Stabilization of vortex-soliton beams in nematic liquid crystals

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We study the interaction of two optical beams of different wavelengths (colors) in a nematic liquid crystal. We consider the case for which one component carries an optical vortex and the other component describes a localized beam. It is shown that a beam in one color can stabilize a vortex in the other color, the vortex being unstable in the absence of the second beam. We also show that the bright vortex can guide the beam in a stable manner, provided that the nonlocality is large enough. In this context we find that a different type of solitary wave (nematicon) instability can arise, one for which a ring structure develops at its peak. The results of approximate modulation solutions for the interaction between the vortex and the beam are found to be in good quantitative agreement with direct numerical simulations.

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I. INTRODUCTION

Optical vortices can be generated in both linear and nonlinear media as singularities of the phase front of the electromagnetic field [1]. However, in nonlinear media with a local response optical vortices usually become unstable and break up into filaments due to a symmetry-breaking azimuthal instability [2]. However when higher order nonlinearities, such as quintic, dominate over the usual cubic nonlinearity vortices in local media can be stabilized [3]. Furthermore vector vortices in off-resonant saturable Raman media have been shown to be stable [4]. In contrast, in nonlocal nonlinear media such as liquid crystals, vortex solitary waves can become stable if the nonlocality parameter exceeds some critical value [5–9]. This stabilizing effect of a nonlocal medium response was subsequently explained using modulation theory based on an averaged Lagrangian approach [10].

In addition to scalar solitary waves, vector solitary waves can form when multiple fields interact parametrically or through cross-phase modulation [11], and their dynamics is governed by coupled systems of equations describing each field. An example of such a vector solitary wave is the two-color solitary waves (vector nematicons) which form when two beams of different colors (wavelengths) propagate through a nematic liquid crystal and couple due to the intermediary effect of the (nonlocal) nematic. These two-color vector nematicons were observed by Alberucci *et al.* [12] and subsequently given a theoretical explanation [13,14].

This multicomponent generalization of solitary wave dynamics is very natural in the context of atomic Bose-Einstein condensates (BECs) because of the several ways in which such systems can be created, for example, as mixtures with two different atomic species or hyperfine states [15] and as internal degrees of freedom liberated under an optical trap and atom-molecule BECs [16]. These multicomponent condensates present novel and fundamentally different scenarios for their ground states and excitations.

In this paper, we study the propagation of two light beams of different wavelengths (colors) in a nematic liquid crystal and consider the case for which one beam carries a bright vortex and the other beam describes a localized mode, which is here termed "nematicon." We discuss two regimes. In the first regime we study the stabilization of a vortex beam in one color by a nematicon in the other, the vortex being unstable in the absence of the nematicon. The developed modulation theory explains this stabilization in terms of the reduction in the vortex width. In the second regime we study the guiding of a nematicon in one color by a vortex in the other color. A corresponding situation for a dark vortex guiding a bright solitary wave in a saturable local medium has been previously considered [2,11]. In this case one beam undergoes defocusing to form the dark vortex, while the focusing solitary wave is supported as a mode of the resulting waveguide. In the present work we consider a different scenario since a nematic liquid crystal is nonlocal [17] and can support stable pulse and bright vortex structures separately in each color. The interaction between a bright localized vortex in one color and a solitary wave in the other, with the nematicon localized inside the vortex, is studied. It is shown that a different type of instability develops on the nematicon. For sufficiently small amplitude relative to the vortex, the nematicon develops a ringlike instability at its peak. On the other hand, as the nematicon becomes relatively large, this instability disappears.

We analyze both regimes analytically using a modulation theory developed from an averaged Lagrangian based on suitable trial functions. As found earlier for a single nonlocal vortex [5,10], the modulation theory gives an explanation in simple terms of all the major effects observed in numerical solutions, with good agreement being obtained.

The paper is organized as follows. In Sec. II we formulate our problem and introduce the system of equations for describing the vector solitons in nematic liquid crystals. As a matter of fact, this is the same system as was studied earlier in Refs. [12–14], but here it is analyzed for a different type of nonlinear two-color localized mode for which one field carries an optical vortex and the other field describes a localized beam. Section III then studies vortex stabilization by a bright beam based on appropriate trial functions. In Sec. IV the second regime of beam evolution is studied, for which the bright beam is guided by a vortex, and the stability boundary for stabilization is derived. Finally, Sec. V concludes the paper.

