Stabilizing Model Predictive Control of Hybrid Systems

M. Lazar, W. P. M. H. Heemels, S. Weiland, and A. Bemporad

***Abstract*—In this note, we investigate the stability of hybrid systems in closed-loop with model predictive controllers (MPC). *A priori* sufficient conditions for Lyapunov asymptotic stability and exponential stability are derived in the *terminal cost and constraint set* fashion, while allowing for discontinuous system dynamics and discontinuous MPC value functions. For constrained piecewise affine (PWA) systems as prediction models, we present novel techniques for computing a terminal cost and a terminal constraint set that satisfy the developed stabilization conditions. For quadratic MPC costs, these conditions translate into a linear matrix inequality while, for MPC costs based on 1, ∞-norms, they are obtained as norm inequalities. New ways for calculating low complexity piecewise polyhedral positively invariant sets for PWA systems are also presented. An example illustrates the developed theory.**

***Index Terms*—Hybrid systems, Lyapunov stability, model predictive control (MPC), piecewise affine systems.**

I. INTRODUCTION

One of the problems in model predictive control (MPC) that has received an increased attention over the years consists in guaranteeing closed-loop stability for the controlled system. The usual approach to ensure stability in MPC is to consider the value function of the MPC cost as a candidate Lyapunov function. Then, if the system dynamics is continuous, the classical Lyapunov stability theory [1] can be used to prove that the MPC control law is stabilizing. For a comprehensive overview on stability of receding horizon control in discrete-time we refer the reader to [2] and the references therein.

The recent development of MPC for hybrid systems, which are inherently discontinuous and nonlinear, requires a reconsideration of the stability results, as it was also pointed out in the excellent survey [2]. Attractivity was proven for hybrid systems in closed-loop with model predictive controllers in [3] and [4]. However, proofs of Lyapunov stability only appeared in the literature recently; for particular classes of hybrid systems and MPC cost functions, see, for example, [5]–[7]. In these works, either *continuous* piecewise affine (PWA) systems are considered, [5], [6] which are in fact *Lipschitz continuous* systems or, in [7], asymptotic stability is established via the results of [2], where continuity of the MPC value function is assumed. Note that this property does not hold in general for the MPC value function, when hybrid systems, such as PWA systems, are employed as prediction models [4].

In this article we present *a priori* verifiable conditions that guarantee stability of discrete-time nonlinear, *possibly discontinuous*, systems in closed-loop with MPC controllers. We develop a general theorem on asymptotic stability in the Lyapunov sense that unifies most of the previous results on stability of MPC. This theorem applies to a wide class of hybrid systems and MPC cost functions, and it does not require continuity of the MPC value function nor of the system dynamics. Efficient methods for calculating the terminal cost, for both quadratic and 1, ∞-norm MPC costs, and the terminal constraint set are developed for the class of *discontinuous* PWA systems, with the origin not necessarily in the interior of one of the regions in the state–space partition. New algorithms for calculating *low complexity piecewise polyhedral* positively invariant sets for PWA systems are also presented.

The remainder of the manuscript is organized as follows. Section II describes the MPC problem setup. The general stability results for MPC of hybrid systems are presented in Section III. New methods for computing the terminal cost are derived in Section IV, while Section V contains the new algorithms for computing low complexity piecewise polyhedral invariant sets for PWA systems. An example is given in Section VI and conclusions are summarized in Section VII.

A. Notation and Basic Definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of nonnegative reals, the set of integers and the set of nonnegative integers, respectively. We use the notation $\mathbb{Z}_{\geq c}$ to denote the set $\{k \in \mathbb{Z} | k \geq c\}$ for some $c \in \mathbb{Z}$. For a set $\mathcal{P} \subseteq \mathbb{R}^n$, we denote by $\partial\mathcal{P}$ the boundary of $\mathcal{P}$, by $\operatorname{int}(\mathcal{P})$ its interior and by $\operatorname{cl}(\mathcal{P})$ its closure. A polyhedral set is a convex set obtained as the intersection of a finite number of open and/or closed half-spaces. A piecewise polyhedral set is a set obtained as the union of a finite number of polyhedra. For a vector $x \in \mathbb{R}^n$, we use $\|x\|$ to denote an arbitrary Hölder vector p-norm defined for $1 \leq p \leq \infty$. For a positive–definite matrix Z, $Z^{1/2}$ denotes the Cholesky factor, which satisfies $(Z^{1/2})^\top Z^{(1/2)} = Z^{1/2}(Z^{1/2})^\top = Z$ and, $\lambda_{\min}(Z)$ and $\lambda_{\max}(Z)$ denote the smallest and the largest eigenvalue of Z, respectively. For a matrix $Z \in \mathbb{R}^{m \times n}$ with full-column rank, $Z^{-L} \triangleq (Z^\top Z)^{-1} Z^\top$ denotes its left Moore–Penrose inverse, which satisfies $Z^{-L}Z = I_n$ and $\|Z\|$ denotes the corresponding induced matrix norm. A real-valued scalar function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. For definitions of Lyapunov stability, asymptotic stability, and exponential stability, we refer the reader to [1] and [9].

II. SETTING UP THE MPC OPTIMIZATION PROBLEM

Consider the discrete-time nonlinear system

$$x_{k+1} = g(x_k, u_k); \qquad k \in \mathbb{Z}_+ \tag{1}$$

where $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state and $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control action at the discrete-time instant $k \in \mathbb{Z}_+$. $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is an arbitrary, *possibly discontinuous*, nonlinear function. Note that the class of nonlinear systems (1) contains certain classes of hybrid systems, such as PWA systems, due to the fact that $g(\cdot,\cdot)$ may be discontinuous. The sets $\mathbb{X}$ and $\mathbb{U}$ specify state and input constraints and it is assumed that they are compact polyhedral sets that contain the origin in their interior. We assume for simplicity that the origin is an equilibrium state for (1) with $u = 0$, meaning that $g(0,0) = 0$. For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) \triangleq (x_{1|k}, \ldots, x_{N|k})$ denote the state sequence generated by system (1) from initial state $x_{0|k} \triangleq x_k$ and by applying the input sequence $\mathbf{u}_k \triangleq (u_{0|k}, \ldots, u_{N-1|k}) \in \mathbb{U}^N$, where $\mathbb{U}^N \triangleq \mathbb{U} \times \ldots \times \mathbb{U}$. Furthermore, let $\mathbb{X}_T \subseteq \mathbb{X}$ denote a desired target set that contains the origin in its interior. The class of *admissible input sequences* defined with respect to $\mathbb{X}_T$ and state $x_k \in \mathbb{X}$ is $\mathcal{U}_N(x_k) \triangleq \{\mathbf{u}_k \in \mathbb{U}^N | \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N, x_{N|k} \in \mathbb{X}_T\}$.

