STABILIZING CONTROL FOR LINEAR SYSTEMS WITH BOUNDED PARAMETER AND INPUT UNCERTAINTY

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ABSTRACT

We consider dynamical systems with norm-bounded uncertainty in (i) the system parameters (model uncertainty) or in (ii) the input (disturbance).

For case (i), the nominal (null uncertainty) system is linear with constant matrices. Such systems with norm-bounded control as well as with a control penalty are treated. However, in the former the treatment is restricted to single input systems in companion form, and in the latter to second order systems. For case (ii), the system is linear with time-varying matrices and norm-bounded control.

Using some results from the theories of differential games and general dynamical systems, we deduce feedback controls which render the origin uniformly asymptotically stable in the large for all admissible parameter uncertainties or input disturbances; these may be both time and state dependent.

The application of the theory is illustrated by examples.

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1. INTRODUCTION

The problem of designing a feedback control for uncertain systems has been discussed in a series of articles, [1-7]. In principle, we distinguish among three types of uncertainties:

a) Uncertainty in the <u>model</u> (parameter)

b) Uncertainty in the <u>input</u> (disturbance)

c) Uncertainty in the <u>state</u> (measurement)

Here we deal only with the first two types of uncertainty, model and input uncertainties for linear† systems.

To motivate the discussion, consider an aircraft maneuvering at a high angle of attack. It is possible to describe the dynamical behavior by a set of nonlinear differential equations such that the "nominal" part is a set of linear differential equations. Often two difficulties arise:

1) The nonlinear characteristics of the parameters are known but it is impossible to find a "best" controller for achieving desired specifications, e.g. controlling the system asymptotically to rest.

2) Because of lack of experimental data, there is incomplete information about the parameter characteristics, except that their value belong to known sets.

In both cases we approach the difficulty by allowing for the "worst" nonlinear characteristics with respect to an appropriate performance index and for that nonlinearity we seek the "best" controller. This "worst case" philosophy does not imply that the "worst" situation will occur, but rather that a controller capable of achieving the desired end under the "worst" of circumstances will also do so under more favorable ones, and hence under all allowable ones.

The theory of two-person zero-sum games is employed to generate "worst case" controllers. Towards this end, an appropriate performance index is stipulated; it is to be maximized by the uncertainty and minimized by the controller, respectively. If a saddlepoint strategy pair exists, then the controller assures himself a cost (in terms of the assumed performance index) that is no greater than the saddlepoint one, no matter what the strategy of the disturbance.

2. MODEL UNCERTAINTY WITH CONTROL PENALTY

2.1 <u>Problem Statement</u>

Here we treat a class of second order dynamical systems with parameter uncertainty. Consider

†That is, when the "nominal" system (namely, the system without uncertainty) is linear.

$$\dot{x}(t) = [A_o + \sum_{i=1}^{p} A_i \; v_i(t)] \; x(t) + B \; v_{p+1}(t) \; u(t) \tag{1}$$

$$x(t_o) = x_o \; , \quad t \in [t_o, t_1]$$

where

$x(t) \in R^2$ is the state of the system at time t ;

A_i , $i = 0, 1, \ldots, p$, are constant 2×2 matrices, each containing a single non-zero element;

B is a constant $2 \times m$ matrix;

$v_i(t)$, $i = 1, \ldots, p$, with $|v_i(t)| \leq 1$, and

$v_{p+1}(t)$ with $v_{p+1}(t) \in [1,q]$, q = constant > 1 ,

are values of parameter uncertainty at time t ;

$u(t) \in R^m$ is the value of the control at time t .

We are interested in the asymptotic behavior of the system (1) under all possible uncertainties

$v(t) = (v_1(t), v_2(t), \ldots, v_{p+1}(t))'$, $t \in [t_o, \infty)$.

Since we are concerned with the asymptotic stability of the origin $x = \{0\}$, we introduce a measure of deviation from that state subject to a control penalty. That is, we introduce the performance index

$$J = \int_{t_o}^{t_1} [x'(t) \; Q \; x(t) + u'(t) \; R \; u(t)] \; dt \tag{2}$$

where

Q is a constant positive semidefinite symmetric 2×2 matrix;

R is a constant positive definite symmetric $m \times m$ matrix;

and consider the differential game with state equation (1) and cost (2) . That is, we seek a saddlepoint $(p^*(\cdot), e^*(\cdot))$ in a given class of strategies

$p(\cdot) : R^2 \times R^1 \to R^m$, $e(\cdot) : R^2 \times R^1 \to R^{p+1}$

such that

$u(t) = p(x(t),t)$, $v(t) = e(x(t),t)$

Note that we are looking for a feedback control $p^*(\cdot)$ while admitting an uncertainty $e(\cdot)$ that may depend on state and time.

Having found a saddlepoint candidate $(p^*(\cdot), e^*(\cdot))$, we inquire then under what conditions the feedback control $p^*(\cdot)$ renders $x = \{0\}$ uniformly asymptotically stable in the large (in the sense of Lyapunov) against every allowable uncertainty $e(\cdot)$.

2.2 Stability

Before discussing the asymptotic behavior of the system, we invoke necessary conditions for a saddlepoint candidate $(p^*(\cdot), e^*(\cdot))$, e.g. [8-9]. These conditions lead us to consider the following procedure, [10]:

Step 1. For each possible combination

$\{v_i^* : v_i^* = 1 \text{ or } -1 , i = 1, \ldots, p\}$

Compute

$$A = A_o + \sum_{i=1}^{p} A_i v_i^* \tag{3}$$

Let $A_{(k)}$ denote the value of A corresponding to the k-th possible combination.

Step 2. Using (3), compute $P_{(k)}$, the solution of

$$PA + A'P - PBR^{-1} B'P + Q = 0 \tag{4}$$

corresponding to the k-th possible combination of the v_i^* .

Step 3. Define $\sigma_i^k(\cdot) : R^2 \to R^1$ by

$$\sigma_i^k(x) \triangleq x' [P_{(k)} A_i + A_i' P_{(k)}] x \tag{5}$$

Step 4. Define a decomposition of R^2 by the lines given by

$$\sigma_i^k(x) = 0 \tag{6}$$

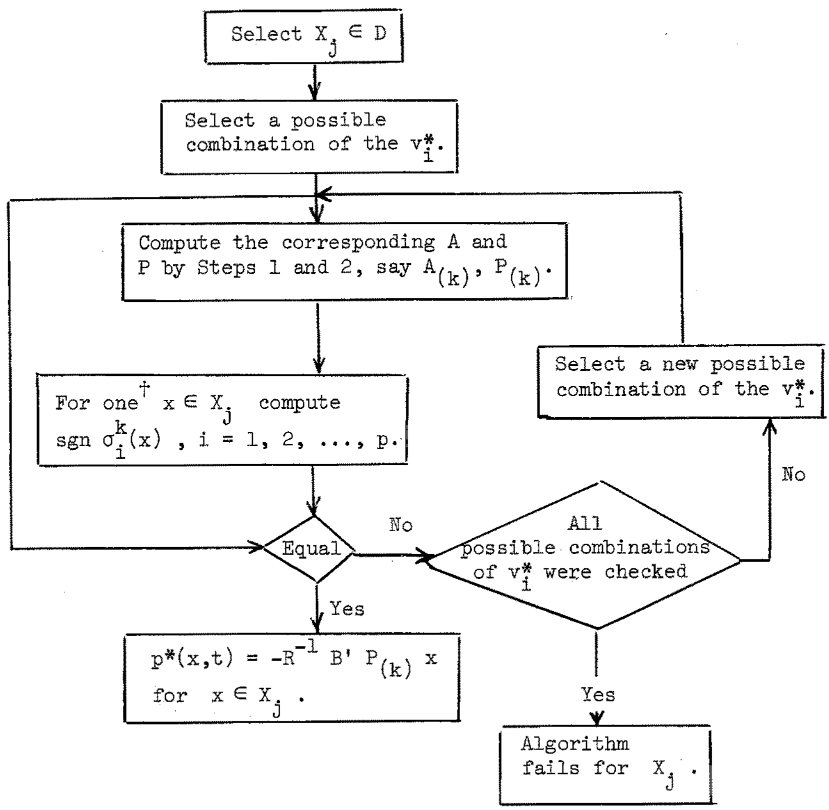
$i = 1, 2, \ldots, p, \quad k = 1, 2, \ldots, 2^p$

and designate the decomposition by $D \triangleq \{X_1, X_2, \ldots, X_\ell\}$.