II. FORMULATION

We consider two coherent polarized light beams of different wavelengths propagating through a cell filled with a nematic liquid crystal. The light propagates in the z direction with the (x,y) plane orthogonal to this. The director of the nematic makes an angle ψ with the z direction. In order to overcome the Freédericz threshold, a static electric field is applied in the x direction to pretilt the nematic at $\hat{\psi}$ to the z direction. Let us set θ to be the perturbation of the director angle from this pretilt so that $\psi=\hat{\psi}+\theta$. The electric fields of the light beams are assumed to be polarized in the x direction. The equations for the envelopes u and v of the light beams are, in dimensionless form,

$$i\frac{\partial u}{\partial z}+\frac{1}{2}D_u\nabla^2 u+A_u u\sin 2\theta=0, \tag{1}$$

$$i\frac{\partial v}{\partial z}+\frac{1}{2}D_v\nabla^2 v+A_v v\sin 2\theta=0, \tag{2}$$

$$\nu\nabla^2\theta-q\sin 2\theta=-2[A_u|u|^2+A_v|v|^2]\cos 2\theta \tag{3}$$

[12,18–20]. Here q measures the strength of the static electric field and ν measures the elasticity of the nematic. The parameters D_u and D_v are the diffraction coefficients for the two wavelengths and A_u and A_v are the coupling coefficients between the electric fields of the light and the nematic director for the two, generally different, wavelengths.

The usual experimental regime is the so-called nonlocal regime in which ν is large [17,20]. In this regime the reorientation of the nematic extends far beyond the waist of the electric field(s). It can be seen from the director [Eq. (3)] that for large ν, θ is small, so that in the highly nonlocal limit the nematicon equations can be approximated by

$$i\frac{\partial u}{\partial z}+\frac{1}{2}D_u\nabla^2 u+2A_u u\theta=0, \tag{4}$$

$$i\frac{\partial v}{\partial z}+\frac{1}{2}D_v\nabla^2 v+2A_v v\theta=0, \tag{5}$$

$$\nu\nabla^2\theta-2q\theta=-2A_u|u|^2-2A_v|v|^2. \tag{6}$$

These nonlocal two-color nematicon equations have the Lagrangian

$$L=i(u^*u_z-uu_z^*)-D_u|\nabla u|^2+4A_u\theta|u|^2-\nu|\nabla\theta|^2-2q\theta^2 \\ +i(v^*v_z-vv_z^*)-D_v|\nabla v|^2+4A_v\theta|v|^2, \tag{7}$$

where the superscript * denotes the complex conjugate.

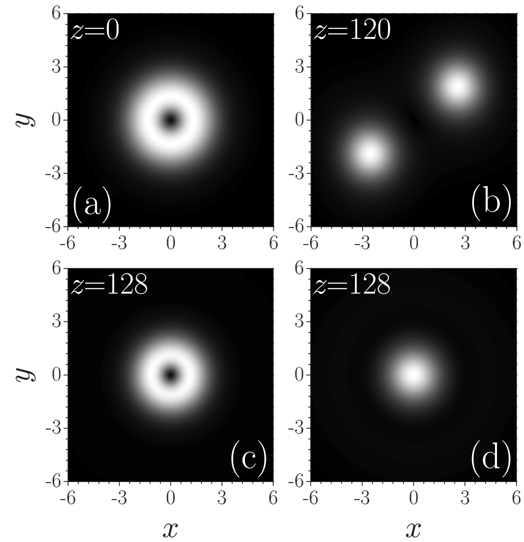

FIG. 1. [(a) and (b)] Input and output intensities for an unstable vortex in color v with no beam in color u. [(c) and (d)] Vortex in color v (c) stabilized by nematicon in color u. (d) Here $\nu=4$ and $a_u^T=0.9$.

In order to study the vortex stabilization and a bright beam guided by a vortex, Lagrangian (7) will be averaged using trial functions suitable for each regime. This analysis will be based on previous work [10], so that only the relevant modifications will be reported here. The first regime considered is that of the stabilization of a vortex by a localized bright beam.

III. VORTEX STABILIZATION

Let us begin by considering the effect of a nematicon (bright beam) on a larger vortex. It is found from numerical solutions, as shown in Fig. 1, that in the weakly nonlocal limit $\nu=O(1)$ the addition of a nematicon in one color eliminates the strong $n=2$ azimuthal instability mode of a vortex in the other color, leading to a stable configuration.