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Problem II.1: Let the target set $\mathbb{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{Z}_{\geq 1}$ be given and let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0,0) = 0$ be mappings. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be given and minimize the cost function $J(x_k, \mathbf{u}_k) \triangleq F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k})$, with prediction model (1), over all input sequences $\mathbf{u}_k \in \mathcal{U}_N(x_k)$.

In the MPC literature, $F(\cdot)$, $L(\cdot,\cdot)$ and N are called the terminal cost, the stage cost and the prediction horizon, respectively. We call an initial state $x_0 \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x_0) \neq \emptyset$. Similarly, Problem II.1 is said to be *feasible* for $x \in \mathbb{X}$ if $\mathcal{U}_N(x) \neq \emptyset$. Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible states* with respect to Problem II.1 and let

$$V_{\mathrm{MPC}} : \mathbb{X}_f(N) \to \mathbb{R}_+ \quad V_{\mathrm{MPC}}(x_k) \triangleq \inf_{\mathbf{u}_k \in \mathcal{U}_N(x_k)} J(x_k, \mathbf{u}_k) \quad (2)$$

denote the MPC value function corresponding to Problem II.1. We assume that there exists an optimal sequence of controls $\mathbf{u}_k^* \triangleq (u_{0|k}^*, u_{1|k}^*, \ldots, u_{N-1|k}^*)$ for Problem II.1 and any state $x_k \in \mathbb{X}_f(N)$. Hence, the infimum in (2) is a minimum and $V_{\mathrm{MPC}}(x_k) = J(x_k, \mathbf{u}_k^*)$. Then, *the MPC control law* is defined as

$$u^{\mathrm{MPC}}(x_k) \triangleq u_{0|k}^*, \qquad k \in \mathbb{Z}_+. \quad (3)$$

The stability results presented in this note also hold when the optimum is not unique in Problem II.1, i.e., all results apply irrespective of which optimal sequence is selected.

III. Stabilization Conditions for MPC of Hybrid Systems

In this section, we investigate the stabilization of the *discontinuous* nonlinear system (1) using MPC. We will employ a *terminal cost and constraint set* method, as the one used for *smooth* nonlinear systems in [2], to guarantee stability for the closed-loop system (1)–(3). Typically, this method relies on continuity of $V_{\mathrm{MPC}}(\cdot)$ and of the system dynamics (e.g., see [2, Sec. III]). This property is no longer guaranteed in the case of discontinuous dynamical systems, such as hybrid systems [4]. Actually, in the survey [2] it was pointed out that all the concepts and ideas used in MPC should be reconsidered in the hybrid context.

A. Main Result

Let $h : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary, possibly discontinuous, nonlinear function with $h(0) = 0$ and let $\mathbb{X}_{\mathbb{U}} \triangleq \{x \in \mathbb{X} | h(x) \in \mathbb{U}\}$. The following theorem was obtained as a kind of general and unifying result, based on previous results regarding stability of discrete-time nonlinear MPC.

Assumption III.1: There exist $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}$, a neighborhood of the origin $\mathcal{N} \subseteq \mathbb{X}_f(N)$ and a function $h(\cdot)$ such that $\mathbb{X}_T \subseteq \mathbb{X}_{\mathbb{U}}$, with $0 \in \mathrm{int}(\mathbb{X}_T)$, is a positively invariant set [8] for system (1) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$

$$L(x,u) \geq \alpha_1(\|x\|) \quad \text{for all} \quad x \in \mathbb{X}_f(N) \quad \text{and all} \quad u \in \mathbb{U} \quad (4a)$$

$$F(x) \leq \alpha_2(\|x\|) \quad \text{for all} \quad x \in \mathcal{N} \quad \text{and} \quad (4b)$$

$$F(g(x, h(x))) - F(x) + L(x, h(x)) \leq 0 \quad \text{for all} \quad x \in \mathbb{X}_T. \quad (4c)$$

Theorem III.2: Fix $N \in \mathbb{Z}_{\geq 1}$ and suppose that Assumption III.1 holds. Then, the following hold.

i) If Problem II.1 is feasible at time $k \in \mathbb{Z}_+$ for state $x_k \in \mathbb{X}$, then Problem II.1 is feasible at time $k+1$ for state $x_{k+1} = g(x_k, u^{\mathrm{MPC}}(x_k))$. Moreover, $\mathbb{X}_T \subseteq \mathbb{X}_f(N)$.

ii) The origin of the MPC closed-loop system (1)–(3) is asymptotically stable in the Lyapunov sense for initial conditions in $\mathbb{X}_f(N)$.

iii) If Assumption III.1 holds with $\alpha_1(s) \triangleq as^{\lambda}$, $\alpha_2(s) \triangleq bs^{\lambda}$ for some constants $a, b, \lambda > 0$, then the origin of the MPC closed-loop system (1)–(3) is exponentially stable in $\mathbb{X}_f(N)$.

