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STABILIZATION OF UNSTABLE PERIODIC ORBITS FOR DISCRETE TIME CHAOTIC SYSTEMS BY USING PERIODIC FEEDBACK

ÖMER MORGÜL
Bilkent University, Department of Electrical Engineering,
Ankara, Turkey
morgul@ee.bilkent.edu.tr

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We propose a periodic feedback scheme for the stabilization of periodic orbits for discrete time chaotic systems. We first consider one-dimensional discrete time systems and obtain some stability results. Then we extend these results to higher dimensional discrete time systems. The proposed scheme is quite simple and we show that any hyperbolic periodic orbit can be stabilized with this scheme. We also present some simulation results.

Keywords: Chaotic systems; chaos control; delayed feedback; Pyragas controller; periodic feedback.

1. Introduction

The study of chaotic behavior in dynamical systems has received great attention in recent years. The interest in using feedback control in chaotic systems mainly accelerated after the seminal work of [Ott *et al.*, 1990], where the term "controlling chaos" was introduced. Such systems usually have many unstable periodic orbits embedded in their chaotic attractors, and as shown in [Ott *et al.*, 1990], some of these orbits may be stabilized by using small control input. Following this work, various chaos control techniques have been proposed, see e.g. [Shinbrot *et al.*, 1990; Chen & Dong, 1998; Fradkov & Pogromsky, 1998]. Among these, the delayed feedback control (DFC) scheme first proposed in [Pyragas, 1992] and also known as Pyragas scheme, has gained considerable attention due to its various attractive features [Fradkov & Evans, 2002]. In this technique the required control input is basically the difference between the current and one period delayed states, multiplied by a gain. Hence if the system is already in the periodic orbit, this term vanishes. Also if the trajectories asymptotically approach the periodic orbit, this term becomes smaller. DFC has been successfully applied to many systems, including the stabilization of coherent modes of laser [Bielawski *et al.*, 1993; Loiko *et al.*, 1997]; cardiac systems [Brandt *et al.*, 1997]; controlling friction [Elmer, 1998]; chaotic electronic oscillators [Pyragas & Tamaševičius, 1993; Gauthier *et al.*, 1994]; magnetoelastic systems [Hai *et al.*, 1997]. Despite its simplicity, a detailed stability analysis of DFC is very difficult [Pyragas, 2001; Ushio, 1996]. For some recent stability results related to DFC, see [Just *et al.*, 1997; Nakajima, 1997; Nakajima & Ueda, 1998; Schuster & Stemmler, 1997; Pyragas, 2001]. Recently, a set of necessary and/or sufficient conditions for the stability of DFC for discrete time systems has been given in [Morgül, 2003]. For more details as well as various applications of DFC, see [Pyragas, 2001; Fradkov & Evans, 2002] and the references therein.

The DFC scheme has some inherent limitations, i.e. it cannot be applied for the stabilization of some periodic orbits, see e.g. [Ushio, 1996; Nakajima, 1997; Hino *et al.*, 2002; Yamamoto *et al.*,

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2002; Morgül, 2003]. To overcome the limitations of DFC, several modifications have been proposed [Socolar *et al.*, 1994; Kittel *et al.*, 1995; Pyragas, 1995, 2001]. Among these, the periodic feedback law given in [Schuster & Stemmler, 1997] seems to be promising due to its simplicity. This method is known to eliminate the limitations of DFC for period 1 case, and various extensions to higher order periods are possible. In this paper we provide such an extension. We will show that the proposed extension yields stabilization of the corresponding periodic orbits under a mild condition. This condition is related to the hyperbolic behavior of the periodic orbit, and we will show that any hyperbolic periodic orbit can be stabilized with the proposed scheme.

This paper is organized as follows. First we will give the statement of the problem and present some notations which will be used in the sequel. Then we will propose a periodic feedback scheme to solve the stabilization problem. In the following two sections we will provide the stability analysis for one-dimensional and higher dimensional cases, respectively. These analysis show a slight difference between one-dimensional and higher dimensional cases, see Remark 4. Then we will present a simple implementation of the proposed scheme. Following some simulation results, we will provide some concluding remarks.

2. Problem Statement

Let us consider the following discrete time system

$$x(k+1) = f(x(k)), \tag{1}$$

where $k = 1, 2 \ldots$ is the discrete time index, $x \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ is an appropriate function, which is assumed to be differentiable wherever required. We assume that the system given by (1) possesses a T periodic orbit characterized by the set

$$\Sigma_T = \{x_1^*, x_2^*, \ldots, x_T^*\}, \tag{2}$$

i.e. for $x(1) = x_1^*$, the iterates of (1) yield $x(2) = x_2^*, \ldots, x(T) = x_T^*$, $x(k) = x(k-T)$ for $k > T$. Let us call this orbit an uncontrolled periodic orbit (UCPO) for future reference.

Let $x(\cdot)$ be a solution of (1). To characterize the convergence of $x(\cdot)$ to Σ_T, we need a distance measure, which is defined as follows. For x_i^*, we will use circular notation, i.e. $x_i^* = x_j^*$ for $i = j \pmod T$. Let us define the following indices ($j = 1, \ldots, T$):

$$d_k(j) = \sqrt{\sum_{i=0}^{T-1} \|x(k+i) - x_{i+j}^*\|^2}, \tag{3}$$

where $\|\cdot\|$ denotes any norm in $\mathbf{R}^n$. Without loss of generality, we will use standard Euclidean norm in the sequel. We then define the following distance measure

$$d(x(k), \Sigma_T) = \min\{d_k(1), \ldots, d_k(T)\}. \tag{4}$$

Clearly, if $x(1) \in \Sigma_T$, then $d(x(k), \Sigma_T) = 0$, $\forall\, k$. Conversely if $d(x(k), \Sigma_T) = 0$ for some k_0, then it remains 0 and $x(k) \in \Sigma_T$, for $k \geq k_0$. We will use $d(x(k), \Sigma_T)$ as a measure of convergence to the periodic solution given by Σ_T.

Let $x(\cdot)$ be a solution of (1) starting with $x(1) = x_1$. We say that Σ_T is (locally) asymptotically stable if there exists an $\varepsilon > 0$ such that for any $x(1) \in \mathbf{R}$ for which $d(x(1), \Sigma_T) < \varepsilon$ holds, we have $\lim_{k\to\infty} d(x(k), \Sigma_T) = 0$. Moreover if this decay is exponential, i.e. the following holds for some $M \geq 1$ and $0 < \rho < 1$, ($k > 1$):

$$d(x(k), \Sigma_T) \leq M\rho^k d(x(1), \Sigma_T), \tag{5}$$

then we say that Σ_T is (locally) exponentially stable.

