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Stabilization Over Power-Constrained Parallel Gaussian Channels

Zhan Shu and Richard H. Middleton

Abstract—This technical note is concerned with state-feedback stabilization of multi-input systems over parallel Gaussian channels subject to a total power constraint. Both continuous-time and discrete-time systems are treated under the framework of $\mathcal{H}_2$ control, and necessary/sufficient conditions for stabilizability are established in terms of inequalities involving unstable plant poles, transmitted power, and noise variances. These results are further used to clarify the relationship between channel capacity and stabilizability. Compared to single-input systems, a range of technical issues arise. In particular, in the multi-input case, the optimal controller has a separation structure, and the lower bound on channel capacity for some discrete-time systems is unachievable by linear time-invariant (LTI) encoders/decoders.

Index Terms—Channel capacity, $\mathcal{H}_2$ control, networked control systems, parallel Gaussian channels, stabilization, transmitted power.

I. INTRODUCTION

With the rapid development of communication technology, an increasing number of control systems have integrated communication channels or networks to transmit signals. While simplifying installation, facilitating maintenance, and reducing building costs, the use of communication has also raised several issues and challenges. e.g., transmission delay, packet loss, data rate limit, and quantization. These issues are rarely treated in traditional control system analysis and synthesis, and spurs new developments of control theory [1], [2].

As one of the most important communication constraints, data rate has received much attention recently. By employing information theoretic arguments, Nair and Evans have established a series of results on the relationship between stabilizability and data rate [3]–[5]. LQG type control has been considered in [6], [7]. State estimation and related observability/detectability have been investigated in [8]. A comprehensive survey on control with data rate limit is available in [9]. Although these results are nice, they rely heavily on various information theoretic techniques, which are unfamiliar to most researchers in control society, and the resulting complex encoding/decoding schemes are difficult to analyze in traditional control terms such as performances and robustness. In [10] and subsequent work [11], [12], an alternative framework for control system analysis and synthesis with data rate constraints has been established for single-input and single-output LTI systems with Gaussian communication channels. A highlight of this framework lies in its simplicity and intimate relationship with modern control theory. Naturally, one may wonder whether the proposed framework can cope with multi-input and multi-output systems, and this motivates the present study. As for networked control with other constraints such as quantization, packet dropout, communication delay, we refer readers to [13], [14] and references therein

In this technical note, we study the problem of state-feedback stabilization for multi-input systems over parallel Gaussian channels subject to a total power constraint, and establish necessary/sufficient conditions for stabilizability in terms of inequalities involving unstable plant

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poles, transmitted power, and channel noise variances for both continuous-time and discrete-time systems. It is revealed that there exists an inherent lower bound on the required transmitted power if an unstable plant needs to be stabilized. We further obtain lower bounds on required channel capacity for stabilization. The lower bound for the continuous-time case can always be achieved by a pair of memoryless LTI encoder and decoder, while for the discrete-time case its achievability via LTI encoders/decoders is not guaranteed generally. The discrepancy between continuous-time and discrete-time systems may be explained by the different power allocations for Shannon limits.

Notation

Throughout this technical note, $\mathbb{C}^-$, $\mathbb{C}^+$, $\mathbb{D}$, and $\bar{\mathbb{D}}^c$ represent the open-left half of the complex plane, open-right half of the complex plane, open unit disk, and complement of the closed unit disk, respectively; $\mathbf{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure; $\operatorname{tr}(\cdot)$, $\det(\cdot)$, $\operatorname{spec}(\cdot)$ represent the trace, determinant, and spectrum of a matrix, respectively; A square matrix A is said to be Hurwitz (anti-Hurwitz, respectively) if $\operatorname{spec}(A) \subset \mathbb{C}^-$ ($\operatorname{spec}(A) \subset \mathbb{C}^+$, respectively), and Schur (anti-Schur, respectively) if $\operatorname{spec}(A) \subset \mathbb{D}$ ($\operatorname{spec}(A) \subset \bar{\mathbb{D}}^c$, respectively); $\lambda_i(\cdot)$ and $\rho(\cdot)$ stand for the ith smallest eigenvalue and spectral radius of a matrix, respectively; $\|\cdot\|_2$ denotes the $\mathcal{H}_2$ norm of an underlying linear system; $(\cdot)_i$ represents the ith column of a matrix; $A^\dagger$ denotes the unique Moore–Penrose inverse of A.

II. PRELIMINARIES

The general feedback configuration considered in this technical note is depicted in Fig. 1. The plant, encoder, and decoder are all linear, and have the following realizations:

$$\mathcal{G} : \begin{cases} \delta x(t) = Ax(t) + Bu(t) \\ y(t) = x(t) \end{cases} \tag{1}$$

$$\mathcal{K}_e : \begin{cases} \delta x_e(t) = K_{eA} x_e(t) + K_{eB} x(t) \\ s(t) = K_{eC} x_e(t) + K_{eD} x(t) \end{cases} \tag{2}$$

$$\mathcal{K}_d : \begin{cases} \delta x_d(t) = K_{dA} x_d(t) + K_{dB} r(t) \\ u(t) = K_{dC} x_d(t) + K_{dD} r(t) \end{cases} \tag{3}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $s \in \mathbb{R}^m$, and $r \in \mathbb{R}^m$ are the system state, the control input, the channel input, and the channel output, respectively; δ represents the differential operator $\delta x(t) \triangleq dx(t)/dt$, $t \in \mathbb{R}$ for the continuous-time case, and the shift operator $\delta x(t) \triangleq x(t+1)$, $t \in \mathbb{Z}$ for the discrete-time case, respectively; $K_e \triangleq \begin{bmatrix} K_{eA} & K_{eB} \\ K_{eC} & K_{eD} \end{bmatrix}$ and $K_d \triangleq \begin{bmatrix} K_{dA} & K_{dB} \\ K_{dC} & K_{dD} \end{bmatrix}$ are matrices to be determined. The channel input and output have the relation $r(t) = s(t) + n_\sigma(t)$, where n_σ is a Gaussian white noise with power spectral density Σ^2, where $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m) > 0$[1]. We therefore rewrite the channel equation as

$$r(t) = s(t) + \Sigma n(t) \tag{4}$$

where n is a zero mean, circularly symmetric, unit variance additive white Gaussian noise process. Similar to the single input case [10], all signals are required to converge to a stationary distribution, and the channel input has to satisfy the total power constraint

$$\operatorname{POW}(s) \triangleq \mathbf{E}\{\|s(t)\|^2\} < \mathcal{P} \tag{5}$$

for some prescribed $\mathcal{P}$. From (4) and the theory of $\mathcal{H}_2$ control, $\mathbf{E}\{\|s(t)\|^2\} = \|\mathcal{T}_{sn}\|_2^2$, where $\mathcal{T}_{sn}$ represents the resulting

[1]Without loss of generality, any additive white Gaussian noise, with non-degenerate covariance matrix can be transformed to this diagonal case
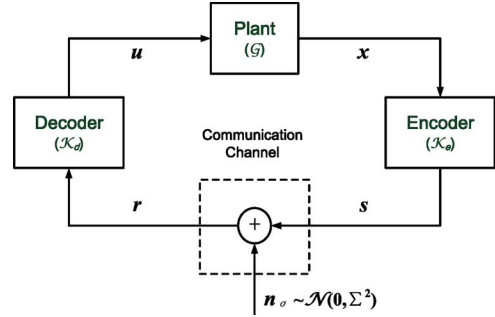

Fig. 1. Feedback configuration over a parallel Gaussian channel.

closed-loop system, and therefore the constraint imposed on the channel input can be reformulated as $\|\mathcal{T}_{sn}\|_2^2 < \mathcal{P}$.