Proof: Since the proof of this theorem can be obtained from Assumption III.1 by following the standard steps indicated, for example, in the survey [2], we only present the Lyapunov stability proof (which differs from the proof given in [2]) and we refer the reader to [9] for a complete proof. First, we recall that statement i) is proven by observing that the shifted sequence of controls $\mathbf{u}_{k+1} \triangleq (u_{1|k}^*, \ldots, u_{N|k}^*, h(x_{N-1|k+1}))$ is feasible at time $k+1$, where $x_{N-1|k+1} \triangleq x_{N|k}^*$ and * denotes the optimum control actions or predicted state-trajectory at time k. By optimality and Assumption III.1, it follows that $V_{\mathrm{MPC}}(x) \geq \alpha_1(\|x\|)$ for all $x \in \mathbb{X}_f(N)$, $V_{\mathrm{MPC}}(x) \leq \alpha_2(\|x\|)$ for all $x \in \widetilde{\mathcal{N}}$, where $\widetilde{\mathcal{N}} \triangleq \mathbb{X}_T \cap \mathcal{N}$, and that $V_{\mathrm{MPC}}(g(x, u^{\mathrm{MPC}}(x))) - V_{\mathrm{MPC}}(x) \leq -\alpha_1(\|x\|)$ for all $x \in \mathbb{X}_f(N)$. Since $\mathbb{X}$ is assumed to be compact and $\mathbb{X}_f(N) \subseteq \mathbb{X}$, it follows that $\mathbb{X}_f(N)$ is bounded. From i), it follows that $\mathbb{X}_f(N)$ is a positively invariant set for the MPC closed-loop system (1)–(3). Let x_k be the solution of (1)–(3), obtained from the initial condition x_0 at time $k = 0$. Choose an $\eta > 0$ such that the ball $\mathcal{B}_\eta \triangleq \{x \in \mathbb{R}^n | \|x\| \leq \eta\}$ satisfies $\mathcal{B}_\eta \subseteq \widetilde{\mathcal{N}}$. Due to $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}$ we can choose for any $0 < \varepsilon \leq \eta$ a $\delta \in (0, \varepsilon)$ such that $\alpha_2(\delta) < \alpha_1(\varepsilon)$. For any $x_0 \in \mathcal{B}_\delta \subseteq \mathbb{X}_f(N)$, due to positive invariance of $\mathbb{X}_f(N)$, we have

$$\ldots \leq V_{\mathrm{MPC}}(x_{k+1}) \leq V_{\mathrm{MPC}}(x_k) \leq \ldots \leq V_{\mathrm{MPC}}(x_0) \leq \alpha_2(\|x_0\|) \leq \alpha_2(\delta) < \alpha_1(\varepsilon).$$

Since $V_{\mathrm{MPC}}(x) \geq \alpha_1(\varepsilon)$ for all $x \in \mathbb{X}_f(N) \setminus \mathcal{B}_\varepsilon$ it follows that $x_k \in \mathcal{B}_\varepsilon$ for all $k \in \mathbb{Z}_+$. Hence, the origin of the MPC closed-loop system (1)–(3) is *Lyapunov stable*. ■

Note that the hypothesis of Theorem III.2 does not require that $V_{\mathrm{MPC}}(\cdot)$ or $g(\cdot,\cdot)$ are continuous, not even on a neighborhood of the origin. It only implies continuity at the point $x = 0$.

B. The Class of Piecewise Affine Systems

The remainder of the article focuses on discrete-time discontinuous piecewise affine systems, i.e.,

$$x_{k+1} = g(x_k, u_k) \triangleq A_j x_k + B_j u_k + f_j, \qquad \text{if } x_k \in \Omega_j, \ j \in \mathcal{S} \quad (5)$$

which is a sub-class of the discontinuous nonlinear system (1). Also, we take the nonlinear function $h(\cdot)$ as a piecewise linear (PWL) state-feedback, i.e.,

$$h(x) \triangleq K_j x, \qquad \text{if } x_k \in \Omega_j, \ j \in \mathcal{S}. \quad (6)$$

Here, $A_j \in \mathbb{R}^{n \times n}$, $B_j \in \mathbb{R}^{n \times m}$, $f_j \in \mathbb{R}^n$, $K_j \in \mathbb{R}^{m \times n}$, and $j \in \mathcal{S}$ with $\mathcal{S} \triangleq \{1, 2, \ldots, s\}$ a *finite set* of indexes and s denotes the number of affine sub-systems in (5). The collection $\{\Omega_j | j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j \in \mathcal{S}} \Omega_j = \mathbb{X}$ and $\mathrm{int}(\Omega_i) \cap \mathrm{int}(\Omega_j) = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron (not necessarily closed). Let $\mathcal{S}_0 \triangleq \{j \in \mathcal{S} | 0 \in \mathrm{cl}(\Omega_j)\}$ and let $\mathcal{S}_1 \triangleq \{j \in \mathcal{S} | 0 \notin \mathrm{cl}(\Omega_j)\}$, so that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. We assume that the origin is an equilibrium state for (5) with $u = 0$ and we require that $f_j = 0$ for all $j \in \mathcal{S}_0$. This implies continuity at the point $x = 0$ and *not* on a neighborhood of the origin. The class of hybrid systems described by (5) contains PWA systems which *may be discontinuous* over the boundaries of the regions Ω_j and it allows that the origin lies on the boundaries of multiple regions Ω_j, $j \in \mathcal{S}$.

If the PWA system (5) is employed as prediction model, the optimization problem corresponding to Problem II.1 is a mixed integer quadratic programming (MIQP) problem in the case of quadratic MPC costs, and a mixed integer linear programming (MILP) problem in the case of $1, \infty$-norm MPC costs. These problems can be solved using the hybrid toolbox (HT) [10] or the multi parametric toolbox (MPT) [11]. Note that if a MIQP (MILP) problem is feasible, the global optimum is attained because, in principle, a MIQP (MILP) consists of a finite number of QP (LP) problems (see, for example, [4]). Then, due to the fact that each QP (LP) (with bounded feasible set) attains its optimum, the existence of an optimum for the MIQP (MILP) problem is guaranteed (although it may not be unique). Hence, the standing assumption employed in Section II on existence of optimal control sequences holds for PWA prediction models and the result of Theorem III.2 applies.

Although we focus on PWA systems of the form (5), the results presented in the sequel have a wider applicability since it is known [12] that PWA systems are equivalent under certain mild assumptions with other relevant classes of hybrid systems, such as mixed logical dynamical systems and linear complementarity systems, and they can approximate nonlinear systems arbitrarily well.

C. The Problem Statement Reconsidered

For a given stage cost $L(\cdot,\cdot)$, the fundamental stability result for MPC of hybrid systems provided by Theorem III.2 comes down to computing a terminal cost $F(\cdot)$, a function $h(\cdot)$ and a terminal set $\mathbb{X}_T$ such that Assumption III.1 holds. This is a nontrivial problem, which depends on the type of system dynamics and MPC cost.

For example, in the particular case of PWA systems and quadratic MPC costs this problem has only been solved partially, in [7], i.e., by employing a common quadratic Lyapunov function for system (5) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$. This is known to be conservative (see, for example, [13]) because there are PWA systems which only admit a piecewise quadratic (PWQ) Lyapunov function (see also [9, Sec. 3.7] for an example where the method of [7] fails). In the case of MPC costs based on 1, ∞-norms, to the authors' knowledge, there is no systematic method available for solving this problem. A complete solution to the problem of calculating the terminal cost is presented in Section IV.