To investigate this stabilization of a vortex by a nematicon in the other color we shall use modulation theory, as in [10]. Since numerical solutions show no instability in the nematicon, we proceed as in [10] with a trial function of a perturbed vortex in one color and a nematicon in the second color. The trial function for the vortex is then

$$v=a_v re^{-r/w_v}e^{i\phi+i\sigma_v z}+ige^{i\phi+i\sigma_v z}, \tag{8}$$

while the nematicon trial function is

$$u=a_u\,\mathrm{sech}(r/w_u)\,e^{i\sigma_u z}\quad\text{and}\quad\psi_u=\alpha_u\,\mathrm{sech}^2(r/\beta_u), \tag{9}$$

as in previous studies [10,21]. The director perturbation is then $\theta=\psi_u+\psi_v$, with ψ_v to be calculated later. Here r and ϕ are the polar coordinates corresponding to the Cartesian coordinates (x,y). It should be noted that for reasonable choices of trial function, the results of the variational analysis are largely independent of this choice [13].

Using trial functions (8) and (9) and the results of Minzoni *et al.* [10,21] we can obtain an averaged Lagrangian for the two-color nematicon vortex of the form

$$\mathcal{L} = \mathcal{L}_u + \mathcal{L}_v + \mathcal{L}_{uv} \tag{10}$$

on taking $D_u = D_v = 1$ and $A_u = A_v = 1$ to leading order. In this regard it is reported by Alberucci *et al.* [12] that the diffraction coefficients in the liquid crystal E6 are 0.805 for red light and 0.823 for near-infrared light. Here

$$\mathcal{L}_u = -4\pi I_{22} a_u^2 - 16\pi I_{42} \nu a_u^2 - 8\pi q I_4 \alpha_u^2 \beta_u^2 + \frac{8\pi R^2 S^2 \alpha_u^2 w_u^2 \beta_u^2}{R^2 \beta_u^2 + S^2 w_u^2} \tag{11}$$

and

$$\mathcal{L}_v = -\left(\frac{3}{4} a_v^2 w_v^4 + 2\Lambda_1 g^2\right)\sigma_{vz} - 4a_v w_v^3 g_z - \frac{3}{8} a_v^2 w_v^2 - \frac{3}{8} a_v^2 w_{v\phi}^2 + 2a_v g w_{v\phi} - 2a_v w_v g_\phi - \Lambda_2 (g_\phi^2 + g^2) + \frac{a_v^4 w_v^6}{32\sqrt{\nu}} + \frac{2a_v^2 w_v^4}{\nu} \int_0^{2\pi} \int_0^{2\pi} \sum_{l\neq 0}^{\infty} \frac{405 e^{-2} w_v^3}{256|l|} e^{il(\phi-\phi')} \times g^2(r,\phi',z)\, d\phi'\, d\phi \tag{12}$$

on using the results of Minzoni *et al.* [10]. Here

$$R = \frac{\sqrt{2I_2}}{\sqrt{I_{x32}}}, \quad S = \sqrt{2I_2}, \quad \Lambda_1 = w_v R, \quad \text{and} \quad \Lambda_2 = \ln\left(\frac{r_{\max}}{r_{\min}}\right). \tag{13}$$

The terms Λ_1 and Λ_2 in Eq. (13) describe the effect of the shelf of radiation under the vortex [22]. This radiation is assumed to be radially symmetric in space, centered about the vortex peak at $r = w_v$, as follows from numerical solutions. Hence g is only nonzero in the region $r_{\min} < r < r_{\max}$, where $r_{\min,\max} = w_v \mp R/2$. The integrals I_i and I_{ij} are

$$I_{22} = \int_0^\infty x\, \mathrm{sech}^2 x \tanh^2 x\, dx = \frac{1}{3}\ln 2 + \frac{1}{6},$$

$$I_{42} = \frac{1}{4}\int_0^\infty x\left[\frac{d}{dx}\mathrm{sech}^2 x\right]^2 dx = \frac{2}{15}\ln 2 + \frac{1}{60},$$

$$I_4 = \int_0^\infty x\, \mathrm{sech}^4 x\, dx = \frac{2}{3}\ln 2 - \frac{1}{6},$$

$$I_{x32} = \int_0^\infty x^3\, \mathrm{sech}^2 x\, dx = 1.352\,301\,002\cdots. \tag{14}$$

