Stabilizing Output Feedback Nonlinear Model Predictive Control: An Extended Observer Approach

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***Abstract*— Nonlinear Model Predictive Control (NMPC), generally based on nonlinear state space models, needs knowledge of the full state for feedback. However, in practice knowledge of the full state is usually not available. Therefore, an asymptotically stabilizing MPC scheme for a class of nonlinear discrete-time systems is proposed, which only requires knowledge of the output of the system for feedback. The presented output based NMPC scheme consists of an extended observer interconnected with an NMPC controller which represents a possibly discontinuous state feedback control law. Sufficient conditions for asymptotic stability of the system in closed-loop with the NMPC observer interconnection are derived using the discrete-time input-to-state stability framework. Moreover, it is shown that there always exist NMPC tuning parameters and observer gains, such that the derived sufficient stabilization conditions can be satisfied.**

***Keywords*— Output feedback, Observers, Nonlinear model predictive control, Input-to-state stability, Asymptotic stability**

I. Introduction

One of the problems in Nonlinear Model Predictive Control (NMPC) that receives an increased attention and has reached a relatively mature stage, consists in guaranteeing closed-loop stability. The approach usually used to ensure nominal closed-loop stability in NMPC is to consider the value function of the NMPC cost as a candidate Lyapunov function, see the surveys [1], [2] for an overview. The stability results heavily rely on state space models of the system, and the assumption that the full state of the real system is available for feedback. However, in practice it is often the case that the full state of the system is not available for feedback. A possible solution to this problem is the use of an observer. An observer can generate an estimate of the full state using knowledge of the output and input of the system. However, nominal stability results for NMPC usually do not guarantee closed-loop stability of an interconnected NMPC-observer combination. Moreover, there exist examples, see for example [3], of zero robustness of nominally stabilizing NMPC controllers in the presence of disturbances, such as estimation errors. One of the potential approaches to guarantee closed-loop stability in the presence of estimation errors in the state, is to employ (inherent) robustness of the model predictive controller. In [4] asymptotic stability of state feedback NMPC is examined in the face of asymptotically decaying disturbances. As stated by the authors of [4], their results are also useful for the solution of the output feedback problem, although a formal proof is missing. A stability result on Observer Based Nonlinear Model Predictive Control (OBNMPC) is reported in [5], under the standing assumption that the NMPC value function and the resulting NMPC control law are Lipschitz continuous. However, no observer which satisfies the assumptions is given in [5]. The stability problem of OBNMPC is revisited in [6], where only continuity of the NMPC value function is assumed. Still, a general observer design methodology is also missing in [6]. Other related results on OBNMPC can be found in [7]. However, there the problem is considered from a continuous-time perspective, while we focus on discrete-time nonlinear systems.

In this paper we investigate stability of an OBNMPC scheme. The novelty of the proposed approach consists in providing an observer design method and using the Input-to-State Stability (ISS) framework, e.g. see [8], [9] and the references therein, to study the stability of the resulting closed-loop system. The extended observer design methodology from [10], [11] is considered. This extended observer approach has the drawback that future information of the controls applied to the system are needed, which results in a causality problem. Since in the NMPC framework *predicted* future controls are available, this framework is suitable to be employed with the proposed observer theory. This idea has been pointed out in [12]. Still, general conditions to guarantee a priori closed-loop stability are lacking. Resolving this issue is one of the main contributions of the current paper.

The remainder of the paper is organized as follows. First, some notations, basic definitions and NMPC notions are introduced in Section II. The observer theory of [10] is summarized in Section III. In Section IV it is shown how one can deal with the causality problem present in the proposed observer by employing the observer in an OBNMPC environment. The definition of the stability problem follows as a consequence of it. In Section V we present a result which enables us to infer ISS with respect to estimation errors (introduced by an observer) from ISS with respect to additive disturbance inputs. This result, which is the first contribution of the paper, enables

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one to employ existing NMPC scenarios, with a priori ISS guarantee with respect to additive disturbances, in an observer based NMPC scenario. Next, in Section VI we prove ISS of the estimation error dynamics of the observer with respect to disturbances due to imperfection of the predicted future control inputs fed to the observer, i.e. the predicted future input sequence does not coincide in general with the real input sequence applied to the system. This is the second contribution of the paper. In Section VII the stability property of the NMPC-observer interconnection is investigated. The ISS results obtained for the NMPC controller and the ISS result of the estimation error dynamics of the observer, together with small gain arguments from [9], are used to prove asymptotic stability of the proposed OBNMPC closed-loop system, which is the main result of this paper. Conclusions are summarized in Section VIII.

II. Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. $\mathbb{Z}_{\geq i}$ denotes the set $\{k \in \mathbb{Z} | k \geq i\}$ for some $i \in \mathbb{Z}$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a *$\mathscr{K}$-function* if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a *$\mathscr{KL}$-function* if, for each fixed $k \in \mathbb{R}_+$, the function $\beta(\cdot, k)$ is a *$\mathscr{K}$-function*, and for each fixed $s \in \mathbb{R}_+$, the function $\beta(s, \cdot)$ is non-increasing and $\beta(s, k) \to 0$ as $k \to \infty$. Composition of two functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^v \to \mathbb{R}^n$ is denoted by $f \circ g$. A function $f(x)$ is called *smooth* if it is infinitely many times differentiable. A smooth function which has the property that the Taylor series at any point x_0 in its domain is convergent for x close enough to x_0 and its value equals $f(x)$ is called an *analytic* function. The class of analytic functions is denoted by C^ω. A function $\phi : \mathbb{R}_+ \to \mathbb{R}^n$, i.e. $\phi(k)$, is for shorthand notational purposes also denoted as ϕ_k. Further $\overline{\lim}_{k\to\infty}\phi_k$ is a shorthand notation for $\lim_{k\to\infty} \sup \phi_k$. For any $x \in \mathbb{R}^n$, x_i with $i \in \{1, 2, ..., n\}$ stands for the i^{th} component of x and $|x|$ stands for its Euclidean norm. For a $n \times m$ matrix A, $|A|$ stands for its induced matrix norm. For any function $\phi : \mathbb{Z}_+ \to \mathbb{R}^n$, we denote $\|\phi\| = \sup\{|\phi_k| : k \in \mathbb{Z}_+\}$. For a set $\mathscr{S} \subseteq \mathbb{R}^n$, we denote by $\partial\mathscr{S}$ the boundary of $\mathscr{S}$, by $\text{int}(\mathscr{S})$ its interior and by $\text{cl}(\mathscr{S})$ its closure. For two arbitrary sets $\mathscr{S} \subseteq \mathbb{R}^n$ and $\mathscr{P} \subseteq \mathbb{R}^n$, $\mathscr{S} \sim \mathscr{P} \triangleq \{x \in \mathbb{R}^n \mid x + \mathscr{P} \subseteq \mathscr{S}\}$ denotes, their *Pontryagin* difference.

Definition II.1 A function $q : \mathbb{X} \times \mathbb{S} \to \mathbb{R}^n$ with $\mathbb{X} \subseteq \mathbb{R}^{n_x}$ and $\mathbb{S} \subseteq \mathbb{R}^{n_s}$ is Lipschitz continuous with respect to x in the domain $\mathbb{X} \times \mathbb{S}$, if there exists a constant $0 \leq L_q < \infty$ such that for all x_1, $x_2 \in \mathbb{X}$ and for all $s \in \mathbb{S}$,

$$|q(x_1, s) - q(x_2, s)| \leq L_q |x_1 - x_2|. \tag{1}$$

The constant L_q is called the Lipschitz constant of q with respect to x.

A. Systems theory notions

Consider the following discrete-time nonlinear system

$$\begin{aligned} \xi_{k+1} &= F(\xi_k, v_k) \\ \zeta_k &= G(\xi_k, v_k) \end{aligned}, \quad \xi_{k=0} = \xi_0, \quad k \in \mathbb{Z}_+, \tag{2}$$

where $\xi_k \in \mathbb{R}^n$ is the state, $\zeta_k \in \mathbb{R}^l$ the output and $v_k \in \mathbb{V} \subseteq \mathbb{R}^m$ the input at discrete time $k \in \mathbb{Z}_+$. The input v_k can be an unknown disturbance at time $k \in \mathbb{Z}_+$. $\mathbb{V}$ is assumed to be a known compact set with $0 \in \text{int}(\mathbb{V})$. Further, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$ are nonlinear, possibly discontinuous, functions. For simplicity we assume that $\xi_e = 0$ is an equilibrium of the 0-input system, i.e. $F(0,0) = 0$, and that $G(0,0) = 0$. A solution to the difference equation (2) for a given input function v and initial condition ξ_0 is denoted as $\xi(\cdot, \xi_0, v)$.

Definition II.2 A set $\mathscr{P} \subseteq \mathbb{R}^n$ is called a *Robust Positively Invariant* (RPI) set for system (2) if for all $\xi_k \in \mathscr{P}$ it holds that $F(\xi_k, v_k) \in \mathscr{P}$ for all $v_k \in \mathbb{V}$ and $k \in \mathbb{Z}_+$.