To stabilize the periodic orbits of (1), let us apply the following control law:

$$x(k+1) = f(x(k)) + u(k) \tag{6}$$

where $u(\cdot)$ is the control input. In classical DFC, the following feedback law is used ($k > T$):

$$u(k) = K(x(k) - x(k-T)), \tag{7}$$

where $K \in \mathbf{R}^{n\times n}$ is a constant gain matrix to be determined. It is known that the scheme given by (6)–(7) has certain inherent limitations, see e.g. [Ushio, 1996]. For example, assume that $n = 1$ and let $\Sigma_1 = \{x_1^*\}$ be a period 1 UCPO of (1) and set $a_1 = f'(x_1^*)$, where a prime denotes the derivative. It can be shown that Σ_1 can be stabilized with this scheme if $-3 < a_1 < 1$ and cannot be stabilized if $a_1 > 1$, see [Ushio, 1996]. For Σ_T, let us set $a_i = f'(x_i^*)$. It can be shown that Σ_T cannot be stabilized with this scheme if $\prod_{i=1}^T a_i > 1$, see e.g. [Morgül, 2003]. A set of necessary and sufficient conditions to guarantee exponential stabilization for $n = 1$ can be found in [Morgül, 2003].

3. Double Period Delayed Feedback Scheme

To overcome the limitations of DFC scheme, various modifications have been proposed. One of these

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schemes is the so-called periodic, or oscillating feedback, see [Schuster & Stemmler, 1997]. For period 1 case, the corresponding feedback law is given by:

$$u(k) = \epsilon(k)(x(k) - x(k-1)), \tag{8}$$

where $\epsilon(k)$ is given as:

$$\epsilon(k) = \begin{cases} K & k \pmod 2 = 0 \\ 0 & k \pmod 2 \neq 0 \end{cases} \tag{9}$$

where $K \in \mathbf{R}^{n\times n}$ is a constant gain matrix to be determined. In the next section we will show that this scheme eliminates the limitations of classical DFC.

The idea given in (8)–(9) can be generalized to the case $T = m > 1$. One particular generalization is given in [Schuster & Stemmler, 1997]. However, as noted in [Pyragas, 2001], the stability analysis given in [Schuster & Stemmler, 1997] is not clear. In the sequel, we will provide a different generalization along with a simple stability analysis.

As a generalization of the control law given by (8), (9), we propose the following control law:

$$u(k) = \epsilon(k)(x(k-m+1) - x(k-2m+1)), \tag{10}$$

where $\epsilon(k)$ is given as:

$$\epsilon(k) = \begin{cases} K & k \pmod{2m} = 0 \\ 0 & k \pmod{2m} \neq 0 \end{cases} \tag{11}$$

Clearly, for $m = 1$, both (10) and (11) reduces to (8), and (9), respectively. For the sake of clarity, we will call the scheme given by (10) and (11) as double period delayed feedback scheme (DPDFC).

To see the relation between the control laws given by (8), (9) and (10), (11), let Σ_m given by (2) be a period m solution of (1). Let us define the m-iterate map F as $F = f^m$. Clearly period m orbits of f are equivalent to period 1 orbits of F, i.e. $F(x_i^*) = x_i^*$, $i = 1, 2, \ldots, m$. Let us set

$$z(j) = x((j-1)m+1), \quad j = 1, 2, \ldots. \tag{12}$$

If j is odd, by using (10)–(11) in (6), we obtain:

$$x(jm+1) = f(x(jm)) = f^m(x((j-1)m+1)), \tag{13}$$

which is the same as

$$z(j+1) = F(z(j)). \tag{14}$$

On the other hand, if j is even, similarly we obtain:

$$\begin{aligned} x(jm+1) &= f^m(x((j-1)m+1)) + K(x((j-1)m+1) \\ &\quad - x((j-2)m+1)), \end{aligned} \tag{15}$$

which is the same as

$$z(j+1) = F(z(j)) + K(z(j) - z(j-1)). \tag{16}$$

By combining (14) and (16), we see that in terms of the variable z as defined in (12), we have the following dynamics:

$$z(j+1) = F(z(j)) + u(j) \tag{17}$$

where $u(j)$ is given by:

$$u(j) = \epsilon(j)(z(j) - z(j-1)), \tag{18}$$

and $\epsilon(\cdot)$ is given by (9). Hence, DPDFC scheme given by (10), (11) is equivalent to the scheme given by (8), (9) in the variable z given in (12). We will use this equivalence in the stability analysis which will be given in the sequel.

4. One-Dimensional Case

To motivate our analysis, let us consider the case $n = 1$, i.e. the one-dimensional case. First, let us consider the period 1 orbits of (1). Let $\Sigma_1 = \{x_1^*\}$ be the period 1 orbit of (1), and define the error as $e(k) = x(k) - x_1^*$. By using the first two iterations of (6), (8), (9) and $x_1^* = f(x_1^*)$, after linearization and considering only the first order terms, we obtain

$$e(2) = a_1 e(1), \tag{19}$$

$$\begin{aligned} e(3) &= (a_1 + K)e(2) - Ke(1) \\ &= (a_1^2 + K(a_1 - 1))e(1), \end{aligned} \tag{20}$$

where $a_1 = f'(x_1^*)$. Note that this corresponds to the linearization of the function $f(\cdot)$ around the periodic point x_1^*. Continuing in the same manner, we obtain for the next two iterates as:

$$e(4) = a_1 e(3), \tag{21}$$

$$\begin{aligned} e(5) &= (a_1 + K)e(4) - Ke(3) \\ &= (a_1(a_1 + K) - K)e(3) \\ &= (a_1^2 + K(a_1 - 1))^2 e(1), \end{aligned} \tag{22}$$

Repeating the same procedure, by mathematical induction we can show that the following holds

$$e(2k+1) = (a_1^2 + K(a_1 - 1))^k e(1), \tag{23}$$

To prove (23), note that from (20) and (22), it follows that (23) holds for $k = 1$ and $k = 2$, respectively. Now, assume that (23) holds for $k - 1$, i.e. the following is true:

$$e(2k-1) = (a_1^2 + K(a_1 - 1))^{(k-1)} e(1), \tag{24}$$

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Similar to (19)–(22), the next two iterates can be easily found as follows:

$$e(2k) = a_1 e(2k-1), \tag{25}$$

$$\begin{aligned} e(2k+1) &= (a_1+K)e(2k) - Ke(2k-1) \\ &= (a_1(a_1+K)-K)e(2k-1) \\ &= (a_1^2 + K(a_1-1))^k e(1), \end{aligned} \tag{26}$$

By mathematical induction, this proves that (23) holds for any k.

Clearly, if $|a_1^2 + K(a_1-1)| < 1$, then Σ_1 is (locally exponentially) stabilizable. If $a_1 \neq 1$, then by using the latter inequality one can easily find a range of K for which the (locally exponential) stabilization is possible, see Corollary 1 given below. This simple analysis shows that for the case $T = 1$, the inherent limitation of DFC (i.e. $a_1 > 1$) can be avoided by using the periodic feedback law given by (8)–(9).