The problem of computing the terminal set $\mathbb{X}_T$ boils down to computing positively invariant sets for PWA systems, which is a notoriously difficult problem. An algorithm for calculating the *maximal* positively invariant set [8] for PWA systems was recently presented in [14]. However, it is known that the maximal positively invariant set inside a given compact set is a piecewise polyhedral set for PWA systems, which can be very complex (i.e., it may consist of the union of a very large number of polyhedra, which in principle can be infinite, if the algorithm does not converge). This in turn influences the computational complexity of the MIQP (MILP) MPC optimization problem. Hence, it would be desirable to obtain a tradeoff between the size of the terminal constraint set and its complexity. Section V deals with this issue.

Note that once a quadratic or 1, ∞-norm terminal cost $F(\cdot)$, a nonlinear function $h(\cdot)$ of the form (6) and a piecewise polyhedral terminal set $\mathbb{X}_T$ that satisfy Assumption III.1 for system (5) have been calculated, it is well-known [4] that the set of feasible states $\mathbb{X}_f(N)$ can be obtained explicitly for a fixed value of the prediction horizon $N \in \mathbb{Z}_{\geq 1}$ using either the HT [10] or the MPT [11]. This may help in selecting a suitable prediction horizon N.

IV. Computation of the Terminal Cost

In this section, we provide solutions to the problem of computing a terminal cost $F(\cdot)$ and a function $h(\cdot)$ of the form (6) for both quadratic and 1, ∞-norm MPC costs.

A. Quadratic MPC Costs

Consider the case when quadratic forms are used to define the cost function, i.e., $F(x) = \|P_j^{1/2}x\|_2^2 = x^\top P_j x$ if $x \in \mathbb{X}_T \cap \Omega_j$ and $L(x,u) = \|Q^{1/2}x\|_2^2 + \|R^{1/2}u\|_2^2 = x^\top Qx + u^\top Ru$. Without significant loss of generality, for quadratic MPC costs we assume that $\mathbb{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$. In this case, $P_j, Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are assumed to be positive–definite and symmetric matrices. For the aforementioned stage and terminal costs, it holds that $L(x,u) \geq x^\top Qx \geq \lambda_{\min}(Q)\|x\|_2^2$ for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$, and $F(x) \leq \max_{j\in\mathcal{S}_0}\lambda_{\max}(P_j)\|x\|_2^2$ for all $x \in \mathbb{R}^n$. Therefore, conditions (4a) and (4b) are trivially satisfied with $\alpha_1(\|x\|) \triangleq \lambda_{\min}(Q)\|x\|_2^2$ and $\alpha_2(\|x\|) \triangleq \max_{j\in\mathcal{S}_0}\lambda_{\max}(P_j)\|x\|_2^2$. Next, we provide methods for calculating the matrices $\{(P_j, K_j) | j \in \mathcal{S}_0\}$ such that inequality (4c) is satisfied for the PWA system (5).

Let $\mathcal{Q}_{ji} \triangleq \{x \in \Omega_j | \exists u \in \mathbb{U} : A_j x + B_j u + f_j \in \Omega_i\}$, $(j,i) \in \mathcal{S}_0 \times \mathcal{S}_0$, let[1] $\mathcal{S}_{t0} \triangleq \{(j,i) \in \mathcal{S}_0 \times \mathcal{S}_0 | \mathcal{Q}_{ji} \neq \emptyset\}$ and consider the PWL sub-system of the PWA system (5)

$$x_{k+1} = A_j x_k + B_j u_k, \qquad \text{if } x_k \in \mathbb{X}_T \cap \Omega_j,\ j \in \mathcal{S}_0. \tag{7}$$

Letting u_k be the state-feedback (6) in (7), $A_j^{cl} \triangleq A_j + B_j K_j$ and substituting the resulting closed-loop system and $F(\cdot)$ in (4c) yields that it is sufficient to find $P_j > 0$, K_j, $j \in \mathcal{S}_0$ that satisfy

$$P_j - \left(A_j^{cl}\right)^\top P_i A_j^{cl} - Q - K_j^\top R K_j > 0 \qquad \forall (j,i) \in \mathcal{S}_{t0} \tag{8}$$

for (4c) to be satisfied with strict inequality. Next, we present three methods that can be used to solve efficiently the nonlinear matrix inequality (8) via semidefinite programming.

Lemma IV.1: Let $\{(P_j, K_j, Z_j, Y_j, G_j) | j \in \mathcal{S}_0\}$ with Z_j, P_j positive definite and symmetric, and G_j invertible for all $j \in \mathcal{S}_0$ denote unknown variables that are related according to $Z_j = P_j^{-1}$ in (10) and (11), and $Y_j = K_j G_j$ in (12), $Y_j = K_j P_j^{-1}$ and $K_j = Y_j G_j^{-1}$, $j \in \mathcal{S}_0$. Furthermore, let $\Gamma_j \triangleq A_j Z_j + B_j Y_j$ and let $\Xi_j \triangleq A_j G_j + B_j Y_j$.

Then, the following matrix inequalities are equivalent:

$$\begin{pmatrix} P_i & 0 \\ 0 & P_j - \left(A_j^{cl}\right)^\top P_i A_j^{cl} - Q - K_j^\top R K_j \end{pmatrix} > 0 \qquad \forall (j,i) \in \mathcal{S}_{t0} \tag{9}$$

$$\begin{pmatrix} Z_j & Z_j & Y_j^\top & \Gamma_j^\top \\ Z_j & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ \Gamma_j & 0 & 0 & Z_i \end{pmatrix} > 0 \qquad \forall (j,i) \in \mathcal{S}_{t0}; \tag{10}$$

$$\begin{pmatrix} Z_j & \Gamma_j^\top & \left(R^{\frac{1}{2}}Y_j\right)^\top & \left(Q^{\frac{1}{2}}Z_j\right)^\top \\ \Gamma_j & Z_i & 0 & 0 \\ R^{\frac{1}{2}}Y_j & 0 & I & 0 \\ Q^{\frac{1}{2}}Z_j & 0 & 0 & I \end{pmatrix} > 0 \qquad \forall (j,i) \in \mathcal{S}_{t0} \tag{11}$$

$$\begin{pmatrix} G_j + G_j^\top - Z_j & G_j^\top & Y_j^\top & \Xi_j^\top \\ G_j & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ \Xi_j & 0 & 0 & Z_i \end{pmatrix} > 0 \qquad \forall (j,i) \in \mathcal{S}_{t0}. \tag{12}$$