The X_i are the open subsets of the decomposition, where $X_i \cap X_j = \emptyset$ for $i \neq j$ and $R^2 = \bigcup_{i=1}^{\ell} \bar{X}_i$.

Step 5. Determine a control candidate $p^*(\cdot)$ by the following algorithm:

Algorithm 1:



Definition 1. Algorithm 1 is said to be positively satisfied on X_j iff there is at least one possible combination of the v_i^* such that the equality test is answered in the affirmative.

Remark 1. If there is a region of R^2 on which the equality test is met by more than one possible combination of the v_i^* , then one may be able to introduce an altered decomposition of R^2 , on each of whose members $p^*(\cdot)$ takes on values corresponding to one of the combinations meeting the sign test; for instance, see Example 1. If this has been done, we still denote the members of the decomposition by X_i , $i \in \{1, 2, \ldots, \ell\}$.

†Since $\sigma_i^k(\cdot)$ is continuous on R^2 , the sgn $\sigma_i^k(x)$, $i = 1, 2, \ldots, p$, remain constant on X_j .

Before continuing, we introduce some definitions.

Definition 2. The set $Z \subset R^2$ is positively invariant with respect to $(p(\cdot), e(\cdot))$ iff $x_o \in Z \Rightarrow x(t) \in \bar{Z}$ $\forall\, t \in [t_o, \infty)$, where $x(\cdot) : [t_o, \infty) \to R^2$ is a solution of (1) generated by $(p(\cdot), e(\cdot))$ and $x(t_o) = x_o$.

Definition 3. The origin is eventually uniformly asymptotically stable in the large iff given any $(x_o, t_o) \in R^2 \times R^1_+$ there exists at least one solution $x(\cdot) : [t_o, \infty) \to R^2$, $x(t_o) = x_o$, and for every such solution there is a $T \geq t_o$ such that the origin is uniformly asymptotically stable (in the sense of Lyapunov) with respect to $x(\cdot)\,\big|_{[T,\infty)}$.

Now suppose that Algorithm 1 is positively satisfied on X_i and $X_j \in D^{\dagger}$, $\bar{X}_i \cap \bar{X}_j \neq \phi$, $i \neq j$, for the k_i-th and k_j-th possible combinations of the v_i^*, respectively. Consider $\tilde{p}(\cdot) : R^2 \times R^1_+ \to R^m$ satisfying

$$\tilde{p}(x,t) \quad \begin{cases} = -R^{-1} B' P_{(k_i)} x & \forall (x,t) \in X_i \times R^1_+ \\ = -R^{-1} B' P_{(k_j)} x & \forall (x,t) \in X_j \times R^1_+ \\ \in \{-R^{-1} B'[\alpha P_{(k_i)} + (1-\alpha) P_{(k_j)}]\, x : \alpha \in [0,1]\} & \\ \forall (x,t) \in \bar{X}_j \cap \bar{X}_j \times R^1_+ . & \end{cases} \tag{7}$$

Now we need one more definition.

Definition 4. Let $\hat{x} \in \bar{X}_i \cap \bar{X}_j \neq \phi$, $i \neq j$, $\Rightarrow \hat{x} \in \{x : d'x = 0,\ x \in R^2\}$. Let $h_i(x,t)$ and $h_j(x,t)$ be the r. h. s. of (1) corresponding to $(\tilde{p}(\cdot), e(\cdot))$ on $\bar{X}_i$ and $\bar{X}_j$, respectively. The boundary $\bar{X}_i \cap \bar{X}_j$ is attractive iff $\forall\, (\tilde{p}(\cdot), e(\cdot))$ and $\forall\, \hat{x} \in \bar{X}_i \cap \bar{X}_j$ there is a ball $B(\hat{x})$ in R^2 with center at $\hat{x}$ such that

$$d'h_i(x,t) \geq 0 \quad \forall (x,t) \in B(\hat{x}) \cap \bar{X}_i \times R^1_+ , \quad \text{and}$$

$$d'h_j(x,t) \leq 0 \quad \forall (x,t) \in B(\hat{x}) \cap \bar{X}_j \times R^1_+ ,$$

where d points into X_j. The boundary is a transition one iff $\forall(\tilde{p}(\cdot), e(\cdot))$

$$d'h_i(x,t) > 0 \quad \text{and} \quad d'h_j(x,t) > 0 \quad \text{or}$$

$$d'h_i(x,t) < 0 \quad \text{and} \quad d'h_j(x,t) < 0$$

for all $(x,t) \in \bar{X}_i \cap \bar{X}_j \times R^1_+$.

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†Decomposition D may be an altered decomposition; see Remark 1.

Consider the following assumptions.

Assumptions 1.

(i) Admissible uncertainty $e(\cdot) : R^2 \times R^1_+ \to R^{p+1}$ is continuous on R^2 and piecewise continuous on any compact subinterval of R^1_+.

(ii) Algorithm 1 is positively satisfied on every $X_j \in D$.

(iii) Triple $\{C, A, B\}$, where $Q = C'C$ and A is given by (3), is completely controllable and observable for all possible combinations of the v_i^*.

(iv) Every boundary $\bar{X}_i \cap \bar{X}_j \neq \phi$, $i \neq j$, is either an attractive or a transition one.

(v) Decomposition D is such that there exists at least one $k \in \{1, 2, \ldots, \ell\}$ such that, given $\tilde{p}(\cdot)$, X_k is positively invariant with respect to $(\tilde{p}(\cdot), e(\cdot))$ for all admissible $e(\cdot)$.

(vi) If a solution $x(\cdot) : [t_o, \infty) \to R^2$ generated by $(\tilde{p}(\cdot), e(\cdot))$ reaches an attractive boundary at $x(T)$, then the origin is uniformly asymptotically stable with respect to $x(\cdot)|_{[T,\infty)}$; see Remark 3.

Now we are ready to state a stability theorem.

Theorem 1. Consider system (1). If Assumptions 1 are met there exists a feedback control $\tilde{p}(\cdot)$ satisfying (7) such that the origin is eventually uniformly asymptotically stable in the large for all admissible uncertainties $e(\cdot)$.

Proof. Since $\tilde{p}(\cdot)$ is discontinuous and hence considered not unique, (1) becomes a generalized dynamical system, [11-16],

$$\dot{x}(t) \in C(x(t),t) \tag{8}$$

where the set valued function $C(\cdot)$ is given by

$$C(x,t) = \{[A_o + \sum_{i=1}^{p} A_i\, e_i(x,t)]x + B\, e_{p+1}(x,t)\, u : u = \tilde{p}(x,t)\}$$

We show first that, given any $(x_o, t_o) \in R^2 \times R^1_+$, there exists at least one solution of (8) and that such a solution can be continued on any compact subset of $R^2 \times R^1_+$.[†] This can be done by showing, [10],

(i) $C(x,t)$ is convex for all $(x,t) \in R^2 \times R^1_+$.

(ii) $C(x,t)$ is compact on any compact subset of $R^2 \times R^1_+$.

(iii) $C(\cdot)$ is upper semicontinuous on $R^2 \times R^1_+$.

(iv) Every member of $C(x,t)$ satisfies a linear growth condition.

To prove the eventual uniform asymptotic stability of the origin, we show that the origin is eventually uniformly asymptotically stable with respect to every solution, and, as indicated above, at least one solution exists and is continuable for every initial point (x_o, t_o).

[†]At points of discontinuity of $e(x,\cdot)$, solutions can be joined in the usual way.

First consider any $X_j \in D$. By Assumption 1 (ii), Algorithm 1 is positively satisfied; suppose this is accomplished by the k-th possible combination of the v_i^*. Let $V_k(\cdot) : X_j \to R^1$ be given by

$$V_k(x) = x'P_{(k)} x \tag{9}$$

where $P = P_{(k)}$ is the solution of

$$P A_{(k)} + A'_{(k)} P - P B R^{-1} B'P + Q = 0 \tag{10}$$

with

$$A_{(k)} = A_o + \sum_{i=1}^{p} A_i \operatorname{sgn} \sigma_i^k (x)$$

$$\sigma_i^k(x) = x'[P A_i + A_i' P] x$$

In view of (iii) of Assumptions 1, $P_{(k)}$ is positive definite and symmetric, [17].

