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Stable and unstable sets for evolution equations of parabolic and hyperbolic type

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ABSTRACT. Some roles in the global dynamics of so called stable and unstable sets will be given for semilinear heat equations and semilinear wave equations with dissipative terms.

1. Introduction

Let $\Omega \subset R^N$ be a bounded domain with smooth boundary $\partial\Omega$. We are concerned with the following two mixed problems:

$$u_t - \Delta u = |u|^{p-1}u, \qquad x \in \Omega, t > 0, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

$$u(t,x)|_{\partial\Omega} = 0 \qquad \text{for } t \geq 0, \tag{1.3}$$

and

$$u_{tt} - \Delta u + \delta u_t = |u|^{p-1}u, \qquad x \in \Omega, t > 0, \tag{1.4}$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \qquad x \in \Omega, \tag{1.5}$$

$$u(t,x)|_{\partial\Omega} = 0 \qquad \text{for } t \geq 0. \tag{1.6}$$

Here $p > 1$, $\delta \geq 0$ and Δ is the Laplacian in R^N.

For these problems, many authors investigated their dynamics. In particular, since Sattinger [21] has constructed so called stable set in 1968, the method of stable set (potential well) was used in order to construct global solutions (Ebihara et al. [3], Ikehata [9], Ishii [11], Lions [14], Nakao et al. [16], Ōtani [17] and Tsutsumi [22, 23] e.g.). Furthermore, with respect to the blowing-up properties, there is a work of Payne et al. [19]. Namely, roughly speaking, if initial data u_0 belongs to so called unstable set, then the associated weak solution blows up in a finite time. Of related interest is the works of Ikehata et al. [10], Ishii [11] and Ōtani [17, 18].

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Now the first purpose of this paper is to characterize those stable and unstable sets by the asymptotic behaviour of solutions to the problems (1.1)–(1.3), adopting the arguments of Dynamical System (see Henry [7]). Our method may be topological in this sense. The second purpose of this paper is to consider the same problems for (1.4)–(1.6). In particular, we can characterize stable set of the equation (1.4) by the asymptotic behaviour of solutions as $t \to \infty$ and give sufficient conditions of initial data in order to blow up in finite time by the energy method. Although the method is different from that for heat equation (1.1), we will be able to say that the wave equation (1.4) with $\delta > 0$ has a similar property to (1.1). However, unfortunately we can not characterize unstable set right now because of lack of 'smoothing effect' in (1.4).

The contents of this paper are as follows. In section 2 we prepare several facts on the local existence of solutions to (1.1)–(1.3) or (1.4)–(1.6) and basic results of stable and unstable sets. In section 3 we state the main results (Theorem 3.1) to the problem (1.1)–(1.3). In section 4 we assert the main theorems to the problem (1.4)–(1.6) (Theorems 4.1 and 4.2). Section 5 is devoted to the proof of Theorem 4.1 and in section 6 we prove Theorem 4.2.

After our work has been completed, we are noticed that Kawanago [12] studied the dynamics of the Cauchy problem of (1.1) in R^N with $|u|^{p-1}u$ replaced by u^p. This is closely related to our study, as he investigated the set K, introduced by Lions [15], of initial values for the existence of global solution, in detail. In addition, the use of an argument of Giga [4] is in the same way as ours.

2. Preliminaries

Throughout this paper the functions considered are all real valued and the notations for their norms are adopted as usual ones (e.g., Lions [14]). Furthermore, $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$.

We shall describe some lemmas.

LEMMA 2.1 (SOBOLEV-POINCARÉ). *If* $2 \leq q \leq \dfrac{2N}{N-2}$, *then*

$$\|u\|_q \leq C(\Omega, q)\|\nabla u\|_2$$

for $u \in H_0^1(\Omega)$, *where* $\|u\|_q$ *means the usual* $L^q(\Omega)$*-norm.*

The next two local existence theorems are given by Hoshino et al. [8] and Haraux [6], respectively. In particular, Theorem 2.3 is easily proved by using Banach's fixed point theorem.

THEOREM 2.2 (HEAT EQUATION). *Assume either* $1 < p < \dfrac{N+2}{N-2}$ $(N \geq 3)$ *or*

$1 < p < +\infty$ $(N = 1, 2)$. *Then for any* $u_0 \in H_0^1(\Omega)$, *there exists a real number* $T_m > 0$ *such that the problem* (1.1)–(1.3) *has a unique local solution* $u \in C([0, T_m); H_0^1(\Omega)) \cap C^1((0, T_m); L^2(\Omega))$. *Furthermore, u becomes a classical solution of* (1.1)–(1.3) *for* $t > 0$ *and if* $T_m < +\infty$, *then*

$$\lim_{t \uparrow T_m} \|\nabla u(t, \cdot)\|_2 = +\infty \qquad \textit{and} \qquad \lim_{t \uparrow T_m} \|u(t, \cdot)\|_\infty = +\infty.$$

THEOREM 2.3 (WAVE EQUATION). *Let* $\delta \geq 0$ *and suppose either* $1 < p \leq \dfrac{N}{N-2}$ $(N \geq 3)$ *or* $1 < p < +\infty$ $(N = 1, 2)$. *Then for any* $u_0 \in H_0^1(\Omega)$ *and* $u_1 \in L^2(\Omega)$, *there exists a real number* $T_m > 0$ *such that the problem* (1.4)–(1.6) *admits a unique local weak solution u(t,x) which belongs to the class*:

$$C([0, T_m); H_0^1(\Omega)) \cap C^1([0, T_m); L^2(\Omega)) \cap C^2([0, T_m); H^{-1}(\Omega)),$$

and if $T_m < +\infty$, *then*

$$\lim_{t \uparrow T_m} [\|\nabla u(t, \cdot)\|_2 + \|u_t(t, \cdot)\|_2] = +\infty.$$