Definition II.3 Let $\mathscr{Z}$ be a subset of $\mathbb{R}^n$, with $0 \in \text{int}(\mathscr{Z})$. Then,

i) system (2) is called locally *input-to-state stable* (ISS) if there exist a *$\mathscr{KL}$-function* β_ξ and a *$\mathscr{K}$-function* γ_ξ^v such that, for each bounded input function v and each initial condition $\xi_0 \in \mathscr{Z}$, it holds that for each $k \in \mathbb{Z}_+$

$$|\xi(k, \xi_0, v)| \leq \beta_\xi(|\xi_0|, k) + \gamma_\xi^v(\|v\|), \tag{3}$$

ii) system (2) is called locally *input-to-output stable* (IOS) if there exist a *$\mathscr{KL}$-function* β_ζ and a *$\mathscr{K}$-function* γ_ζ^v such that, for each bounded input v and each initial condition $\xi_0 \in \mathscr{Z}$, it holds that for each $k \in \mathbb{Z}_+$

$$|\zeta(k, \xi_0, v)| \leq \beta_\zeta(|\xi_0|, k) + \gamma_\zeta^v(\|v\|). \tag{4}$$

Another formulation of the ISS notion in Definition II.3 is one where (3) is replaced by

$$|\xi(k, \xi_0, v)| \leq \max\left\{\tilde{\beta}_\xi(|\xi_0|, k), \tilde{\gamma}_\xi^v(\|v\|)\right\}, \tag{5}$$

for some *$\mathscr{KL}$-function* $\tilde{\beta}_\xi$ and *$\mathscr{K}$-function* $\tilde{\gamma}_\xi^v$. Note that if trajectory $\xi(k, \xi_0, v)$ of (2) satisfies property (3) in Definition II.3, then it is sufficient to take $\tilde{\beta}_\xi = 2\beta_\xi$ and $\tilde{\gamma}_\xi^v = 2\gamma_\xi^v$ for (5) to hold. This also holds for the IOS property (4) in Definition II.3.

Definition II.4 Let $\mathscr{Z} \subseteq \mathbb{R}^n$, with $0 \in \text{int}(\mathscr{Z})$. We say that the system (2) has the *asymptotic gain* (AG) property if for all $\xi_0 \in \mathscr{Z}$ and each bounded input function v there exists some *$\mathscr{K}$-function* γ_{AG} (asymptotic gain) such that

$$\overline{\lim_{k\to\infty}} |\xi(k, \xi_0, v)| \leq \gamma_{AG}\left(\overline{\lim_{k\to\infty}} |v_k|\right). \tag{6}$$

In Lemma 3.8 given in [9] it is stated that if the system trajectories admit property (5), then $\tilde{\gamma}_\xi^v$ can be taken as the asymptotic gain γ_{AG} of the system.

Definition II.5 Let $\mathscr{X} \subseteq \mathbb{R}^n$, with $0 \in \text{int}(\mathscr{X})$. System (2) is called (locally) *0-asymptotically stable* (0-AS) if for all $\xi_0 \in \mathscr{X}$ and input $v = 0$ there exists a $\mathscr{KL}$*-function* β_ξ such that for each $k \in \mathbb{Z}_+$

$$|\xi(k,\xi_0,0)| \le \beta_\xi(|\xi_0|,k). \tag{7}$$

Lemma II.6 [9], [13] *Let $\mathscr{X}$ be an RPI set for system* (2) *with $0 \in \text{int}(\mathscr{X})$ and let $\alpha_1(|\xi|) \triangleq a|\xi|^\lambda$, $\alpha_2(|\xi|) \triangleq b|\xi|^\lambda$ and $\alpha_3(|\xi|) \triangleq c|\xi|^\lambda$ for some $a,b,c,\lambda > 0$ and $\sigma \in \mathscr{K}$. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function with $V(0) = 0$. Consider now the following inequalities:*

$$\alpha_1(|\xi|) \le V(\xi) \le \alpha_2(|\xi|), \tag{8a}$$
$$V(F(\xi,v)) - V(\xi) \le -\alpha_3(|\xi|) + \sigma(|v|). \tag{8b}$$

If inequalities (8) *hold for all $\xi \in \mathscr{X}$, $v \in \mathbb{V}$, then the perturbed system* (2) *is (locally) ISS with respect to input v in $\mathbb{V}$ for initial conditions ξ_0 in $\mathscr{X}$. Moreover, the ISS property of Definition II.3 is satisfied with $\beta_\xi(|\xi_0|,k) \triangleq \alpha_1^{-1}(2\rho^k\alpha_2(|\xi_0|))$ and $\gamma_\xi^v(\|v\|) \triangleq \alpha_1^{-1}(2\sigma(\|v\|)\frac{1}{1-\rho})$, where $\rho \triangleq \frac{c}{b} \in [0,1)$.*

The proof of Lemma II.6 can be based on the proof of Lemma 3.5 in [9]. A complete proof, including how the specific form of the β_ξ and γ_ξ^v functions are obtained, is given in [13]. Note that, the conditions (8a), (8b) imply Lyapunov asymptotic stability for the 0-input system.

Definition II.7 A function $V : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies the hypothesis of Lemma II.6 is called an *ISS Lyapunov function*.

B. MPC notions

Consider the following nominal and perturbed discrete-time nonlinear systems

$$x_{k+1} = f(x_k,u_k), \quad k \in \mathbb{Z}_+, \tag{9a}$$
$$\tilde{x}_{k+1} = f(\tilde{x}_k,u_k) + w_k, \quad k \in \mathbb{Z}_+, \tag{9b}$$

where x_k, $\tilde{x}_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the state and the input at discrete-time $k \in \mathbb{Z}_+$, respectively. Further $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear Lipschitz continuous function with respect to x in the domain of interest with $f(0,0) = 0$. The vector $w_k \in \mathbb{W} \subseteq \mathbb{R}^n$ denotes an unknown additive disturbance and $\mathbb{W}$ is assumed to be a known compact set with $0 \in \text{int}(\mathbb{W})$. The nominal discrete-time nonlinear system (9a) will be used in a NMPC scheme to make an N time steps ahead prediction of the systems behavior. The system given by (9b) represents a perturbed discrete-time system to which the NMPC controller based on the nominal model (9a) will be applied to. Throughout the paper we assume that the state and the controls are constrained for both systems (9a) and (9b) to some *compact* subsets $\mathbb{X}$ of $\mathbb{R}^n$ and $\mathbb{U}$ of $\mathbb{R}^v$, respectively, which contain the origin in their interior.

For a fixed $N \in \mathbb{Z}_{\ge 1}$, let $\mathbf{x}_k^{[1,N]}(\tilde{x}_k,\mathbf{u}_k^{[0,N-1]}) \triangleq [x_{k+1|k}^\top,\ldots,x_{k+N|k}^\top]^\top$ denote the state sequence generated by the nominal system (9a) from initial state $x_{k|k} \triangleq \tilde{x}_k$ and by applying the input sequence $\mathbf{u}_k^{[0,N-1]} \triangleq [u_{k|k}^\top,\ldots,u_{k+N-1|k}^\top]^\top \in \mathbb{U}^N$, where $\mathbb{U}^N \triangleq \mathbb{U} \times \ldots \times \mathbb{U}$. Furthermore, let $\mathscr{X}_T \subseteq \mathbb{X}$ denote a desired target set that contains the origin. The class of *admissible input sequences* defined with respect to $\mathscr{X}_T$ and state $x_k \in \mathbb{X}$ is $\mathscr{U}_N(\tilde{x}_k) \triangleq \{\mathbf{u}_k^{[0,N-1]} \in \mathbb{U}^N \mid \mathbf{x}_k^{[1,N]}(\tilde{x}_k,\mathbf{u}_k^{[0,N-1]}) \in \mathbb{X}^N, x_{k+N|k} \in \mathscr{X}_T\}$.

Problem II.8 Let the target set $\mathscr{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{Z}_{\ge 1}$ be given and let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_+$ with $L(0,0) = 0$ be continuous bounded mappings. At time $k \in \mathbb{Z}_+$, let $\tilde{x}_k \in \mathbb{X}$ be given and minimize the cost $J(\tilde{x}_k,\mathbf{u}_k^{[0,N-1]}) \triangleq F(x_{k+N|k}) + \sum_{i=0}^{N-1} L(x_{k+i|k},u_{k+i|k})$, with prediction model (9a), over all $\mathbf{u}_k^{[0,N-1]} \in \mathscr{U}_N(\tilde{x}_k)$.

In the NMPC literature F, L and N are called the terminal cost, the stage cost and the prediction horizon, respectively. We call a state $\tilde{x}_k \in \mathbb{X}$ *feasible* if $\mathscr{U}_N(\tilde{x}_k) \ne \emptyset$. Similarly, Problem II.8 is said to be *feasible* for $\tilde{x}_k \in \mathbb{X}$ if $\mathscr{U}_N(\tilde{x}_k) \ne \emptyset$. Let $\mathscr{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible initial states* with respect to Problem II.8 and let $V_{\text{MPC}} : \mathscr{X}_f(N) \to \mathbb{R}_+$,

$$V_{\text{MPC}}(\tilde{x}_k) \triangleq \inf_{\mathbf{u}_k^{[0,N-1]} \in \mathscr{U}_N(\tilde{x}_k)} J(\tilde{x}_k,\mathbf{u}_k^{[0,N-1]}) \tag{10}$$

denote the NMPC value function corresponding to Problem II.8. The existence of a minimum in (10) is usually guaranteed by assuming compactness of $\mathbb{U}$ and continuity of the dynamics (9a), the stage and terminal costs [14]. We assume in the sequel that there exists a feasible sequence of controls $\overline{\mathbf{u}}_k^{[0,N-1]} \triangleq [\overline{u}_{k|k}^\top,\overline{u}_{k+1|k}^\top,\ldots,\overline{u}_{k+N-1|k}^\top]^\top$, possibly sub-optimal, for Problem II.8 and any state $\tilde{x}_k \in \mathscr{X}_f(N)$. Then, $V_{\text{MPC}}(\tilde{x}_k) = J(\tilde{x}_k,\overline{\mathbf{u}}_k^{[0,N-1]})$ denotes the NMPC value function and the *the NMPC control law* is defined as

$$u_k = \kappa^{\text{MPC}}(\tilde{x}_k) \triangleq \overline{u}_{k|k}, \quad k \in \mathbb{Z}_+. \tag{11}$$

Substituting (11) in (9b) yields the closed-loop system

$$\tilde{x}_{k+1} = f(\tilde{x}_k,\kappa^{\text{MPC}}(\tilde{x}_k)) + w_k, \quad w_k \in \mathbb{W} \subseteq \mathbb{R}^n, \quad k \in \mathbb{Z}_+. \tag{12}$$

Since the model predictive control law (11) is discontinuous in general, even when simple continuous feedback stabilizers and Lyapunov functions exist [3], (12) is discontinuous in general too. In literature there are various NMPC schemes with an a priori guarantee that (12) is ISS with respect to w as additive disturbance input, see for example [4], [13], [15], [16], [17]. In all these approaches it is shown that there are conditions under which a candidate ISS Lyapunov function, mostly V_{MPC}, satisfies the hypothesis of Lemma II.6 for initial conditions $\tilde{x}_0$ in some subset $\mathscr{X}_f(N)$ with 0 in its interior. In Section V we will elaborate on this issue.

