decompose (cfr. Zuazua [26]) the solution u of problem (1.9)–(1.13) as $u = w + \tilde{w}$, where w solves (4.14)–(4.16) with initial condition

$$w(x,0)=u_0(x), \quad w_t(x,0)=u_1(x) \quad \text{in } \Omega,$$

and $\tilde{w}$ satisfies

$$\tilde{w}_{tt}-\Delta\tilde{w}=-a(x)\Big[\mu_0 u_t(x,t)+\int_{\tau_1}^{\tau_2}\mu(s)u_t(x,t-s)ds\Big] \quad \text{in} \quad \Omega\times(0,+\infty), \tag{4.20}$$

$$\tilde{w}(x,t)=0 \quad \text{on} \quad \Gamma_0 \times (0,+\infty) \tag{4.21}$$

$$\frac{\partial \tilde{w}}{\partial \nu}(x,t)=0 \quad \text{on} \quad \Gamma_1 \times (0,+\infty) \tag{4.22}$$

$$\tilde{w}(x,0)=0 \quad \text{and} \quad \tilde{w}_t(x,0)=0 \quad \text{in} \quad \Omega. \tag{4.23}$$

Then, from (4.13) and (4.18),

$$\begin{aligned} E_0(0) &= \mathcal{E}(0)+E_I(0) \qquad (4.24)\\ &= \mathcal{E}_w(0)+\frac{1}{2}\int_\Omega a(x)\int_{\tau_1}^{\tau_2} s[\mu(s)+c_1]\int_0^1 u_t^2(x,-\rho s)d\rho ds dx. \end{aligned}$$

If we take $T > T^0 := \max\{\overline{T}, \tau_2\}$, from (4.24) with a change of variable we obtain

$$E_0(0) \leq \mathcal{E}_w(0) + \frac{1}{2}\int_\Omega a(x)\int_0^T \int_{\tau_1}^{\tau_2} [\mu(s) + c_1]u_t^2(x, t-s)dsdtdx,$$

and then, from (4.19),

$$\begin{aligned} E_0(0) &\leq c\int_\Omega a(x)\int_0^T \left\{w_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x, t-s)ds\right\} dtdx \qquad (4.25) \\ &\leq c\int_\Omega a(x)\int_0^T \{\tilde{w}_t^2(x,t) + u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)ds\}dtdx, \end{aligned}$$

for a suitable positive constant c. Therefore, from standard energy estimates for $\tilde{w}$ and Cauchy-Schwarz's inequality, we obtain

$$E_0(0) \leq C_0\int_0^T \int_\Omega a(x)\left\{u_t^2(x,t) + \int_{\tau_1}^{\tau_2} u_t^2(x,t-s)ds\right\} dxdt. \qquad \square$$

Now, using estimates (4.3) and (4.12), we can prove the exponential decay of the energy of solutions of problem (1.9) – (1.13).

Theorem 4.3. *Let assumption* (1.14) *be satisfied. Then, there exist positive constants* β_1, β_2 *such that, for any solution of problem* (1.9) – (1.13),

$$E_0(t) \leq \beta_1 E_0(0)e^{-\beta_2 t}, \quad \forall t \geq 0. \qquad (4.26)$$

Proof. We omit the proof since it is analogous to the proof of Theorem 3.3.

5. INSTABILITY EXAMPLES

Consider the problem with internal feedback and delay concentrated at a point $\tau > 0$, namely

$$u_{tt} - \Delta u + \mu_0 u_t + u_t(t-\tau) = 0 \quad \text{in} \quad \Omega \times (0, +\infty), \qquad (5.1)$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty), \qquad (5.2)$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \qquad (5.3)$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \qquad (5.4)$$

$$u_t(x,-t) = f_0(x,-t) \quad \text{in} \quad \Omega \times (0,\tau). \qquad (5.5)$$

Assume $\mu_0 < 1$. As proved in [21] in this case there are instability phenomena for some delays τ.

Let $\varphi : \mathbb{R} \to [0, +\infty)$ be a continuous function with $\varphi(x) = 0$ on the set $\mathbb{R} \setminus (-1, 1)$ and

$$\int_{\mathbb{R}} \varphi(t)dt = 1\,.$$

Let τ_1, τ_2 be two real numbers with $0 \le \tau_1 < \tau_2$ and

$$\tau = \frac{\tau_1 + \tau_2}{2}. \tag{5.6}$$

Define, for $\varepsilon > 0$, a family $\{\mu_\varepsilon\}_\varepsilon$ of functions $\mu_\varepsilon : \mathbb{R} \to [0, +\infty)$,

$$\mu_\varepsilon(t) := \varepsilon\varphi\Big(\frac{1}{\varepsilon}\frac{t-\tau}{\tau-\tau_1}\Big).$$

It is well known that $\mu_\varepsilon(t) \to \delta(t-\tau)$, for $\varepsilon \to 0$, where $\delta(t-\tau)$ denotes the Dirac delta function centered at $t = \tau$.