Let us consider the case $T = m > 1$, and the period m orbit Σ_m of (1) as given by (2). We will use the previously given notation. Now consider the DPDFC scheme given by (6), (10), (11). It can easily be shown that this system could be transformed into (17), (18) by using (12), where $F = f^m$. Clearly any point x_i^* in the period m orbit Σ_m of (1) is a fixed point of F. Hence, by using the previously given stability analysis, see e.g. (26), we conclude that period 1 orbit $\Sigma_1^i = \{x_i^*\}$ of F is stable for the system given by (17) if $|a^2 + (a-1)K| < 1$ where $a = F'(x_i^*)$. On the other hand, since Σ_m is a period m orbit of (1) and $F = f^m$, by using the chain rule we easily find that $a = \prod_{i=1}^m a_i$, where $a_i = f'(x_i^*)$. We can summarize these results as follows:

Theorem 1. *Let a period m orbit of (1) be given as $\Sigma_m = \{x_1^*, \ldots, x_m^*\}$ and set $a_i = f'(x_i^*)$, $i = 1, 2, \ldots, m$, $a = \prod_{i=1}^m a_i$. The DPDFC scheme given by (6), (10)–(11) is*

(i) *locally exponentially stable if and only if*

$$|a^2 + K(a-1)| < 1, \tag{27}$$

(ii) *not stable if $|a^2 + K(a-1)| > 1$.*
(iii) *This analysis is inconclusive if $|a^2 + K(a-1)| = 1$.*

Proof. Note that the local exponential stability is equivalent to the stability of the linearized system, see e.g. [Khalil, 2002]. The proof of the theorem then easily follows from standard Lyapunov stability arguments and (26). ■

Remark 1. Note that although the stability condition given by (27) is similar to the one given in [Schuster & Stemmler, 1997], the form of both (10) and (11) are different than the ones given in [Schuster & Stemmler, 1997].

Now we consider the problem of determining the stabilizing gains K. This problem could easily be solved by using (27).

Corollary 1. *Let a period m orbit of (1) be given as $\Sigma_m = \{x_1^*, \ldots, x_m^*\}$ and set $a_i = f'(x_i^*)$, $i = 1, 2, \ldots, m$, $a = \prod_{i=1}^m a_i$. Consider the DPDFC scheme given by (6), (10)–(11). There exists a K such that DPDFC is locally exponentially stable if and only if $a \neq 1$.*

Proof. Let $a \neq 1$. Note that (27) is equivalent to

$$-(1+a^2) < K(a-1) < 1 - a^2. \tag{28}$$

If $a > 1$, then we obtain the range of K for stabilization as

$$\frac{(1+a^2)}{(1-a)} < K < -(1+a). \tag{29}$$

Note that in this case the stabilizing gains are negative. If $a < 1$, then we obtain the range of K for stabilization as

$$-(1+a) < K < \frac{(1+a^2)}{(1-a)}. \tag{30}$$

Note that if Σ_m is unstable for (1), then we have $|a| > 1$, hence $a < 1$ implies $a < -1$. In this case, the stabilizing gains are positive.

If $a = 1$, then (27) cannot be satisfied for any gain K, hence by Theorem 1 exponential stability does not hold. ■

Remark 2. We note that the classical DFC cannot achieve stabilization of Σ_m if $a > 1$, see e.g. [Ushio, 1996; Morgül, 2003], and even if $a < 1$, the stabilization is not guaranteed, see [Morgül, 2003]. On the other hand, DPDFC scheme given by (10)–(11) always achieves stabilization when $a \neq 1$. Note that this condition may be considered as a generic case and we may state that almost all periodic orbits can be stabilized by DPDFC. Also note that the condition $a \neq 1$ is related to the hyperbolic behavior of Σ_m, and from Corollary 1 it follows that DPDFC will stabilize any hyperbolic periodic orbit. We will elaborate on this point later, see e.g. Remark 5.

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Remark 3. Note that a measure of the degree of instability of Σ_m is the value of $|a|$, and for higher order periods we expect that this degree becomes large. Indeed, consider the tent map given as $f(x) = rx$ for $0 < x < 0.5$, and $f(x) = r(1-x)$ for $0.5 < x < 1$, where $1 < r \leq 2$. Here we have $a_i = \pm r$, and hence we have $|a| = \prod_{i=1}^{m} |a_i| = r^m$. Hence, for this example we have $|a| \to \infty$ as $m \to \infty$. From this point of view, intuitively we may expect that the stabilization of higher order periodic orbits becomes more difficult, see e.g. [Hunt, 1991]. Indeed, for classical DFC, determining whether a stabilizing gain exists or not becomes increasingly difficult as the order increases, see [Morgül, 2003]. On the other hand, for DPDFC it seems that, contrary to the intuition, the higher order periodic orbits could be stabilized quite easily. In fact, we note that input is applied at every double period, see (11), and for higher order periodic orbits most of the time the input is not applied. On the other hand, note that the stabilization property given in Theorem 1 and Corollary 1 is only local, and for higher order periodic orbits we expect the size of the domain of attraction to be small. Indeed, if $|a| \to \infty$ as $m \to \infty$, from (29) and (30) we see that $|K| \to \infty$ as well. Therefore, to have a bounded $|u|$ in (10) we should have small $|x(k-m+1) - x(k-2m+1)|$, i.e. the trajectory should be sufficiently close to Σ_m. In the limit, as $m \to \infty$, we have $|K| \to \infty$, and the corresponding domain of attraction shrinks to zero.

5. Extension to Higher Dimensional Case

In the previous section we considered the one-dimensional case. In this section we will extend these results to higher dimensional case. We will use the notation introduced in previous sections.

To motivate the analysis, consider the simple period 1 case for the system given by (6), (8), (9). Let $\Sigma_1 = \{x_1^*\}$ be the period 1 orbit of (1), and define the error as $e(k) = x(k) - x_1^*$. Note that in this case we have $x(k) \in \mathbf{R}^n$, and $K \in \mathbf{R}^{n\times n}$ is a gain matrix. By using the first two iterations of (6), (8), (9) and $x_1^* = f(x_1^*)$, after linearization and considering only the first order terms, we obtain

$$e(2) = J_1 e(1), \tag{31}$$

$$\begin{aligned} e(3) &= (J_1 + K)e(2) - Ke(1) \\ &= (J_1^2 + K(J_1 - I))e(1), \end{aligned} \tag{32}$$

where I is the identity matrix and J_1 is the Jacobian of f evaluated at x_1^*, i.e.

$$J_1 = \left.\frac{\partial f}{\partial x}\right|_{x=x_1^*}. \tag{33}$$

Note that, as in Sec. 3, this corresponds to the linearization of the function $f(\cdot)$ around the periodic point x_1^*. Again, by mathematical induction, as in Sec. 3, one can easily show that the following holds for the linearized error dynamics:

$$e(2k+1) = (J_1^2 + K(J_1 - I))^k e(1), \tag{34}$$

for any k. The proof of this fact is omitted here, since it can easily be done by a similar analysis given in (21)–(26). From (34), it follows easily that the linearized error dynamics given by (34) is locally exponentially stable if and only if the matrix A_1 given below

$$A_1 = J_1^2 + K(J_1 - I), \tag{35}$$

is stable, i.e. all eigenvalues of A_1 are inside the unit disc.

Now consider the problem of finding an appropriate gain matrix K such that A_1 is stable. It can be shown that such a K exists if and only if $J_1 - I$ is invertible, see Corollary 2 below. This condition is satisfied if $\lambda = 1$ is not an eigenvalue of J_1. This condition is related to the hyperbolic behavior of periodic orbits, and we will elaborate on this point later.