A well known property, which is often employed to prove stability of NMPC schemes, see for example [4], is regularity.

Definition II.9 Regularity is obtained when the future inputs *predicted* by the model predictive controller satisfy

$$|u_{k+i|k}| \le \theta_i|x_{k|k}| \quad \text{for} \quad i = 0,\ldots,N-1, \tag{13}$$

for some constants $\theta_i > 0$.

III. Extended Observer Theory: An Introductory Survey

In this paper we use the extended observer theory proposed in [10], [11]. For notational brevity we consider the theory for the single input single output case, although the theory applies in the multiple input output case as well. Consider the following system

$$\begin{aligned} x_{k+1} &= f(x_k,u_k) \\ y_k &= g(x_k) \end{aligned}, \quad x_{k=0}=x_0, \quad k\in\mathbb{Z}_+, \tag{14}$$

where $x_k\in\mathbb{R}^n$, $u_k\in\mathbb{R}$ and $y_k\in\mathbb{R}$ is the state, the control and the output at discrete-time $k\in\mathbb{Z}_+$, respectively. Further $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ and $g:\mathbb{R}^n\to\mathbb{R}$ are analytic i.e. $f,g\in C^\omega$ with $f(0,0)=0$ and $g(0)=0$. The observer problem for (14) deals with the question how to reconstruct the state trajectory $x(\cdot,x_0,u)$ on the basis of the knowledge of the input u and the output y of the system. The problem of observer design in its full generality is a problem that is not yet fully solved for nonlinear systems of the form (14). A proposed potential observer candidate for a broad class of discrete-time nonlinear systems is studied in this paper. To be more precise, observer design for a class of systems that can be expressed in the so called *Extended Nonlinear Observer Canonical Form* (ENOCF) is considered. Observers that are based on the system form in ENOCF are denoted by *extended* observers. One of the major characteristics that distinguishes *extended* observers form "conventional" observers, is that not only the output (input) at the current time y_k is employed to obtain an estimate of the state trajectory x, but, as in the receding horizon observer approach [18], additional knowledge of past information present in the output trajectory y is also taken into account.

A. Observers in ENOCF

A system representation in ENOCF, or the z-dynamics for brevity, reads as

$$\begin{aligned} z_{k+1} &= A_z z_k + f_z(\mathbf{y}_k^{[1-n,0]},\mathbf{u}_k^{[1-n,0]},\mathbf{u}_k^{[1,n]}) \\ y_k &= h_z(C_z z_k,\mathbf{u}_k^{[1-n,0]}) \end{aligned}, \quad z_{k=0}=z_0, \quad k\in\mathbb{Z}_+, \tag{15}$$

with $A_z \triangleq \begin{bmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}, \quad C_z \triangleq [0\dots 0\ 1],$

$$f_z(\mathbf{y}_k^{[1-n,0]},\mathbf{u}_k^{[1-n,0]},\mathbf{u}_k^{[1,n]}) \triangleq \begin{bmatrix} f_{z,0}(y_k,u_k,\mathbf{u}_k^{[1,n]}) \\ f_{z,1}(\mathbf{y}_k^{[-1,0]},\mathbf{u}_k^{[-1,0]},\mathbf{u}_k^{[1,n-1]}) \\ \vdots \\ f_{z,n-1}(\mathbf{y}_k^{[1-n,0]},\mathbf{u}_k^{[1-n,0]},u_{k+1}) \end{bmatrix},$$

where $\mathbf{y}_k^{[1-n,0]} \triangleq [y_{k-n+1},\dots,y_k]^\top$, $\mathbf{u}_k^{[1-n,0]} \triangleq [u_{k-n+1},\dots,u_k]^\top$, $\mathbf{u}_k^{[1,n]} \triangleq [u_{k+1},\dots,u_{k+n}]^\top$, $z_k\in\mathbb{R}^n$ represent the past output, input, future input and state in z-coordinates at discrete time $k\in\mathbb{Z}_+$, respectively. Further, $f_z:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $h_z:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ are nonlinear functions, where h_z is, for fixed input sequence, an invertible output function for the system in ENOCF. Except for the future input sequence, all other sequences are known at time k if input and output variables (measurements) are buffered. The dependence on the future input variables corresponds (or can be compared to) the appearance of (also unknown) time derivatives of the input in the generalized continuous-time observer from [19]. Why a system representation in ENOCF is future input dependent, in the considered discrete-time context, will become clear later when details on the existence of a system representation in ENOCF are discussed. First we focus on the existence of observers for the system representation in ENOCF. Observer candidates based on the system descriptions in ENOCF were proposed in [10]. One of the observer candidates simply consists of a "copy" of the z-dynamics (15) added with an output injection term (also known as an "innovation" term) $[\ell_1,\dots,\ell_n]^\top(h^{-1}_{z,\mathbf{u}\text{fixed}}(y_k,\mathbf{u}_k^{[1-n,0]})-\hat z_{n,k})$, i.e.

$$\begin{aligned} \hat z_{k+1} = A_z\hat z_k + f_z(\mathbf{y}_k^{[1-n,0]},\mathbf{u}_k^{[1-n,0]},\mathbf{u}_k^{[1,n]}) + \\ [\ell_1,\dots,\ell_n]^\top(\underbrace{h^{-1}_{z,\mathbf{u}\text{fixed}}(y_k,\mathbf{u}_k^{[1-n,0]})}_{z_{n,k}} - \hat z_{n,k}), \end{aligned} \tag{16}$$

with $\hat z_{n,k}=C_z\hat z_k$, $\hat z_{k=0}=\hat z_0$, $k\in\mathbb{Z}_+$ and $h^{-1}_{z,\mathbf{u}\text{fixed}}$ represents for a fixed input sequence $\mathbf{u}_k^{[1-n,0]}$ the inverse function of h_z in (15). Further, $\ell_1,\dots,\ell_n$ represent the observer gains. The observer gains can be used to assign a certain dynamic behavior of the observer z-error dynamics. The z-error dynamics is the dynamics which describes the evolution of the z-error defined at each time $k\in\mathbb{Z}_+$ as $e_{z,k}\triangleq z_k-\hat z_k$. Due to the fact that the state z_k of a system representation in ENOCF appears linearly in the system equations and all nonlinearity enters the state equations via the nonlinear function f_z, depending only on input and output sequences of the system, linear autonomous z-error dynamics is obtained. The z-error dynamics for (15) and (16) reads as

$$e_{z,k+1}=A_\ell e_{z,k}, \quad \text{with} \quad A_\ell \triangleq \left(A_z-[\ell_1,\dots,\ell_n]^\top C_z\right). \tag{17}$$

In the next subsection it will become clear that a system description in ENOCF is not unique. To some extent there is freedom in choosing the structure for the function f_z (and also h_z). A possible structure for the function f_z is $f_z=[0,\dots,0,f_{z,n-1}]^\top$. Based on this structure of f_z, another observer candidate in ENOCF is proposed in [10] and is given by

$$\hat z_{k+1} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_{z,n-1} \end{bmatrix}}_{f_z} + \begin{bmatrix} q_1\hat z_{1,k} \\ \vdots \\ q_{n-1}\hat z_{n-1,k} \\ q_n(\hat z_{n,k}-h^{-1}_{z,\mathbf{u}\text{fixed}}(y_k,\mathbf{u}_k^{[1-n,0]})) \end{bmatrix}, \text{ with} \tag{18}$$

$\hat z_{k=0}=\hat z_0$, $k\in\mathbb{Z}_+$ and where $[q_1,\dots,q_n]^\top\in\mathbb{R}^n$ denote the observer gains. The z-error dynamics of the observer in

(18) then reads as

$$e_{z,k+1} \triangleq z_{k+1} - \hat{z}_{k+1} = \begin{bmatrix} 0 \\ z_{1,k} \\ \vdots \\ z_{n-1,k} \end{bmatrix} - \begin{bmatrix} q_1 \hat{z}_{1,k} \\ \vdots \\ q_{n-1}\hat{z}_{n-1,k} \\ q_n(\hat{z}_{n,k} - z_{n,k}) \end{bmatrix}. \quad (19)$$

Then, after $n-1$ time iterations we have that $z_{1,k} = 0, \ldots, z_{n-1,k} = 0$ irrespective of the initial condition z_0. This follows directly from the structure of the dynamics (15) and the fact that $f_{z,0}, \ldots, f_{z,n-2}$ have been chosen to be identically zero. This observation implies that after $n-1$ time steps the z-error dynamics given by (19) behaves linearly according to

$$e_{z,k+1} = A_q e_{z,k}, \quad \text{with} \quad A_q \triangleq \text{diag}\{[q_1, \ldots, q_n]\}. \quad (20)$$

The matrices defining the z-error dynamics of both observers (16) and (18) can always be rendered Schur by choosing appropriate observer gains such that the proposed observers are asymptotically stable. Therefore, systems that can be transformed into ENOCF admit the design of several suitable observers for (15).

B. Existence of equivalence relation

Previously, we showed that if the dynamics of a system is given in the extended nonlinear observer canonical form (15), then it is always possible to design an observer for this system. However, the following question remains open: What can be done if the dynamics are not in the form of (15) but in the more common form given by (14)? In this subsection we will therefore show under which condition system (14) can be transformed into (15).