Next we show that $V_k \circ x(t)$ decreases along a solution $x(\cdot)$ of (3) generated by $(\tilde{p}(\cdot), e(\cdot))$ for all $x(t) \in X_j$. For all $x(t) \in X_j$

$$W(t) = \operatorname{grad} V_k(x(t))\, \dot{x}(t)$$
$$= 2\, x'(t)\, P_{(k)} \left[(A_o + \sum_{i=1}^{p} A_i v_i(t)\, x(t) + B\, v_{p+1}(t)\, \tilde{p}(x(t),t)\right]$$

where

$$v_i(t) = e_i(x(t),t)$$

$$\tilde{p}(x(t),t) = -\, R^{-1} B'P_{(k)}\, x(t)$$

However, since $|v_i(t)| \leq 1,\ i \in \{1, 2, \ldots, p\}$,

$$\sigma_i^k (x(t)) \operatorname{sgn} \sigma_i^k (x(t)) \geq v_i(t)\, \sigma_i^k(x(t)) \tag{11}$$

so that

$$W(t) \leq x'(t) [P_{(k)} A_{(k)} + A'_{(k)} P_{(k)}] x(t)$$
$$- 2\, x'(t) [P_{(k)} B R^{-1} B'P_{(k)}]\, x(t)\, v_{p+1}(t)$$

Since $P_{(k)} B R^{-1} B'P_{(k)}$ is positive semidefinite and $v_{p+1}(t) \in [1, q]$,

$$W(t) \leq x'(t) [P_{(k)} A_{(k)} + A'_{(k)} P_{(k)} - P_{(k)} B R^{-1} B'P_{(k)}]\, x(t)$$
$$- x'(t) [P_{(k)} B R^{-1} B'P_{(k)}]\, x(t)$$

In view of (10) we have

$$W(t) \leq -\, x'(t)\, Q\, x(t) - x'(t) [P_{(k)} B R^{-1} B'P_{(k)}]\, x(t) \tag{12}$$

Now we have two possibilities:

(i) $x(t) \in X_j$, $t \in [t', t'']$, and "Nature" does not use her "optimal" strategy. Then (11), and hence (12), is a strict inequality; thus, $W(t) < 0$.

(ii) $x(t) \in X_j$, $v_i(t) = \operatorname{sgn} \sigma_i^k(x(t))$, $i = 1, 2, \ldots, p$, $v_{p+1}(t) = 1$,

$t \in [t', t'']$. Then the system is linear with constant coefficients (linear, time-invariant). Further more,

$$- W(t) = x'(t)\, Q\, x(t) + x'(t)\, P_{(k)}\, B\, R^{-1}\, B' P_{(k)}\, x(t)$$
$$= x'(t)\, C'C\, x(t) + u'(t)\, R\, u\, (t) \equiv 0 \qquad (13)$$

on $[t', t'']$. Since both terms in (13) are non-negative
$x'(t)\, C'C\, x(t) \equiv 0$, $u'(t)\, R\, u(t) \equiv 0$.
Since R is positive definite, $u(t) \equiv 0$, and the system is
$\dot{x}(t) = A_{(k)}\, x(t)$, $t \in [t', t'']$.
But, since $\{C, A_{(k)}, B\}$ is assumed to be observable, $x'(t)\, C'C\, x(t) \equiv 0$ cannot occur and so neither can $W(t) \equiv 0$. We conclude that $V_{(k)} \circ x(t)$ decreases along a solution $x(\cdot)$ for all t such that $x(t) \in X_j \in D$.

Finally we note:

a) If a solution $x(\cdot) : [t_o, \infty) \to R^2$ remains in an $X_j \in D$ for all $t \in [T, \infty)$, $T \geq t_o$, the origin is eventually uniformly asymptotically stable with respect to $x(\cdot)$ since the requirements for Lyapunov stability are met with respect to $x(\cdot)|_{[T, \infty)}$

b) If a solution leaves an $X_j \in D$ it cannot return to it by Assumptions 1 (iv) and (v). Since the decomposition D is finite, a solution must remain in some X_j (case a)), or enter an attractive boundary, or reach an invariant set $X_k \in D$. If it enters an attractive boundary, Assumption 1 (vi) assures eventual uniform asymptotic stability. If it enters an invariant X_k , it must remain in X_k (case a)) or reach an attractive boundary. In either case, eventual uniform asymptotic stability is assured, since X_k exists by Assumption 1(v).

Remarks

2. Assumptions 1 are sufficient but not necessary to assure that $\tilde{p}(\cdot)$ is stabilizing, [10].

3. Assumptions 1 (iv) - (vi) depend on the properties of boundaries $\bar{X}_i \cap \bar{X}_j$. For some cases, for instance single input systems in companion form, these properties are readily checked, [10].

4. Feedback control $\tilde{p}(\cdot)$ is defined almost everywhere on $R^n \times R^1_+$. Due to a real controller's delay in switching, chattering across an attractive boundary occurs, [11 - 13]; see also Example 1.

2.3 Example 1

Here we consider a simple example to illustrate the theory developed in Section 2.2, namely, a second order system with a single input and a single uncertainty:

$$\dot{x}_1(t) = x_2(t) \tag{14}$$

$$\dot{x}_2(t) = 1.6\ v(t)\ x_2(t) + u(t)$$

with uncertainty $v(t) \in [-1,1]$, and control penalty matrix $R = 1$. Furthermore, let matrix

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

With the system so specified, we have only two possible combinations for v^*. These, together with the pertinent $P_{(k)}$, $\sigma^k(x)$ and $\tilde{p}(x,t)$ are listed below. The decomposition induced by $\sigma^k(x) = 0$, $k = 1, 2$, is shown in Fig. 1.

$v^* = 1$ | $v^* = -1$

$$P_{(1)} = \begin{bmatrix} 2.1 & 1 \\ 1 & 3.7 \end{bmatrix} \qquad P_{(2)} = \begin{bmatrix} 2.1 & 1 \\ 1 & 0.5 \end{bmatrix}$$

$$\sigma^1(x) = x_2\ (3.2\ x_1 + 12\ x_2) \qquad \sigma^2 = x_2\ (3.2\ x_1 + 1.7\ x_2)$$

$$\tilde{p}(x,t) = -x_1 - 3.7\ x_2 \qquad \tilde{p}(x,t) = -x_1 - 0.5\ x_2$$

The algorithm is positively satisfied on each member of the decomposition induced by $\sigma^k(x) = 0$, $k = 1, 2$; the corresponding switching functions are indicated on Fig. 1. As can be seen, on two members of the decomposition, the algorithm is positively satisfied with both possible combinations. Furthermore, both combinations satisfy the algorithm positively on two pairs of adjacent members of the decomposition. Thus, these adjacent members can be combined into a single one; e.g., the ones for $\operatorname{sgn} \sigma^2 = -1$. Recalling that one assumption underlying Theorem 2 requires that every boundary of the decomposition be either attractive or a transition one, we verify readily that the boundaries given by $x_2 = 0$ are transition ones; however, the boundaries given by $3.2\ x_1 + 12\ x_2 = 0$ are neither. Thus, we alter the decomposition by rotating this line until we obtain boundaries satisfying the above assumption, in this case attractivity. The final decomposition is shown in Fig. 2.

If we denote system (14) by

$$\dot{x}(t) = A\ x(t) + b\ u(t)$$

then, upon setting $u(t) = \tilde{p}(x(t),t)$ and $v(t) = \operatorname{sgn} \sigma^k(x(t))$ for $k = 1, 2$, we get†

$$\dot{x}(t) = A_{CL}\ x(t)$$

where matrix

$$A_{CL} = \begin{bmatrix} 0 & 1 \\ -1 & -2.1 \end{bmatrix}$$

Line AOA contains one eigenvector of A_{CL}.

†For special features of single input systems in companion form see [10].

Finally, Fig. 3 shows some typical solution curves of system (14) subject to a parameter uncertainty that is a random piecewise constant function of time. Note that the solution curves reach the attractive boundary $\bar{X}_2 \cap \bar{X}_3$ and then move along it towards the origin.

3. INPUT DISTURBANCE WITH BOUNDED CONTROL

3.1 Problem Statement

Now we treat a class of dynamical systems with input disturbance. Consider

$$\dot{x}(t) = A(t)\ x(t) + B(t)\ u(t) + B(t)\ v(t) \tag{15}$$

$$x(t_o) = x_o\ ,\ t \in [t_o,\ t_1]$$

where

$x(t) \in R^n$ is the state of the system at time t ;

$A(\cdot)$ is an $n \times n$ matrix, continuous on R^1 ;

$B(\cdot)$ is an $n \times m$ matrix, continuous on R^1 ;

$u(t) \in U = \{u \in R^m : \| u \| \leqslant \rho_u = \text{constant} \in (0,\infty)\}$ is the control;

$v(t) \in V = \{v \in R^m : \| v \| \leqslant \rho_v = \text{constant} \in (0,\infty)\}$ is the disturbance.