Now we define some functionals as follows:

$$J(u) \equiv \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \qquad \text{for } u \in H_0^1(\Omega), \tag{2.1}$$

$$I(u) \equiv \|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1} \qquad \text{for } u \in H_0^1(\Omega). \tag{2.2}$$

And also we define so called 'Nehari manifold' and 'potential depth', respectively as follows (see Payne et al. [19]):

$$\begin{gathered} \mathcal{N} \equiv \{u \in H_0^1(\Omega); I(u) = 0, u \neq 0\}, \\ d \equiv \inf\left\{\sup_{\lambda \geq 0} J(\lambda u); u \in H_0^1(\Omega), u \neq 0\right\}. \end{gathered} \tag{2.3}$$

Then with the aid of Lemma 2.1, we have (see Ikehata et al. [10] and Payne et al. [19])

$$0 < d = \inf_{u \in \mathcal{N}} J(u). \tag{2.4}$$

Furthermore, if we set:

$$E \equiv \{u \in H_0^1(\Omega); -\Delta u = |u|^{p-1}u, u|_{\partial\Omega} = 0\},$$

$$E^* \equiv \{u \in E; J(u) = d\},$$

then we have (see Payne et al. [19])

$$E^* = \{u \in \mathcal{N}; J(u) = d\} \neq \phi.$$

Now let us define so called stable set W^* and unstable set V^* (see Sattinger [21], Payne et al. [19]):

$$W^* \equiv \{u \in H_0^1(\Omega); J(u) < d, I(u) > 0\} \cup \{0\}, \tag{2.5}$$

$$V^* \equiv \{u \in H_0^1(\Omega); J(u) < d, I(u) < 0\}. \tag{2.6}$$

Then we have

LEMMA 2.4.
(1) *W^* is a bounded neighbourhood of* 0 *in* $H_0^1(\Omega)$,
(2) $0 \notin \bar{V}^*$,
(3) $\bar{W}^* \cap \bar{V}^* = E^*$,
(4) $E^* \subset \mathcal{N}$.
Here $\bar{U}$ means the closure of U in $H_0^1(\Omega)$.

PROOF. For (1), see Lions [14, p. 31]. Let us show (2). Suppose $0 \in \bar{V}^*$. Then there exists a sequence $\{v_n\} \subset V^*$ such that $v_n \to 0$ as $n \to \infty$ in $H_0^1(\Omega)$. (1) means that if n is sufficiently large, then $v_n \in W^*$. These contradict to the fact $W^* \cap V^* = \phi$. Since (4) is trivial, we finally prove (3). Indeed, if $v \in \bar{W}^* \cap \bar{V}^*$, then $I(u) = 0$ and $J(u) \leq d$. Further, (2) implies $v \neq 0$. Therefore, we get $v \in \mathcal{N}$, $J(v) \leq d$. Noting (2.4), we obtain $v \in E^*$. Conversely, if $v \in E^*$, then we have $J(u) = d$ and $I(u) = 0$ with $v \neq 0$. This implies $v \in \bar{W}^* \cap \bar{V}^*$. ■

Finally, we shall prepare energy identities associated with the problems (1.1)–(1.3) and (1.4)–(1.6), respectively:

LEMMA 2.5 (HEAT EQUATION). *Let $u(t,x)$ be a local solution to* (1.1)–(1.3) *on* $[0, T_m)$ *with initial data* $u_0 \in H_0^1(\Omega)$. *Then*

$$J(u(t,\cdot)) + \int_0^t \|u_t(s,\cdot)\|_2^2 \, ds = J(u_0) \qquad \text{on } [0, T_m).$$

LEMMA 2.6 (WAVE EQUATION). *Let $\delta \geq 0$ and let $u(t,x)$ be a local solution to* (1.4)–(1.6) *on* $[0, T_m)$ *with initial data* $u_0 \in H_0^1(\Omega)$ *and* $u_1 \in L^2(\Omega)$. *Then*

$$E(u(t,\cdot), u_t(t,\cdot)) + \delta \int_0^t \|u_t(s,\cdot)\|_2^2 \, ds = E(u_0, u_1) \qquad \text{on } [0, T_m),$$

where $E(u,v) \equiv \frac{1}{2}\|v\|_2^2 + J(u)$ *is a Liapnov functional corresponding to the equation* (1.4).

3. Heat equation and stable-unstable set

Throughout this section, we shall concentrate our interest on an analysis of the problem (1.1)–(1.3). Of course, we assume the unique local existence Theorem 2.2. Then our results read as follows:

THEOREM 3.1. *Let $u(t,x)$ be a local solution to the problem* (1.1)–(1.3) *on* $[0, T_m)$ *with initial data* $u_0 \in H_0^1(\Omega)$. *Then there exists a real number* $t_0 \in [0, T_m)$ *such that* $u(t_0, \cdot) \in W^*$ *if and only if* $T_m = +\infty$ *and* $\lim_{t\to\infty} \|\nabla u(t,\cdot)\|_2 = 0$.

THEOREM 3.2. *Suppose that either* $u_0 \geq 0$ *or* Ω *is a convex set. Let* $u(t,x)$ *be a local solution to the problem* (1.1)–(1.3) *on* $[0, T_m)$ *with initial data* $u_0 \in H_0^1(\Omega)$. *Then there is a real number* $t_0 \in [0, T_m)$ *such that* $u(t_0, \cdot) \in V^*$ *if and only if* $T_m < +\infty$.

REMARK 3.3. *It has been known that if* $u_0 \in W^*$, *then* $T_m = +\infty$ *and* $u(t,\cdot) \to 0$ *in* $H_0^1(\Omega)$ *as* $t \to \infty$ *or if* $u_0 \in V^*$, *then* $T_m < +\infty$ *(see Ishii* [11], *Ōtani* [17], *Payne et al.* [19] *and Tsutsumi* [22]). *However, all of their results depend on the energy method differently from ours.*

To prove Theorems 3.1–3.2, we need some lemmas.