In order to answer the posed question, we have to recall the *strongly local observability* notion [20]. For convenience we first introduce the *observability map* for non-autonomous discrete-time nonlinear systems, which was already defined for discrete-time nonlinear autonomous systems in [21], [22].

Definition III.1 The observability map ψ of the system given by (14) is defined as:

$$\psi(x_k, \mathbf{u}_k^{[0,n-2]}) \triangleq \begin{bmatrix} g(f^0(x_k)) \\ g(f^1(x_k, u_k)) \\ \vdots \\ g(f^{n-1}(x_k, [u_k, \ldots, u_{k+n-2}]^\top)) \end{bmatrix}, \quad (21)$$

where $f^0(x_k) = x_k$, $f^i(x_k, [u_k, \ldots, u_{k+i-1}]^\top) = f(f(\ldots f(f(x_k, u_k), u_{k+1}), \ldots,), u_{k+i-1})$, with $i \geq 1$.

Next, *strongly locally observability* is introduced.

Definition III.2 i) Let $\mathscr{N} \in \mathbb{X}$ be an open neighborhood around some state $x_0 \in \mathbb{X}$ and let $\breve{x}_0$ be a state in $\mathscr{N}$. Then, system (14) is *strongly locally observable in* $\breve{x}_0$, if for all states $\breve{x}_0 \in \mathscr{N}$ resulting in the same output sequence as obtained by x_0, i.e.

$$\psi(x_0, \mathbf{u}_0^{[0,n-2]}) = \psi(\breve{x}_0, \mathbf{u}_0^{[0,n-2]}), \quad (22)$$

for all admissible input sequences $\mathbf{u}_k^{[0,n-2]}$, implies that $x_0 = \breve{x}_0$.

ii) System (14) is *strongly locally observable*, if i) holds for all $x_0 \in \mathbb{X}$.

The word *locally* refers to the fact that two states must be distinguishable in some neighborhood $\mathscr{N}$. And the word *strongly* refers to the distinguishability of the states after observing the output trajectory for a finite number of time steps (n time steps). A sufficient condition for system (14) to be *strongly locally observable* in x_0 is the following rank condition,

$$\text{rank}\left\{ \left. \frac{\partial \psi(x, \mathbf{u}_k^{[0,n-2]})}{\partial x_k} \right|_{x=x_0} \right\} = n, \quad \forall \mathbf{u}_k^{[0,n-2]} \in \mathbb{U}^{n-1}, \quad (23)$$

where $\mathbb{U}^{n-1} \subseteq \mathbb{R}^{n-1}$ and ψ is defined as in (21). Condition (23) is sufficient[1] for the existence of a one-to-one smooth invertible map of the observability map for fixed input sequences. This follows from the fact that ψ is analytic (because f and h are analytic). The inverse function of ψ for fixed input sequences is denoted as $\psi_{\mathbf{u}\text{fixed}}^{-1}$. If system (14) is *strongly locally observable*, then ψ in (21) acts for fixed inputs, as a (local) diffeomorphism relating state x_k satisfying (14) to a state s_k satisfying another representation of system (14) having the form

$$s_{k+1} = \begin{bmatrix} s_{2,k} \\ \vdots \\ s_{n,k} \\ f_s(s_k, \mathbf{u}_k^{[0,n-1]}) \end{bmatrix}, \; y_k = s_{1,k}, \; s_{k=0} = s_0, \quad (24)$$

where

$$s_k = \psi(x_k, \mathbf{u}_k^{[0,n-2]}) \;\Leftrightarrow\; x_k = \psi_{\mathbf{u}\text{fixed}}^{-1}(s_k, \mathbf{u}_k^{[0,n-2]}), \quad (25)$$

$$f_s(s_k, \mathbf{u}_k^{[0,n-1]}) = g(f^n(\psi_{\mathbf{u}\text{fixed}}^{-1}(s_k, \mathbf{u}_k^{[0,n-2]}), \mathbf{u}_k^{[0,n-1]})). \quad (26)$$

Note that system (24) is obtained by defining s_k as $s_k \triangleq [y_{k-1}, \ldots, y_{k-n+1}]^\top$. By defining s_k in this manner *future* input sequence dependence, as is encountered in the previous subsection, is introduced. Next it will be shown that from state s_k one can obtain state z_k satisfying (15) employing for fixed input and output sequences a map $\Omega : \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, i.e.

$$z_k = \Omega(s_k, \mathbf{y}_k^{[1-n,-1]}, \mathbf{u}_k^{[1-n,0]}, \mathbf{u}_k^{[1,n-1]}), \quad (27)$$

where $\mathbf{y}_k^{[1-n,-1]} \triangleq [y_{k-n+1}, \ldots, y_{k-1}]^\top$, and $\mathbf{u}_k^{[1,n-1]} \triangleq [u_{k+1}, \ldots, u_{k+n-1}]^\top$. The motivation for (27) can be explained by considering the structure of the system representation in ENOCF (15) and taking into account that $s_k \triangleq [y_{k-1}, \ldots, y_{k-n+1}]^\top$. Taking the inverse of the output equation of (15) for fixed input sequence $\mathbf{u}_k^{[1-n,0]}$ yields $z_{n,k} = h_{z,\mathbf{u}\text{fixed}}^{-1}(y_k, \mathbf{u}_k^{[1-n,0]})$. Substitution of y_k by $s_{1,k}$ results in $z_{n,k} = h_{z,\mathbf{u}\text{fixed}}^{-1}(s_{1,k}, \mathbf{u}_k^{[1-n,0]})$. From the last component of

[1] Note that the rank condition (23) is sufficient for invertibility. Take for example $\psi = x^3$. The rank condition is obviously not satisfied. However, a one-to-one inverse function exists.

the state equation (15), it follows that $z_{n-1,k} = z_{n,k+1} - f_{z,n-1}(\mathbf{y}_k^{[1-n,0]}, \mathbf{u}_k^{[1-n,0]}, u_{k+1})$ where $z_{n,k+1}$ can be replaced by $h^{-1}_{z,\mathbf{u}\text{fixed}}(y_{k+1}, \mathbf{u}_k^{[2-n,1]})$ and y_{k+i}, $i = 0, 1, \ldots, n-1$, by $s_{i+1,k}$. Continuing in this way, one obtains the following structure for Ω in (27):

$$
\begin{aligned}
z_{1,k} \triangleq & h^{-1}_{z,\mathbf{u}\text{fixed}}(s_{n,k}, \mathbf{u}_k^{[0,n-1]}) \\
& - \sum_{j=1}^{n-1} f_{z,j}(y_{k-1}, s_{1,k}, \ldots, s_{j,k}, \mathbf{u}_k^{[-1,n-1]}) \\
& \vdots \\
z_{n-2,k} \triangleq & h^{-1}_{z,\mathbf{u}\text{fixed}}(s_{3,k}, \mathbf{u}_k^{[3-n,2]}) \\
& - f_{z,n-1}(\mathbf{y}_k^{[2-n,-1]}, s_{1,k}, s_{2,k}, \mathbf{u}_k^{[2-n,2]}) \\
& - f_{z,n-2}(\mathbf{y}_k^{[2-n,-1]}, s_{1,k}, \mathbf{u}_k^{[2-n,2]}) \\
z_{n-1,k} \triangleq & h^{-1}_{z,\mathbf{u}\text{fixed}}(s_{2,k}, \mathbf{u}_k^{[2-n,1]}) \\
& - f_{z,n-1}(\mathbf{y}_k^{[1-n,-1]}, s_{1,k}, \mathbf{u}_k^{[1-n,1]}) \\
z_{n,k} \triangleq & h^{-1}_{z,\mathbf{u}\text{fixed}}(s_{1,k}, \mathbf{u}_k^{[1-n,0]}).
\end{aligned} \tag{28}
$$

The following composition of Ω and ψ, i.e.

$$
z_k = \Xi(x_k, \mathbf{y}_k^{[1-n,-1]}, \mathbf{u}_k^{[1-n,0]}, \mathbf{u}_k^{[1,n-1]}) \triangleq \Omega \circ \psi, \tag{29}
$$

then acts, for fixed input and output sequences ($\Xi_{\mathbf{uy}\text{fixed}}$), as a local diffeomorphism around x_0 relating the state x_k from (14) and z_k from (15), iff (14) is strongly locally observable at x_0. The interested reader can find a detailed proof in [10].

Note that the observer design based on the system representation in ENOCF is thus based on the selection of two functions, namely f_z and h_z. A question still unanswered is, what criteria the functions f_z and h_z must satisfy in order to obtain a system representation in ENOCF and its coordinate transformation (29) relating the system representation in ENOCF to (14). In [10] it is shown that this criteria for f_z and h_z is given by

$$
\begin{aligned}
& h^{-1}_{z,\mathbf{u}\text{fixed}}\big(f_s(s_k, \mathbf{u}_k^{[0,n-1]}), \mathbf{u}_k^{[1,n]}\big) = \\
& \qquad \sum_{j=0}^{n-1} f_{z,j}(s_{1,k}, s_{2,k}, \ldots, s_{j+1,k}, \mathbf{u}_k^{[0,n]}).
\end{aligned} \tag{30}
$$

Note that there are various possibilities to choose the functions f_z and h_z. This means that given system (14), there can exist multiple system representations of this system in ENOCF. We can now summarize the previous in the following result

Theorem III.3 [10] *Let* (14) *be* strongly locally observable at x_0. *Then, for all functions* $f_z : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ *and* $h_z : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ *satisfying* (30) Ξ*, defined in* (29)*, acts, for fixed input and output sequences as a local diffeomorphism relating state x_k satisfying* (14) *and a state z_k satisfying a system representation in ENOCF* (18).