Since we are again concerned with the asymptotic stability of the origin $x = \{0\}$, we introduce a measure of deviation

$$J = \int_{t_o}^{t_1} x'(t)\ Q(t)\ x(t)\ dt \tag{16}$$

where $Q(\cdot)$ is a symmetric $n \times n$ matrix, continuous on R^1 , and consider the differential game with state equation (15) and cost (16). That is, we seek a saddlepoint $(p^*(\cdot),\ e^*(\cdot))$ in a given class of strategies

$p(\cdot) : R^n \times R^1 \to R^m$, $e(\cdot) : R^n \times R^1 \to R^m$

such that

$u(t) = p(x(t),\ t)$, $v(t) = e(x(t),\ t)$.

Again, we look for a feedback control $p^*(\cdot)$ while admitting a disturbance that may depend on state and time.

Having found a saddlepoint $(p^*(\cdot),\ e^*(\cdot))$, we inquire under what conditions feedback control $p^*(\cdot)$ renders $x = \{0\}$ uniformly asymptotically stable in the large (in the sense of Lyapunov) <u>against every allowable disturbance</u> $e(\cdot)$.

3.2 Saddlepoint Strategy

On invoking necessary conditions for a saddlepoint, e.g. [8-9], and then sufficient conditions, e.g. [18-19], we find the following saddlepoint for the case $\rho_u = \rho_v = \rho$:

$$p^*(x,t) = -\ e^*(x,t) = \begin{cases} -\dfrac{B'(t)P(t)x}{\|B'(t)P(t)x\|}\,\rho & \forall (x,t) \notin N \\ \text{any admissible value} & \forall (x,t) \in N \end{cases} \tag{17}$$

where

$$N = \{(x,t) \in R^n \times R^1 : B'(t)\, P(t)\, x = 0\}$$

and matrix $P(\cdot)$ is the solution of

$$\dot{P}(t) + P(t)\, A(t) + A'(t)\, P(t) + Q(t) = 0 \tag{18}$$
$$P(t_1) = 0$$

The details of the derivation can be found in Chapter 2 of [10].

3.3 Stability

Having deduced a saddlepoint, we ask now whether the controller's saddlepoint strategy $p^*(\cdot)$ results in asymptotic stability of the origin against any allowable disturbance strategy $e(\cdot)$. Thus, consider a feedback control $\tilde{p}(\cdot) : R^n \times R^1_+ \to U$ given by

$$\tilde{p}(x,t) = \begin{cases} -\dfrac{B'(t)P(t)x}{\|B'(t)P(t)x\|}\, \rho_u & \forall (x,t) \notin N \\ u \in U = \{u \in R^m : \|u\| \leqslant \rho_u\} & \forall (x,t) \in N \end{cases} \tag{19}$$

with, [20],

$$P(t) = \int_t^\infty \Phi'(\tau - t)\, Q(\tau)\, \Phi(\tau - t)\, d\tau \tag{20}$$

which is a particular solution of (18), where $\Phi(\cdot)$ is the transition matrix of $\dot{x}(t) = A(t)\, x(t)$.

Consider the following assumptions.

Assumptions 2.

(i) Admissible disturbance $e(\cdot) : R^n \times R^1_+ \to V \subset R^m$ is continuous on R^n and piecewise continuous on any compact subinterval of R^1_+.

(ii) $\exists\ c_1, c_2 \in (0, \infty)$ such that $\|A(t)\| \leqslant c_1$, $\|B(t)\| \leqslant c_2$ $\forall t \in R^1_+$.

(iii) $Q(t)$ is positive definite (symmetric); that is, $\exists\ c_3, c_4 \in (0, \infty)$, $c_3 \leqslant c_4$, such that $c_3 I \leqslant Q(t) \leqslant c_4 I$ $\forall t \in R^1_+$.

(iv) $A(t)$ is uniformly asymptotically stable.

(v) $\rho_u \geqslant \rho_v$.

Now we are ready to state a stability theorem.

Theorem 2. Consider system (15). If Assumptions 2 are met there exists a feedback control $\tilde{p}(\cdot)$ satisfying (19) such that the origin is uniformly asymptotically stable in the large (Lyapunov) for all admissible disturbances $e(\cdot)$.

Proof. Since $\tilde{p}(\cdot)$ is discontinuous and hence considered not unique, (15) becomes a generalized dynamical system, [11-16],

$$\dot{x}(t) \in C(x(t), t) \tag{21}$$

where the set valued function $C(\cdot)$ is given by

$$C(x,t) = \{A(t)x + B(t)u + B(t)e(x,t) : u = \tilde{p}(x,t)\}$$

As in the proof of Theorem 1, it can again be shown, [10], that, given any $(x_o, t_o) \in R^n \times R^1_+$, there exists at least one solution of (21) and that such a solution can be continued on any compact subset of $R^n \times R^1_+$.

To demonstrate the uniform asymptotic stability of the origin, we consider the function $V(\cdot) : R^n \times R^1_+ \to R^1$ given by

$$V(x,t) = x'P(t)x \tag{22}$$

where $P(t)$ is defined by (20).

Since $A(t)$ is uniformly asymptotically stable and $Q(t)$ is positive definite according to (iii) and (iv) of Assumptions 2, matrix $P(t)$ is positive definite, [20]. In particular, there exist $c_5, c_6 \in (0, \infty)$, $c_6 \geq c_5$, such that $c_5\|x\|^2 \leq V(x,t) \leq c_6\|x\|^2 \quad \forall (x,t) \in R^n \times R^1_+$.
Thus, $V(\cdot)$ is a Lyapunov function candidate.

Finally, we observe that $V \circ x(t)$ decreases along a solution $x(\cdot)$ of (21) generated by $(\tilde{p}(\cdot), e(\cdot))$. Namely, for $(x(t), t) \notin N$,

$$\begin{aligned}
W(t) &= \operatorname{grad}_x V(x(t), t)\dot{x}(t) + \frac{\partial V(x(t),t)}{\partial t} \\
&= 2x'(t)P(t)\Big[A(t)\,x(t) - B(t)\,\frac{B'(t)P(t)x(t)}{\|B'(t)P(t)x(t)\|}\,\rho_u + B(t)e(x(t),t)\Big] \\
&\qquad + x'(t)\dot{P}(t)x(t) \\
&= x'(t)[\dot{P}(t) + P(t)A(t) + A'(t)P(t)]\,x(t) \\
&\qquad -2\rho_u\|B'(t)P(t)x(t)\| + 2x'(t)P(t)B(t)e(x(t),t) \\
&= -x'(t)Q(t)x(t) - 2\rho_u\|B'(t)P(t)x(t)\| + 2x'(t)P(t)B(t)e(x(t),t) \\
&\leq -x'(t)Q(t)x(t) - 2(\rho_u - \rho_v)\|B'(t)P(t)x(t)\| \\
&< 0 \quad \forall\, \rho_u \geq \rho_v .
\end{aligned}$$

For $x(t) \in N$ but $x(t) \neq 0$,
$W(t) = -x'(t)Q(t)x(t) < 0$.
This concludes the proof.

Theorem 2 has an immediate corollary.

<u>Corollary 1</u>. The average measure of deviation from the origin along a solution $x(\cdot) : [t_o, \infty] \to R^n$, $x(t_o) = x_o$, generated by $(\tilde{p}(\cdot), e(\cdot))$ is

$$\int_{t_o}^{\infty} x'(t)Q(t)x(t)dt \leq x_o'\, P(t_o)x_o .$$

<u>Proof</u>. In view of (ii) - (iv) of Assumptions 2, $P(t)$ is bounded on R^1_+, [20]. Thus, the result follows upon integration of $W(t)$.

<u>Remarks</u>

5. If matrices $A(t)$ and $Q(t)$ are constant and $t_1 \to \infty$, then $P(\cdot)$ is the constant matrix solution of the Lyapunov equation, [20],
$$PA + A'P + Q = 0 \tag{23}$$
6. Chattering across the singular manifold N is possible, [11-13].

7. In the scalar input case, the control $\tilde{p}(x,t)$ is bang-bang.
8. If the matrix A is not stable but $\{A,B\}$ is stabilizable, Theorem 2 is applicable,[21].
9. The results of this section, in particular Theorem 2, remain unaltered if input matrix B is state and time-dependent; i.e., $B(\cdot)$ may be continuous on $R^n \times R^1$.
10. For state-independent input matrix $B(\cdot)$, control $\tilde{p}(x,t)$ is only output-dependent for outputs $y = C(t)x$ where $C(t) = B'(t)P(t)$ depends on $Q(\cdot)$.