For $\varepsilon > 0$ let us now consider the problem

$$u_{tt} - \Delta u + \mu_0 u_t + \int_{\tau_1}^{\tau_2} \mu_\varepsilon(s) u_t(t-s)ds = 0 \quad \text{in} \quad \Omega \times (0, +\infty), \tag{5.7}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times (0, +\infty), \tag{5.8}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \times (0, +\infty), \tag{5.9}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{5.10}$$

$$u_t(x,-t) = f_0(x,-t) \quad \text{in} \quad \Omega \times (0, \tau_2). \tag{5.11}$$

We look for a solution of problem (5.7) – (5.11) in the form

$$u_\varepsilon(x,t) = e^{\lambda_\varepsilon t}\psi_\varepsilon(x). \tag{5.12}$$

Then, ψ_ε has to be a solution of the problem

$$\begin{cases} \Delta\psi = \tilde{\lambda}\psi \quad \text{in} \quad \Omega \\ \psi = 0 \quad \text{on} \quad \Gamma_0 \\ \frac{\partial\psi}{\partial\nu} = 0 \quad \text{on} \quad \Gamma_1 \end{cases}$$

with

$$\tilde{\lambda} = \lambda^2 + \ \Big(\mu_0 + \int_{\tau_1}^{\tau_2} \mu_\varepsilon(s)e^{-\lambda s}ds\Big)\lambda.$$

Therefore, the constant λ_ε in (5.12) has to solve the equation

$$\lambda^2 + \Big(\mu_0 + \int_{\tau_1}^{\tau_2} \mu_\varepsilon(s)e^{-\lambda s}ds\Big)\lambda = -\Lambda^2, \tag{5.13}$$

where we denote by Λ^2 a generic eigenvalue for the Laplace operator with a mixed Dirichlet-Neumann boundary condition.

Our aim is to prove that for ε small enough there is a solution of problem (5.7) – (5.11) in the form (5.12) with Re $\lambda_\varepsilon > 0$, that is, a solution whose energy is not decaying to zero.

We are not able to do this directly, since the integral equation (5.13) is not easy to deal with. However, as we have proved in [21], for suitable delays τ (arbitrarily small or large) problem (5.1) – (5.5) admits a solution in the form $u(x,t) = e^{\lambda t}\psi(x)$ with Re $\lambda > 0$.

Indeed, for suitable delays, one proves that there exists a solution λ with Re $\lambda > 0$, of the equation

$$\lambda^2 + (\mu_0 + e^{-\lambda\tau})\lambda = -\Lambda^2. \tag{5.14}$$

Now, fix τ such that we have a solution λ with Re $\lambda > 0$ of (5.14) and choose τ_1 and τ_2 in (5.7) – (5.11) such that (5.6) holds. We can rewrite (5.13) and (5.14) as

$$F_\varepsilon(\lambda) = 0, \tag{5.15}$$

$$F_0(\lambda) = 0, \tag{5.16}$$

respectively. It is easy to verify that $F_\varepsilon(\lambda) \to F_0(\lambda)$ as $\varepsilon \to 0$. Then, (5.15) and (5.16) can be rewritten as

$$F(\lambda, \varepsilon) = 0, \tag{5.17}$$

$$F(\lambda, 0) = 0, \tag{5.18}$$

with $F : \mathbb{C} \times [0,1] \to \mathbb{R}$ continuous. Moreover, we can easily verify that F is analytic in $\mathbb{C}$.

We know that (5.18) admits a solution λ_0 with Re $\lambda_0 > 0$. Let $B \subset \mathbb{C}$ be a ball centered at λ_0 with no zeroes of (5.18) on ∂B and Re $\lambda > 0$ for all $\lambda \in B$. Then, as a consequence of Rouché's theorem (see (9.17.4) of [8]), for ε small enough, equation (5.17) admits a root $\lambda_\varepsilon \in B$.

Then, for ε small enough, problem (5.7) – (5.11) admits a solution in the form (5.12) with Re $\lambda_\varepsilon > 0$. This proves that, for ε small enough, system (5.7) – (5.11) is not stable.

Remark 5.1. We can not repeat here the detailed analysis of [21] in order to give instability examples, for both boundary or internal feedbacks. Now, the presence of an integral term in the equation (5.13) for λ makes the problem more difficult. However, the above argument shows that, at least in the case of internal feedback, instability phenomena occur when our assumption (1.14) does not hold. We expect to have analogous phenomena also in the case of boundary feedback when assumption (1.8) is not satisfied. But the analysis in this case is more complicated, even for a delay concentrated at a time.

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