In Subsection III-A we showed that (16) and (18) are observers for a system representation in ENOCF. Then, via the result established in this subsection one can conclude that under the condition that the system (14) is *locally strongly observable* the observers given by (16) and (18) are (local) observers for (14). Via the coordinate transformation map (29) the estimated state in z-coordinates can be mapped to estimates of the state in x-coordinates. By continuity of the transformation map (29), it can be argued that the behavior of the estimation error in x-coordinates $e_{x,k} = x_k - \hat{x}_k$ is representable for the behavior assigned for in the z-coordinates. Although the observers seem to be global observers in the z-coordinates, the observers are locally defined in x-coordinates. This follows from the fact that the equivalence relation between the z-dynamics and the x-dynamics denoted by (29) is not globally defined in general due to the local nature of the strongly locally observability notion. Therefore the observer candidates are in general only locally well defined. However, if the equivalence relation between the system representation (15) and (14) is defined globally also the observer candidates will be global observer candidates for (14).

IV. Problem Formulation

Consider the system dynamics given by (14). The full state x_k is assumed not to be available for feedback. For feedback, an estimate of the state $\hat{x}_k$ is fed to an NMPC controller instead, i.e. $u_k = \kappa^{\text{MPC}}(\hat{x}_k)$. The state estimate $\hat{x}_k$ is generated by, for example, observer (16) or (18) in combination with the map $\Xi^{-1}_{\mathbf{uy}\text{fixed}}$ defined in (29). The observer candidates appear to be (local) observers for a broad class of systems of the form (14) under the assumption that the future input sequence $\mathbf{u}_k^{[1,n]}$ is known a priori. Still, the future input sequence is not known a priori. Under the assumption that the prediction horizon of the NMPC controller is sufficiently long ($N \geq n$), one can employ a part of the *predicted* future input sequence obtained by the NMPC controller at every time step k, denoted by $\bar{\mathbf{u}}_k^{[1,n]}$, and feed this sequence to the observer as an educated guess for the unknown sequence $\mathbf{u}_k^{[1,n]}$. In Fig. 1 a block diagram of the resulting control scheme is presented. The major question that must be answered in

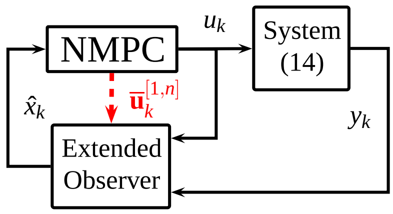

Fig. 1. Proposed OBNMPC Scheme.

order to show that the proposed OBNMPC scheme can work, is whether the resulting closed-loop system can be rendered (locally) asymptotically stable to the origin ($x = 0$ and $e_x \triangleq x - \hat{x} = 0$). An outline of the reasoning used to answer this question is given in the sequel.

A. Outline of the approach

In order not to destabilize the model predictive controller with the estimation error presented in the state information introduced by the observer, we want to synthesize a model predictive controller which is robust to the estimation error. Notions of input-to-state-stability (ISS) are used for this purpose. Once the controller in closed-loop with system (14), e.g.

$$x_{k+1} = f(x_k, \kappa^{\text{MPC}}(x_k + e_{x,k})), \quad e_{x,k} \in \mathbb{E}_x \subseteq \mathbb{R}^n, \quad k \in \mathbb{Z}_+, \tag{31}$$

is (locally) ISS with respect to the estimation error e_x, it is known that if the estimation error vanishes, e.g. $e_{x,k} \to 0$ for $k \to \infty$ also $x_k \to 0$ for $k \to \infty$. This follows directly from the ISS system property given in Definition II.3. Following this approach, we in fact decouple the observer design problem from the controller design problem. An approach to synthesize an NMPC controller that renders (31) ISS with respect to e_x is given in the next section.

As pointed out before, under the assumption that system (31) is ISS, a sufficient condition which will lead to asymptotic stability of the OBNMPC scheme is that the estimation error vanishes, i.e. $e_{x,k} \to 0$ for $k \to \infty$. The condition under which the error of the observer candidate vanishes is easy to find if the future *predicted* input sequence $\overline{\mathbf{u}}_k^{[1,n]}$, from the NMPC controller, coincides with the actual future input sequence $\mathbf{u}_k^{[1,n]}$. In that case, as shown in Section III, the error dynamics of the observer is asymptotically stable ($e_{x,k} \to 0$ for $k \to \infty$), and thus with the ISS assumption on (31) we also have $x_k \to 0$ for $k \to \infty$. For this situation (perfect future input sequence predictions), The major question considered in this paper is thus trivially answered. However, since the *predicted* and real future input sequences do not coincide in general, the asymptotic stability result of the estimation error dynamics pointed out in Section III cannot be applied for this scenario. A closer study to the error dynamics of the observer in the case of an imperfect *predicted* future input sequence is therefore necessary.

In case the predicted future input sequence $\overline{\mathbf{u}}_k^{[1,n-1]}$ fed to the proposed observer candidates (16) or (18) does not coincide with the real future input sequence $\mathbf{u}_k^{[1,n-1]}$ in the dynamics in ENOCF (15), cancellation of the nonlinearity in the derivation of the z-error dynamics as in Section III is not realized. Taking this fact into account and defining the future predicted input error sequence as $\mathbf{e}_{\mathbf{u},k}^{[1,n]} \triangleq \overline{\mathbf{u}}_k^{[1,n]} - \mathbf{u}_k^{[1,n]}$, the error dynamics of the observer in z-coordinates is given by

$$e_{z,k+1} = A_i e_{z,k} + \Delta f_z(\mathbf{y}_k^{[1-n,0]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n]}, \mathbf{e}_{\mathbf{u},k}^{[1,n]}), \tag{32}$$

where $A_i \triangleq A_\ell$ or $A_i \triangleq A_q$ (depending on which observer structure, either (16) or (18), is used) and Δf_z is of the form

$$\begin{aligned}\Delta f_z(\mathbf{y}_k^{[1-n,0]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n]}, \mathbf{e}_{\mathbf{u},k}^{[1,n]}) \triangleq \\ f_z(\mathbf{y}_k^{[1-n,0]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n]} - \mathbf{e}_{\mathbf{u},k}^{[1,n]}) \\ - f_z(\mathbf{y}_k^{[1-n,0]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n]}),\end{aligned}$$

with $\Delta f_z(\cdot,\cdot,\cdot,0) = 0$.

Remark IV.1 If the predicted future input sequence would coincide with the actual future input sequence ($\mathbf{e}_{\mathbf{u},k}^{[1,n]} = 0$), one recovers the linear autonomous description of the error dynamics defined by either (16) or (18).

The estimation error in x-coordinates manifests itself via the coordinate transformation map given by (29). The influence of the mismatch between the predicted future input sequence and the actual future input sequence on the estimation error in x-coordinates (e_x) can be studied using the coordinate transformation map given in (29). Note that

$$\begin{aligned}e_{x,k} &= \Delta\Xi(e_{z,k}, \hat{z}_k, \mathbf{y}_k^{[1-n,-1]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n-1]}, \mathbf{e}_{\mathbf{u},k}^{[1,n-1]}) \\ &\triangleq \underbrace{\Xi_{\mathbf{uy}\text{fixed}}^{-1}(\hat{z}_k - e_{z,k}, \mathbf{y}_k^{[1-n,-1]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n-1]} - \mathbf{e}_{\mathbf{u},k}^{[1,n-1]})}_{x_k} \\ &\quad - \underbrace{\Xi_{\mathbf{uy}\text{fixed}}^{-1}(\hat{z}_k, \mathbf{y}_k^{[1-n,-1]}, \mathbf{u}_k^{[1-n,0]}, \overline{\mathbf{u}}_k^{[1,n-1]})}_{\hat{x}_k},\end{aligned} \tag{33}$$

where $\Delta\Xi(0,\cdot,\cdot,\cdot,0) = 0$. Equations (32) and (33) define the x-error dynamics of the observer candidates in case of feeding an imperfect predicted future input sequence to the observers. The error dynamics has now become a non-autonomous system. In Section VI it will be proven that the z-error dynamics given by (32) is ISS with respect to future input prediction errors $\mathbf{e}_{\mathbf{u},k}^{[1,n]}$.

In Section VII the cascade, as depicted in Fig. 2, of the ISS (IOS) observer error dynamics (32), (33) and the ISS NMPC controller in closed-loop with (14) is considered.

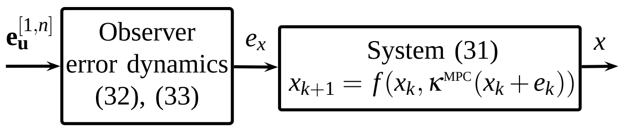

Fig. 2. ISS, IOS observer error dynamics (with respect to $\mathbf{e}_{\mathbf{u}}^{[1,n]}$ as input) cascaded with the ISS NMPC controller in closed-loop with (14) (with respect to e_x as input).

The state and input of the cascaded system in Fig. 2 is denoted by $(e_{z,k}, x_k)$ and $\mathbf{e}_{\mathbf{u}}^{[1,n]}$, respectively. By assuming *regularity* of the NMPC controller, it is proven that one can tune the NMPC controller and observer tuning parameters such that the prediction error $\mathbf{e}_{\mathbf{u}}^{[1,n]}$, present in the *predicted* future input $\overline{\mathbf{u}}_k^{[1,n]}$, will asymptotically decay to zero, i.e. $\mathbf{e}_{\mathbf{u}}^{[1,n]} \to 0$ for $k \to \infty$. Due to the asymptotic decaying property of $\mathbf{e}_{\mathbf{u}}^{[1,n]}$ and the IOS and ISS property of (32), (33) and the ISS NMPC controller in closed-loop with (14), respectively, one can conclude that the cascade in Fig. 2 is (locally) asymptotically stable. This implies (local) asymptotic stability of the proposed OBNMPC scheme as presented in Fig. 1.