3.11 Example 2

As an example illustrating the theory of Section 3.2 consider the third order system

$$\dot{x}(t) = A\,x(t) + B\,u(t) + B\,v(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2.9 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Q = 2I$$

The solution of (15) is

$$P = \begin{bmatrix} 4.6 & 3.8 & 1 \\ 3.8 & 6.3 & 1.6 \\ 1 & 1.6 & 0.9 \end{bmatrix}$$

and

$$B'Px = \begin{bmatrix} x_1 + 1.6\,x_2 + 0.9\,x_3 \\ 3.8\,x_1 + 6.3\,x_2 + 1.6\,x_3 \end{bmatrix}$$

Note that N is of dimension $n - 2 = 1$.

Figure 4 shows the response of this system under a random piecewise constant disturbance and a control given by (19).

3.5 Example 3

Finally, as another illustration of the stabilization of a system with input disturbance consider the second order single input system

$$\dot{x}(t) = A\,x(t) + b\,u(t) + b\,v(t) \tag{25}$$

$$A = \begin{bmatrix} 0 & 1 \\ -22 & -24 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 22 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & \frac{23}{24} \end{bmatrix}$$

Following Section 3.2, it is readily shown that

$$\tilde{p}(x,t) = -\mathrm{sgn}(x_1 + x_2) \quad \text{for} \quad x_1 + x_2 \neq 0 \tag{26}$$

Here, the singular manifold N is of dimension $n - 1 = 1$, and chattering occurs due to delay in digital computation.

Figures 5 and 6 show the system's response under four types of disturbance -- constant, sinusoidal, random piecewise constant, and "worst" -- and control (26). For comparison, Figure 7 shows the analog computer solution for zero as well as sinusoidal disturbance. As expected, the analog solution is smoother than the digital computer one (of the discretized system); the response slides along N rather than chattering across it.

4. MODEL UNCERTAINTY WITH BOUNDED CONTROL

4.1 Problem Statement

Now we return to a class of model uncertainty problems. Here we treat n-th order single input systems in companion form. Consider

$$\dot{x}(t) = [A_o + \sum_{i=1}^{p} A_i v_i(t)]\, x(t) + b\, u(t) \tag{27}$$

$$x(t_o) = x_o \;, \quad t \in [t_o, t_1]$$

where

$x(t) \in R^n$ is the state of the system at time t;

A_o is a constant $n \times n$ matrix of the form

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_n \end{bmatrix}$$

with α_i = constant, $i = 1, 2, \ldots, n$;

A_i , $i = 1, 2, \ldots, p$, are constant $n \times n$ matrices of form

$$A_i = \begin{bmatrix} & \mathbf{0} & \\ \hline 0 \;\cdots & a_i & \cdots\; 0 \end{bmatrix}$$

with a_i = constant ≥ 0 , $b = [0 \;.\;.\;.\; 0 \;\; 1]' \in R^n$;

$v_i(t)$, $i = 1, 2, \ldots, p$, with $|v_i(t)| \leq 1$, are values of parameter uncertainty at time t;

$u(t) \in R^1$, with $|u(t)| \leq \rho_u \in (0, \infty)$, is the value of control at time t .

Again, we are interested in the asymptotic behavior of system (27) under all possible parameter uncertainties. Towards that end we introduce a performance index

$$J = \int_{t_o}^{t_1} x'(t)\ Q\ x(t)dt \tag{28}$$

where Q is a constant positive definite symmetric $n \times n$ matrix.

4.2 Stability

Before proceeding we note that the system may be converted into an equivalent input disturbance one:

$$\dot{x}(t) = A_o x(t) + b\ u(t) + b\ \tilde{v}(t) \tag{29}$$

with

$$\tilde{v}(t) = c'(t)x(t)$$

where

$c'(t) = [a_1 v_1(t)\quad a_2 v_2(t)\ .\ .\ .\ a_n v_n(t)] \in R^n$, and

$$|\tilde{v}(t)| \leqslant \|c(t)\|\ \|x(t)\| \leqslant \left(\sum_{i=1}^{p} a_i^2\right)^{\frac{1}{2}} \|x(t)\|$$

Thus, we allow disturbances subject to

$$|\tilde{v}(t)| \leqslant \rho_v = \left(\sum_{i=1}^{p} a_i^2\right)^{\frac{1}{2}} \|x(t)\| \tag{30}$$

We see now that the equivalent input disturbance problem is of the type treated in Section 3, with the sole exception of the state dependence of the disturbance constraint.

Upon applying necessary conditions for a saddlepoint $(p^*(\cdot), e^*(\cdot))$, with

$$\rho_u = \rho_v = \left(\sum_{i=1}^{p} a_i^2\right)^{\frac{1}{2}} \|x\| = \rho(\|x\|) \tag{31}$$

one finds the results of Section 3 unchanged. Hence, we can state a stability theorem for the equivalent input disturbance problem.

Theorem 3. Consider system (29). If Assumptions 2(i) and (iv) are met there exists a feedback control $\tilde{p}(\cdot) : R^n \times R^1_+ \to R^1$ satisfying

$$\tilde{p}(x,t) = \begin{cases} -\dfrac{b'Px}{|b'Px|}\ \rho(\|x\|) & \forall (x,t) \in \{(x,t) : b'Px \neq 0\} \\ \text{an admissible value} & \forall (x,t) \in \{(x,t) : b'Px = 0\} \end{cases}$$

where

$$PA_o + A'_o P + Q = 0$$

such that the origin is uniformly asymptotically stable in the large for all admissible disturbances $e(\cdot)$.

Remarks

11. If, in addition to the parameter uncertainties, there is also an input disturbance, say w with $|w| \leqslant \rho_w$, then $\tilde{p}(\cdot)$ is stabilizing provided $\rho(\|x\|) = \rho_w + \left(\sum_{i=1}^{p} a_i^2\right)^{\frac{1}{2}} \|x\|$.

12. The results are readily extended to the case of time-varying matrix A_o by means of Section 3,[10] .

4.3 Example 4

To illustrate the preceding results let us consider a second order system (27) with

$$A_o = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \quad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} , \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Of course, here

$$\rho\,(\|x\|) = \|x\|\,\sqrt{10}$$

Then it is readily shown that

$$\tilde{p}(x,t) = -\,(x_1 + 2.4\,x_2) - \rho(\|x\|)\ \mathrm{sgn}\,(x_1 + 2.4\,x_2)$$

for all $(x,t) \notin \{(x,t) : x_1 + 2.4\,x_2 = 0\}$.

Here, matrix A_o is not stable but $\{A_o, b\}$ is stabilizable by linear feedback; this accounts for the first term in the expression for $\tilde{p}(\cdot)$; see Remark 8.

Finally, the digital computer response of the system under the indicated parameter uncertainty and control $\tilde{p}(\cdot)$ is shown in Figure 8.