For technical simplicity, we take the following assumptions on A, B, and Σ.

Assumption 1: The system matrix $A \in \mathbb{R}^{n \times n}$ and the input matrix $B \in \mathbb{R}^{n \times m}$ have the structure

$$A = \operatorname{diag}(A_u, A_s), \ B = \begin{bmatrix} B_u^T & B_s^T \end{bmatrix}^T \tag{6}$$

where $A_s \in \mathbb{R}^{n_s \times n_s}$ is Hurwitz (continuous-time systems) or Schur (discrete-time systems), and $A_u \in \mathbb{R}^{n_u \times n_u}$ is anti-Hurwitz (continuous-time systems) or anti-Schur (discrete-time systems). Note that A has no eigenvalues on the stability boundary[2].

Assumption 2: Σ satisfies

$$\sigma_1 \le \sigma_2 \le \cdots \le \sigma_m. \tag{7}$$

Assumption 3: (A_u, B_u) is controllable (continuous-time) or reachable (discrete-time).

Note that if A has no eigenvalues on the stability boundary, one can always use appropriate coordination transformation and renumber the control input such that (6) and (7) hold. The controllability or reachability of (A_u, B_u) is essential for the stabilization problem considered in this technical note. We end this section by giving a lemma which will be used frequently in the sequel.

Lemma 1 ([15]): For any positive semidefinite matrices $M, N \in \mathbb{R}^{n \times n}$, $\sum_{i=1}^n \lambda_i(M) \lambda_{n-i+1}(N) \le \operatorname{tr}(MN) \le \sum_{i=1}^n \lambda_i(M) \lambda_i(N)$.

III. CONTINUOUS-TIME CASE

A. Single Encoder and Inequalities for Stabilizability

In this subsection, we consider the case without decoder, i.e., $K_d = I$, and derive necessary/sufficient conditions for stabilizability. Direct manipulations give the closed-loop system

$$\mathcal{T}_{sn} : \quad \delta x_{cl}(t) = A_{cl} x_{cl}(t) + B_{cl} \Sigma n(t), \ s(t) = C_{cl} x_{cl}(t) \tag{8}$$

where $x_{cl}(t) = \begin{bmatrix} x^T(t) & x_e^T(t) \end{bmatrix}^T$ and

$$A_{cl} = \begin{bmatrix} A + BK_{eA} & BK_{eC} \\ K_{eB} & K_{eA} \end{bmatrix}, \ B_{cl} = \begin{bmatrix} B \\ 0 \end{bmatrix},$$
$$C_{cl} = \begin{bmatrix} K_{eD} & K_{eC} \end{bmatrix}. \tag{9}$$

Thus, the $\mathcal{H}_2$ norm from n to s, provided that A_{cl} is Hurwitz, can be evaluated as $\|\mathcal{T}_{sn}\|_2^2 = \operatorname{tr}(\Sigma B_{cl}^T Q B_{cl} \Sigma)$, where Q satisfies

$$QA_{cl} + A_{cl}^T Q + C_{cl}^T C_{cl} = 0. \tag{10}$$

[2]"Stability boundary" stands for the imaginary axis for continuous-time systems and the unit circle for discrete-time systems.

The following theorem establishes a relationship between stabilizability and transmitted power.

Theorem 1: The system in (1), subject to Assumptions 1–3 and power constraint (5), is stabilizable via an encoder in (2) only if

$$2 \sum_{i=1}^{n_u} \mathrm{Re}\,\{p_i\} < \frac{\mathcal{P}}{\sigma_1^2} \qquad (11)$$

where $\mathrm{spec}\,(A_u) = \{p_i : i = 1, 2, \ldots, n_u\}$. Conversely, suppose that

$$2 \sum_{i=1}^{n_u} \mathrm{Re}\,\{p_i\} < \frac{\mathcal{P}}{\sigma_m^2} \qquad (12)$$

then the system in (1) is stabilizable, and the static encoder $K_e = -\begin{bmatrix} B_u^T P_u & 0 \end{bmatrix}$ is optimal[3].

Proof: First, lower and upper bounds on the infimum of $\|\mathcal{T}_{sn}\|_2^2$ achieved by a *stabilizing* encoder will be derived. Since the Hamiltonian matrix $\begin{bmatrix} A_u & -B_u B_u^T \\ 0 & -A_u^T \end{bmatrix}$ has no eigenvalues on the $j\omega$-axis, and $(A_u, -B_u B_u^T)$ is controllable (by the controllability of (A_u, B_u)), the Riccati equation $P_u A_u + A_u^T P_u - P_u B_u B_u^T P_u = 0$ has a unique symmetric solution P_u such that $A_u - B_u B_u^T P_u$ is Hurwitz. The fact that $(0, A_u)$ has no stable unobservable mode further implies that $P_u > 0$ [16, Theorem 13.7]. Now, define $\bar{P} \triangleq \mathrm{diag}(P_u, 0)$, where the partition is compatible with (6). It is easy to show that

$$\bar{P} A + A^T \bar{P}^T - \bar{P} B B^T \bar{P} = 0. \qquad (13)$$

Furthermore, define $P \triangleq \mathrm{diag}(\bar{P}, 0)$, where the partition is compatible with (9). Then, simple algebraic manipulations together with (10) and (13) give that

$$(Q - P) A_{cl} + A_{cl}^T (Q - P) + \left(C_{cl} + B_{cl}^T P \right)^T \left(C_{cl} + B_{cl}^T P \right) = 0.$$

Since A_{cl} is Hurwitz, one has that $Q - P \geq 0$, which implies that $\|\mathcal{T}_{sn}\|_2^2 \geq \mathrm{tr}\left(\Sigma B_{cl}^T P B_{cl} \Sigma \right)$. The equality can be achieved by setting $C_{cl} = -B_{cl}^T P$, for which the corresponding encoder becomes $s(t) = -\begin{bmatrix} B_u^T P_u & 0 \end{bmatrix} x(t)$, and A_{cl} remains Hurwitz due to the stability of $A_u - B_u B_u^T P_u$. With this relationship, one has that

$$\inf_{K_e \text{is stabilizing}} \|\mathcal{T}_{sn}\|_2^2 = \mathrm{tr}\left(\Sigma B_{cl}^T P B_{cl} \Sigma \right)$$
$$= \mathrm{tr}\left(\Sigma B_u^T P_u B_u \Sigma \right).$$