Let us consider the case $T = m > 1$, and the period m orbit Σ_m of (1) as given by (2). Now consider the DPDFC scheme given by (6), (10), (11). As explained in Sec. 3, this system could be transformed into (17), (18) by using (12), where $F = f^m$. Clearly any point x_i^* in the period m orbit Σ_m of (1) is a fixed point of F. Hence, by repeating the previous stability analysis given in Sec. 4, we conclude that period 1 orbit $\Sigma_1^i = \{x_i^*\}$ of F is stable for the system given by (17) if the matrix A_i given as

$$A_i = J_i^2 + K(J_i - I), \tag{36}$$

is stable, where J_i is the Jacobian of F at x_i^*, i.e.

$$J_i = \left.\frac{\partial F}{\partial z}\right|_{z=x_i^*}. \tag{37}$$

Now let us define the the matrices D_j as Jacobian matrices of f at periodic points of Σ_m, i.e.

$$D_j = \left.\frac{\partial f}{\partial x}\right|_{x=x_j^*}, \quad j = 1, \ldots, m. \tag{38}$$

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Since $F = f^m$, by using chain rule we obtain the following relation

$$J_i = D_i D_{i+1} \dots D_{i+m-1}, \tag{39}$$

see e.g. [Devaney, 1987]. Note that here we employ the circular notation, i.e. $D_i = D_j$ if $i = j (\mathrm{mod}\ m)$.

Remark 4. Note that since Σ_m is a period m solution of (1), although in general we may have $D_i \neq D_j$ for $i \neq j$, the set of eigenvalues of J_i are the same for any i, where D_i and J_i are given in (38) and (39), respectively, see e.g. [Devaney, 1987; Alligood *et al.*, 1997]. Hence, when $K = 0$, i.e. when DPDFC is not applied, the set of matrices A_i given by (36) have the same set of eigenvalues for any i. In other words, in the uncontrolled case the stability property of (17) is the same at any fixed point $\Sigma_1^i = \{x_i^*\}$. However, in the controlled case this symmetry does not hold in general and for a given $K \neq 0$, the set of eigenvalues of A_i and A_j may be different for $i \neq j$. As a result, the stability properties of A_i and A_j may be different for $i \neq j$. This appears to be an interesting difference between one-dimensional and higher dimensional cases. Indeed, for the one-dimensional case we have $J_i = J_j$ and hence $A_i = A_j$ for any i and j due to the scalar nature of these coefficients. Since

$$\Sigma_m = \Sigma_1^1 \cup \Sigma_1^2 \cdots \cup \Sigma_1^m, \tag{40}$$

for the stability of Σ_m, we require that at least one of the matrices A_i be stable.

Recall that a matrix is called stable if all its eigenvalues are inside the unit disc, unstable if at least one of its eigenvalues is outside the unit disc, and marginally stable if at least one of its eigenvalues is on the unit disc while the rest of its eigenvalues are inside the unit disc. We can summarize these results as follows.

Theorem 2. *Let a period m orbit of (1) be given as $\Sigma_m = \{x_1^*, \dots, x_m^*\}$ and let us define the matrices D_i and J_i as given in (38) and (39), respectively. The DPDFC scheme given by (6), (10)–(11) is*

(i) *locally exponentially stable if and only if at least one of the matrices A_i given by (36) is stable,*
(ii) *not stable if all of the matrices A_i are unstable.*
(iii) *This analysis is inconclusive if all of the matrices A_i are marginally stable.*

Proof. Note that the local exponential stability is equivalent to the stability of the linearized system, see e.g. [Khalil, 2002]. The proof of the theorem then easily follows from standard Lyapunov stability arguments, (34)–(39), and Remark 4. ■

Now let us consider the problem of finding an appropriate gain matrix K for the stabilization of Σ_m. Although for a given K the stability properties of A_i may be different, see Remark 4, the solvability of this problem depends only on the eigenvalues of J_i. Also note that the eigenvalues of J_i are the same for all i, see Remark 4. The solution of the problem of finding appropriate gain K is given in the following Corollary.

Corollary 2. *Let a period m orbit of (1) be given as $\Sigma_m = \{x_1^*, \dots, x_m^*\}$ and let us define the matrices D_i and J_i as given in (38) and (39), respectively. There exists a gain matrix K such that the DPDFC scheme given by (6), (10)–(11) is locally exponentially stable if and only if $\lambda = 1$ is not an eigenvalue of J_i for any (hence for all) $i = 1, 2, \dots, m$.*

Proof. Assume that $\lambda = 1$ is not an eigenvalue of J_i. Hence, $J_i - I$ is invertible. Let $X \in \mathbf{R}^{n\times n}$ denote an arbitrary stable matrix. Let us choose K as

$$K = (-J_i^2 + X)(J_i - I)^{-1}. \tag{41}$$

Substituting (41) in (36) we obtain $A_i = X$; hence with this choice A_i becomes a stable matrix. In fact, (41) gives all possible choices of K, i.e. A_i becomes stable for a gain matrix K if and only if K has the form given by (41).

Now assume that $\lambda = 1$ is an eigenvalue of J_i. Let $\phi \in \mathbf{R}^n$ be the corresponding eigenvector of J_i. By using the fact $J_i\phi = \phi$, we obtain

$$A_i\phi = J_i^2\phi + K(J_i - I)\phi = J_i^2\phi = \phi. \tag{42}$$

Hence $\lambda = 1$ is then an eigenvalue of A_i, independent of K. Since the eigenvalues of J_i are the same, see Remark 4, it follows that independent of K, none of the matrices A_i is stable. Therefore, by Theorem 2, there cannot be a K such that the DPDFC is locally exponentially stable. ■

Remark 5. To see the improvement we obtained by using DPDFC over the classical DFC for the stabilization of Σ_m, let us consider the latter, see (6), (7). It is known that classical DFC scheme has some inherent limitations, and it can be shown that it cannot stabilize Σ_m if the number of real eigenvalues of J_i greater than 1 is odd, see e.g. [Ushio, 1996; Morgül, 2003]. Note that this condition is satisfied

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in many chaotic orbits, and classical DFC cannot be used in their stabilization. Also note that even if this necessary condition is satisfied, stabilization by classical DFC is not guaranteed, see [Morgül, 2003]. On the other hand, DPDFC scheme presented in this paper always yields stabilization provided that $\lambda = 1$ is not an eigenvalue of J_i. Since we are mainly concerned with the stabilization of unstable periodic orbits, this condition most likely holds in most of the periodic orbits, hence we can safely state that practically all periodic orbits can be stabilized with this approach. We also note that having an eigenvalue at $\lambda = 1$ may be considered as a nongeneric case, hence from this point of view we may also argue that almost all the unstable periodic orbits can be stabilized by DPDFC. Note that this property is related to the hyperbolic behavior of periodic orbits. Recall that a periodic orbit Σ_m is called hyperbolic if none of the eigenvalues of J_i are on the unit disc, see e.g. [Devaney, 1987]. Hence, Corollary 2 implies that any hyperbolic periodic orbit can be stabilized by DPDFC. Since the only limitation is the exclusion of $\lambda = 1$ as an eigenvalue, some nonhyperbolic periodic orbits can also be stabilized by DPDFC.