V. Controller Design: ISS w.r.t. additive disturbances implies ISS w.r.t. estimation errors for Lipschitz continuous systems

As explained in the previous section, we seek for NMPC schemes that can render (31) ISS with respect to estimation error e_x. Rendering system (31) ISS with respect to the estimation error e_x by using NMPC is however difficult. The problem was considered in [5], where robustness to estimation errors is shown under the assumption of Lipschitz continuity of the NMPC value function and control law. A similar result was obtained more recently in [6], under the milder assumption of continuity of the NMPC value function. To the authors knowledge, besides the result of [6], no general practically applicable NMPC schemes are available in literature that can a priori guarantee ISS of (31) with respect to the estimation error e_x as input. However, due to the result obtained in this section we can infer ISS of (31) with respect to e_x from ISS of (12) with respect to additive disturbances w. This result then allows us to employ all existing NMPC schemes that can a priori guarantee ISS of (12) to also establish a priori ISS of (31). Note that there are several MPC schemes for nonlinear systems that have an a priori ISS guarantee with respect to additive disturbances, see for example, [1], [4], [13], [16], [17].

The standing assumption is Lipschitz continuity of the function f, with respect to x with Lipschitz constant L_f on the domain $\mathbb{X}\times\mathbb{U}$. Moreover, we assume we have an NMPC scheme, which renders (12) locally ISS with respect to *additive* disturbance w_k and initial conditions $\tilde{x}_0$ in $\widetilde{\mathscr{X}}_f(N)$. Assume $\widetilde{\mathscr{X}}_f(N)$ is an RPI set of system (12) and has the origin in its interior, then the following result can be obtained:

Theorem V.1 *Suppose $u_k = \kappa^{\text{MPC}}(\tilde{x}_k)$ is an NMPC control law which renders system (12) locally ISS for initial conditions $\tilde{x}_0$ in $\widetilde{\mathscr{X}}_f(N)$ and additive disturbances w in $\mathbb{W}\triangleq\{w\in\mathbb{R}^n \mid \|w\|\leq\mu\}$ for some $\mu>0$ and let L_f be the Lipschitz constant of the system dynamics f with respect to x. Then, the NMPC control law $u_k=\kappa^{\text{MPC}}(x_k+e_{x,k})$, $k\in\mathbb{Z}_+$, renders (31) locally ISS for initial conditions x_0 in $\widetilde{\mathscr{X}}_f(N)$, and estimation errors e_x in $\mathbb{E}_x\triangleq\{e_x\in\mathbb{R}^n \mid \|e_x\|\leq\nu\triangleq\frac{\mu}{L_f+1}\}$, i.e.*

$$|x(k,x_0,e_x)|\leq\beta_x(|x_0|,k)+\gamma_x^{e_x}(\|e_x\|), \tag{34}$$

with $\beta_x(|x_0|,k)\triangleq\alpha_1^{-1}(2\rho^k\alpha_2(2|x_0|))$ *and* $\gamma_x^{e_x}(\|e_x\|)\triangleq\alpha_1^{-1}(2\alpha_2(2\|e_x\|))+\alpha_1^{-1}(2\sigma((L_f+1)\|e_x\|)\frac{1}{1-\rho})+\|e_x\|$.

Proof: Consider system (31). We perform the following coordinate change on (31), i.e.

$$x_k=\tilde{x}_k-e_{x,k},\quad \forall k\in\mathbb{Z}_+, \tag{35}$$

which gives

$$\tilde{x}_{k+1}=f(\tilde{x}_k-e_{x,k},\kappa^{\text{MPC}}(\tilde{x}_k))+e_{x,k+1}. \tag{36}$$

Rewriting (36) as

$$\begin{aligned}\tilde{x}_{k+1}=f(\tilde{x}_k-e_{x,k},\kappa^{\text{MPC}}(\tilde{x}_k))+f(\tilde{x}_k,\kappa^{\text{MPC}}(\tilde{x}_k))\\ -f(\tilde{x}_k,\kappa^{\text{MPC}}(\tilde{x}_k))+e_{x,k+1}\end{aligned} \tag{37}$$

yields

$$\tilde{x}_{k+1}=f(\tilde{x}_k,\kappa^{\text{MPC}}(\tilde{x}_k))+w_k, \tag{38}$$

where $w_k\triangleq f(\tilde{x}_k-e_{x,k},\kappa^{\text{MPC}}(\tilde{x}_k))-f(\tilde{x}_k,\kappa^{\text{MPC}}(\tilde{x}_k))+e_{x,k+1}$. Using the Lipschitz property of $f(\cdot,u)$ for all fixed u in a compact space $\mathbb{U}$ leads to

$$|f(\tilde{x}_k-e_{x,k},u_k)-f(\tilde{x}_k,u_k)|\leq L_f|e_{x,k}|. \tag{39}$$

Thus, for all $k\in\mathbb{Z}_+$ it holds that

$$\begin{aligned}|w_k|=&|f(\tilde{x}_k-e_{x,k},\kappa^{\text{MPC}}(\tilde{x}_k))-f(\tilde{x}_k,\kappa^{\text{MPC}}(\tilde{x}_k))+e_{x,k+1}|\\ \leq&|f(\tilde{x}-e_{x,k},\kappa^{\text{MPC}}(\tilde{x}_k))-f(\tilde{x}_k,\kappa^{\text{MPC}}(\tilde{x}_k))|+\|e_x\|\\ \leq&(L_f+1)\|e_x\|.\end{aligned} \tag{40}$$

From the hypothesis we have that (38) is locally ISS with respect to w in $\mathbb{W}$ for $\tilde{x}_0$ in $\widetilde{\mathscr{X}}_f(N)$. Moreover, from (40) it follows that (38) is thus locally ISS with respect to e_x in $\mathbb{E}_x$ for all $k\in\mathbb{Z}_+$, i.e.

$$|\tilde{x}_k(k,\tilde{x}_0,e_x)|\leq\beta_{\tilde{x}}(|\tilde{x}_0|,k)+\gamma_{\tilde{x}}^{e_x}(\|e_x\|), \tag{41}$$

where $\beta_{\tilde{x}}(|\tilde{x}_0|,k)\triangleq\alpha_1^{-1}(2\rho^k\alpha_2(|\tilde{x}_0|))$ and $\gamma_{\tilde{x}}^{e_x}(\|e_x\|)=\gamma_{\tilde{x}}^{w}\circ(L_f+1)\|e_x\|\triangleq\alpha_1^{-1}(2\sigma((L_f+1)\|e_x\|)\frac{1}{1-\rho})$. Utilizing the proposed coordinate change (35) and property (41), we obtain that for all x_0, $\tilde{x}_0\in\widetilde{\mathscr{X}}_f(N)$, $k\in\mathbb{Z}_+$,

$$\begin{aligned}|x(k,x_0,e_x)|=&|\tilde{x}_k(k,\tilde{x}_0,e_x)-e_{x,k}|\leq|\tilde{x}_k(k,\tilde{x}_0,e_x)|+|e_{x,k}|\\ &\leq\beta_{\tilde{x}}(|x_0+e_{x,0}|,k)+\gamma_{\tilde{x}}^{e_x}(\|e_x\|)+|e_{x,k}|\\ &\leq\beta_{\tilde{x}}(|x_0|+|e_{x,0}|,k)+\gamma_{\tilde{x}}^{e_x}(\|e_x\|)+\|e_x\|\\ &\leq\beta_{\tilde{x}}(2|x_0|,k)+\beta_{\tilde{x}}(2|e_{x,0}|,k)+\gamma_{\tilde{x}}^{e_x}(\|e_x\|)+\|e_x\|\\ &\leq\beta_{\tilde{x}}(2|x_0|,k)+\beta_{\tilde{x}}(2\|e_x\|,0)+\gamma_{\tilde{x}}^{e_x}(\|e_x\|)+\|e_x\|\\ &\leq\beta_x(|x_0|,k)+\gamma_x^{e_x}(\|e_x\|),\end{aligned} \tag{42}$$

where $\beta_x(|x_0|,k)\triangleq\beta_{\tilde{x}}(2|x_0|,k)=\alpha_1^{-1}(2\rho^k\alpha_2(2|x_0|))$ and $\gamma_x^{e_x}(\|e_x\|)\triangleq\beta_{\tilde{x}}(2\|e_x\|,0)+\gamma_{\tilde{x}}^{e_x}(\|e_x\|)+\|e_x\|=\alpha_1^{-1}(2\alpha_2(2\|e_x\|))+\alpha_1^{-1}(2\sigma((L_f+1)\|e_x\|)\frac{1}{1-\rho})+\|e_x\|$. Expression (42) implies (local) ISS of (31) with respect to e_x as input. ■

VI. OBSERVER DESIGN

In Section IV we derived the error dynamics (32), (33) of the observer candidates (16) and (18). In this section we will prove that error dynamics (32), (33) is (locally) ISS and IOS with respect to $\mathbf{e}_{\mathbf{u},k}^{[1,n]}$ as input. Recall that $\mathbf{e}_{\mathbf{u},k}^{[1,n]}$ represents the prediction error present in the *predicted* future input sequence $\overline{\mathbf{u}}_k^{[1,n]}$ fed from the NMPC controller to the observer candidates at discrete time $k\in\mathbb{Z}_+$. The standing assumption for the result in this section is $f,g\in C^\omega$, and Lipschitz continuity of f_z with respect to $\mathbf{u}_k^{[1,n]}$.