To calculate the averaged Lagrangian for the interaction between the vortex and the nematicon $\mathcal{L}_{uv}$ we denote by ψ_u and ψ_v the director distributions in the u and v colors, which are determined by

$$\nu\nabla^2\psi_u - 2q\psi_u = -2|u|^2 \quad \text{and} \quad \nu\nabla^2\psi_v - 2q\psi_v = -2|v|^2. \tag{15}$$

These can be solved, due to reciprocity, in terms of a symmetric Green's function $G(\mathbf{x},\mathbf{x}')$ in the form

$$\psi_u = -2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{x},\mathbf{x}')|u(\mathbf{x}',z)|^2\, dx'\, dy',$$

$$\psi_v = -2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{x},\mathbf{x}')|v(\mathbf{x}',z)|^2\, dx'\, dy' \tag{16}$$

[21]. We then find that

$$\mathcal{L}_{uv} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (4\psi_u|v|^2 + 4\psi_v|u|^2)\, dx dy. \tag{17}$$

Due to the symmetry of the Green's function this can be re-expressed, on rearranging the integration, as

$$\mathcal{L}_{uv} = 16\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_v |u(\mathbf{x},z)|^2\, d\mathbf{x} \tag{18}$$

to eliminate the dependence on ψ_u.

To calculate ψ_v, we use the same approximation as in [10] and solve

$$\nu\frac{d^2\psi_v}{dr^2} - 2q\psi_v = -2a_v^2 r^2 e^{-2r/w_v} \tag{19}$$

with $\psi_v'(0) = 0$ and $\psi_v \to 0$ as $r \to \infty$. Since in the nonlocal limit $w_v \ll \nu$, the right-hand side of this equation can be approximated by a δ function at $r = w_v$. Then, as in [10], we obtain

$$\psi_v = \begin{cases} A, & r \le w_v \\ A e^{-\sqrt{2q/\nu}\,(r-w_v)}, & r > w_v, \end{cases} \tag{20}$$

where

$$A = \frac{a_v^2 w_v^3}{32\sqrt{2q\nu}}. \tag{21}$$

This gives finally

$$\mathcal{L}_{uv} = 16 A a_u^2 w_u^2. \tag{22}$$

Since the vortex and the nematicon are assumed to have comparable amplitudes (powers) $\mathcal{L}_{uv} = O(\nu^{-1/2})$ on using Eq. (21). The influence of the vortex on the nematicon is then small and from [21] and expression (22) it follows that the amplitude-width relation for the nematicon is the same as that in [21], so that

$$a_u^2 = \frac{2\nu I_{22} I_{42}}{R^4 S^6 w_u^6 \beta_u^2} \tag{23}$$

in the limit of large ν. On the other hand since the term in $\mathcal{L}_v$ responsible for the amplitude-width relation for the vortex is also $O(\nu^{-1/2})$, the $\mathcal{L}_{uv}$ contribution in averaged Lagrangian (10) will change the width of the vortex due to the presence of the nematicon. This has been shown, in fact, to be due to the amplitude-width relation for the vortex being sensitive to the size of the tilt of the optical axis in the region where the vortex is small [10]. In the present scenario, it is in this region that the nematicon in the second color is large. The amplitude-width relation for the vortex is then modified. Variations in averaged Lagrangian (10) with respect to the parameters a_v and w_v give the amplitude-width relation for the vortex in the presence of the nematicon as the solution of the quadratic equation

$$\frac{e^2 H_v^2}{8\sqrt{\nu}} w_v^2 + (a_u w_u)^2 w_v - \frac{3}{2} = 0. \tag{24}$$

In this equation the amplitude H_v of the vortex at its maximum is given by

$$H_v = a_v w_v e^{-1}. \tag{25}$$

The equations for the azimuthal perturbations of the vortex are the same as in [10], where now w_v satisfies Eq. (24). In the limit $a_u=0$ we recover the results of Minzoni *et al.* [10] for a vortex. The results of this work give that the vortex is destabilized when

$$D_2 = \frac{405}{128\nu} H_v^2 w_v^4 \tag{26}$$

reaches the threshold value $D_2=14.42$. It is clear from stability relation (26) that as w_v decreases the vortex is stabilized. This vortex width contraction is displayed in Fig. 1. The remarkable result is that the threshold value for vortex stability has been decreased by a factor of order 25 due to the presence of the nematicon over that when there is no nematicon [10]. Moreover it follows from amplitude-width relation (24) that as $a_u^2 w_u^2$ increases the vortex will stabilize for smaller values of ν. This stability analysis then gives a qualitative explanation of the numerical results showing vortex stabilization due to the presence of a nematicon in the other color.