6. A Simple Implementation

Note that the DPDFC scheme given by (10)–(11) achieves only local stabilization, i.e. it achieves stabilization only when the solutions of (6) are sufficiently close to the periodic orbit in certain sense. Hence, from implementation point of view, it is reasonable to apply DPDFC only when the solutions are sufficiently close to Σ_m. Let $\rho(k)$ denote an appropriate function which measures the closeness of trajectories to Σ_m, and let $\epsilon_m > 0$ denote a constant related to the size of the domain of attraction of Σ_m. A reasonable implementation of DPDFC, which we will use in our simulations, is given as follows:

$$x(k+1) = f(x(k)) + u(k), \tag{43}$$

$$u(k) = \epsilon(k)(x(k-m+1) - x(k-2m+1)), \tag{44}$$

$$\epsilon(k) = \begin{cases} K & k \pmod{2m} = 0 \ \& \ \rho(k) < \epsilon_m \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

Since the solutions of (43) are chaotic for $u = 0$, eventually the trajectories of the uncontrolled system will enter into the domain of attraction of Σ_m, i.e. $\rho(k) < \epsilon_m$ will be satisfied for some k, and hence afterwards the DPDFC given by (43)–(45) will be effective. Also, with this modification DPDFC will achieve stabilization for any initial condition in the domain of attraction of the chaotic attractor of (6). Obviously, for higher order periodic orbits, the time required till the trajectories enter into the domain of attraction of Σ_m will be larger.

Now let us consider the selection of $\rho(k)$ in (45). The distance measure given by (3) is not suitable from implementation point of view, since T iterates of (1) starting from $x(k)$ are compared with Σ_T, whereas to compute $u(k)$ we could only use the past iterates. For this reason, instead of (3), we modify $d_k(j)$ in this section as follows ($j = 1, 2, \ldots, T$):

$$d_k(j) = \sqrt{\sum_{i=0}^{T-1} \|x(k-T+1+i) - x^*_{i+j}\|^2}. \tag{46}$$

For the case $n = 1$, we can choose $\rho(k)$ as any $d_k(j)$ as given by (46), or as $d(x(k), \Sigma_m)$ as given by (4). We choose the latter in our simulations. For higher dimensional case, since the stability properties of A_j are different, particular care should be given to the selection of $\rho(k)$. If K is chosen so that a particular A_j becomes stable, we should choose $\rho(k) = d_k(j)$, see (46). We will use this approach in our simulations. In general, let us define the following index set $\mathcal{I}$

$$\mathcal{I} = \{j \mid A_j \text{ is stable}\}. \tag{47}$$

Then we may choose $\rho(k)$ as

$$\rho(k) = \min\{d_k(j) \mid j \in \mathcal{I}\}. \tag{48}$$

Finally let us consider the size of the control input given by (44). By using (44) and (45), we obtain:

$$\|u(k)\| \le \|K\| \|x(k-m+1) - x(k-2m+1)\|, \tag{49}$$

where $\|K\|$ denotes the operator norm of K. Clearly, if ϵ_m in (45) is small, the term $\|x(k-m+1) - x(k-2m+1)\|$ will be small as well. Following this idea, we can use (49) to bound the size of the control input. Hence, by using the implementation given by (43)–(45), it follows that we may stabilize Σ_m with arbitrary small input. However, this point deserves further research.

7. Simulation Results

In the simulations, we used the system given by (43)–(45) for various well known chaotic maps.

For the first order case, we consider the logistic map given by $f(x) = 4x(1-x)$. It

is well known that this map has chaotic solutions and periodic orbits of all orders. Two true period 3 orbits of this map can be computed as $\Sigma_{3-} = \{0.413175, 0.969846, 0.116977\}$, $\Sigma_{3+} = \{0.611260, 0.950484, 0.188255\}$. For Σ_{3-}, we have $a_1 = 4 - 8x_1^* = 0.6952$, $a_2 = 4 - 8x_2^* = -3.7584$, $a_3 = 4 - 8x_3^* = 3.0648$, and hence $a = -8$, and by using (30), it follows that exponential stability holds for $7 < K < 7.22$. Note that, although the necessary condition ($a < 1$) is satisfied, it can be shown that this orbit cannot be stabilized by classical DFC, see [Morgül, 2003]. For the implementation we use (43)–(45), with $\rho(k) = d(x(k), \Sigma_{3-})$ where the latter is defined in (4). We choose $K = 7.11$, which is in the middle of the interval of stabilizing gains. To estimate the size of the domain of attraction for Σ_{3-} is very difficult, and by extensive simulations we observed that we could choose $\epsilon_m = 0.04$. Since the solutions of (1) are chaotic, eventually the proposed control law will be effective and stabilization will be achieved for any $x(1) \in (0, 1)$. Our simulations show exponential stabilization for any $x(1) \in (0 \quad 1)$, which is not shown here due to space limitation. We simulated this system for $x(1) = 0.3$, and the results are shown in Fig. 1. In Fig. 1(a), we show $d(x(k), \Sigma_{3-})$ versus k, and as can be seen the decay is exponential for $k \geq 400$; apparently the solutions enter the domain of attraction for $k \geq 400$ in this simulation. The required input $u(k)$ is shown in Fig. 1(b); as can be seen $u(k) \to 0$ as $k \to \infty$. Figures 1(c) and 1(d) show the behavior of $x(k)$ after transients, where the $x(k)$ versus k is plotted in Fig. 1(c) for $950 \leq k \leq 965$, and $x(k)$ versus $x(k-3)$ plot of Fig. 1(d) is plotted for $k \geq 400$. As can be seen from these figures, the solutions converge to the period 3 orbit characterized by Σ_{3-}.