Noting that P_u satisfies the Riccati equation mentioned above, one obtains

$$\mathrm{tr}\left(B_u^T P_u B_u \right) = \mathrm{tr}\left(P_u^{1/2} B_u B_u^T P_u^{1/2} \right)$$
$$= 2 \mathrm{tr}\left(P_u^{-1/2} A_u^T P_u^{1/2} \right) = 2 \sum_{i=1}^{n_u} \mathrm{Re}\,\{p_i\}.$$

Therefore, applying Lemma 1 to matrix $\left(B_u^T P_u B_u \right) \Sigma^2$ yields that

$$2 \sigma_1^2 \sum_{i=1}^{n_u} \mathrm{Re}\,\{p_i\} \leq \inf_{K_e \text{is stabilizing}} \|\mathcal{T}_{sn}\|_2^2 \leq 2 \sigma_m^2 \sum_{i=1}^{n_u} \mathrm{Re}\,\{p_i\}. \qquad (14)$$

If the system in (1) is stabilizable, then from this inequality it follows that (11) holds. If (12) holds, then, from this inequality, there must exist an encoder such that the closed-loop system is stable and $\mathrm{POW}\,(s) < \mathcal{P}$. ∎

Corollary 1: If the system in (1), subject to Assumptions 1–3 and power constraint (5), is stabilizable for $\Sigma = \epsilon I$, it is stabilizable for any $\Sigma \leq \epsilon I$ as well. If the system is not stabilizable for $\Sigma = \epsilon I$, then it is not stabilizable for any $\Sigma \geq \epsilon I$ as well.

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[3]The "optimality" considered here means that the required transmitted power is minimal.

Proof: Stabilizability for $\Sigma = \epsilon I$ is equivalent to $2 \epsilon^2 \sum_{i=1}^{n_u} \mathrm{Re}\,\{p_i\} = \inf_{K_e \text{is stabilizing}} \|\mathcal{T}_{sn}\|_2^2 < \mathcal{P}$. This implies that, for $\Sigma \leq \epsilon I$, $2 \sum_{i=1}^{n_u} \mathrm{Re}\,\{p_i\} < \mathcal{P}/\sigma_m^2$, which by (12) implies the stabilizability. The proof of instabilizability is similar. ∎

The lower bound in (14) is closely related to the Shannon limit of parallel Gaussian channels, as will be clear later, but generally unachievable. It is natural to ask whether it is possible to make the bound always achievable. In the next subsection, we shall give a positive answer to this question, and show that adding a static decoder after receiving signals from the channels is sufficient for this.

B. Achievability of Optimal Transmitted Power via Encoder Plus Decoder

We first restrict attention to static decoders. Generally speaking, simultaneous design of K_e and K_d may be complex, since it is equivalent to an $\mathcal{H}_2$ problem subject to a structural constraint. However, the weak separation property that the optimal encoder is uniquely determined if a decoder is given enables one to split the design into two steps: K_d first and K_e later. In the remaining part of this subsection, two achievability lemmas will be given first, and then a theorem showing that the lower bound in (14) is always achievable will be established. Before proceeding, define

$$\boldsymbol{M}_d \triangleq \left\{ k \mid \sigma_k^2 = \sigma_1^2, \, k = 1, 2, \ldots, m \right\}.$$

Lemma 2 (Sufficient Condition for Achievability): If there exists a K_d such that $(B_u K_d)_i = 0$ for $i \notin \boldsymbol{M}_d$, and $(A_u, B_u K_d)$ is controllable, then the lower bound in (14) is achievable.

Proof: The controllability of $(A_u, B_u K_d)$ ensures the existence of the optimal encoder $s(t) = -\begin{bmatrix} B_u^T P_u & 0 \end{bmatrix} x(t)$, and the corresponding transmitted power is $\mathrm{tr}\left(K_d^T B_u^T P_u B_u K_d \Sigma^2 \right)$. Noting that the ith diagonal position, $i \notin \boldsymbol{M}_d$, of $K_d^T B_u^T P_u B_u K_d$ is zero, one has that

$$\mathrm{tr}\left(K_d^T B_u^T P_u B_u K_d \Sigma^2 \right) = \sigma_1^2 \mathrm{tr}\left(K_d^T B_u^T P_u B_u K_d \right)$$
$$= 2 \sigma_1^2 \sum_{i=1}^{n_u} \mathrm{Re}\,\{p_i\}. \qquad \blacksquare$$

Lemma 3 (Necessary Condition for Achievability): If the lower bound in (14) is achievable by some K_d and corresponding K_e, then $(A_u, B_u K_d)$ is controllable and $(B_u K_d)_i = 0$ for $i \notin \boldsymbol{M}_d$.

Proof: Represent $K_d^T B_u^T P_u B_u K_d$ as

$$K_d^T B_u^T P_u B_u K_d = \begin{bmatrix} z_1 & * & \cdots & * \\ * & z_2 & \cdots & * \\ \vdots & \vdots & \ddots & * \\ * & * & * & z_m \end{bmatrix}. \qquad (15)$$

Obviously, $z_j \geq 0, j = 1, 2, \ldots, m$, and $\mathrm{tr}\left(\Sigma K_d^T B_u^T P_u B_u K_d \Sigma \right) = \sum_{i=1}^{m} z_i \sigma_i^2$. If there exists $i \notin \boldsymbol{M}_d \neq \{1, 2, \ldots, m\}$ such that $z_i \neq 0$, then $\Sigma \neq \sigma I$ for some σ, and thus $\sum_{i=1}^{m} z_i \sigma_i^2 > \sigma_1^2 \sum_{i=1}^{m} z_i = 2 \sigma_1^2 \sum_{i=1}^{n_u} \mathrm{Re}\,\{p_i\}$, which contradicts with the fact that $\sum_{i=1}^{m} z_i \sigma_i^2 = 2 \sigma_1^2 \sum_{i=1}^{n_u} \mathrm{Re}\,\{p_i\}$. Hence $z_i = 0$ for $i \notin \boldsymbol{M}_d$. On the other hand, $z_i = (B_u K_d)_i^T P_u (B_u K_d)_i$. Therefore, $(B_u K_d)_i = 0$ for $i \notin \boldsymbol{M}_d$ due to $P_u > 0$. If $\boldsymbol{M}_d = \{1, 2, \ldots, m\}$, then $\Sigma = \sigma I$ for some σ, and there is no $i \notin \boldsymbol{M}_d$. The result is obvious. In addition, the stability of $A_u + B_u K_d K_e$ implies that $(A_u, B_u K_d)$ is stabilizable, and thus $(A_u, B_u K_d)$ is controllable due to $\mathrm{Re}\,\{p_i\} > 0$. ∎

These two achievability lemmas establish a necessary and sufficient condition for stabilizability with optimal transmitted power, and show that optimal stabilization can be achieved only when all subchannels with noise variance strictly larger than σ_1^2 are abandoned. As will be shown later, this transmission scheme also maximizes the mutual information between channel input and output, and therefore achieves the Shannon limit of (continuous-time) parallel Gaussian channels.