When $T_m = +\infty$, we can define so called ω-limit set $\omega(u_0)$ associated with (1.1)–(1.3) as follows: Let $u(t,x)$ be a global solution to (1.1)–(1.3) with $T_m = +\infty$ in Theorem 2.2. Then

$$\omega(u_0) \equiv \{u \in H_0^1(\Omega); \text{there is a sequence } \{t_n\} \text{ with } t_n \to \infty \text{ as } n \to \infty \text{ such that } u(t_n, \cdot) \to u \text{ in } H_0^1(\Omega)\}.$$

The following proposition will be given by Henry [7]:

PROPOSITION 3.4. *Suppose* $T_m = +\infty$ *in Theorem* 2.2. *Then*

(1) $\omega(u_0) \neq \phi$ *is compact in* $H_0^1(\Omega)$,

(2) $\omega(u_0)$ *is connected in* $H_0^1(\Omega)$,

(3) $\omega(u_0) \subset E$,

(4) $\operatorname{dist}(u(t,\cdot), \omega(u_0)) \to 0$ *as* $t \to +\infty$.

Here $\operatorname{dist}(u, \omega(u_0))$ *means the distance from* u *to* $\omega(u_0)$.

Next we can prove the following lemma in the same way as in Tsutsumi [22].

LEMMA 3.5. *Let* $u(t,\cdot)$ *be a local solution to* (1.1)–(1.3) *on* $[0, T_m)$ *and let* $S(t)$ *be a 'dynamical system' corresponding to the problem* (1.1)–(1.3), *i.e.,* $S(t)$ *is a mapping* $u_0 \mapsto u(t,\cdot)$. *Then*

$$S(t)W^* \subset W^* \qquad \text{and} \qquad S(t)V^* \subset V^* \quad \text{on } [0, T_m).$$

Now we are in a position to prove Theorems 3.1–3.2.

PROOF OF THEOREM 3.1. First we shall prove Theorem 3.1. Suppose that there is a real number $t_0 \in [0, T_m)$ such that $u(t_0, \cdot) \in W^*$. Then Lemma 3.5 means $u(t,\cdot) \in W^*$ for all $t \in [t_0, T_m)$. Therefore, by (1) of Lemma 2.4 there exists $M > 0$ such that $\|\nabla u(t,\cdot)\|_2 \leq M$ which implies $T_m = +\infty$ in Theorem

2.2. And also, it follows from Proposition 3.4 that $\omega(u_0) \neq \phi$ is connected and $\omega(u_0) \subset E$.

To begin with, assume $0 \notin \omega(u_0)$. Then there are $\omega \in \omega(u_0)$ with $\omega \neq 0$ and a sequence $\{t_n\}$ with $t_n \to \infty$ such that $u(t_n, \cdot) \to \omega$ in $H_0^1(\Omega)$. Futher, for sufficient large n we also have $u(t_n, \cdot) \in W^*$. So we obtain $\omega \in \omega(u_0) \cap \overline{W}^*$. On the other hand, since $\omega \neq 0$, we get $\omega \in \omega(u_0) \subset E \backslash \{0\} \subset \mathcal{N}$. This implies $\omega \in \mathcal{N} \cap \overline{W}^*$. Thus, we obtain $I(\omega) = 0$ and $J(\omega) \leq d$. It follows from the definition of d (see (2.4)) that

$$J(\omega) = d. \tag{3.1}$$

Moreover, since $u(t_0, \cdot) \in W^*$, it follows from Lemma 2.5 that

$$J(u(t_n, \cdot)) \leq J(u(t_0, \cdot)) < d$$

for sufficiently large n. Letting $n \to \infty$ above, we get

$$J(\omega) \leq J(u(t_0, \cdot)) < d, \tag{3.2}$$

which contradicts to (3.1). So it must hold $0 \in \omega(u_0)$.

Next let $B \neq \phi$ be a subset of $H_0^1(\Omega)$ such that $\omega(u_0) = \{0\} \cup B$. Then B must be closed in $H_0^1(\Omega)$. In fact, let $b_n \in B$ be a sequence such that $b_n \to b$ in $H_0^1(\Omega)$ for some $b \in H_0^1(\Omega)$. (1) of Proposition 3.4 means $b \in \omega(u_0)$. Suppose $b = 0$. Then we obtain from (1) of Lemma 2.4 that $b_n \in W^*$ for n large enough. On the other hand, since $b_n \neq 0$, it follows from (3) of proposition 3.4 that $b_n \in E0\} \subset \mathcal{N}$. So we get $b_n \in W^* \cap \mathcal{N} = \phi$ for n large enough. This is a contradiction. Thus, $b \neq 0$. This implies $b \in B$, i.e., B is closed. Finally, it follows from (2) and (4) of Proposition 3.4 that $B = \phi$, $\omega(u_0) = \{0\}$ and $u(t, \cdot) \to 0$ in $H_0^1(\Omega)$.

Conversely, if $T_m = +\infty$ and $\lim_{t \to \infty} \|\nabla u(t, \cdot)\|_2 = 0$, then from (1) of Lemma 2.4 that there is a number $t_0 \in [0, \infty)$ such that $u(t_0, \cdot) \in W^*$. ■

PROOF OF THEOREM 3.2. Second we shall prove Theorem 3.2. Since the proof of 'if' part of Theorem 3.2 is almost the same as that of Theorem 3.1, we will state only the outline of proof.

Assume that there exists a real number $t_0 \in [0, T_m)$ such that $u(t_0, \cdot) \in V^*$. Then we have from Lemma 3.5 that $u(t, \cdot) \in V^*$ for all $t \in [t_0, T_m)$. In the following, we suppose $T_m = +\infty$ (see Ōtani [17]). By using V^* instead of W^* in the proof of Theorem 3.1, first of all we get $\omega(u_0) = \{0\}$ and $u(t, \cdot) \to 0$ in $H_0^1(\Omega)$. From (1) of Lemma 2.4 we obtain that $u(t, \cdot) \in W^*$ for t large enough. Therefore, we get $u(t, \cdot) \in W^* \cap V^*$ for sufficiently large $t > t_0$ which contradicts to $W^* \cap V^* = \phi$. So we get $T_m < +\infty$.