At a physical level the vortex is stabilized by the beam as the beam tilts the nematic in the center of the vortex, resulting in stability for sufficiently large added tilt. In this regard the incoherent nature of the two-color interaction is vital as it allows the vortex and beam to rotate the nematic independently. Finally the stabilization mechanism is extremely effective as it results in a drastic reduction in the stability threshold for the vortex, independent of the actual parameters involved. Now that the mechanism for stabilizing a vortex by a beam has been determined, it is clear that any mechanism, such as other beams or a collection of beams, which tilts the nematic in the center of the vortex will stabilize it in a manner similar to that determined here for a beam of another color.

Let us now consider quantitative comparisons between the modulation theory and numerical results. The numerical results were obtained by solving two-color nematicon Eqs. (1)–(3). While approximations (4)–(6) to full nematicon Eqs. (1)–(3) in the limit of small director deviation θ have been the basis of the modulation theory, these limiting equations are known to give an excellent approximation to the full equations in the nonlocal regime for which $|\theta|$ is small [21]. A stationary vortex was used as an initial condition. This stationary solution was obtained by numerically solving nematicon Eqs. (1)–(3) for an isolated vortex in the color v, $v=f(r)\exp(i\sigma z+i\phi)$ with $u=0$, which results in an eigenvalue problem for σ for a vortex of a given amplitude [5,23]. For sufficiently small nematicon amplitude a_u the vortex was unstable to the $n=2$ azimuthal mode and split. As the amplitude of the nematicon increased the vortex was stabilized, giving a numerical threshold amplitude a_u^T for stability. The corresponding threshold amplitudes a_u^* and widths w_u^* as given by modulation theory were calculated using the theoretical threshold D_2 [Eq. (26)], based on Eq. (24). The comparison is given in Table I. It can be seen that there is excellent agreement between the present stability thresholds and the numerically determined thresholds of Xu *et al.* [23]. This is especially so as the modulation theory analysis was based on the assumption that ν is large, while for reasonable amplitudes the values of the nonlocality used were $\nu=O(1)-O(10)$.

TABLE I. Comparison between stability thresholds as given by the present modulation theory and the numerical results for a_u^T in [23].

ν	H_v	a_u^T	a_u^*	w_u^*	$D_2(a_u^T)$
2	3.59	1.4	1.47	1.0	18.33
3	2.77	1.2	1.27	1.2	16.96
4	2.39	0.9	0.85	2.2	13.8
5	2.02	0.45	0.28	0.3	5.74
6	1.96	0	0	0	

IV. STABILITY OF A VECTOR NEMATICON

In contrast to Sec. III where the vortex and nematicon have similar powers, let us consider the interaction of a nematicon (solitary wave) in the u color and a relatively large vortex in the v color, so that the power of the vortex is large relative to that of the nematicon. As by Minzoni *et al.* [10,21], Skuse and Smyth [13], and Assanto *et al.* [14], this interaction will be investigated using suitable trial functions in Lagrangian (7). In Fig. 2 numerical solutions are displayed which show that the nematicon in the u color becomes unstable due to a vortex mode developing around its peak. Suitable trial functions which include this dominant instability mode are then

$$u = \begin{cases} [a_u \operatorname{sech}(r/w_u) + B(\mu^2 r^2 - r^4)]e^{i\psi_u} \\ + i\gamma_u e^{i\psi_u}, & r < \mu \\ a_u \operatorname{sech}(r/w_u)e^{i\sigma_u} + i g_u e^{i\sigma_u}, & r \geq \mu \end{cases} \tag{27}$$

and

$$\theta_u = \alpha_u \operatorname{sech}^2(r/\beta_u) \tag{28}$$

for the u color nematicon with the vortex instability mode and

$$v = a_v r e^{-r/w_v} e^{i\sigma_v + i\phi} + i g_v e^{i\sigma_v + i\phi} \tag{29}$$

for the v color vortex [10]. The variables (r,ϕ) are polar coordinates in the (x,y) plane. The form of the director θ_v for the v color will be calculated below. Also the instability amplitude $B(\phi,z)$ will be determined from the modulation equations. The parameters of the nematicon and the vortex are functions of z and ϕ. The first terms in these trial functions