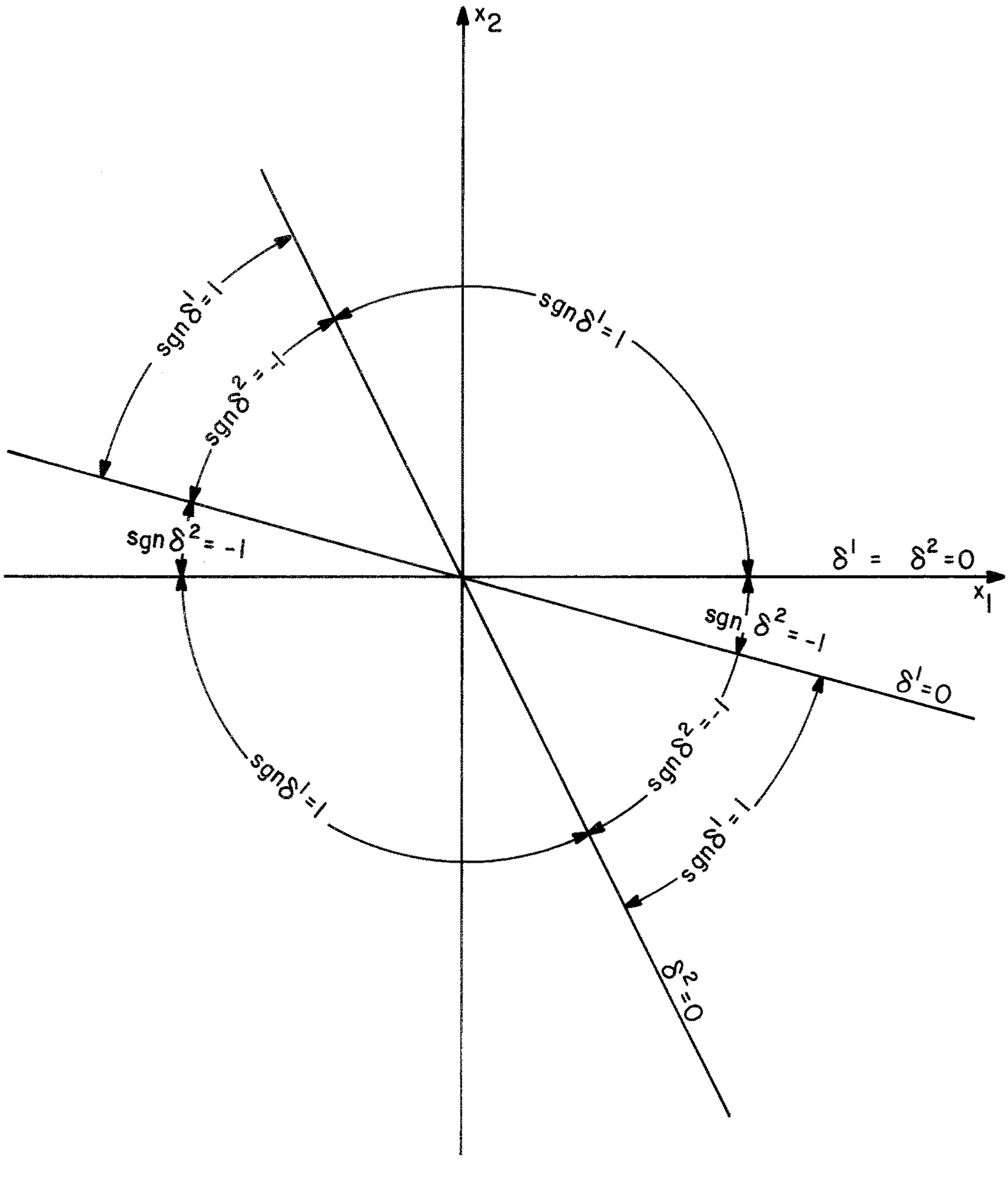


FIGURE 1, EXAMPLE 1

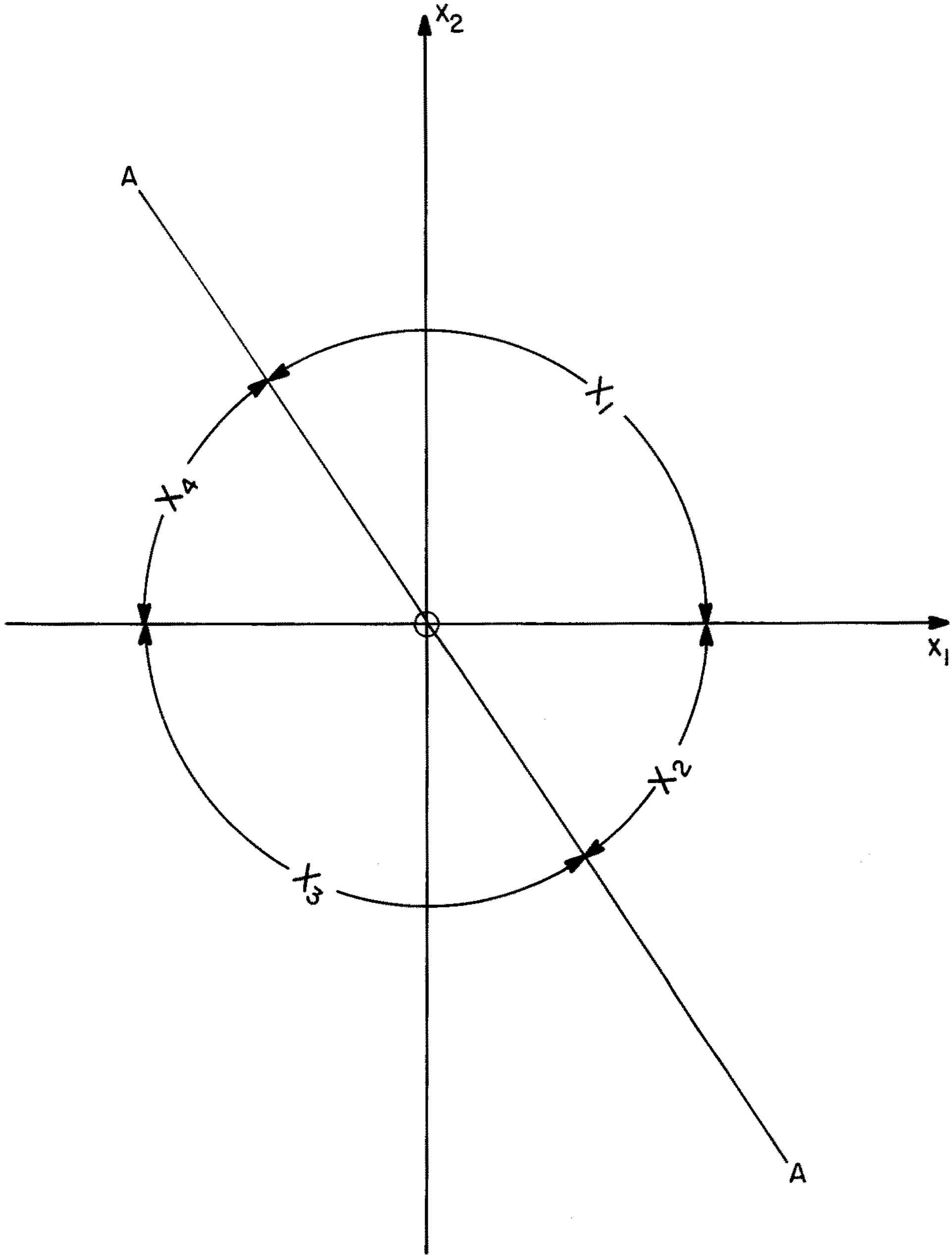


FIGURE 2, EXAMPLE 1

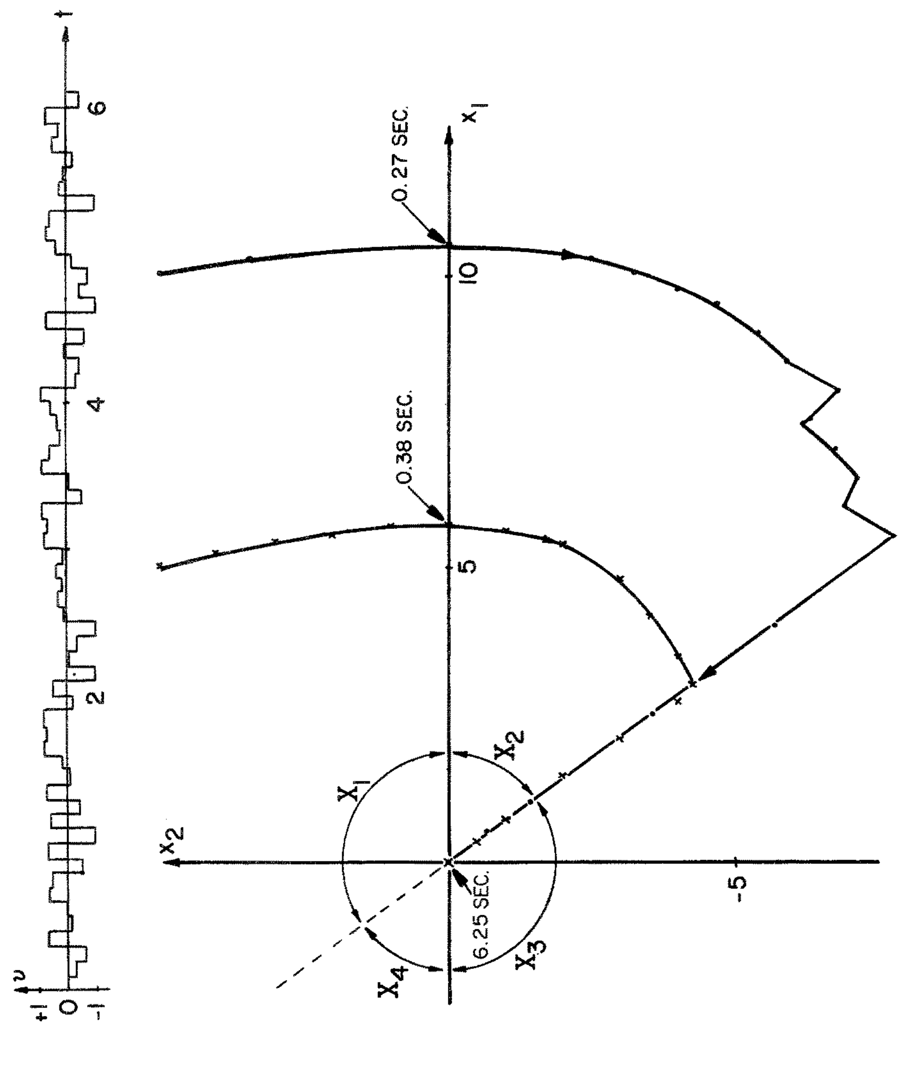