Proof: The equivalences (9) ⇔ (10) and (9) ⇔ (11) are proven by applying the Schur complement to (10) and (11), respectively, making the change of variables $Z_j = P_j^{-1}$, $Y_j = K_j P_j^{-1}$, and pre- and post-multiplying with $\begin{pmatrix} P_i & 0 \\ 0 & P_j \end{pmatrix}$ (see [9, Sec. 3.4] for a complete proof). The equivalence (9) ⇔ (12) is proven in a similar way by applying the Schur complement to (12) and exploiting the matrix inequality $G_j^\top Z_j^{-1} G_j \geq G_j + G_j^\top - Z_j$ for all $j \in \mathcal{S}_0$. ■

After solving any of the previous LMIs, the terminal weights P_j and the feedbacks K_j are simply recovered as $P_j \triangleq Z_j^{-1}$ and $K_j \triangleq$

[1]The set of pairs of indexes $\mathcal{S}_{t0}$ can be easily determined offline by solving s_0^2 linear programs, where s_0 is the number of elements of $\mathcal{S}_0$

$Y_j Z_j^{-1}$, $j \in \mathcal{S}_0$ for (10) and (11), and as $P_j \triangleq Z_j^{-1}$ and $K_j \triangleq Y_j G_j^{-1}$, $j \in \mathcal{S}_0$ for (12).

Note that solving any of the LMIs of Lemma IV.1 boils down to searching for a PWQ Lyapunov function. Conservativeness of (9) can be further reduced by employing an S-procedure technique, see [9, Sec. 3.4] for details.

B. MPC Costs Based on 1, ∞-Norms

Consider the case when 1, ∞-norms are used to define the cost function, i.e., $F(x) = \|P_j x\|$ if $x \in \mathbb{X}_T \cap \Omega_j$ and $L(x,u) = \|Qx\| + \|Ru\|$, where $\|\cdot\|$ denotes the 1-norm or the ∞-norm, for brevity of notation. Here, $P_j \in \mathbb{R}^{p\times n}$, $Q \in \mathbb{R}^{q\times n}$, and $R \in \mathbb{R}^{r\times n}$ are assumed to be matrices that have full-column rank. In this setting, we no longer require that $\mathbb{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$. Since Q has full-column rank there exists a positive number γ such that $\|Qx\| \geq \gamma\|x\|$ for all $x \in \mathbb{R}^n$. Then, it follows that $L(x,u) \geq \|Qx\| \geq \gamma\|x\|$ for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$, and that $F(x) \leq \max_{j\in\mathcal{S}} \|P_j\|\|x\|$ for all $x \in \mathbb{R}^n$. Hence, conditions (4a) and (4b) are trivially satisfied with $\alpha_1(\|x\|) \triangleq \gamma\|x\|$ and $\alpha_2(\|x\|) \triangleq \max_{j\in\mathcal{S}} \|P_j\|\|x\|$.

Let $\mathcal{Q}_{ji} \triangleq \{x \in \Omega_j | \exists u \in \mathbb{U} : A_j x + B_j u + f_j \in \Omega_i\}$, $(j,i) \in \mathcal{S}\times\mathcal{S}$ and let $\mathcal{S}_t \triangleq \{(j,i) \in \mathcal{S}\times\mathcal{S} | \mathcal{Q}_{ji} \neq \emptyset\}$. Substituting (5) and $F(\cdot)$ in (4c) yields that it is sufficient to find $\{(P_j,K_j) | j \in \mathcal{S}\}$ that satisfy for all $x \in \mathbb{X}_T$ and all $(j,i) \in \mathcal{S}_t$

$$\|P_i((A_j + B_jK_j)x + f_j)\| - \|P_jx\| + \|Qx\| + \|RK_jx\| \leq 0 \quad (13)$$

for (4c) to be satisfied. Consider now the following norm inequalities, for all $(j,i) \in \mathcal{S}_t$:

$$\left\|P_i(A_j + B_jK_j)P_j^{-L}\right\| + \left\|QP_j^{-L}\right\| + \left\|RK_jP_j^{-L}\right\| \leq 1 - \gamma_{ji} \quad (14)$$

and

$$\|P_if_j\| \leq \gamma_{ji}\|P_jx\| \qquad \forall x \in \mathbb{X}_T \cap \Omega_j \quad (15)$$

where $\gamma_{ji} \in [0,1)$, $(j,i) \in \mathcal{S}_t$. Note that, because $f_j = 0$ for all $j \in \mathcal{S}_0$, (15) trivially holds if $\mathcal{S} = \mathcal{S}_0$.

Theorem IV.2: Suppose (14), (15) is solvable in (P_j, K_j, γ_{ji}) where P_j has full-column rank and $\gamma_{ji} \in [0,1)$ for $(j,i) \in \mathcal{S}_t$. Then, (P_j, K_j) with $j \in \mathcal{S}$ is a solution of the norm inequality (13).

Proof: Since $\{(P_j, K_j, \gamma_{ji}) | (j,i) \in \mathcal{S}_t\}$ satisfy (14) we have that for all $(j,i) \in \mathcal{S}_t$

$$\left\|P_i(A_j+B_jK_j)P_j^{-L}\right\| + \left\|QP_j^{-L}\right\| + \left\|RK_jP_j^{-L}\right\| + \gamma_{ji} - 1 \leq 0. \quad (16)$$

Right multiplying the inequality (16) with $\|P_jx\|$ and using the inequality (15) yields

$$\begin{aligned} 0 &\geq \left\|P_i(A_j + B_jK_j)P_j^{-L}\right\| \|P_jx\| + \left\|QP_j^{-L}\right\| \|P_jx\| \\ &\quad + \gamma_{ji}\|P_jx\| + \left\|RK_jP_j^{-L}\right\| \|P_jx\| - \|P_jx\| \\ &\geq \left\|P_i(A_j + B_jK_j)P_j^{-L}P_jx\right\| + \left\|QP_j^{-L}P_jx\right\| \\ &\quad + \|P_if_j\| + \left\|RK_jP_j^{-L}P_jx\right\| - \|P_jx\| \\ &\geq \|P_i(A_j + B_jK_j)x + P_if_j\| + \|RK_jx\| \\ &\quad + \|Qx\| - \|P_jx\|. \end{aligned} \quad (17)$$