Conversely, suppose $T_m < +\infty$. It follows from Lemma 2.5 that

$$(p+1)J(u_0) \geq (p+1)J(u(t, \cdot)) = \frac{p-1}{2} \|\nabla u(t, \cdot)\|_2^2 + I(u(t, \cdot)).$$

Since $\lim_{t\uparrow T_m} \|\nabla u(t,\cdot)\|_2 = +\infty$, the above inequality gives

$$\lim_{t\uparrow T_m} I(u(t,\cdot)) = -\infty. \tag{3.3}$$

Furthermore, since we also get $\lim_{t\uparrow T_m} \|u(t,\cdot)\|_\infty = +\infty$, when the initial data satisfies $u_0 \geq 0$, it follows from the results of Giga [4] that

$$\lim_{t\uparrow T_m} J(u(t,\cdot)) = -\infty. \tag{3.4}$$

(3.3) and (3.4) imply that there is a number $t_0 \in [0, T_m)$ such that $u(t_0,\cdot) \in V^*$.

Next, we shall rely on the results of Giga-Kohn [5] in order to prove (3.4) when Ω is convex. Indeed, if $T_m < +\infty$, then there exists a "blowup point" $a \in \Omega$ such that

$$v(t,y) \equiv (T_m - t)^\beta u(t, a + (T_m - t)^{1/2} y) \to \pm \beta^\beta \qquad \text{as } t \uparrow T_m,$$

where $\beta \equiv \dfrac{1}{p-1}$ and the convergence is uniform on every compact subset of R^N. If we set $s \equiv \log \dfrac{1}{T_m - t}$ and $w(s,y) \equiv \pm v(t,y)$, then $w(s,y)$ satisfies

$$w_s - \Delta w + \frac{1}{2} y \cdot \nabla w + \beta w = |w|^{p-1} w$$

on $\left(\log \dfrac{1}{T_m}, +\infty\right) \times \Omega_s$ with $\Omega_s \equiv \exp\left(\dfrac{s}{2}\right)(\Omega + \{-a\})$. Here, it is known that $w(s,y) \to \pm\beta^\beta$ as $s \to +\infty$ so that $\nabla w \to 0$ and $w_s \to 0$ as $s \to +\infty$, where the convergence is uniform on every compact subset of R^N. Under the above preliminaries, we can calculate as follows:

$$\begin{aligned} K(t) &\equiv \int_{t_1}^{t} \int_\Omega |u_t(t,x)|^2 dx\, dt \\ &= \int_{\log(1/T_m - t_1)}^{\log(1/T_m - t)} \exp\left\{\left(2\beta + 1 - \frac{N}{2}\right)s\right\} ds \int_{\Omega_s} \left|\beta w + w_s + \frac{1}{2} y \cdot \nabla w\right|^2 dy, \end{aligned}$$

where $\alpha \equiv 2\beta + 1 - \dfrac{N}{2} > 0$ by the conditions of p. So there is a real number $R > 0$ such that

$$K(t) \geq \int_{\log(1/T_m - t_1)}^{\log(1/T_m - t)} \exp(\alpha s) ds \int_{|y|<R} \left|\beta w + w_s + \frac{1}{2} y \cdot \nabla w\right|^2 dy.$$

On the other hand, by letting $s \to \infty (t \uparrow T_m)$, it follows that for any $\varepsilon > 0$ there is a number $s_0 > 0$ such that if $s_0 \leq s$, then

$$\int_{|y|<R} \left|\beta w + \frac{1}{2} y \cdot \nabla w + w_s\right|^2 dy > \beta^{2(\beta+1)} |B_R| - \varepsilon,$$

where $|B_R|$ is a volume of the set $\{y \in R^N; |y| < R\}$. Thus, there exist a constant $C_0 > 0$ and an another number $s_0 > 0$ such that if $s_0 \le s$, then

$$\int_{|y|<R} \left| \beta w + \frac{1}{2} y \cdot \nabla w + w_s \right|^2 dy \ge C_0.$$

Let $t_0 \equiv T_m - \exp(-s_0)$. Then for all $t \in [t_0, T_m)$ we get from Lemma 2.5 that

$$\begin{aligned} J(u(t)) &= J(u(t_0)) - \int_{t_0}^{t} \int_{\Omega} |u_t|^2 dx\, dt \\ &\le J(u(t_0)) - c_0 \int_{\log(1/T_m - t_0)}^{\log(1/T_m - t)} \exp(\alpha s) ds \\ &= J(u(t_0)) + \frac{C_0}{\alpha} \frac{1}{(T_m - t_0)^{\alpha}} - \frac{C_0}{\alpha} \frac{1}{(T_m - t)^{\alpha}}. \end{aligned}$$

Letting $t \uparrow T_m$, we get

$$\lim_{t \uparrow T_m} J(u(t)) = -\infty.$$

Taking (3.3) into consideration, we obtain the converse statement of Theorem 3.2. ∎

COROLLARY 3.6. *Suppose that either $u_0 \ge 0$ or Ω is a convex set. Let $u(t,x)$ be as in Theorems 3.1–3.2. Then the followings are equivalent each other:*

(1) $T_m = +\infty$ *and* $0 \notin \omega(u_0)$,
(2) $J(u(t,\cdot)) \ge d$ *for all* $t \in [0, T_m)$,
(3) $u(t,\cdot) \notin W^* \cup V^*$ *for all* $t \in [0, T_m)$.

PROOF. First it is easy to show that (2) is equivalent to (3). Next let us prove the equivalence of (1) with (2).