Theorem VI.1 *Let (14) be strongly locally observable at x_0 and A_i in (32) be Schur. Suppose the sequences $\mathbf{y}_k^{[1-n,0]}$, $\mathbf{u}_k^{[1-n,0]}$ and $\overline{\mathbf{u}}_k^{[1,n]}$ are bounded for all $k\in\mathbb{Z}_+$. Then, the z-error dynamics (32) is ISS with respect to $\mathbf{e}_{\mathbf{u}}^{[1,n]}$ as input, i.e. for all $k\in\mathbb{Z}_+$*

$$|e_z(k,e_{z,0},\mathbf{e}_{\mathbf{u}}^{[1,n]})|\leq\max\left\{\tilde{\beta}_{e_z}(|e_{z,0}|,k),\tilde{\gamma}_{e_z}^{\mathbf{e}_{\mathbf{u}}}(\|\mathbf{e}_{\mathbf{u}}^{[1,n]}\|)\right\}, \tag{43}$$

where $\tilde{\beta}_{e_z}(|e_{z,0}|,k) \triangleq 2\hbar\eta^k|e_{z,0}|$, $\tilde{\gamma}^{\mathbf{e_u}}_{e_z}(\|\mathbf{e_u}^{[1,n]}\|) \triangleq \frac{2\hbar}{1-\eta}L_{f_z}\|\mathbf{e_u}^{[1,n]}\|$ *with* $\hbar > 0$ *and* $\eta \in [0,1)$ *such that* $|A^k| \leq \hbar\eta^k$ *holds*[2]*. Moreover, the x-error dynamics defined by* (32), (33) *is (locally) IOS with respect to input* $\mathbf{e_u}^{[1,n]}$ *with*

$$|e_x(k,e_{z,0},\mathbf{e_u}^{[1,n]})| \leq \max\left\{\tilde{\beta}_{e_x}(|e_{z,0}|,k), \tilde{\gamma}^{\mathbf{e_u}}_{e_x}(\|\mathbf{e_u}^{[1,n]}\|)\right\}, \tag{44}$$

where $\tilde{\beta}_{e_x}(|e_{z,0}|,k) \triangleq L_\Xi\tilde{\beta}_{e_z}(|e_{z,0}|,k)$, $\tilde{\gamma}^{\mathbf{e_u}}_{e_x}(\|\mathbf{e_u}^{[1,n]}\|) \triangleq L_\Xi\max\{\frac{2\hbar}{1-\eta}L_{f_z},1\}\|\mathbf{e_u}^{[1,n]}\|$ *and* L_Ξ *is the Lipschitz constant of the function* $\Xi^{-1}_{\mathbf{uy}fixed}$ *with respect to the arguments* z_k *and* $\mathbf{u}_k^{[1,n-1]}$.

Proof: The z-error dynamics defined by (32) can be seen as a non-autonomous linear system, i.e.

$$e_{z,k+1} = A_ie_{z,k} + v_k, \quad e_{z,k=0} = e_{z,0}, \tag{45}$$

where input v_k is defined as

$$v_k \triangleq \Delta f_z(\mathbf{y}_k^{[1-n,0]},\mathbf{u}_k^{[1-n,0]},\overline{\mathbf{u}}_k^{[1,n]},\mathbf{e}_{\mathbf{u},k}^{[1,n]}).$$

The function f_z is Lipschitz continuous with respect to $\mathbf{u}_k^{[1,n]}$, therefore there exists for all fixed bounded sequences $\mathbf{y}_k^{[1-n,0]}$, $\mathbf{u}_k^{[1-n,0]}$ and $\mathbf{u}_k^{[1,n]}$ a Lipschitz constant L_{f_z} such that for all $k \in \mathbb{Z}_+$

$$|v_k| = |\Delta f_z(\mathbf{y}_k^{[1-n,0]},\mathbf{u}_k^{[1-n,0]},\overline{\mathbf{u}}_k^{[1,n]},\mathbf{e}_{\mathbf{u},k}^{[1,n]})| \leq L_{f_z}\|\mathbf{e_u}^{[1,n]}\|. \tag{46}$$

Since A_i in (45) is Schur, there exist constants $\hbar > 0$ and $\eta \in [0,1)$ such that $|A_i^k| \leq \hbar\eta^k$ holds, e.g. see [9]. From (45), we have that

$$e_{z,k+1} = A_i^{k+1}e_{z,0} + \sum_{j=0}^{k}A_i^{k-j}v_j, \tag{47}$$

which yields that the ISS property in Definition II.3 holds with

$$\beta_{e_z}(|e_{z,0}|,k) \triangleq \hbar\eta^k|e_{z,0}|,\ \gamma^v_{e_z}(\|v\|) \triangleq \sum_{j=0}^{\infty}\hbar\eta^j\|v\| \triangleq \frac{\hbar}{1-\eta}\|v\|.$$

Via (46) we obtain (local) ISS of (32) (in the sense of (5)) with respect to $\mathbf{e_u}^{[1,n]}$ as input, for

$$\tilde{\beta}_{e_z}(|e_{z,0}|,k) \triangleq 2\hbar\eta^k|e_{z,0}|,\ \tilde{\gamma}^{\mathbf{e_u}}_{e_z}(\|\mathbf{e_u}^{[1,n]}\|) \triangleq \frac{2\hbar}{1-\eta}L_{f_z}\|\mathbf{e_u}^{[1,n]}\|.$$

Next, since the map $\Xi^{-1}_{\mathbf{uy}\text{fixed}}(\cdot,\mathbf{y}_k^{[1-n,0]},\mathbf{u}_k^{[1-n,0]},\cdot)$ is Lipschitz continuous, for all fixed bounded sequences $\mathbf{y}_k^{[1-n,-1]}$ and $\mathbf{u}_k^{[1-n,0]}$, with respect to the first and last argument with Lipschitz constant L_Ξ, we obtain that for all $k \in \mathbb{Z}_+$

$$\begin{aligned}|e_{x,k}| &= |\Delta\Xi(e_{z,k},\hat{z}_k,\mathbf{y}_k^{[1-n,-1]},\mathbf{u}_k^{[1-n,0]},\overline{\mathbf{u}}_k^{[1,n-1]},\mathbf{e}_{\mathbf{u},k}^{[1,n-1]})| \\ &\leq L_\Xi\max\left\{|e_{z,k}|,\|\mathbf{e_u}^{[1,n-1]}\|\right\},\end{aligned} \tag{48}$$

Substitution of (43) in (48) results in (44) with the $\mathscr{KL}$- and $\mathscr{K}$-*functions* $\tilde{\beta}_{e_x}$ and $\tilde{\gamma}^{\mathbf{e_u}}_{e_x}$, respectively, as stated in Theorem VI.1. ∎

[2] L_{f_z} is the Lipschitz constant of the function f_z in (15) with respect to the argument $\mathbf{u}_k^{[1,n]}$

Next it will be shown that there exists an explicit bound on e_x satisfying (32), (33) for all $\mathbf{e_u}^{[1,n]}$ in some compact set $\mathbb{E}_{\mathbf{e_u}}$ with zero in its interior.

Lemma VI.2 *Let* (14) *be strongly locally observable at* x_0 *and* A_i *in* (32) *be Schur. Suppose the sequences* $\mathbf{y}_k^{[1-n,0]}$, $\mathbf{u}_k^{[1-n,0]}$, $\overline{\mathbf{u}}_k^{[1,n]}$ *are bounded and* $\mathbf{e_u}^{[1,n]} \in \mathbb{E}_{\mathbf{e_u}}$ *where* $\mathbb{E}_{\mathbf{e_u}} \triangleq \{\mathbf{e_u}^{[1,n]} \in \mathbb{R}^n \mid \|\mathbf{e_u}^{[1,n]}\| \leq \varepsilon_{\mathbf{e_u}}\}$ *with* $\varepsilon_{\mathbf{e_u}} > 0$*. Then, for initial conditions* $e_{z,0}$ *in* $\mathscr{E}_z$ *with* $\mathscr{E}_z \triangleq \{e_{z,0} \in \mathbb{R}^n \mid |e_{z,0}| \leq \frac{1}{1-\eta}L_{f_z}\varepsilon_{\mathbf{e_u}}\}$*, the trajectory* $e_x(k,e_{z,0},\mathbf{e_u}^{[1,n]})$*, satisfying the x-error dynamics* (32), (33)*, satisfies*

$$|e_x(k,e_{z,0},\mathbf{e_u}^{[1,n]})| \leq L_\Xi\varepsilon_{\mathbf{e_u}}\max\left\{\frac{2\hbar}{1-\eta}L_{f_z},1\right\}, \quad \forall k \in \mathbb{Z}_+. \tag{49}$$

Moreover, for observer (18) *there exist observer gains* $q_1,\ldots,q_n$ *such that*

$$|e_x(k,e_{z,0},\mathbf{e_u}^{[1,n]})| \leq L_\Xi\varepsilon_{\mathbf{e_u}}, \quad \forall k \in \mathbb{Z}_+. \tag{50}$$

Proof: From the hypothesis we have that expression (43) in Theorem VI.1 holds, so that for all $k \in \mathbb{Z}_+$

$$\begin{aligned}|e_z(k,e_{z,0},\mathbf{e_u}^{[1,n]})| &\leq \max\left\{\tilde{\beta}_{e_z}(|e_{z,0}|,k),\tilde{\gamma}^{\mathbf{e_u}}_{e_z}(\|\mathbf{e_u}^{[1,n]}\|)\right\} \\ &\leq \max\left\{\tilde{\beta}_{e_z}(|e_{z,0}|,k),\frac{2\hbar}{1-\eta}L_{f_z}\varepsilon_{\mathbf{e_u}}\right\} \\ &\leq \max\left\{\tilde{\beta}_{e_z}(|e_{z,0}|,0),\frac{2\hbar}{1-\eta}L_{f_z}\varepsilon_{\mathbf{e_u}}\right\}.\end{aligned} \tag{51}$$

For initial conditions $e_{z,0} \in \mathscr{E}_z$ we have that

$$\tilde{\beta}_{e_z}(|e_{z,0}|,0) \leq \frac{2\hbar}{1-\eta}L_{f_z}\varepsilon_{\mathbf{e_u}}. \tag{52}$$

Thus, for $e_{z,0} \in \mathscr{E}_z$, inequality (51) yields

$$\|e_z\| \leq \frac{2\hbar}{1-\eta}L_{f_z}\varepsilon_{\mathbf{e_u}}. \tag{53}$$

Substituting (53) in (48) and taking into account the fact that $\mathbf{e_u}^{[1,n-1]} \subseteq \mathbb{E}_{\mathbf{e_u}}$ the first statement in Lemma VI.2, i.e. (49), follows. The second statement of Lemma VI.2, i.e. (50), follows from the fact that the diagonal structure of the matrix A_q, defining the z-error dynamics (32) of observer candidate observer (18), allows to render the term $\frac{\hbar}{1-\eta}$ in (49) arbitrary small by choosing appropriate observer gains $q_1,\ldots,q_n$. This can be concluded by employing the relation $|A_q^k| \leq \hbar\eta^k$ from Theorem VI.1. ∎

VII. Interconnection Results

So far, we have *separately* designed an NMPC controller which is robust (ISS) to estimation errors (e_x) and obtained an observer for which the error dynamics is robust (IOS) with respect to prediction errors $\mathbf{e_u}^{[1,n]}$ present in the predicted future input sequence $\overline{\mathbf{u}}^{[1,n]}$ fed to the observer. In this section we investigate the properties of the IOS observer error dynamics interconnected with the ISS NMPC system (31) according to Fig. 2.