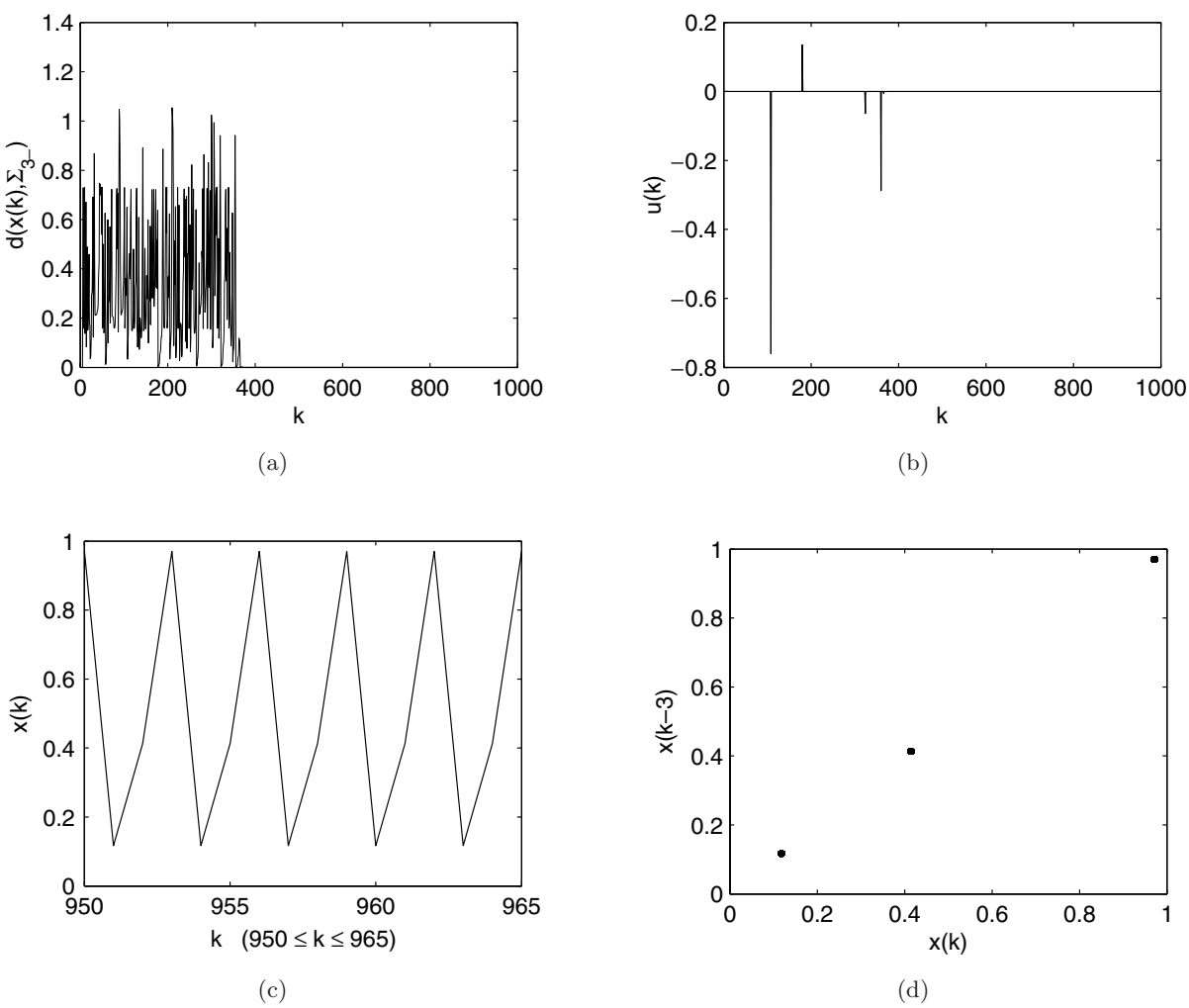

Fig. 1. DPDFC applied to logistic map, (a) $d(x(k), \Sigma_{3-})$ versus k, (b) $u(k)$ versus k, (c) $x(k)$ versus k for $950 \leq k \leq 965$, (d) $x(k)$ versus $x(k-3)$ for $k \geq 400$.

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For Σ_{3+}, we have $a_1 = 4 - 8x_1^* = -0.8896$, $a_2 = 4 - 8x_2^* = -3.6032$, $a_3 = 4 - 8x_3^* = 2.4944$, and hence $a = 8$. Since $a > 1$, this orbit cannot be stabilized by DFC [Ushio, 1996; Morgül, 2003]. By using (29), it follows that exponential stability holds for $-9.28 < K < -9$. For this case, we choose $K = -9.14$, and by extensive numerical simulations we find that we have $\epsilon_m = 0.05$ as an indicator for the size of the domain of attraction for Σ_{3+}. For the implementation we use (43)–(45), with $\rho(k) = d(x(k), \Sigma_{3+})$ where the latter is defined in (4). Our simulations show exponential stabilization for any $x(1) \in (0 \quad 1)$, which is not shown here due to space limitation. We simulated this system for $x(1) = 0.3$, and the results are shown in Fig. 2. In Fig. 2(a), we show $d(x(k), \Sigma_{3+})$ versus k, and as can be seen the decay is exponential for $k \geq 1000$; apparently the solutions enter the domain of attraction for $k \geq 1000$ in this simulation. The required input $u(k)$ is shown in Fig. 2(b); as can be seen $u(k) \to 0$ as $k \to \infty$. Figures 2(c) and 2(d) show the behavior of $x(k)$ after transients, where the $x(k)$ versus k is plotted in Fig. 2(c) for $1950 \leq k \leq 1965$, and $x(k)$ versus $x(k-3)$ plot of Fig. 2(d) is plotted for $k \geq 1000$. As can be seen from these figures, the solutions converge to the period 3 orbit characterized by Σ_{3+}.

For the second order case, we considered the well known Hénon map given as

$$f(w) = \begin{pmatrix} 1 + y - 1.4x^2 \\ 0.3x \end{pmatrix}, \tag{50}$$

where $w = (x \quad y)^T \in \mathbf{R}^2$, and the superscript T denotes the transpose. Note that instead of the notation x in (43)–(45) to denote the state variable, we use w here since it is customary to use the

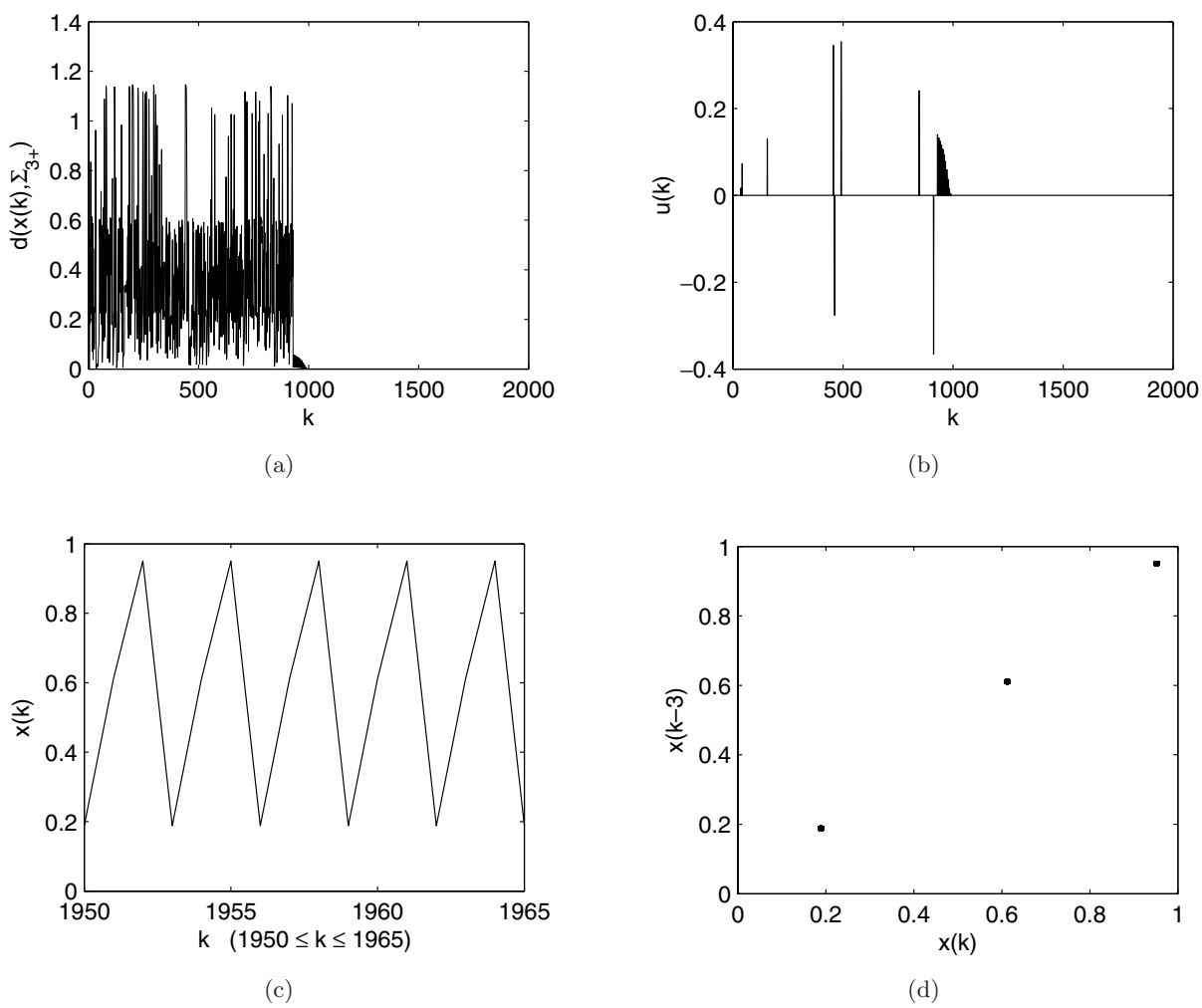

Fig. 2. DPDFC applied to logistic map, (a) $d(x(k), \Sigma_{3+})$ versus k, (b) $u(k)$ versus k, (c) $x(k)$ versus k for $1950 \leq k \leq 1965$, (d) $x(k)$ versus $x(k-3)$ for $k \geq 1000$.