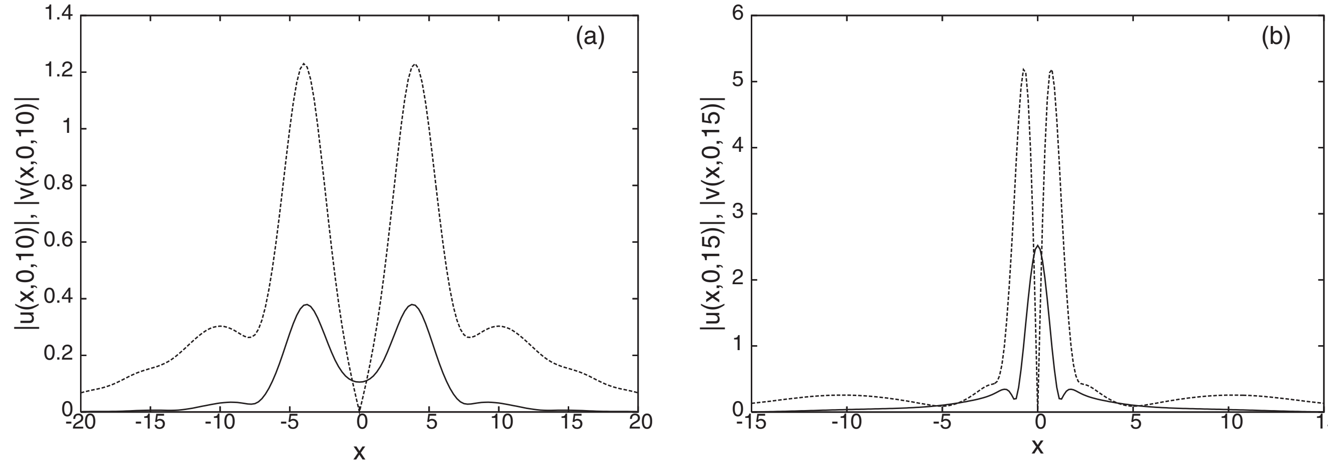

FIG. 2. Cross sections of numerical solutions of two-color nematicon Eqs. (4)–(6) for $|u|$ and $|v|$ at $y=0$ for boundary conditions $a_u=0.55$, $w_u=3.1$, $a_v=0.55$, and $w_v=4.0$ for $\nu=10$, $A_u=1$, $A_v=0.95$, $D_u=1$, and $D_v=0.98$. u color (—); v color (– – –). (a) $z=10$; (b) $z=15$.

are the varying nematicon and vortex, respectively. The u color trial function in $r<\mu$ is an approximation to the instability mode at an early stage of its development. Its amplitude is assumed to be small, so that $|B|$ is small. The second terms represent the effect of the low wave-number diffractive radiation which accumulates under the pulses as they evolve [10,21,22]. An averaged Lagrangian

$$\mathcal{L} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L\, dxdy \tag{30}$$

is then calculated by substituting trial functions (27)–(29) into Lagrangian (7), from which modulation equations for the parameters are derived as variational equations.

As it is observed from numerical solutions that the main effect of the vortex on the nematicon is to produce a vortex-like instability localized around $r=0$, we shall ignore the effects of the shelves under the vortex in the v color and the nematicon so that $g_u=g_v=0$. In addition, as we shall not be interested in the effect of the shed diffractive radiation, these can be ignored [21]. As in [10] it will be further assumed in the present linear stability analysis that, to leading order, the amplitude and width of the v color vortex and the u color nematicon remain at their steady state values. It is observed from numerical solutions that these parameters evolve over a longer z scale than the vortexlike instability. Therefore, for the present analysis, only the parameters B and γ_u will be assumed to evolve, as these are the parameters which evolve on the z scale of the instability and control the vortex-mode instability of the u color nematicon. Under these assumptions the contribution of the v color vortex to the averaged Lagrangian is as in [10].