Theorem 2: For the system in (1), subject to Assumptions 1–3 and power constraint (5), the lower bound in (14) is always achievable by some static K_d and K_e.

Proof: Since (A_u, B_u) is controllable, for any left eigenvector v_i of A_u such that $v_i^* A_u = v_i^* p_i$, $i = 1, 2, \ldots, l$, $v_i^* B_u \neq 0$. Thus, it suffices to prove that there exists a K_d such that $v_i^* B_u K_d \neq 0$ for any left eigenvector v_i. Choose an arbitrary nonzero vector h_0, and write $\begin{bmatrix} v_1 & v_2 & \cdots & v_l \end{bmatrix}^* B_u h_0 = \begin{bmatrix} w_1 & w_2 & \cdots & w_l \end{bmatrix}^*$. If $w_i = 0$ for some i, then perturb h_0 to h_1 such that $w_i \neq 0$ and other nonzero w_i remains nonzero. This is always possible, since $v_i^* B_u \neq 0$. Repeat this process until an h_q is obtained such that all w_i are nonzero. Construct K_d such that $(K_d)_i = 0$, $i \notin \boldsymbol{M}_d$, and $(K_d)_i = h_q$, $i \in \boldsymbol{M}_d$. From the property of h_q, it follows that $v_i^* B_u K_d \neq 0$, and thus $(A_u, B_u K_d)$ is controllable. Applying Lemma 2 yields the desired result. ∎

It is noted that the lower bound in (14) is solely related to the poles in $\mathbb{C}^+$ and the minimal noise variance. If one connects a decoder in (3) to a plant in (1), one could obtain an equivalent plant with the realization $\left(\begin{bmatrix} A & B K_{dC} \\ 0 & K_{dA} \end{bmatrix}, \begin{bmatrix} B K_{dD} \\ K_{dB} \end{bmatrix}, I \right)$. The fact that A_{new} includes all unstable poles of A together with Theorem 2 indicates that dynamic decoders are unnecessary for the stabilization problem considered here.

C. Channel Capacity and Stabilizability

The information capacity of the power constrained parallel Gaussian channel [17] is

$$\mathcal{C} \triangleq \max_{\text{POW}(s) \leq \mathcal{P}} I(s; r) = \max_{\text{POW}(s) \leq \mathcal{P}} \sum_{i=1}^{m} \frac{\mathcal{P}_i}{2\sigma_i^2} \log_2 e$$

where $I(s; r)$ is the mutual information of s and r, $\mathcal{P}_i \geq 0$ is transmitted power of the ith subchannel, and $\sum_{i=1}^{m} \mathcal{P}_i \leq \mathcal{P}$. Noting that $\sum_{i=1}^{m} (\mathcal{P}_i / 2\sigma_i^2) \log_2 e \leq \sum_{i=1}^{m} (\mathcal{P}_i / 2\sigma_1^2) \log_2 e \leq (\mathcal{P} / 2\sigma_1^2) \log_2 e$ where the first equality is achieved when $\mathcal{P}_i = 0$, $i \notin \boldsymbol{M}_d$, one has that

$$\mathcal{C} = \frac{\mathcal{P}}{2\sigma_1^2} \log_2 e \quad \frac{\text{bits}}{\text{second}}.$$

This together with Theorems 1 and 2 yields the following result.

Theorem 3: The system in (1), subject to Assumptions 1–3 and power constraint (5), is stabilizable via a pair of encoder K_e and decoder K_d if and only if the channel capacity satisfies

$$\mathcal{C} > \left(\sum_{i=1}^{n_u} \text{Re}\{p_i\} \right) \log_2 e.$$

All the results mentioned above manifest clearly that memoryless LTI encoders/decoders are sufficient for stabilizing a continuous-time linear systems while achieving the lower bound on required channel capacity. The proof of Theorem 2 has suggested an approach to finding a desired K_d, and, once K_d is fixed, the optimal K_e can be obtained readily by solving corresponding Riccati equation. It should be emphasized here that, unlike the single-input case, both K_e and K_d in general must be utilized in order to achieve the optimal transmitted power.

IV. DISCRETE-TIME CASE

A. Single Encoder and Inequalities for Stabilizability

Similar to the continuous-time case, the closed-loop system with a single encoder is described by (8), and the $\mathcal{H}_2$ norm from n to s, provided that A_{cl} is Schur, can be evaluated as $\|\mathcal{T}_{sn}\|_2^2 = \text{tr}(\Sigma B_{cl}^T Q B_{cl} \Sigma)$, where Q satisfies $A_{cl}^T Q A_{cl} - Q + C_{cl}^T C_{cl} = 0$. Before presenting the main result of this subsection, define $f(s) \triangleq \sqrt[s]{\prod_{i=1}^{s} \sigma_i^2 \prod_{k=1}^{n_u} |\phi_k|^2}$, where

$\text{spec}(A_u) = \{\phi_k : k = 1, 2, \ldots, n_u\}$, and $1 \leq s \leq m$ is a natural number.

Lemma 4: Regarding $f(s)$, one has that
a) If $\sigma_r^2 < f(r)$ for some $r \in \mathbb{N}$, then, for any $s \in \mathbb{N}$, $s < r$: $\sigma_s^2 < f(s)$.
b) $f(s) \leq \sigma_1^2 \prod_{k=1}^{n_u} |\phi_k|^2$, $\forall s \in \mathbb{N}$ such that $\sigma_s^2 < f(s)$.