Suppose (1). If there is a number $t_0 \in [0, +\infty)$ such that $J(u(t_0,\cdot)) < d$, then one of the following three cases hold:

$$\text{(i)}\quad I(u(t_0,\cdot)) > 0, \qquad \text{(ii)}\quad I(u(t_0,\cdot)) < 0, \qquad \text{(iii)}\quad I(u(t_0,\cdot)) = 0.$$

If (i) is true, since $u(t_0,\cdot) \in W^*$, it follows from Theorem 3.1 that $0 \in \omega(u_0)$. This contradicts to the hypothesis. If (ii) is right, then we get $u(t_0,\cdot) \in V^*$ which implies $T_m < +\infty$ by Theorem 3.2. This is also a contradiction. Finally assume (iii). If $u(t_0,\cdot) \ne 0$, then $u(t_0,\cdot) \in \mathcal{N}$ and $J(u(t_0,\cdot)) < d$. This contradicts to (2.4). So we get $u(t_0,\cdot) = 0$ and therefore $J(u(t_0,\cdot)) = 0$. From the monotonicity of the mapping $t \mapsto J(u(t,\cdot))$ (see Lemma 2.5) we have $0 \ge J(u(t,\cdot))$ for all $t \ge t_0$. On the other hand, we can easily see that if $T_m = +\infty$, then $J(u(t,\cdot)) \ge 0$ for all $t \in [0, +\infty)$. Thus, we obtain $J(u(t,\cdot)) = 0$ for all

$t \in [t_0, +\infty)$. Because of Lemma 2.5 with 0 replaced by t_0 we have $\|u_t(t, \cdot)\|_2 = 0$ for all $t \in [t_0, +\infty)$. Since $u(t_0, \cdot) = 0$, this implies $u(t, \cdot) = 0$ for all $t \in [t_0, +\infty)$. This also contradicts to the hypothesis $0 \notin \omega(u_0)$.

Conversely, suppose $T_m < +\infty$ even if $J(u(t, \cdot)) \geq d$ for all $t \in [0, T_m)$. Since it follows from Theorem 3.2 that $u(t_0, \cdot) \in V^*$, this contradicts to hypothesis. So we have $T_m = +\infty$. Finally if $0 \in \omega(u_0)$, then from (2) of Proposition 3.4 we can prove $\omega(u_0) = \{0\}$. This implies $\lim_{t\to\infty} J(u(t, \cdot)) = 0$ which contradicts to the assumptions. ■

REMARK 3.7. *Owing to the 'smoothing effect' of the equation* (1.1), *we can apply the theory of Giga* [4] *in order to prove 'only if' part of Theorem* 3.2. *Therefore, it may be difficult to apply the argument directly to the problem* (1.4)–(1.6). *The fact that the global solutions have their values J bounded is first proved by Ōtani* [17].

4. Wave equation and stable-unstable set

In this section we treat the problem (1.4)–(1.6). To begin with, we shall introduce "modified" unstable set depending on $\delta \geq 0$ as follows (see (2.6)): Suppose

$$0 \leq \delta < \min\{p + 3, (p-1)C(\Omega, 2)^{-2}\}. \tag{4.1}$$

Then we define

$$V_\delta^* \equiv \{u \in H_0^1(\Omega); J(u) < d_\delta, I(u) < 0\}, \tag{4.2}$$

where $d_\delta \equiv d\left(1 - \dfrac{\delta C(\Omega, 2)^2}{p-1}\right)$. Note that $V_0^* = V^*$. Then we obtain the following two main Theorems by using Theorem 2.3 with regard to the existence of local solutions.

THEOREM 4.1. *Let $\delta > 0$ and let $u(t, x)$ be a local solution to the problem* (1.4)–(1.6) *on $[0, T_m)$ with initial data $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there exists a real number $t_0 \in [0, T_m)$ such that $u(t_0, \cdot) \in W^*$ and $E(u(t_0, \cdot), u_t(t_0, \cdot)) < d$ if and only if $T_m = +\infty$ and $\lim_{t\to\infty} \|\nabla u(t, \cdot)\|_2 = \lim_{t\to\infty} \|u_t(t, \cdot)\|_2 = 0$.*

THEOREM 4.2 (BLOWING-UP). *Let δ satisfy* (4.1) *and suppose that $u(t, x)$ be a local solution to* (1.4)–(1.6) *on $[0, T_m)$ with initial data $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. If there is a real number $t_0 \in [0, T_m)$ such that $u(t_0, \cdot) \in V_\delta^*$ and $E(u(t_0, \cdot), u_t(t_0, \cdot)) < d_\delta$, then $T_m < +\infty$.*

REMARK 4.3. *In proving Theorem* 4.1, *we shall get the decay estimates of $\|\nabla u(t, \cdot)\|_2$ or $\|u_t(t, \cdot)\|_2$ as $t \to \infty$ simultaneously. This part is closely related* *to the recent work by Nakao et al.* [16]. *Next, concerning the "only if" part in Theorem* 4.2, *it is still open.*

REMARK 4.4. *In Theorem* 4.1, *we can not take* $\delta = 0$. *This means that the presence of a dissipative term plays an essential role to obtain a decay property of total energy to* (1.4)–(1.6). *And also, the equation* (1.4) *has similar properties to heat equation* (1.1) *in this case of* $\delta > 0$. *On the other hand, taking into consideration to the effect of "damping", it will be natural to restrict a value of coefficient* δ *in Theorem* 4.2 *in order to get the blowing-up properties.*

5. Proof of Theorem 4.1

In this section we shall prove Theorem 4.1. To this end, we prepare several lemmas. Throughout this section, we always assume the local existence Theorem 2.3.

LEMMA 5.1. *Let* $\delta > 0$ *and let* $u(t,x)$ *be a local solution to* (1.4)–(1.6) *on* $[0, T_m)$. *If there is a number* $t_0 \in [0, T_m)$ *such that* $E(u(t_0,\cdot), u_t(t_0,\cdot)) < d$ *and* $I(u(t_0,\cdot)) > 0$, *then* $u(t,\cdot) \in W^*$ and $E(u(t,\cdot), u_t(t,\cdot)) < d$ *for all* $t \in [t_0, T_m)$.