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labels x and y for the variables in Hénon map. This map has period 2 solution characterized by the set $\Sigma_2 = \{x_1^*, x_2^*\}$ where

$$x_1^* = \begin{pmatrix} 0.975800051 \\ -0.142740015 \end{pmatrix}, \quad x_2^* = \begin{pmatrix} -0.475800051 \\ 0.292740015 \end{pmatrix}. \tag{51}$$

Since $m = 2$, as indicated in Theorem 2 and Corollary 2, we have two choices for the stabilizing gain, see (41) for $i = 1$ and $i = 2$. The Jacobians D_1 and D_2 as given in (38) can be computed as:

$$\begin{aligned} D_1 &= \left.\frac{\partial f}{\partial w}\right|_{w=x_1^*} = \begin{pmatrix} -2.7322 & 1 \\ 0.3 & 0 \end{pmatrix}, \\ D_2 &= \left.\frac{\partial f}{\partial w}\right|_{w=x_2^*} = \begin{pmatrix} 1.3322 & 1 \\ 0.3 & 0 \end{pmatrix}. \end{aligned} \tag{52}$$

The matrices J_1 and J_2 given by (39) can be computed as $J_1 = D_1 D_2$ and $J_2 = D_2 D_1$. For the index $i = 1$, by choosing the free matrix X as $X = 0$, from (41) we obtain the following stabilizing gain:

$$K = \begin{pmatrix} 2.5095 & 2.0707 \\ -0.3029 & -0.2492 \end{pmatrix}. \tag{53}$$

We simulated the system given by (43)–(45), (50), (53) with $\rho(k) = d_k(1)$ where the latter is defined in (46). After extensive numerical simulations we find that in this case we have $\epsilon_m = 0.1$ as an indicator for the size of the domain of attraction for Σ_2. Our simulations show exponential convergence to Σ_2 for a wide range of initial conditions. A typical simulation result for $x(1) = 0.3$, $y(1) = 0$ is shown in Fig. 3. In Fig. 3(a), we show $d(w(k), \Sigma_2)$ versus k, and as can be seen the decay is exponential for

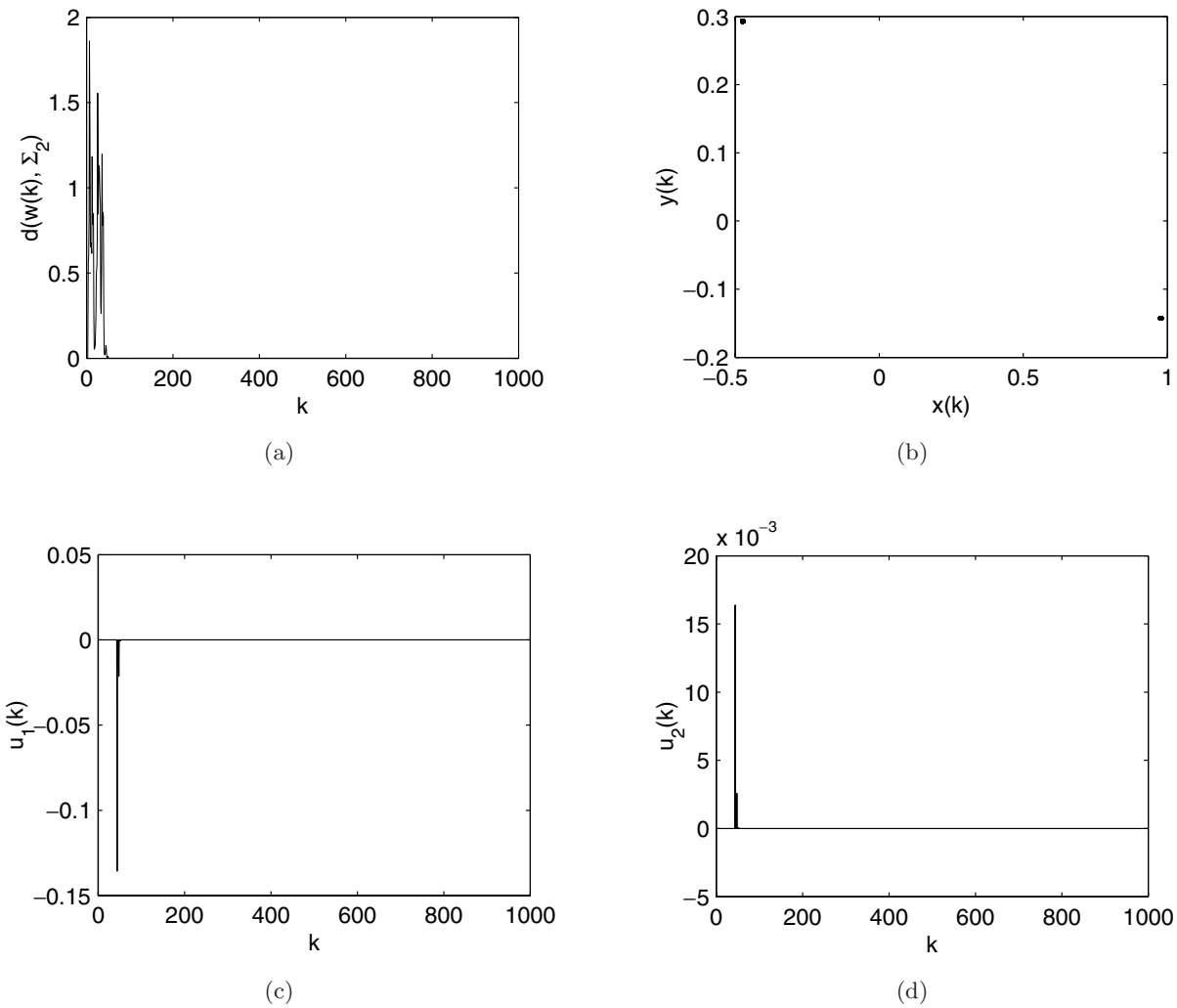

Fig. 3. DPDFC applied to Hénon map for case $i = 1$, (a) $d(w(k), \Sigma_2)$ versus k, (b) $x(k)$ versus $y(k)$ for $k \geq 50$, (c) $u_1(k)$ versus k, (d) $u_2(k)$ versus k.