Proof:
a) $\sigma_r^2 < f(r)$ implies that $(\sigma_r^2)^r < \prod_{i=1}^{r} \sigma_i^2 \prod_{k=1}^{n_u} |\phi_k|^2$, and thus $(\sigma_{r-1}^2)^{r-1} \leq (\sigma_r^2)^{r-1} < \prod_{i=1}^{r-1} \sigma_i^2 \prod_{k=1}^{n_u} |\phi_k|^2$, which means that $\sigma_{r-1}^2 < f(r-1)$. Repeating this argument gives the result.
b) According to a), there must exist an r such that $\sigma_s^2 < f(s)$, $\forall s \leq r$, and $\sigma_s^2 \geq f(s)$, $\forall s > r$. From this observation, it suffices to prove b) for $1 \leq s \leq r$. When $s = 1$, this is obvious. Assume that this holds for $s = l$, i.e., $f(l) \leq \sigma_1^2 \prod_{k=1}^{n_u} |\phi_k|^2$. Then, for $s = l + 1 \leq r$

$$f(l+1) = \sqrt[l+1]{f^l(l)} \sqrt[l+1]{\sigma_{l+1}^2}$$

$$\leq \sqrt[l+1]{\left(\sigma_1^2 \prod_{k=1}^{n_u} |\phi_k|^2 \right)^l} \sqrt[l+1]{\sigma_{l+1}^2}$$

$$< \sqrt[l+1]{\left(\sigma_1^2 \prod_{k=1}^{n_u} |\phi_k|^2 \right)^l} \sqrt[l+1]{f(l+1)}$$

where the second inequality is due to a) and $l + 1 \leq r$. Multiplying both sides of the above inequality by $f(l+1)^{-1/(l+1)}$ yields that b) holds for $s = l + 1$. The result follows by induction. ∎

Theorem 4: The system in (1), subject to Assumptions 1–3 and power constraint (5), is stabilizable via an encoder in (2) only if

$$\sqrt[v]{\prod_{k=1}^{n_u} |\phi_k|^2} < \frac{\mathcal{P} + \sum_{j=1}^{v} \sigma_j^2}{v \sqrt[v]{\prod_{i=1}^{v} \sigma_i^2}} \tag{16}$$

where $v \leq m$ is the largest integer such that $\sigma_v^2 < f(v)$. Conversely, suppose that

$$\prod_{k=1}^{n_u} |\phi_k|^2 < \frac{\mathcal{P}}{\sigma_m^2} + 1 \tag{17}$$

then the system in (1) is stabilizable, and the static encoder $K_e = -\left[(I + B_u^T P_u B_u)^{-1} B_u^T P_u A_u \quad 0 \right]$ is optimal.

Proof: Since the Simplectic matrix $\begin{bmatrix} A_u & -B_u B_u^T (A_u^T)^{-1} \\ 0 & (A_u^T)^{-1} \end{bmatrix}$ has no eigenvalues on the unit circle, and $(A_u, -B_u B_u^T)$ is reachable (by the reachability of (A_u, B_u)), the Riccati equation $A_u^T P_u A_u - P_u - A_u^T P_u B_u (I + B_u^T P_u B_u)^{-1} B_u^T P_u A_u = 0$ has a unique symmetric solution such that $A_u - B_u (I + B_u^T P_u B_u)^{-1} B_u^T P_u A_u$ is Schur. In addition, the fact that $(0, A_u)$ has no stable unobservable mode implies that $P_u > 0$ [16, Theorem 21.11]. Now, define $\bar{P} \triangleq \text{diag}(P_u, 0)$ and $P \triangleq \text{diag}(\bar{P}, 0)$, where the partition is compatible with (6) and (9), respectively. By following a similar line as used in the proof of Theorem 1, one obtains that

$$A_{cl}^T (Q - P) A_{cl} - (Q - P)$$
$$+ (C_{cl} + F)^T (I + B_{cl}^T P B_{cl}) (C_{cl} + F) = 0$$

where $F = \left[(I + B^T \bar{P} B)^{-1} B^T \bar{P} A \quad 0 \right]$. Since A_{cl} is Schur, one obtains that $Q - P \geq 0$, which results in $\|\mathcal{T}_{sn}\|_2^2 \geq \text{tr}(\Sigma B_{cl}^T P B_{cl} \Sigma)$. The equality can be achieved by setting $K_{eD} = -\left[(I + B_u^T P_u B_u)^{-1} B_u^T P_u A_u \quad 0 \right]$, $K_{eC} = 0$, for which A_{cl} remains Schur due to the stability of $A_u - B_u (I + B_u^T P_u B_u)^{-1} B_u^T P_u A_u$. Noting that P_u satisfies the Riccati equation mentioned above, the spectrum of the closed-loop

system can be decomposed as

$$\text{spec}\left(A - B\left(I + B^T \bar{P} B\right)^{-1} B^T \bar{P} A\right)$$
$$= \text{spec}\left(P_u^{-1} A_u^{-T} P_u\right) \cup \text{spec}\left(A_s\right)$$
$$= \text{spec}\left(A_u^{-1}\right) \cup \text{spec}\left(A_s\right). \tag{18}$$

According to the Matrix Inverse Lemma [18], $A - B\left(I + B^T \bar{P} B\right)^{-1} B^T \bar{P} A = \left(I + BB^T \bar{P}\right)^{-1} A$. This together with (18) implies that

$$\det\left(\left(I + B_u^T P_u B_u\right)^{-1}\right)\det\left(A_u\right)\det\left(A_s\right)$$
$$= \det\left(A_u^{-1}\right)\det\left(A_s\right)$$

and thus $\det\left(I + B_u^T P_u B_u\right) = \prod_{k=1}^{n_u} |\phi_k|^2$. With this and Lemma 1, one obtains that

$$\sum_{i=1}^m \sigma_i^2 \lambda_{m-i+1}\left(B_u^T P_u B_u\right) \leq \inf_{\mathcal{K}_e \text{is stabilizing}} \|\mathcal{T}_{sn}\|_2^2$$
$$\leq \sum_{i=1}^m \sigma_i^2 \lambda_i\left(B_u^T P_u B_u\right) \tag{19}$$

where $\lambda_i\left(B_u^T P_u B_u\right) \geq 0$ and $\prod_{i=1}^m \left(1 + \lambda_i\left(B_u^T P_u B_u\right)\right) = \prod_{k=1}^{n_u} |\phi_k|^2$. Let us now evaluate both sides of (19).

1) By using the arithmetic and geometric mean inequality, one has that

$$\sum_{i=1}^m \sigma_i^2 \lambda_{m-i+1}\left(B_u^T P_u B_u\right)$$
$$\geq m\sqrt[m]{\prod_{i=1}^m \left\{\sigma_i^2 \left[\lambda_{m-i+1}\left(B_u^T P_u B_u\right) + 1\right]\right\}} - \sum_{i=1}^m \sigma_i^2$$
$$= mf\left(m\right) - \sum_{i=1}^m \sigma_i^2$$

where the equality is achieved when

$$\sigma_1^2\left(1 + \lambda_m\left(B_u^T P_u B_u\right)\right)$$
$$= \sigma_2^2\left(1 + \lambda_{m-1}\left(B_u^T P_u B_u\right)\right)$$
$$= \ldots = \sigma_m^2\left(1 + \lambda_1\left(B_u^T P_u B_u\right)\right) = f\left(m\right).$$

However, since $\lambda_i\left(B_u^T P_u B_u\right)$ cannot be negative, the lower bound may not be achieved if σ_i^2 is too large. More specifically, if $\sigma_m^2 \geq f\left(m\right)$, then the optimal $\lambda_1\left(B_u^T P_u B_u\right)$ is 0. Repeat this process until $\sigma_v^2 < f\left(v\right)$ for some v. Then, the possibly achieved bound is

$$\sum_{i=1}^m \sigma_i^2 \lambda_{m-i+1}\left(B_u^T P_u B_u\right) \geq vf\left(v\right) - \sum_{i=1}^v \sigma_i^2. \tag{20}$$

One may worry about that whether $\lambda_i\left(B_u^T P_u B_u\right)$ would have to exceed its maximum $\prod_{k=1}^{n_u} |\phi_k|^2 - 1$ if σ_i^2 is very small. Applying Lemma 4 yields that $\forall i \leq v$, $\sigma_i^2 < f\left(i\right)$ and $f\left(i\right)/\sigma_i^2 \leq \sigma_1^2 \prod_{k=1}^{n_u} |\phi_k|^2 /\sigma_i^2 = \prod_{k=1}^{n_u} |\phi_k|^2$, and thus this situation would never occur.