PROOF. Since the proof is almost the same as that of Tsutsumi [22], we shall omit it. ■

The next lemma plays an important role to derive the decay estimate of the total energy $E(u(t,\cdot), u_t(t,\cdot))$ as $t \to \infty$. Although the proof is almost the same as that of Ishii [11], we will describe it for the sake of completeness.

LEMMA 5.2. *Let* $u(t,x)$ *be a local solution to* (1.4)–(1.6) *on* $[0, T_m)$. *If there exists a number* $t_0 \in [0, T_m)$ *such that* $u(t_0,\cdot) \in W^*$ *and* $E(u(t_0,\cdot), u_t(t_0,\cdot)) < d$, *then*

$$\|u(t,\cdot)\|_{p+1}^{p+1} \le (1-\gamma)\|\nabla u(t,\cdot)\|_2^2 \qquad \text{on } [t_0, T_m),$$

where $\gamma \equiv 1 - C(\Omega, p+1)^{p+1}\left(2\dfrac{p+1}{p-1}\right)^{(p-1)/2} E(u(t_0,\cdot), u_t(t_0,\cdot))^{(p-1)/2} > 0.$

PROOF. In general, if $u \in H_0^1(\Omega)$ satisfies $I(u) > 0$, then

$$(p+1)J(u) = \frac{p-1}{2}\|\nabla u\|_2^2 + I(u) \ge \frac{p-1}{2}\|\nabla u\|_2^2.$$

So we have

$$\|\nabla u\|_2^2 \le 2\frac{p+1}{p-1}J(u).$$

Therefore, it follows from Lemma 2.1 with $q = p+1$ that

$$(5.1) \qquad \|u\|_{p+1}^{p+1} \le C(\Omega, p+1)^{p+1}(\|\nabla u\|_2^2)^{(p-1)/2}\|\nabla u\|_2^2$$
$$\le C(\Omega, p+1)^{p+1}\left(2\frac{p+1}{p-1}\right)^{(p-1)/2}\|\nabla u\|_2^2 J(u)^{(p-1)/2}$$

for $u \in H_0^1(\Omega)$ with $I(u) > 0$. Since $J(u) \le E(u,v)$, from (5.1), Lemma 2.6 and Lemma 5.1 we get

$$\|u(t,\cdot)\|_{p+1}^{p+1} \le C(\Omega, p+1)^{p+1}\left(2\frac{p+1}{p-1}\right)^{(p-1)/2}\|\nabla u(t,\cdot)\|_2^2 E(u(t_0,\cdot), u_t(t_0,\cdot))^{(p-1)/2}.$$

Taking $\gamma \equiv 1 - C(\Omega, p+1)^{p+1}\left(2\frac{p+1}{p-1}\right)^{(p-1)/2} E(u(t_0,\cdot), u_t(t_0,\cdot))^{(p-1)/2}$, we obtain the desired inequality. ■

LEMMA 5.3. *Under the same assumptions as in Lemma 5.2, it holds that there exists a constant $M > 0$ such that*

$$\|\nabla u(t,\cdot)\|_2 + \|u_t(t,\cdot)\|_2 \le M,$$
$$\int_{t_0}^t \|u_t(s,\cdot)\|_2^2\, ds < \frac{d}{\delta} \qquad \text{on } [t_0, T_m).$$

PROOF. The first inequality is a direct consequence of Lemma 2.1, (1) of Lemma 2.4 and Lemma 5.1. Next noting that $u \in W^*$ implies $J(u) \ge 0$, from Lemma 2.6 with 0 replaced by t_0 we get the desired inequality. ■

LEMMA 5.4. *Under the same assumptions as in Lemma 5.2, it holds that there is a real number $M > 0$ such that*

$$\int_{t_0}^t I(u(s,\cdot))ds \le M, \qquad \int_{t_0}^t \|\nabla u(s,\cdot)\|_2^2\, ds \le M \qquad \text{on } [t_0, \infty).$$

PROOF. Note that under the hypothesis we get $T_m = +\infty$ by Theorem 2.3 and Lemma 5.3. Since we obtain

$$\frac{d}{dt}(u'(t), u(t)) - \|u'(t)\|_2^2 + I(u(t)) + \frac{\delta}{2}\frac{d}{dt}\|u(t)\|_2^2 = 0,$$

by integrating the above equality on $[t_0, t]$ and using the Schwarz inequality it follows that

$$\int_{t_0}^t I(u(s))ds + \frac{\delta}{2}\|u(t)\|_2^2 \le \frac{\delta}{2}\|u(t_0)\|_2^2 + \int_{t_0}^t \|u'(s)\|_2^2\, ds$$
$$+ \|u'(t_0)\|_2\|u(t_0)\|_2 + \|u'(t)\|_2\|u(t)\|_2 \qquad \text{on } [t_0, \infty).$$

Here $u(t) \equiv u(t,\cdot)$ and $u'(t) \equiv u_t(t,\cdot)$. Therefore, from Lemmas 2.1 and 5.3 we have

$$\int_{t_0}^{t} I(u(s))ds \leq \frac{\delta}{2}\|u(t_0)\|_2^2 + \frac{d}{\delta} + \|u'(t_0)\|_2\|u(t_0)\|_2 \\ + C(\Omega,2)\|u'(t)\|_2\|\nabla u(t)\|_2. \tag{5.2}$$

Since Lemma 5.2 implies

$$\gamma\|\nabla u(t)\|_2^2 \leq I(u(t)),$$

it follows from (5.2) and Lemma 5.3 that

$$\gamma\int_{t_0}^{t} \|\nabla u(s)\|_2^2\,ds \leq \int_{t_0}^{t} I(u(s))ds \leq M$$

with a constant $M > 0$. ■

LEMMA 5.5. *Under the same assumptions as in Lemma 5.2, it holds that*

$$E(u(t,\cdot),u_t(t,\cdot)) \leq \frac{M}{1+t} \qquad (t \geq t_0)$$

with a constant $M > 0$.