The standing assumptions for the results in this section are

Assumption VII.1

- $f, g \in C^{\omega}$;
- Lipschitz continuity of f with respect to x on the domain $\mathbb{X} \times \mathbb{U}$;
- Lipschitz continuity of f_z with respect to $\mathbf{u}_k^{[1,n]}$;
- *Regularity* of the NMPC controller, in the sense of Definition II.9, w.r.t the $\hat{x}$, i.e. $|u_{k+i|k}| \leq \theta_i |\hat{x}_k|$.

Lemma VII.2 *Suppose $N \geq n$ and Assumption VII.1 holds. Then, there exist $\mathscr{K}$-functions $\gamma_{\mathbf{e_u}}^x$ and $\gamma_{\mathbf{e_u}}^{e_x}$ such that the sequence $\mathbf{e}_{\mathbf{u},k}^{[1,n]}$ satisfies*

$$|\mathbf{e}_{\mathbf{u}}^{[1,n]}(k,x,e_x)| \leq \max\left\{\tilde{\gamma}_{\mathbf{e_u}}^x(\|x\|), \tilde{\gamma}_{\mathbf{e_u}}^{e_x}(\|e_x\|)\right\}, \quad \forall k \in \mathbb{Z}_+, \tag{54}$$

where $\tilde{\gamma}_{\mathbf{e_u}}^x$ and $\tilde{\gamma}_{\mathbf{e_u}}^{e_x}$ are defined as $\tilde{\gamma}_{\mathbf{e_u}}^x(\|x\|) \triangleq 2(\theta_0 + \overline{\theta})\|x\|$ and $\tilde{\gamma}_{\mathbf{e_u}}^{e_x}(\|e_x\|) \triangleq 2(\theta_0 + \overline{\theta})\|e_x\|$, with $\overline{\theta} = \max_{i \in \{1,2,\ldots,n\}}\{\theta_i\}$.

Proof: Using regularity (Definition II.9) and the triangle inequality, the induced norm of the difference between the predicted future inputs and the real inputs can be upper bounded for all $k \in \mathbb{Z}_+$ and $i = 1, \ldots, n$, i.e.

$$\begin{aligned}|u_{k+i} - u_{k+i|k}| &\leq |u_{k+i}| + |u_{k+i|k}| \leq \theta_0|\hat{x}_{k+i}| + \theta_i|\hat{x}_k|, \\ &\leq \theta_0|\hat{x}_{k+i}| + \overline{\theta}|\hat{x}_k|.\end{aligned} \tag{55}$$

Since (55) holds for all $k \in \mathbb{Z}_+$ and $i = 1, \ldots, n$ we have that

$$\begin{aligned}\|\mathbf{e}_{\mathbf{u}}^{[1,n]}\| &\leq (\theta_0 + \overline{\theta})\|\hat{x}\| \\ &\leq (\theta_0 + \overline{\theta})(\|x\| + \|e_x\|) \\ &\leq \max\left\{2(\theta_0 + \overline{\theta})\|x\|, 2(\theta_0 + \overline{\theta})\|e_x\|\right\},\end{aligned} \tag{56}$$

which concludes the proof of the statement. ■

Regularity thus leads to property (54). Due to this property the following result can be obtained.

Theorem VII.3 *Let* (14) *be strongly locally observable on the domain $\mathbb{X} \times \mathbb{U}$ and A_i in* (32) *be Schur. Suppose the NMPC control law κ^{MPC}, with $N \geq n$, renders* (31) *(locally, i.e. for initial conditions $x_0 \in \widetilde{\mathscr{X}}_f(N)$) ISS with respect to input $e_x \in \mathbb{E}_x \triangleq \{e_x \in \mathbb{R}^n \mid \|e_x\| \leq \nu\}$ with $\nu \geq L_{\Xi}\varepsilon_{\mathbf{e_u}} \max\{\frac{2\hbar}{1-\eta}L_{f_z}, 1\}$ and assume Assumption VII.1 holds. Then, if*

$$4(\theta_0 + \overline{\theta})\gamma_x^{e_x}\left(L_{\Xi}\max\left\{\frac{2\hbar}{1-\eta}L_{f_z}, 1\right\}\right) \leq 1, \tag{57}$$

it holds that

$$\overline{\lim_{k\to\infty}}|\mathbf{e}_{\mathbf{u},k}^{[1,n]}| = 0, \tag{58}$$

for all initial conditions $e_{z,0} \in \mathscr{E}_z \triangleq \{e_{z,0} \in \mathbb{R}^n \mid |e_{z,0}| \leq \frac{1}{1-\eta}L_{f_z}\varepsilon_{\mathbf{e_u}}\}$.

Proof: By the hypothesis we have that property (34), (44) and (54) of Theorem V.1, VI.1, and Lemma VII.2, respectively, hold. Thus, we know that

$$\overline{\lim_{k\to\infty}}|e_{x,k}| \leq \tilde{\gamma}_{e_x}^{\mathbf{e_u}}\left(\overline{\lim_{k\to\infty}}|\mathbf{e}_{\mathbf{u},k}^{[1,n]}|\right), \tag{59a}$$

$$\overline{\lim_{k\to\infty}}|x_k| \leq 2\gamma_x^{e_x}\left(\overline{\lim_{k\to\infty}}|e_{x,k}|\right), \tag{59b}$$

$$\overline{\lim_{k\to\infty}}|\mathbf{e}_{\mathbf{u},k}^{[1,n]}| \leq \max\left\{\tilde{\gamma}_{\mathbf{e_u}}^x\left(\overline{\lim_{k\to\infty}}|x_k|\right), \tilde{\gamma}_{\mathbf{e_u}}^{e_x}\left(\overline{\lim_{k\to\infty}}|e_{x,k}|\right)\right\}. \tag{59c}$$

Substitution of (59a) and (59b) in (59c) yields

$$\begin{aligned}\overline{\lim_{k\to\infty}}|\mathbf{e}_{\mathbf{u},k}^{[1,n]}| \leq 2(\theta_0 + \overline{\theta})\max\Bigg\{&2\gamma_x^{e_x} \circ \tilde{\gamma}_{e_x}^{\mathbf{e_u}}\left(\lim_{k\to\infty}|\mathbf{e}_{\mathbf{u},k}^{[1,n]}|\right), \\ &\tilde{\gamma}_{e_x}^{\mathbf{e_u}}\left(\overline{\lim_{k\to\infty}}|\mathbf{e}_{\mathbf{u},k}^{[1,n]}|\right)\Bigg\}.\end{aligned} \tag{60}$$

By inspection of the gain function $\gamma_x^{e_x}$ (defined in Theorem V.1) we can conclude that the maximum in (60) is determined by its first argument, so that

$$\overline{\lim_{k\to\infty}}|\mathbf{e}_{\mathbf{u},k}^{[1,n]}| \leq 4(\theta_0 + \overline{\theta})\gamma_x^{e_x} \circ \tilde{\gamma}_{e_x}^{\mathbf{e_u}}\left(\overline{\lim_{k\to\infty}}|\mathbf{e}_{\mathbf{u},k}^{[1,n]}|\right), \tag{61}$$

with $\tilde{\gamma}_{e_x}^{\mathbf{e_u}} \triangleq L_{\Xi}\max\{\frac{2\hbar}{1-\eta}L_{fz}, 1\}$. By the hypothesis of Theorem VII.3 we have that $4(\theta_0 + \overline{\theta})\gamma_x^{e_x} \circ \tilde{\gamma}_{e_x}^{\mathbf{e_u}} < 1$. together with the fact that $\overline{\lim}_{k\to\infty}|\mathbf{e}_{\mathbf{u},k}^{[1,n]}|$ is well defined (due to compactness of $\mathbb{U}$ we know that $\overline{\lim}_{k\to\infty}|\mathbf{e}_{\mathbf{u},k}^{[1,n]}|$ is finite) we have that (61) can only be true if (58) in Theorem VII.3 holds. This concludes the proof of the statement. ■

Under condition (57) in Theorem VII.3, the IOS and ISS property of the observer error dynamics (32), (33) with respect to $\mathbf{e}_{\mathbf{u},k}^{[1,n]}$ as input implies $e_{x,k} \to 0$ for $k \to \infty$. Then, due to the ISS property of the NMPC controller in closed-loop with (14) with respect to $e_{x,k}$ as input implies $x_k \to 0$ for $k \to \infty$. Thus, if condition (57) in Theorem VII.3 is satisfied (local) asymptotic stability of the proposed OBNMPC scheme as depicted in Fig. 1 is guaranteed.

VIII. CONCLUSIONS

In this paper we proposed an observer based nonlinear predictive control approach for a class of nonlinear discrete-time systems. We derived sufficient conditions for (local) asymptotic stability of the closed-loop system trajectory of the observer based nonlinear model predictive control scheme. The sufficient (local) asymptotic stability conditions can in theory always be satisfied by choosing appropriate controller tuning parameters and observer gains.

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