2) It can be shown by using the Lagrangian multiplier method that $\sum_{i=1}^{n_u} \sigma_i^2 \lambda_i\left(B_u^T P_u B_u\right)$ has only one minimum point, and thus $\sum_{i=1}^{n_u} \sigma_i^2 \lambda_i\left(B_u^T P_u B_u\right)$ takes its maximum at boundary points. Simple calculations give that

$$\sum_{i=1}^m \sigma_i^2 \lambda_i\left(B_u^T P_u B_u\right) \leq \sigma_m^2\left(\prod_{k=1}^{n_u} |\phi_k|^2 - 1\right) \tag{21}$$

where the equality holds when $\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = 0$. The results follows immediately from the two bounds. ∎

Corollary 2: If the system in (1), subject to Assumptions 1–3 and power constraint (5), is stabilizable for $\Sigma = \epsilon I$, it is stabilizable for any $\Sigma \leq \epsilon I$ as well. If the system is not stabilizable for $\Sigma = \epsilon I$, then it is not stabilizable for any $\Sigma \geq \epsilon I$ as well.

Proof: Stabilizability for $\Sigma = \epsilon I$ is equivalent to $\epsilon^2 \sum_{i=1}^m \lambda_i\left(B_u^T P_u B_u\right) = \inf_{\mathcal{K}_e \text{is stabilizing}} \|\mathcal{T}_{sn}\|_2^2 < \mathcal{P}$. This implies that, for $\Sigma \leq \epsilon I$

$$\inf_{\mathcal{K}_e \text{is stabilizing}} \|\mathcal{T}_{sn}\|_2^2 = \text{tr}\left(\Sigma B_u^T P_u B_u \Sigma\right)$$
$$\leq \epsilon^2 \sum_{i=1}^m \lambda_i\left(B_u^T P_u B_u\right) < \mathcal{P}$$

which implies the stabilizability. The proof of instabilizability is similar. ∎

The lower bound in (20), similar to the continuous-time case, is closely associated with Shannon limit, but is generally unachievable by a static encoder. One may speculate from the results for continuous-time systems that the bound can always be achieved by adding a decoder. Unfortunately, this conjecture is not true, as will be revealed in the next subsection.

B. Achievability of Optimal Transmitted Power via Encoder Plus Decoder

We first restrict attention to static decoders. Before proceeding, define

$$\mu_i^* = \begin{cases} \frac{f(v)}{\sigma_i^2} - 1 & i \leq v \\ 0 & i > v \end{cases} \tag{22}$$

where v is defined as in Theorem 4. In fact, μ_i^* is the optimal signal-to-noise ratio for the ith subchannel when the lower bound on channel capacity is achieved. This will be clear along with the development. In the subsequent derivations, unless stated particularly, P_u denotes the solution of

$$A_u^T P_u A_u - P_u - A_u^T P_u B_u K_d$$
$$\times \left(I + K_d^T B_u^T P_u B_u K_d\right)^{-1} K_d^T B_u^T P_u A_u = 0.$$

Lemma 5 (Sufficient Condition for Achievability): If there exists a K_d such that $(A_u, B_u K_d)$ is reachable and

$$(B_u K_d)_i^T P_u (B_u K_d)_i = \mu_i^*$$

then the lower bound in (20) is achievable.

Proof: The reachability of $(A_u, B_u K_d)$ ensures the existence of optimal encoder, and thus the corresponding transmitted power is $\text{tr}\left(K_d^T B_u^T P_u B_u K_d \Sigma^2\right) = \sum_{i=1}^m \sigma_i^2 \mu_i^*$. The proof is completed by employing (22). ∎

Lemma 6 (Necessary Condition for Achievability): If the lower bound in (20) is achievable by some K_d and corresponding K_e, then $(A_u, B_u K_d)$ is reachable and $K_d^T B_u^T P_u B_u K_d = \text{diag}\{\mu_i^*\}$.

Proof: The reachability of $(A_u, B_u K_d)$ follows from the stability of the closed-loop system. Write $K_d^T B_u^T P_u B_u K_d$ as (15). Obviously, $z_j \geq 0$, $j = 1, 2, \ldots, m$, and

$$\text{tr}\left(\Sigma K_d^T B_u^T P_u B_u K_d \Sigma\right) = \sum_{i=1}^m z_i \sigma_i^2$$
$$= v\sqrt[v]{\prod_{i=1}^v \sigma_i^2 \prod_{k=1}^{n_u} |\phi_k|^2} - \sum_{i=1}^v \sigma_i^2.$$

From this

$$vf\left(v\right) + \sum_{i=v+1}^m \sigma_i^2 = \sum_{i=1}^m \left(z_i + 1\right)\sigma_i^2. \tag{23}$$

Let us now evaluate the minimum of the right hand side of (23)

$$\sum_{i=1}^{m}(z_i+1)\sigma_i^2 \geq m\sqrt[m]{\prod_{i=1}^{m}\sigma_i^2(z_i+1)}$$

$$\geq m\sqrt[m]{\prod_{i=1}^{m}\sigma_i^2\det\left(I+K_d^T B_u^T P_u B_u K_d\right)}$$

$$= mf(m)$$

where the first and the second inequalities come from the arithmetic and geometric means inequality and Hadamard's inequality, respectively. The equality holds if and only if $K_d^T B_u^T P_u B_u K_d$ is diagonal (Hadamard's inequality) and $(z_1+1)\sigma_1^2 = (z_2+1)\sigma_2^2 = \cdots = (z_m+1)\sigma_m^2 = f(m)$ (arithmetic and geometric means inequality). However, it has been pointed out in the proof of Theorem 4 that $\sigma_m^2 \geq \sqrt[m]{\prod_{i=1}^{m}\sigma_i^2\prod_{k=1}^{n_u}|\phi_k|^2}$. Hence, z_m cannot achieve its globally optimal value, and the optimal z_m is $z_m = 0$. Repeating this argument gives that $\sum_{i=1}^{m}(z_i+1)\sigma_i^2 \geq vf(v)+\sum_{i=v+1}^{m}\sigma_i^2$ where the equality holds only if $K_d^T B_u^T P_u B_u K_d$ is diagonal and $(z_1+1)\sigma_1^2 = (z_2+1)\sigma_2^2 = \cdots = (z_v+1)\sigma_v^2 = f(v)$. From this and (23), it follows that $z_i = f(v)/\sigma_i^2-1 = \mu_i^*$, $i = 1, 2, \ldots, v$ and $z_i = 0$, $i = v+1, v+2, \ldots, m$. This completes the proof. ∎

The following theorem and its corollary indicate that the lower bound in (20) is not always achievable by LTI encoders/decoders.