FIGURE 3, EXAMPLE 1

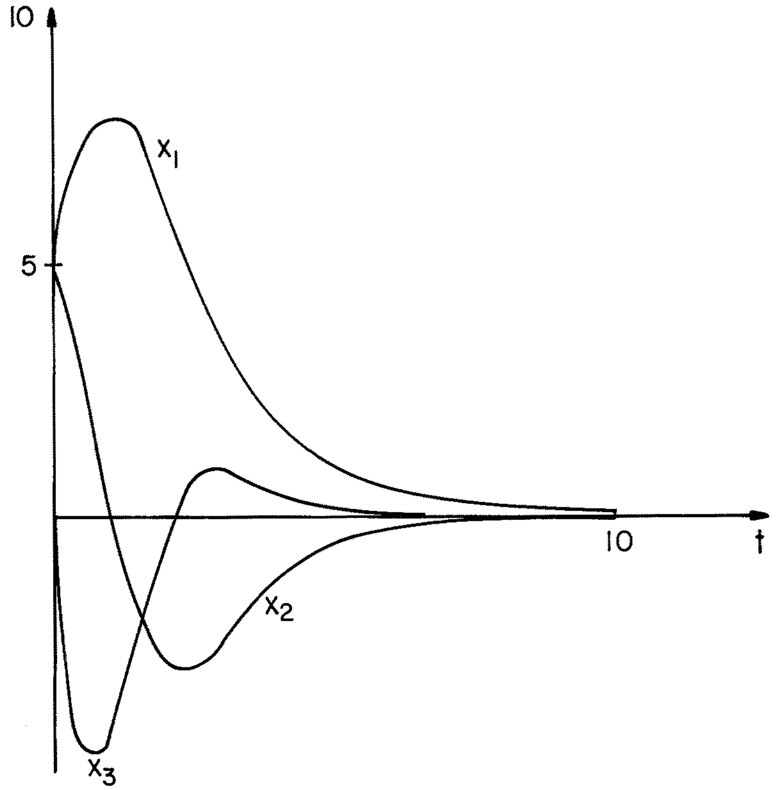


FIGURE 4, EXAMPLE 2

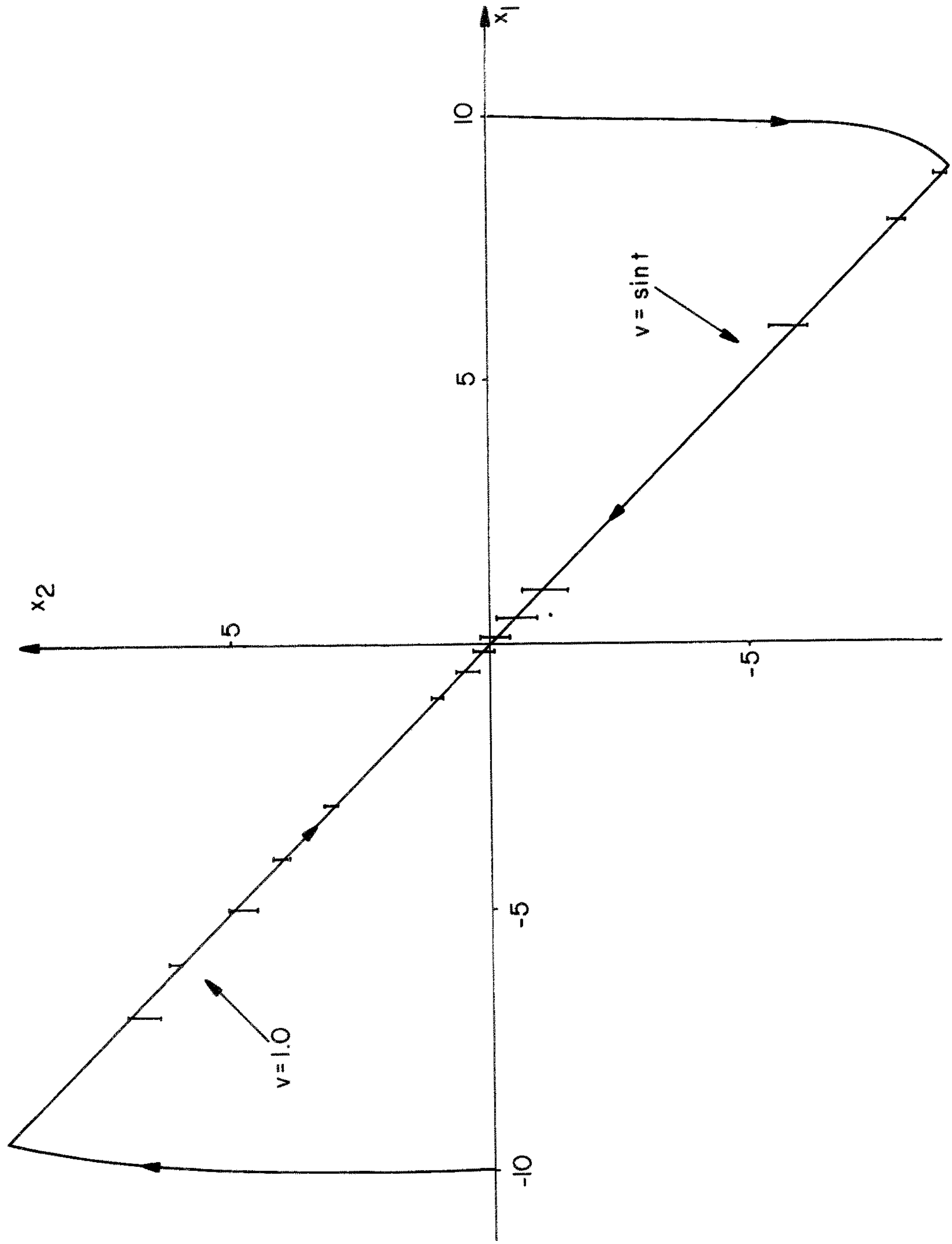
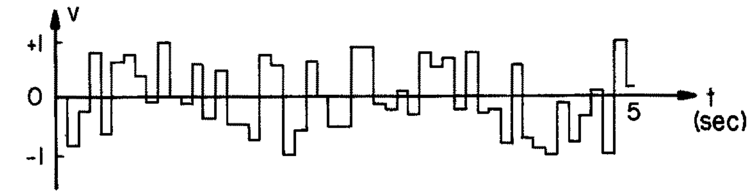