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$k \geq 50$; apparently the solutions enter the domain of attraction for $k \geq 50$ in this simulation. The $x(k)$ versus $y(k)$ plot in Fig. 3(b) is plotted for $k \geq 50$. As can be seen from these figures, the solutions converge to the period 2 orbit characterized by Σ_2. Finally, the required input components $u_1(k)$ and $u_2(k)$, where $u(k) = (u_1(k) \;\; u_2(k))^T$, are shown in Figs. 3(c) and 3(d), respectively. As can be seen from these figures, $u(k) \to 0$ as $k \to \infty$.

For the case $i = 2$, by choosing the free matrix X as $X = 0$, from (41) we obtain the following stabilizing gain:

$$K = \begin{pmatrix} 2.5095 & -1.0097 \\ 0.6212 & -0.2492 \end{pmatrix}. \tag{54}$$

We simulated the system given by (43)–(45), (50), (53) with $\rho(k) = d_k(2)$ where the latter is defined in (46). After extensive numerical simulations we find that in this case we have $\epsilon_m = 0.1$ as an indicator for the size of the domain of attraction for Σ_2. Our simulations show exponential convergence to Σ_2 for a wide range of initial conditions. A typical simulation result for $x(1) = 0.3$, $y(1) = 0$ is shown in Fig. 4. In Fig. 4(a), we show $d(w(k), \Sigma_2)$ versus k, and as can be seen the decay is exponential for $k \geq 400$; apparently the solutions enter the domain of attraction for $k \geq 400$ in this simulation. The $x(k)$ versus $y(k)$ plot in Fig. 4(b) is plotted for $k \geq 400$. As can be seen from these figures, the solutions converge to the period 2 orbit characterized by Σ_2. Finally, the required input components $u_1(k)$ and $u_2(k)$, where $u(k) = (u_1(k) \;\; u_2(k))^T$, are shown in Figs. 4(c) and 4(d), respectively. As can be seen from these figures, $u(k) \to 0$ as $k \to \infty$.

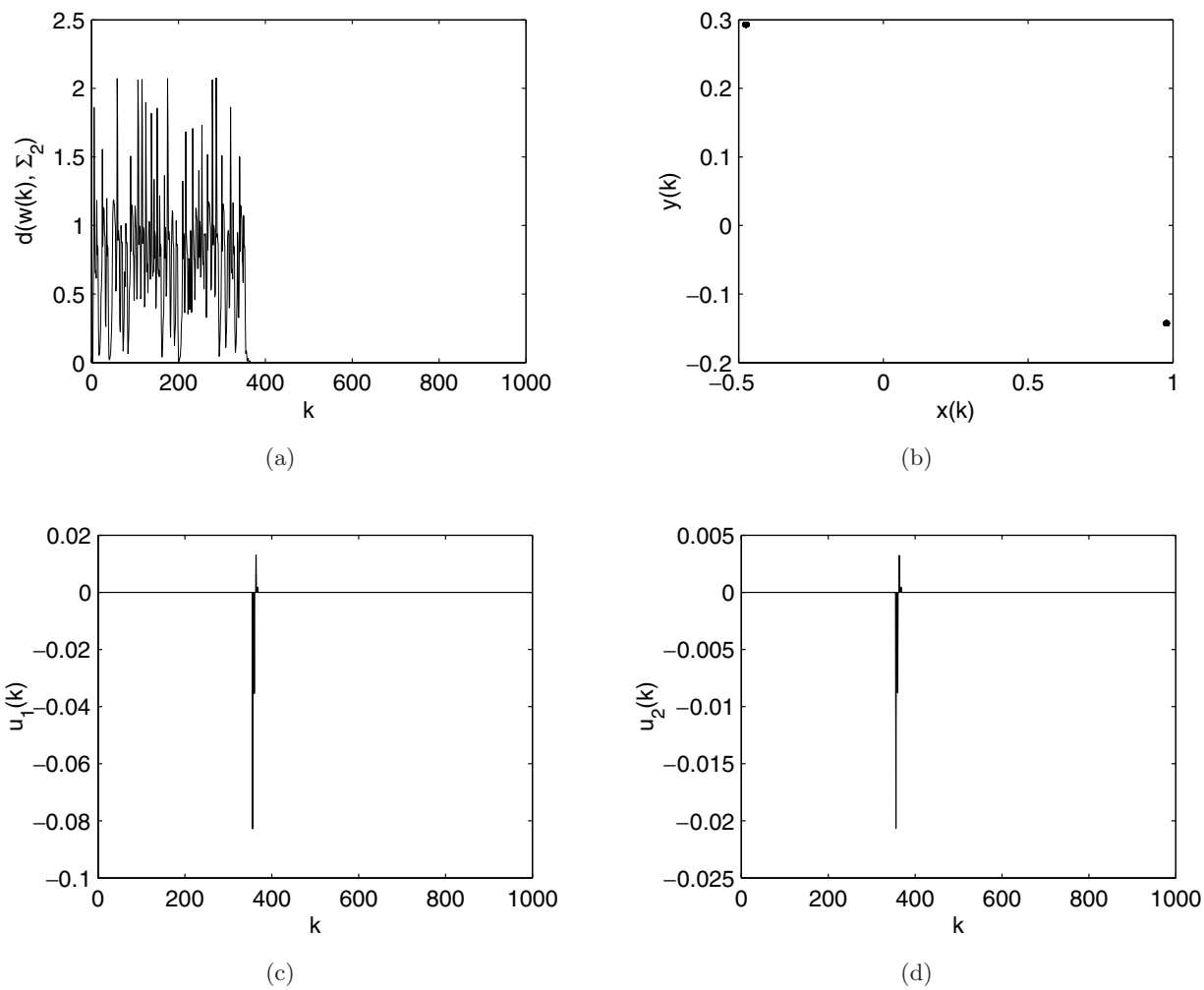

Fig. 4. DPDFC applied to Hénon map for case $i = 2$, (a) $d(w(k), \Sigma_2)$ versus k, (b) $x(k)$ versus $y(k)$ for $k \geq 400$, (c) $u_1(k)$ versus k, (d) $u_2(k)$ versus k.

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8. Conclusion

In this paper, we have considered the periodic delayed feedback law given by (8) for the stabilization of period 1 orbits of one-dimensional discrete time chaotic systems, and proposed a possible generalization of this law for the stabilization of arbitrary periodic orbits of higher dimensional discrete time chaotic systems. The proposed generalization is called as Double Period Delayed Feedback Scheme (DPDFC) since the input is applied at every double period. We proved that the DPDFC scheme can stabilize any hyperbolic periodic orbit of any discrete time chaotic system. We note that some nonhyperbolic periodic orbits can also be stabilized with this approach, see Remark 5. We also argue that the necessary and sufficient condition for the stabilization by DPDFC may be considered as a generic condition, see Corollary 2 and Remark 5, i.e. we expect that this condition holds in almost all cases. Hence, we may state that the inherent limitations of classical DFC may be eliminated by the use of DPDFC for discrete time systems.

We note that the proposed generalization is not the only possible periodic feedback scheme for the stabilization of periodic orbits for discrete time systems. Other generalizations may be possible, and this point deserves further research. Another possibility is the extension of the results presented here to continuous time systems. But such an extension is not obvious, and this point deserves further investigation.

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