Theorem 5: The lower bound in (20) is achievable by some K_d and corresponding K_e if and only if $n_u \geq v$ and there exist unitary matrices M and N such that the following constrained Sylvester equation has a nonsingular solution Z:

$$A_u Z = Z(M\Lambda N) \tag{24a}$$

$$\left(B_u B_u^\dagger - I\right) Z\bar{\Lambda} = 0 \tag{24b}$$

where $\Lambda = \left(I+\bar{\Lambda}\bar{\Lambda}^T\right)^{1/2}$ and $\bar{\Lambda} = [\,\mathrm{diag}\{\sqrt{\mu_i^*}\} \quad 0\,]^T$ $(n_u \geq m)$ or $\mathrm{diag}\{\sqrt{\mu_i^*}\}$ $(n_u < m)$.

Proof: From Lemmas 5, 6, and $0 < P_u \in \mathbb{R}^{n_u \times n_u}$, the lower bound in (20) is achievable if and only if $n_u \geq v$ and there exists a $P_u > 0$ such that

$$A_u^T P_u A_u - P_u$$
$$-A_u^T P_u B_u K_d\left(I+K_d^T B_u^T P_u B_u K_d\right)^{-1}K_d^T B_u^T P_u A_u = 0$$

and $K_d^T B_u^T P_u B_u K_d = \mathrm{diag}\{\mu_i^*\}$. By letting $X_u = P_u^{-1}$, and applying the Matrix Inverse Lemma [18], one obtains an equivalent condition that

$$A_u X_u A_u^T - X_u = B_u K_d K_d^T B_u^T \tag{25a}$$

$$K_d^T B_u^T X_u^{-1} B_u K_d = \mathrm{diag}\{\mu_i^*\}. \tag{25b}$$

Noting that $X_u > 0$ if and only if there exists a nonsingular matrix Z such that $X_u = ZZ^T$. With this decomposition, (25) is equivalent to $\left(Z^{-1}A_u Z\right)\left(Z^{-1}A_u Z\right)^T = I+\left(Z^{-1}B_u K_d\right)\left(Z^{-1}B_u K_d\right)^T$ and $\left(Z^{-1}B_u K_d\right)^T\left(Z^{-1}B_u K_d\right) = \mathrm{diag}\{\mu_i^*\}$, which, by using singular value decomposition, implies that

$$\left(Z^{-1}A_u Z\right)\left(Z^{-1}A_u Z\right)^T = \bar{U}\Lambda^2\bar{U}^T \tag{26a}$$

$$Z^{-1}B_u K_d = \bar{U}\bar{\Lambda}\bar{V} \tag{26b}$$

for some unitary matrices $\bar{U}$ and $\bar{V}$. Using singular value decomposition again yields that (26) is equivalent to

$$\bar{U}^T Z^{-1} A_u Z\bar{U} = M\Lambda N \tag{27a}$$

$$B_u\left(K_d\bar{V}^T\right) = \left(Z\bar{U}\right)\bar{\Lambda} \tag{27b}$$

for some unitary matrices M and N. By re-defining Z and K_d as $Z\bar{U} \rightarrow Z$ and $K_d\bar{V}^T \rightarrow K_d$, and noting that the equation $B_u\left(K_d\bar{V}^T\right) = \left(Z\bar{U}\right)\bar{\Lambda}$ has a solution $K_d\bar{V}^T$ if and only if $B_u B_u^\dagger\left(Z\bar{U}\right)\bar{\Lambda} = \left(Z\bar{U}\right)\bar{\Lambda}$ [18], (27) is further equivalent to (24). ∎

Corollary 3: The lower bound in (20) is achievable only if

$$\rho(A_u) \leq \frac{\sqrt{f(v)}}{\sigma_1} \tag{28}$$

$$\left|\sum_{k=1}^{n_u}\phi_k\right| \leq \sqrt{f(v)}\sum_{i=1}^{v}\frac{1}{\sigma_i}+n_u-v \tag{29}$$

and there exists a matrix Y such that

$$Y-\left(B_u B_u^\dagger-I\right)^\dagger\left(B_u B_u^\dagger-I\right)Y\bar{\Lambda}\bar{\Lambda}^\dagger > 0. \tag{30}$$

Proof: Equation (28) follows immediately from the fact that A_u and $M\Lambda N$ have the same spectrum and $\rho(M\Lambda N) = \rho(N M\Lambda) \leq \|N M\Lambda\| = \sqrt{f(v)}/\sigma_1$. Equation (29) can be proved by noting $\sum_{k=1}^{n_u}\phi_k = \mathrm{tr}\left(Z^{-1}A_u Z\right) = \mathrm{tr}(N M\Lambda)$ and $-\mathrm{tr}(\Lambda) \leq \mathrm{tr}(N M\Lambda) \leq \mathrm{tr}(\Lambda)$, which follows from the diagonal positivity of Λ and the orthogonality of $N M$. To prove (30), noting that $P_u^{-1} = X_u > 0$ in Theorem 5 can be decomposed as $X_u = Z^2$, $Z > 0$ without loss of generality, and Z satisfying (24b) can be parameterized as $Y-\left(B_u B_u^\dagger-I\right)^\dagger\left(B_u B_u^\dagger-I\right)Y\bar{\Lambda}\bar{\Lambda}^\dagger$, where Y is arbitrary, one has the result immediately. ∎

Example 1: Consider a feedback control system with $\phi_1 = 2$, $\phi_2 = 5$, and $\phi_3 = 8$. The channel noise variances are $\sigma_1^2 = 0.4$ and $\sigma_2^2 = 0.5$. Simple calculations give that $\rho(A_u) = 8$, $\left|\sum_{k=1}^{n_u}\phi_k\right| = 15$, $\sqrt{f(v)}/\sigma_1 = 3.1623$, and $\sqrt{f(v)}\sum_{i=1}^{v}(1/\sigma_i)+n_u-v = 10$. Obviously, (28) and (29) are not satisfied, and hence (24) has no nonsingular solution.

As for dynamic decoders, it is not difficult to show that, through some algebraic manipulations, they are not helpful in reducing the lower bound on transmitted power, and thus unnecessary for stabilization if the lower bound can be achieved by static encoders/decoders. However, dynamic decoders may reduce *actual* transmitted power if the lower bound cannot be achieved by static ones.