FIGURE 5, EXAMPLE 3



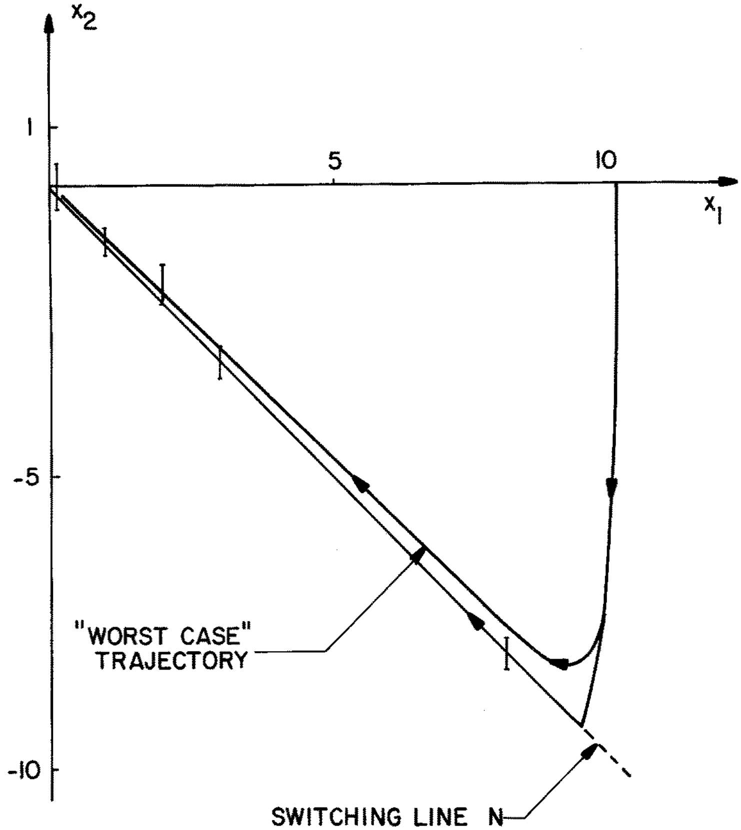


FIGURE 6, EXAMPLE 3

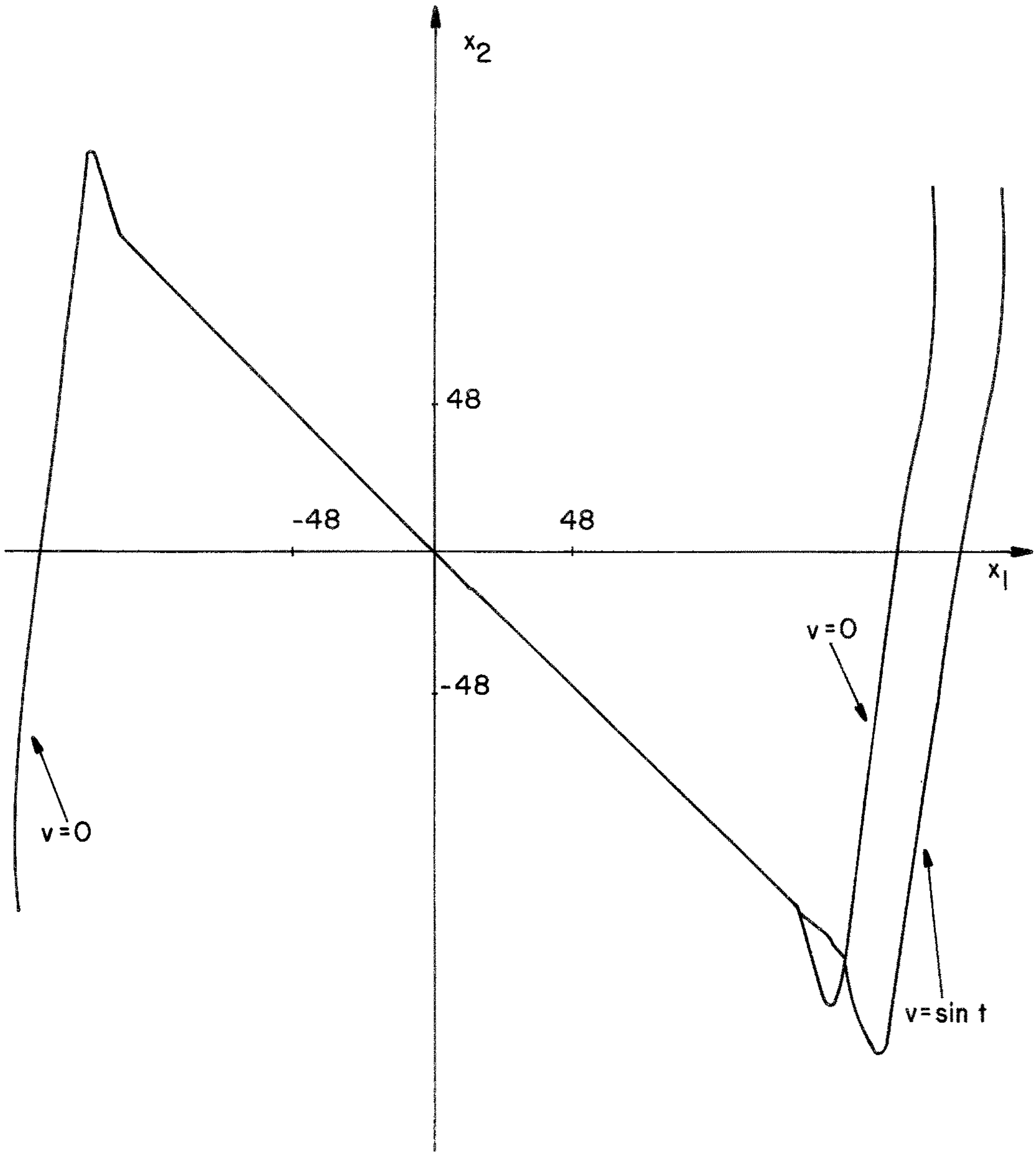


FIGURE 7, EXAMPLE 3

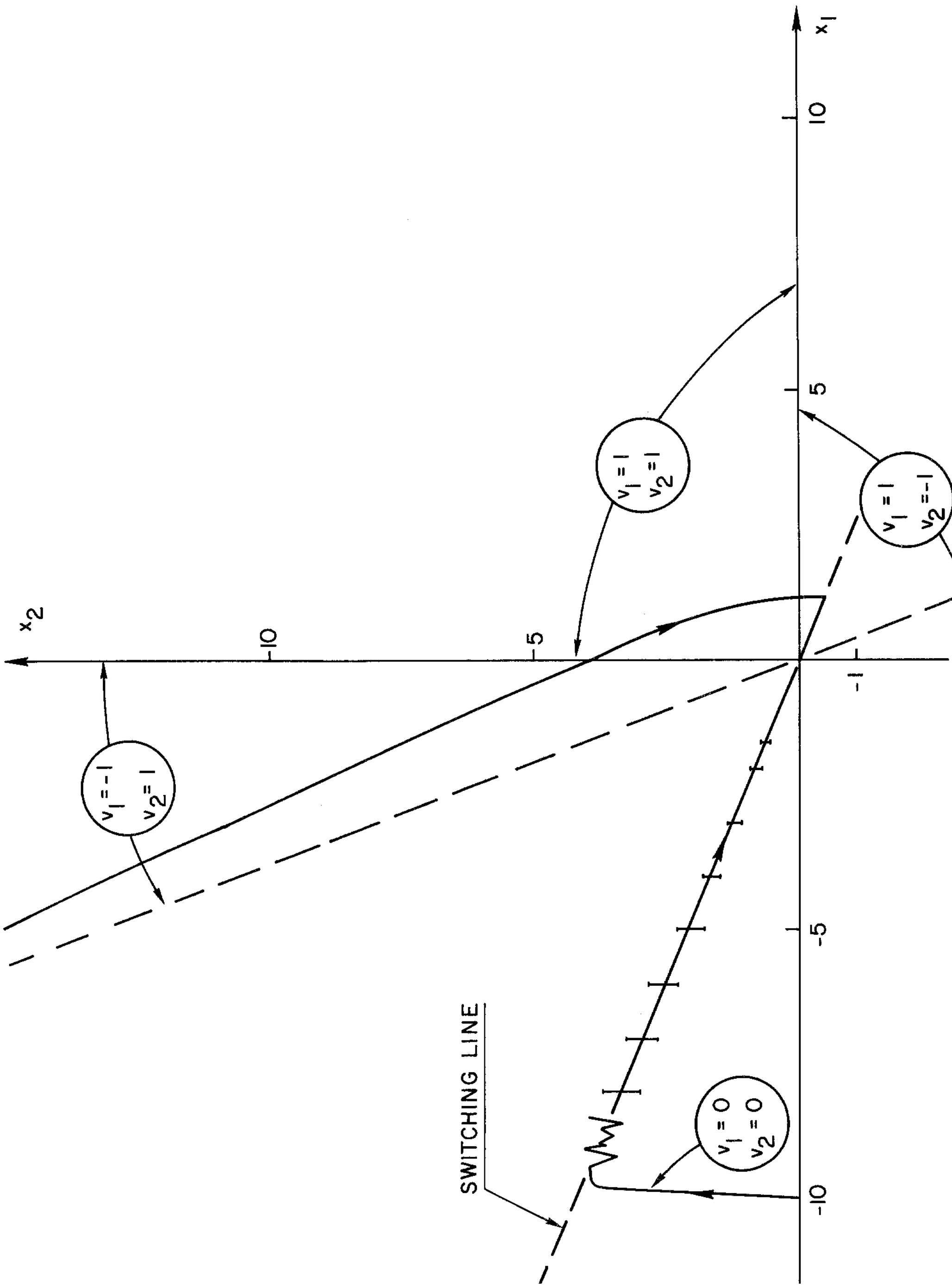


FIGURE 8, EXAMPLE 4

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