Hence, inequality (13) holds. ■

A way to solve the norm inequalities (14) is to minimize the cost

$$J_1(\{P_j, K_j | j \in \mathcal{S}\}) \triangleq \max_{(j,i)\in\mathcal{S}_t} \left(\left\|P_i(A_j + B_jK_j)P_j^{-L}\right\| + \left\|QP_j^{-L}\right\| + \left\|RK_jP_j^{-L}\right\|\right)$$

if the resulting value function is less than 1. Alternatively, one can solve an optimization problem with a zero cost subject to the nonlinear constraint $J_1(\{P_j, K_j | j \in \mathcal{S}\}) < 1$. These are nonconvex nonlinear optimization problems, which can be solved using black-box optimization solvers, such as *fmincon* and *fminunc* of Matlab. The nonlinear nature of these optimization problems is not critical for online implementation, since they are solved offline.

Once the matrices P_j and the numbers γ_{ji} satisfying (14) have been found, one still has to check that they also satisfy inequality (15), provided that $\mathcal{S} \neq \mathcal{S}_0$. For example, this can be verified by checking the inequality $\|P_if_j\| \leq \gamma_{ji} \min_{x\in\mathbb{X}_T\cap\Omega_j} \|P_jx\|$, $(j,i) \in \mathcal{S}_t$. To overcome the difficulty of solving (14) and (15) simultaneously, one can require that $\mathbb{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$ is a positively invariant set only for the PWL sub-system (7), as done for quadratic MPC costs. In this case, Theorem IV.2 can be reformulated as follows.

Corollary IV.3: Suppose that the inequality

$$\left\|P_i(A_j + B_jK_j)P_j^{-L}\right\| + \left\|QP_j^{-L}\right\| + \left\|RK_jP_j^{-L}\right\| \leq 1, \quad (j,i) \in \mathcal{S}_{t0} \quad (18)$$

is solvable in (P_j, K_j) for P_j with full-column rank and that $\mathbb{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$. Then, (P_j, K_j) with $j \in \mathcal{S}_0$ is a solution of the norm inequality (13).

V. Computation of the Terminal Constraint Set

In this section, we present new methods for computing low complexity piecewise polyhedral positively invariant sets for PWA systems. Consider the closed-loop system (7) with the feedback gains calculated using one of the methods from Section IV, i.e.,

$$x_{k+1} = (A_j + B_jK_j)x_k = A_j^{cl}x_k \qquad \text{if } x_k \in \Omega_j, \;\; j \in \mathcal{S}_0. \quad (19)$$

In the first method, we consider the autonomous *switched linear system* corresponding to (19)

$$x_{k+1} = A_j^{cl}x_k, \qquad j \in \mathcal{S}_0. \quad (20)$$

Note that we removed the switching rule from (19), turning the PWL system (19) into a switched linear system (20) under arbitrary switching. It is easy to prove that a set which is positively invariant for the switched linear system (20) under arbitrary switching is also a positively invariant set for the PWL system (19). Since we require that $\mathbb{X}_T \subseteq \mathbb{X}_{\mathbb{U}} \cap \{\cup_{j\in\mathcal{S}_0}\Omega_j\}$ and $\mathbb{X}_{\mathbb{U}}$ is not convex in general, we consider a new set instead, $\widetilde{\mathbb{X}}_{\mathbb{U}}$, taken as a reasonably large compact polyhedral set (that contains the origin in its interior) inside $\mathbb{X}_{\mathbb{U}} \cap \{\cup_{j\in\mathcal{S}_0}\Omega_j\}$. For an arbitrary set $\mathbb{X}$ we define $\mathcal{Q}_j^1(\mathbb{X}) \triangleq \{x \in \mathbb{R}^n | A_j^{cl}x \in \mathbb{X}\}$. Note that if $\mathbb{X}$ is a polyhedron that contains the origin, then $\mathcal{Q}_j^1(\mathbb{X})$ has the same properties and, if $\mathbb{X}$ is compact, then $\mathcal{Q}_j^1(\mathbb{X})$ is closed (see [8] for proofs). Consider now the sequence of sets

$$\mathbb{X}_0 = \widetilde{\mathbb{X}}_{\mathbb{U}} \quad \mathbb{X}_i = \bigcap_{j\in\mathcal{S}_0} \mathbb{X}_i^j, \qquad i = 1, 2, \ldots \quad (21)$$

where $\mathbb{X}_i^j \triangleq \mathcal{Q}_j^1(\mathbb{X}_{i-1}) \bigcap \mathbb{X}_{i-1}$, $i = 1, 2, \ldots$.

Theorem V.1: The following properties hold with respect to the sequence of sets (21).

i) The maximal positively invariant set contained in the safe set $\widetilde{\mathbb{X}}_{\mathbb{U}}$ for system (20) under arbitrary switching is a convex set that contains the origin and is given by

$$\mathcal{P} = \bigcap_{i=0}^{\infty} \mathbb{X}_i = \lim_{i\to\infty} \mathbb{X}_i. \quad (22)$$

ii) If an algorithm based on the recurrent sequence of sets (21) terminates in a finite number of iterations then the set $\mathcal{P}$ defined in (22) is a polyhedral set.

iii) If there exists a λ-contractive set [8] with $0 < \lambda < 1$ for system (20) under arbitrary switching that contains the origin in its interior, then an algorithm based on the recurrent sequence of sets (21) terminates in a finite number of iterations and $0 \in \mathrm{int}(\mathcal{P})$.

iv) The set $\mathcal{P}$ defined in (22) is a positively invariant set for the PWL system (19).