PROOF. First note that the following identity holds:

$$\frac{d}{dt}\{(1+t)E(u(t),u'(t))\} + \delta(1+t)\|u'(t)\|_2^2 = E(u(t),u'(t)).$$

By integrating this equality on $[t_0,t]$ we have

$$(1+t)E(u(t),u'(t)) \leq (1+t_0)E(u(t_0),u'(t_0)) \\ + \frac{1}{2}\int_{t_0}^{t} \|u'(s)\|_2^2\,ds + \int_{t_0}^{t} J(u(s))ds.$$

Since $(p+1)J(u(t)) = \frac{p-1}{2}\|\nabla u(t)\|_2^2 + I(u(t))$, the above inequality gives:

$$(1+t)E(u(t),u'(t)) \leq (1+t_0)E(u(t_0),u'(t_0)) + \frac{1}{2}\int_{t_0}^{t} \|u'(s)\|_2^2\,ds \\ + \frac{p-1}{2(p+1)}\int_{t_0}^{t} \|\nabla u(s)\|_2^2\,ds + \frac{1}{p+1}\int_{t_0}^{t} I(u(s))ds.$$

Finally, by using Lemmas 5.3 and 5.4 we obtain the desired inequality. ■

PROOF OF THEOREM 4.1. First suppose that there exists a real number $t_0 \in [0,T_m)$ such that $u(t_0,\cdot) \in W^*$ and $E(u(t_0,\cdot),u_t(t_0,\cdot)) < d$. Then it follows from Theorem 2.3 and Lemma 5.3 that $T_m = +\infty$. In addition, Lemma 5.5 implies

$$\lim_{t\to\infty} E(u(t,\cdot), u_t(t,\cdot)) = 0.$$

So we get

$$\|u_t(t,\cdot)\|_2^2 \to 0 \qquad \text{and} \qquad J(u(t,\cdot)) \to 0 \qquad \text{as } t \to \infty.$$

Since $u \in W^*$ implies $J(u) \geq 0$ and $I(u) > 0$, the following inequality:

$$(p+1)J(u(t,\cdot)) \geq \frac{p-1}{2}\|\nabla u(t,\cdot)\|_2^2$$

means $\lim_{t\to\infty} \|\nabla u(t,\cdot)\|_2^2 = 0$.

Conversely, if $\|\nabla u(t,\cdot)\|_2 \to 0$, $\|u_t(t,\cdot)\|_2 \to 0$ as $t \to \infty$, it follows from Lemma 2.1 that $\lim_{t\to\infty} \|u(t,\cdot)\|_{p+1} = 0$ which implies

$$\lim_{t\to\infty} E(u(t,\cdot), u_t(t,\cdot)) = 0.$$

Therefore, from (1) of Lemma 2.4 and the above mentioned results we get: $u(t_0,\cdot) \in W^*$ and $E(u(t_0,\cdot), u_t(t_0,\cdot)) < d$ for some $t_0 \in [0,\infty)$. ■

6. Prooof of Theorem 4.2

Throughout this section, we always assume (4.1). First we shall prepare two lemmas:

LEMMA 6.1. *Let $u(t,x)$ be a local solution to (1.4)–(1.6) with initial data $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. If there exists a number $t_0 \in [0, T_m)$ such that $u(t_0,\cdot) \in V_\delta^*$ and $E(u(t_0,\cdot), u_t(t_0,\cdot)) < d_\delta$, then $u(t,\cdot) \in V_\delta^*$ and $E(u(t,\cdot), u_t(t,\cdot)) < d_\delta$ for all $t \in [t_0, T_m)$.*

PROOF. Proof is almost the same as that of Tsutsumi [22]. ■

LEMMA 6.2. *If $u \in H_0^1(\Omega)$ satisfies $I(u) < 0$, then*

$$\|\nabla u\|_2^2 \geq 2d\frac{p+1}{p-1}. \tag{6.1}$$

PROOF. From the definition of d, we know that

$$d = \inf\left\{\left(\frac{1}{2} - \frac{1}{p+1}\right)\|\nabla u\|_2^{2(p+1)/(p-1)}\|u\|_{p+1}^{-2(p+1)/(p-1)}; u \in H_0^1(\Omega)\backslash\{0\}\right\}.$$

Therefore,

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\frac{1}{d} \geq \left(\frac{\|u\|_{p+1}^{p+1}}{\|\nabla u\|_2^{p+1}}\right)^{2/(p-1)}.$$

Since $\|\nabla u\|_2^2 < \|u\|_{p+1}^{p+1}$ by assumption, we obtain the desired inequality. ■

Now we are just in a position to prove Theorem 4.2. The proof will be done by the modifications of Ikehata [9].