The averaged Lagrangian for the u color nematicon has two contributions. The first comes from the nematicon itself and is as in [21]. The contribution which comes from the interaction of the term of amplitude $B(\phi,z)$ and the main pulse is zero since $B(\phi,z)$ is taken to have zero angular average. For linear stability, $|B|$ is assumed to be small and only quadratic terms in B in the averaged Lagrangian are retained. The contribution of the shelf γ_u under the instability vortex is confined to the region around its maximum at $r=\mu/2$. Under these assumptions the contribution of the nematicon to the averaged Lagrangian is

$$\mathcal{L} = \mathcal{L}_u + \mathcal{L}_v + \mathcal{L}_{vQ} + \mathcal{L}_{v\gamma}, \tag{31}$$

where

$$\mathcal{L}_u = \hat{\mathcal{L}}_u + \left(a_u^2 w_u^2 \ln 2 + \frac{\mu^6}{20}B^2\right)\psi_{uz} + \frac{\mu^6}{20}gB_z - \frac{\mu^4}{24}B_\phi^2 - \frac{\Lambda_\mu}{w_u}\gamma_{u\phi}^2 - \frac{3}{35}\mu^8 B^2. \tag{32}$$

Here $\hat{\mathcal{L}}_u$ is Eq. (11) evaluated at the fixed point. The averaged Lagrangian for the vortex in $\mathcal{L}_v$ is Eq. (12). The averaged Lagrangians $\mathcal{L}_{vQ}$ and $\mathcal{L}_{v\gamma}$ give the interaction of the main vortex with the shelf of the small cap vortex on the nematicon.

The averaged Lagrangian contribution due to the interaction between the optical axis for the v color vortex and the vortex perturbation Q is given by $\mathcal{L}_{vQ}$ in the form

$$\mathcal{L}_{vQ} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 4(A_u+A_v)B^2|Q(\mathbf{x}')|^2\theta_v\, d\mathbf{x}' = \frac{1}{60}AB^2\mu^{10} \tag{33}$$

on again using the symmetry of the Green's function.

Finally we consider the interaction between the distortions of the optical axis caused by the shelf γ_u under the vortex instability in the u color and the v color vortex. The corresponding term in the averaged Lagrangian is

$$\mathcal{L}_{v\gamma} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 4(A_u+A_v)\theta_v|\gamma_u|^2\, d\mathbf{x}, \tag{34}$$

where θ_v is the solution given in Eq. (20). This gives

$$\mathcal{L}_{v\gamma} = 4(A_u+A_v)A\Lambda_\mu, \tag{35}$$

where Λ_μ is the area of the annulus below the maximum of the vortex perturbation in the u color.

The modulation equations governing the evolution of the two-color vortex nematicon are then derived from the full averaged Lagrangian

$$\mathcal{L} = \int_{z_0}^{z_1} \int_0^{2\pi} [\mathcal{L}_{vQ} + \mathcal{L}_{v\gamma} + \mathcal{L}_u + \mathcal{L}_v + \mathcal{L}_{uv}]\, d\phi dz. \quad (36)$$

Let us now examine averaged Lagrangian (36) to study the stability of the vortex-nematicon system. To this end, it is only necessary to study the associated Hamiltonian and show that it is positive definite. Now since the nonlocality ν is assumed to be large, $\mathcal{L}_{uv}$ can be neglected. In this case we have that the quadratic form, which is the Hamiltonian of the vortex perturbation, is positive definite. Examination of this Hamiltonian shows that perturbations of the form $e^{-in\phi}$ will be stable for n large, as in [10] for a single vortex, and that the most unstable mode will be proportional to $e^{-i\phi}$. For this mode to be stable, we require

$$\frac{3}{7}\mu^8 - \frac{A}{12}\mu^{10} \geq 0 \quad (37)$$

and

$$\Lambda_\mu w_u^{-1} - \Lambda_\mu A \geq 0. \quad (38)$$

From expression (21) for A it is clear that both of these inequalities will be satisfied, provided that the nonlocality ν is sufficiently large. Moreover, the shelf area Λ_μ plays no role to this order and need not be determined. The two-color vortex nematicon is then stable, provided that the nematic is highly nonlocal. The parameter μ, which gives the width of the vortex perturbation on the nematicon, can be taken as $\mu = w_u$ [10].

To study the stability quantitatively, let us consider small perturbations about the steady state for the vortex nematicon which is determined from the variational equations for averaged Lagrangian (36), $\mathcal{L}_{a_u}=0$, $\mathcal{L}_{a_v}=0$, $\mathcal{L}_{w_u}=0$, and $\mathcal{L}_{w_v}=0$. For $\nu \gg 1$, these variational equations give the amplitude-width relations for the vortex nematicon as

$$a_u^2 = \frac{2\nu I_{22} I_{42}}{R^4 S^6 w_u^6 \beta_u^2} \quad \text{and} \quad a_v^2 = 48\sqrt{\nu} w_u^{-4}. \quad (39)$$

These fixed point expressions are the same as for an uncoupled vortex and nematicon [10,21] since $\mathcal{L}_{vQ}$ is of smaller order. Moreover, smaller nematicon widths correspond to larger nematicon amplitudes. When steady states [Eq. (39)] are used in stability relations (37) and (38), we find for a given value of the nonlocality ν that larger nematicons are more stable than smaller ones. As the nematicon amplitude decreases, the vortex nematicon destabilizes. The unusual aspect of the vortex destabilizing the beam is that an optical vortex, which is usually unstable, has destabilized a solitary wave, which is usually stable.