Proof:

i) If $x \in \mathcal{P}$, then $x \in \mathbb{X}_i$ for all i. Hence, we have that $A_j^{cl} x \in \mathbb{X}_{i-1}$ for all $j \in \mathcal{S}_0$ and all $i \geq 1$. Then, $A_j^{cl} x \in \mathcal{P}$ for all $j \in \mathcal{S}_0$. So, $\mathcal{P}$ is a positively invariant set for system (20) with arbitrary switching. In order to prove that the set $\mathcal{P}$ is maximal let $\widetilde{\mathcal{P}} \subseteq \widetilde{\mathbb{X}}_{\mathbb{U}} = \mathbb{X}_0$ be a positively invariant set for system (20) with arbitrary switching. In order to use induction, we assume that $\widetilde{\mathcal{P}} \subseteq \mathbb{X}_i$ for some i (note that this holds for $i = 0$). Due to the positive invariance of $\widetilde{\mathcal{P}}$, for any $x \in \widetilde{\mathcal{P}}$ we have that $A_j^{cl} x \in \widetilde{\mathcal{P}} \subseteq \mathbb{X}_i$ for all $j \in \mathcal{S}_0$. Hence, $x \in \mathbb{X}_{i+1}$. Thus, $\widetilde{\mathcal{P}} \subset \mathbb{X}_{i+1}$ and by induction $\widetilde{\mathcal{P}} \subset \mathbb{X}_i$ for all i, which yields $\widetilde{\mathcal{P}} \subset \bigcap_{i=0}^{\infty} \mathbb{X}_i = \mathcal{P}$. Now, we prove that $\mathcal{P}$ is a convex set. Assume that $\mathcal{P}$ is the maximal positively invariant set for system (20) with arbitrary switching. Then, we have that $\mathcal{P}$ is a positively invariant set for any linear subsystem in (20) and thus, it follows from [16] that the convex hull of $\mathcal{P}$ is also a positively invariant set for any linear system in (20). Hence, the convex hull of $\mathcal{P}$ is a positively invariant set for system (20) under arbitrary switching. Since $\widetilde{\mathbb{X}}_{\mathbb{U}}$ is a convex set, it follows that the convex hull of $\mathcal{P}$ is included in $\widetilde{\mathbb{X}}_{\mathbb{U}}$. By maximality, the convex hull of $\mathcal{P}$ is also included in $\mathcal{P}$ and thus, $\mathcal{P}$ is convex. As the origin is an equilibrium for $x_{k+1} = A_j^{cl} x, \forall j \in \mathcal{S}_0$, $\mathcal{P}$ contains the origin.

ii) Assume that the algorithm (21) terminates in i^* steps. Then, it follows directly from $\mathbb{X}_i \subseteq \mathbb{X}_{i-1}$ for all $i > 0$ that $\mathbb{X}_i = \mathbb{X}_{i^*}$ for all $i \geq i^*$ and $\mathcal{P} = \mathbb{X}_{i^*}$. Since $\widetilde{\mathbb{X}}_{\mathbb{U}}$ is a polyhedral set and from the fact that the intersection of polyhedra produces polyhedra, it follows that the sets $\mathbb{X}_0^j \triangleq \mathcal{Q}_j^1(\widetilde{\mathbb{X}}_{\mathbb{U}}) \bigcap \widetilde{\mathbb{X}}_{\mathbb{U}}$ are polyhedra for all $j \in \mathcal{S}_0$. Then it follows that the set $\mathbb{X}_1$ is a polyhedral set and, for the same reason, $\mathbb{X}_i, i = 2, 3, \ldots$, are polyhedral sets. Then, it follows that $\mathcal{P}$ is also a polyhedral set.

iii) Let $\mathcal{E}$ denote a λ-contractive set with $0 < \lambda < 1$ for system (20) under arbitrary switching that contains the origin in its interior. Then there exist $c_2 > c_1 > 0$ such that $c_1 \mathcal{E} \subsetneq \widetilde{\mathbb{X}}_{\mathbb{U}} \subsetneq c_2 \mathcal{E}$. Since $c_2 \mathcal{E}$ is λ-contractive, we have that any state trajectory starting on the boundary or in the interior of $c_2 \mathcal{E}$ reaches in i discrete-time steps the set $\lambda^i c_2 \mathcal{E}$. Hence, there exists an i^* such that all the state trajectories starting inside $\widetilde{\mathbb{X}}_{\mathbb{U}} \subsetneq c_2 \mathcal{E}$ lie in $c_1 \mathcal{E}$ within i^* discrete-time steps. Since $c_1 \mathcal{E}$ is λ-contractive and thus, positively invariant, it follows that if a state trajectory stays i^* discrete-time steps inside $\widetilde{\mathbb{X}}_{\mathbb{U}}$, then it stays in forever. Hence, $\mathbb{X}_{i^*} \subseteq \mathcal{P}$ and thus, $\mathbb{X}_{i^*} = \mathcal{P}$. As $c_1 \mathcal{E} \subseteq \mathcal{P}$, $\mathcal{P}$ contains the origin in its interior.

iv) This follows directly from i). ■

Note that an algorithm based on (21) comes down to computing s_0 one-step controllable sets $\mathcal{Q}_j^1(\mathbb{X}_{i-1})$ at each iteration, which is computationally more efficient than computing the maximal positively invariant set.

Next, we present another method for computing low complexity *piecewise polyhedral* positively invariant sets for PWA systems, which relies on the result of Theorem IV.2. Let

$$\mathcal{P} \triangleq \cup_{j \in \mathcal{S}} \{x \in \Omega_j | \|P_j x\| \leq c\} \tag{23}$$

for some $c > 0$, where $\{P_j | j \in \mathcal{S}\}$ are the weights of the terminal cost and $\|\cdot\|$ denotes the $1, \infty$-norm. If $\mathcal{P}$ is used as the terminal set in Problem II.1, c must be taken less than or equal to $\sup\{\mu > 0 | \{x \in \Omega_j | \|P_j x\| \leq \mu\} \subseteq \mathbb{X}_{\mathbb{U}}\}$ to satisfy Assumption III.1.

Lemma V.2: Suppose that the hypothesis of Theorem IV.2 is satisfied. Then, the piecewise polyhedral set $\mathcal{P}$ defined in (23) is a positively invariant set for the PWA system (5) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, and with $h(\cdot)$ as defined in (6).

Note that the set $\mathcal{P}$ defined in (23) is a sublevel set of the terminal cost $F(x) = \|P_j x\|$ if $x \in \Omega_j$, and it consists of the union of s polyhedra (this is because each region Ω_j is assumed to be a polyhedron). If a common terminal weight is used, i.e. $P_j = P$ for all $j \in \mathcal{S}$, then the set $\mathcal{P}$ defined in (23) is a polyhedral set.