PROOF OF THEOREM 4.2. Suppose that $T_m = +\infty$. Then the proof is based on the identity:

$$\frac{1}{2}\frac{d^2}{dt^2}\|u(t,\cdot)\|_2^2 - \|u_t(t,\cdot)\|_2^2 = \langle u_{tt}(t,\cdot), u(t,\cdot)\rangle_{X^*X} \qquad \text{on } [0,\infty), \tag{6.2}$$

where $\langle\,,\rangle_{X^*X}$ means the usual duality of X^* and X with $X \equiv H_0^1(\Omega)$. Next multiplying (1.4) by $u(t,x)$ in the duality $\langle\,,\rangle_{X^*X}$ we get

$$\begin{aligned}\langle u_{tt}(t,\cdot), u(t,\cdot)\rangle_{X^*X} = &\|u(t,\cdot)\|_{p+1}^{p+1} - \|\nabla u(t,\cdot)\|_2^2 \\ &- \delta(u_t(t,\cdot), u(t,\cdot)),\end{aligned} \tag{6.3}$$

where $(\,,)$ means the usual $L^2(\Omega)$-inner product. Furthermore, it follows from Lemma 2.6 that

$$\begin{aligned}&\frac{p+1}{2}\|u_t(t,\cdot)\|_2^2 + I(u(t,\cdot)) + \frac{p-1}{2}\|\nabla u(t,\cdot)\|_2^2 \\ &\quad + (p+1)\delta\int_{t_0}^t \|u_t(s,\cdot)\|_2^2\,ds = (p+1)E_0 \qquad \text{on } [0,\infty),\end{aligned} \tag{6.4}$$

where $E_0 \equiv E(u(t_0,\cdot), u_t(t_0,\cdot))$. From (6.2)–(6.4) we can estimate as follows:

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\|u(t,\cdot)\|_2^2 &= \|u_t(t,\cdot)\|_2^2 - I(u(t,\cdot)) - \delta(u_t(t,\cdot), u(t,\cdot)) \\ &= \|u_t(t,\cdot)\|_2^2 + \frac{p+1}{2}\|u_t(t,\cdot)\|_2^2 + \frac{p-1}{2}\|\nabla u(t,\cdot)\|_2^2 \\ &\quad + (p+1)\delta\int_{t_0}^t \|u_t(s,\cdot)\|_2^2\,ds - (p+1)E_0 - \delta(u_t(t,\cdot), u(t,\cdot)) \\ &\geq \frac{p+3}{2}\|u_t(t,\cdot)\|_2^2 + \frac{p-1}{2}\|\nabla u(t,\cdot)\|_2^2 - (p+1)E_0 \\ &\quad - \delta(u_t(t,\cdot), u(t,\cdot)).\end{aligned}$$

Here, from the Schwarz inequality and Lemma 2.1 we have

$$2(u_t(t,\cdot),u(t,\cdot)) \le 2\|u_t(t,\cdot)\|_2\|u(t,\cdot)\|_2 \le \|u_t(t,\cdot)\|_2^2 + \|u(t,\cdot)\|_2^2$$
$$\le \|u_t(t,\cdot)\|_2^2 + C(\Omega,2)^2\|\nabla u(t,\cdot)\|_2^2.$$

Therefore, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\|u(t,\cdot)\|_2^2 \ge{}& \frac{p+3}{2}\|u_t(t,\cdot)\|_2^2 + \frac{p-1}{2}\|\nabla u(t,\cdot)\|_2^2 \\ &- (p+1)E_0 - \frac{\delta}{2}\|u_t(t,\cdot)\|_2^2 - \frac{C(\Omega,2)^2\delta}{2}\|\nabla u(t,\cdot)\|_2^2 \\ ={}& \frac{1}{2}\{(p+3)-\delta\}\|u_t(t,\cdot)\|_2^2 + \frac{1}{2}\{(p-1) \\ &- C(\Omega,2)^2\delta\}\|\nabla u(t,\cdot)\|_2^2 - (p+1)E_0.\end{aligned}$$

Here we know from Lemmas 6.1 and 6.2:

$$\|\nabla u(t,\cdot)\|_2^2 \ge 2d\frac{p+1}{p-1} \qquad \text{for all } t\in[t_0,\infty).$$

So we have

$$\begin{aligned}\frac{d^2}{dt^2}\|u(t,\cdot)\|_2^2 \ge{}& \{(p+3)-\delta\}\|u_t(t,\cdot)\|_2^2 - 2(p+1)E_0 \\ &+ 2(p+1)d - \frac{2(p+1)C(\Omega,2)^2\delta d}{p-1}.\end{aligned} \tag{6.5}$$

Let $K_1 \equiv (p+3)-\delta > 0$ (see (4.1)) and $K_2 \equiv 2(p+1)\left\{d - \dfrac{C(\Omega,2)^2\delta d}{p-1} - E_0\right\}$. Then we have $K_2 > 0$ since $E_0 < d\left(1 - \dfrac{\delta C(\Omega,2)^2}{p-1}\right)$ by assumption. Thus, it follows from (6.5) that

$$\frac{d^2}{dt^2}\|u(t,\cdot)\|_2^2 \ge K_1\|u_t(t,\cdot)\|_2^2 + K_2 \qquad \text{on } [t_0,\infty). \tag{6.6}$$

Integrating (6.6) on $[t_0,t]$ $(t_0 < t < \infty)$, we get

$$\frac{d}{dt}\|u(t,\cdot)\|_2^2 \ge 2(u(t_0,\cdot),u_t(t_0,\cdot)) + K_2(t-t_0) \qquad \text{on } [t_0,\infty).$$

This implies that there is $t_1 > t_0$ such that

$$\frac{d}{dt}\|u(t,\cdot)\|_2^2 > 0 \qquad \text{on } (t_1,\infty).$$

Consequently, $P(t) \equiv \|u(t,\cdot)\|_2^2$ never vanish on (t_1, ∞). On the other hand, it follows from (6.6) that

$$\begin{aligned} P(t)P''(t) - \frac{K_1}{4}[P'(t)]^2 &\geq K_1\|u(t,\cdot)\|_2^2\|u_t(t,\cdot)\|_2^2 + K_2\|u(t,\cdot)\|_2^2 \\ &\quad - K_1|(u(t,\cdot), u_t(t,\cdot))|^2 \\ &\geq K_2\|u(t,\cdot)\|_2^2 > 0 \qquad \text{on } (t_1, \infty); \end{aligned}$$

in the last step we have used the Schwarz inequality. According to the standard "concavity argument" (see Levine [13]) we can find $T_0 > 0$ such that

$$\lim_{t \uparrow T_0} \|u(t,\cdot)\|_2 = +\infty,$$

which contradicts to $T_m = +\infty$. ■

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