For values of ν below a critical threshold, the two-color vortex nematicon is unstable, with the nematicon peak developing a dip. This dip expands and increases in depth until the nematicon evolves into a vortex. The early stage of this development is governed by the $\delta\psi_u$ variational equation, which gives mass conservation for the vortex-nematicon system at the nematicon peak. Varying averaged Lagrangian (36) with respect to ψ_u gives this mass equation at the nematicon peak as

$$\frac{d}{dz}\left(a_u^2 w_u^2 \ln 2 + \frac{\mu^6}{20} B^2 \right) = 0. \quad (40)$$

This mass equation shows the decay in the amplitude of the nematicon as the vortex perturbation grows as a result of the interaction with the optical axis of the main vortex in the v color. Numerical solutions show that the center of the nematicon then rebounds, reforming the original beam shape. This rebound is, of course, not governed by the present linearized stability equations. This process repeats itself until ultimate collapse occurs due to the instability of the vortex and nematicon for the low director deviation Eqs. (4)–(6) for small values of the nonlocality ν. For small values of ν the vortex itself is unstable [5,10]. Also, since there is no saturation effect in the small director deviation nematicon Eqs. (4)–(6), as there is in full Eqs. (1)–(3), the nematicon itself is also unstable for small ν, the local limit [19,24]. For values of ν above a critical, the vortex-nematicon bound state is stable, with the nematicon sitting inside the vortex. In this context it should be noted that for large enough ν, so that the nematic has a large enough nonlocal response, a vortex is stable [5,10]. Figure 2 shows numerical solutions of nematicon Eqs. (4)–(6) for $\nu=10$, which is a nonlocality parameter value well below the stability threshold for the nematicon to be stable to vortex-type perturbations. The vortex-type instability can clearly be seen in Fig. 2(a), which is at $z=10$. In Fig. 2(b) the solution is shown at $z=15$, at which distance the center of the nematicon has rebounded. For the low value of ν used the vortex and nematicon are unstable [5,10,19,24], as can be seen in this figure.

Let us now examine a quantitative comparison between the modulation theory and numerical solutions of two-color Eqs. (4)–(6). For the steady state near $a_u=0.5$, $w_u=3.5$, $a_v=1.0$, and $w_v=4.5$, numerical solutions of the nematicon equations gave the change in stability threshold for the vortex nematicon to be about $\nu=171$, while the zero of stability criterion (37) gave about $\nu=150$, a 12% difference. It should be noted that inequality (38) is also satisfied at $\nu=150$. Given the number of approximations which were made to derive stability criteria (37) and (38), this agreement is quite good.

V. CONCLUSIONS

We have studied the interaction of two beams of different colors (wavelengths), one a vortex and the other a bright beam, in a nematic liquid crystal. In particular, we have developed a modulation theory which gives analytical results for this interaction, including the stability of the corresponding two-color vector solitons. We have shown that a nematicon in one color can stabilize a vortex in the other color, the isolated vortex being unstable, provided that the nematicon's amplitude is above a threshold which depends on the degree of nonlocality and the vortex amplitude. Moreover, we have shown that a bright vortex can guide a localized mode of a different color in a nematic liquid crystal, provided that the nonlocality of the liquid crystal is large enough. It has been further shown that a low amplitude guided mode will develop a ringlike instability at its peak due to the interaction between the peak and the flat optical axis produced by the vortex at its core. This interaction decreases in strength as the nonlocality increases, so stabilizing the vector soliton.

The simple modulation theory developed to describe the two stability behaviors has been found to give predictions in good quantitative agreement with numerical solutions for the stability threshold and furthermore gives details of how the stability boundary depends on the vortex and nematicon parameters. The results obtained here suggest a very rich family of vector vortex solutions which exchange stability or undergo bifurcations to periodic solutions. This could form the basis for detailed further work.

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