C. Channel Capacity and Stabilizability

The information capacity of the discrete-time power constrained parallel Gaussian channel [17] is

$$\mathcal{C} = \max_{\mathrm{POW}(s)\leq\mathcal{P}} I(s;r) = \max_{\mathrm{POW}(s)\leq\mathcal{P}}\frac{1}{2}\log_2\prod_{i=1}^{m}\left(1+\frac{\mathcal{P}_i}{\sigma_i^2}\right)$$

where $\mathcal{P}_i \geq 0$ is the transmitted power of the ith subchannel, and $\sum_{i=1}^{m}\mathcal{P}_i \leq \mathcal{P}$. Noting that, when $\sum_{i=1}^{m}\mathcal{P}_i = \mathcal{P}$

$$\prod_{i=1}^{m}\left(1+\frac{\mathcal{P}_i}{\sigma_i^2}\right) \leq \left(\prod_{i=1}^{m}\frac{1}{\sigma_i^2}\right)\left(\frac{\sum_{j=1}^{m}\left(\sigma_j^2+\mathcal{P}_j\right)}{m}\right)^m$$

$$= \left(\prod_{i=1}^{m}\frac{1}{\sigma_i^2}\right)\left(\frac{\sum_{j=1}^{m}\sigma_j^2+\mathcal{P}}{m}\right)^m$$

where the first equality holds when $\sigma_1^2+\mathcal{P}_1 = \sigma_2^2+\mathcal{P}_2 = \cdots = \sigma_m^2+\mathcal{P}_m = (\sum_{i=1}^{m}\sigma_i^2+\mathcal{P})/m$. If $\sigma_m^2 \geq (\sum_{i=1}^{m}\sigma_i^2+\mathcal{P})/m$, then, to maximize mutual information, $\mathcal{P}_m$ should be set to zero. Likewise, if $\sigma_{m-1}^2 \geq (\sum_{i=1}^{m-1}\sigma_i^2+\mathcal{P})/(m-1)$, $\mathcal{P}_{m-1}$ should be set to zero.

Repeat this process until $\sigma_w^2 < (\sum_{i=1}^w \sigma_i^2 + \mathcal{P})/w$. Then, the optimal power allocation is

$$\mathcal{P}_i^* = \begin{cases} \dfrac{\sum_{j=1}^w \sigma_j^2 + \mathcal{P}}{w} - \sigma_i^2 & i \le w \\ 0 & i > w \end{cases} \tag{31}$$

and, therefore, the information capacity can be evaluated as

$$\mathcal{C} = \frac{1}{2}\log_2 \frac{\left(\mathcal{P} + \sum_{j=1}^w \sigma_j^2\right)^w}{\prod_{i=1}^w \sigma_i^2} \quad \frac{\text{bits}}{\text{transmission}}.$$

The power allocation process for discrete-time parallel Gaussian channels is usually referred to as *water filling* [17] due to its similarity to distribution of water in a vessel, and is regarded as a major difference from continuous-time channels. Based on this, we have the following result.

Theorem 6: The system in (1), subject to Assumptions 1–3 and power constraint (5), is stabilizable via a pair of encoder K_e and decoder K_d only if the channel capacity satisfies

$$\mathcal{C} > \sum_{k=1}^{n_u} \log_2 |\phi_k| + \frac{v}{2}\log_2 v \triangleq \mathcal{C}_{inf}. \tag{32}$$

Proof: When the lower bound in (20) is achieved, it is easy to verify that $\sigma_v^2 < (\sum_{i=1}^v \sigma_i^2 + \mathcal{P}_{inf})/v$ and $\sigma_{v+1}^2 \ge (\sum_{i=1}^{v+1} \sigma_i^2 + \mathcal{P}_{inf})/(v+1)$, where v is defined as in Theorem 4 and $\mathcal{P}_{inf} = vf(v) - \sum_{j=1}^v \sigma_j^2$. Hence, w in (31) is equal to v, and corresponding capacity is $\mathcal{C}_{inf}$. From this, the result follows immediately. If the lower bound is unachievable, it can be shown readily that corresponding capacity $\mathcal{C}$ is greater than $\mathcal{C}_{inf}$. This completes the proof. ∎

If the lower bound in (20) is unachievable for certain discrete-time systems, $\mathcal{C}_{inf}$ is no longer sufficient for stabilizability, and extra channel capacity would be required. As for the computation of optimal or suboptimal encoders/decoders, one may transform it into a bilinear matrix inequality (BMI) optimization problem by using some techniques developed in $\mathcal{H}_2$ control, and we do not intend to include this here due to page length consideration. Further study on computation will be reported elsewhere. The following example shows that, for some systems, the lower bound in (20) is achievable by static LTI encoders/decoders.

Example 2: Consider a system with

$$A = A_u = \begin{bmatrix} 3 & 0 & 5 \\ -2 & 1 & -1 \\ 3 & 2.5 & 1 \end{bmatrix}, \quad B = B_u = \begin{bmatrix} 0.15 & 0.3 \\ -0.3 & 1 \\ 0 & -1 \end{bmatrix}.$$

The channel noise is characterized by $\Sigma = \text{diag}(2.9, 4.8)$. Simple calculations give $v = 2$, $f(2) = 410.6400$, $\mu_1^* = 47.8276$, and $\mu_2^* = 16.8229$. By solving the corresponding BMI, a desired pair of encoder/decoder can be obtained as

$$K_e = \begin{bmatrix} -38.5074 & -26.1067 & -40.8811 \\ -39.7752 & -29.2173 & -45.0964 \end{bmatrix}$$

$$K_d = \begin{bmatrix} 0.0374 & -1.4169 \\ -0.6016 & 0.2569 \end{bmatrix}.$$

The actual transmitted power is thus computed as $\text{POW}(s) = 789.8300$, which is exactly the same as $vf(v) - \sum_{i=1}^v \sigma_i^2 = 789.8300$. Furthermore, computing transmitted power of each channel

gives $\text{POW}(s_1) = 402.2300$ and $\text{POW}(s_2) = 387.6000$, which are also consistent with the water filling in (31).

V. CONCLUSION

We have investigated the state-feedback stabilization problem of multi-input LTI systems over parallel Gaussian channels subject to a total power constraint. It has been shown that to stabilize an LTI plant transmitted power must be greater than a bound related to unstable plant poles and channel noises. In terms of this, lower bounds on required channel capacity for feedback stabilization have been obtained for both continuous-time and discrete-time systems. For continuous-time systems with LTI encoders/decoders, the derived capacity bound is always tight, while, for discrete-time systems with LTI encoders/decoders, the tightness of the capacity bound is not guaranteed. In addition, to achieve the optimal transmitted power, both encoder and decoder in general must be utilized.

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