Another method for computing low complexity positively invariant sets for PWA systems that admit a (local) PWQ Lyapunov function is presented in [15].

VI. Illustrative Example

Consider the following open-loop unstable discontinuous 3-D PWA system with four linear sub-systems:

$$x_{k+1} = A_j x_k + B_j u_k, \qquad \text{if } x_k \in \Omega_j, \ \ j = 1,2,3,4 \tag{24}$$

subject to the constraints $x_k \in \mathbb{X} = [-5, 5]^3$ and $u_k \in \mathbb{U} = [-2.5, 2.5]$, where $\Omega_j = \{x \in \mathbb{X} | E_j x > 0\}$ for $j = 1, 3$, $\Omega_j = \{x \in \mathbb{X} | E_j x \geq 0\}$ for $j = 2, 4$ (here, all inequalities hold componentwise)

$$A_1 = \begin{bmatrix} -0.2523 & 0.4856 & 0.6467 \\ 0.5290 & -0.2616 & 0.3128 \\ -0.4415 & -0.2713 & -0.6967 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0.5656 \\ 0.5460 \\ 0.9389 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.0647 & 0.1729 & -0.6542 \\ -0.3131 & -0.6691 & -0.6516 \\ -0.3085 & 0.0613 & 0.0099 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0.6543 \\ 0.5266 \\ -0.0558 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0.6402 & -0.5409 & -0.5629 \\ -0.6693 & -0.6874 & 0.1748 \\ -0.2812 & 0.4898 & -0.3526 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0.7580 \\ -0.8050 \\ -0.4059 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -0.3501 & 0.2590 & 0.6695 \\ -0.4808 & 0.1905 & 0.3865 \\ -0.1217 & -0.2631 & -0.0013 \end{bmatrix} \quad B_4 = \begin{bmatrix} 0.6961 \\ -0.7619 \\ -0.2590 \end{bmatrix}$$

$$E_1 = -E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = -E_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The state-space partition $\{\Omega_j | j = 1, 2, 3, 4\}$ corresponding to system (24) is plotted in Fig. 1. The weights of the MPC cost are $Q = 0.02 I_3$ and $R = 0.01$ and the cost is defined using ∞-norms. The following solution to the inequality (14) was found using a min–max formulation and the Matlab `fminunc` solver

$$P = \begin{bmatrix} 0.7029 & 3.8486 & 1.1501 \\ 4.1796 & 0.5642 & 1.6656 \\ -1.4275 & 1.5026 & 5.3197 \\ -1.3717 & 2.5343 & -1.5468 \end{bmatrix}$$

$$K_1 = [0.4699 \quad 0.1750 \quad 0.1591]$$

$$K_2 = [0.4039 \quad 0.4239 \quad 1.1529]$$

$$K_3 = [-0.7742 \quad -0.1436 \quad -0.1603]$$

$$K_4 = [-0.0800 \quad -0.0405 \quad -0.2867].$$

The terminal set (see Fig. 2 for a plot) has been obtained as in (23) for $c = 4$. The resulting terminal set satisfies $\mathbb{X}_T \subset \mathbb{X}_{\mathbb{U}} = \cup_{j \in \mathcal{S}} \{x \in \Omega_j | K_j x \in \mathbb{U}\}$ for the gains given above. The simulation results are plotted in Fig. 2 for initial state $x_0 = [3.6 \ 2 \ 1]^\top$ and system (24) in closed-loop with the MPC control (3), calculated for the matrices

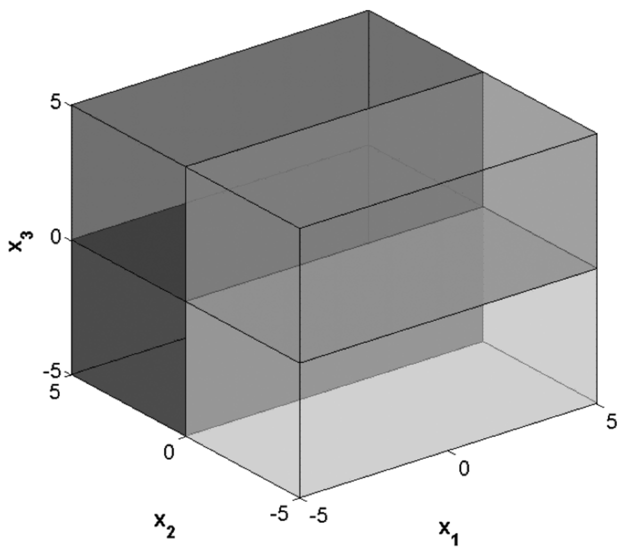

Fig. 1. State–space partition for the example.

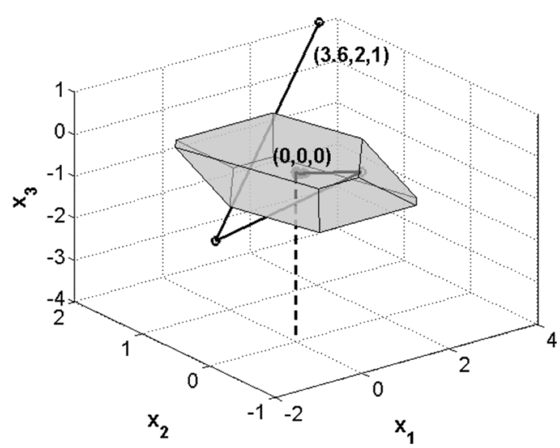

Fig. 2. Simulation results: Closed-loop trajectory—circle solid line and terminal constraint set—gray polyhedron.

P, Q and R given previously, $N = 3$ and with $\mathbb{X}_T$ as terminal set. As guaranteed by Theorem III.2, the MPC control law (3) stabilizes the open-loop unstable discontinuous system (24) while satisfying the constraints.

VII. Conclusion

In this note, we derived sufficient *a priori* verifiable conditions for Lyapunov asymptotic stability of model predictive control of hybrid systems. We developed a general theory which shows that Lyapunov stability can be achieved even if the considered Lyapunov function and the system dynamics are discontinuous. In the particular case of constrained PWA systems and quadratic forms or 1, ∞-norms cost functions, new procedures for calculating the terminal cost and the terminal constraint set have been developed. Novel methods for calculating low complexity piecewise polyhedral positively invariant sets for PWA systems have also been presented.

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