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Stable Bargained Equilibria for Assignment Games Without Side Payments[1]

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Abstract: We consider NTU assignment games, which are generalizations of two-sided markets. Matched pairs bargain over feasible allocations; the disagreement outcome is endogenuously determined, taking in account outside options which are based on the current payoff of other players. An allocation is in equilibrium if and only if each pair is in equilibrium (no player wishes to rebargain). The set of equilibria is not empty and it naturally generalizes the intersection of the core and prekernel of TU assignment games. A set with similar properties does not exist for general NTU games. The main source of technical difficulties is the relatively complicated structure of the core in NTU games. We make a strong use of reduced games and consistency requirements. We generalize also the results obtained by Rochford (1984) for TU assignment games.

0 Introduction

Most of the solution concepts for games in characteristic function form were devised primarily for TU (transferable utility) games. There is by now a long tradition of attempts to generalize these concepts to the NTU (non-transferable utility) case, but usually something is lost on the way. For example, the Shapley value is not point-valued, or the Bargaining set may be empty.

It is not at all clear how one should generalize concepts like the kernel and nucleolus because their definition involves the notion of "excess", which ist based on the TU assumption. Kalai (1975) defined a kernel and nucleolus for NTU games using "excess functions", but his concepts are not independent of equivalent utility representations.

The Kernel of a TU game (Davis, Maschler (1965)) is a solution based on pairwise considerations. Its intuition is conveyed by the following argument due to Harsanyi: A particular payoff vector "will represent the equilibrium outcome of a

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bargaining among the n-players only if no pair of players i and j has any incentive to redistribute their payoffs between them, as long as the other players' payoffs are kept constant" (Harsanyi (1977), p. 196).

The intersection of the core and the kernel (or prekernel) has very interesting geometric properties (Maschler, Peleg, Shapley (1979)). Combining their results with the axiomatization of this intersection (Peleg (1985b)) one can indeed show that a payoff vector is in the intersection of the core and the prekernel if and only if each pair gets the "standard solution" in its reduced game. The standard solution is the only efficient, symmetric, covariant solution of a two person TU game, and most solutions for TU games or bargaining problems coincide with it on the class of super-additive, two-person, TU games.

Reduced games and related consistency (or "stability") properties are important tools in the analysis, comparision, and axiomatization of solution concepts. The main idea is one of stability of solutions under partial implementations by subgroups of players which consider their outside opportunities (or expectations). Various versions have been used in Sobolev (1975), Aumann, Dreze (1974), Aumann, Maschler (1985), Maschler, Owen (1989), Hart, Mas-Collel (1989), Peleg (1985, 1986, 1989), Thomson, Lensberg (1989). The reader is referred to the papers of Peleg and the book of Thomson and Lensberg for discussions.

By using these ideas one can avoid the notion of "excess", but an analogue of the intersection of the core and prekernel, having the same geometric and axiomatic properties, does not seem to exist for general NTU games. The equations determining such a solution may be inconsistent.

A natural candidate for applications of kernel-like solutions is the class of two sided markets (called also marriage, matching or assignment games). This is a class of games where the essential coalitions are those consisting of exactly one player from each side of the market, so it makes sense to look for solutions which emphasize equilibria of pairs.

Following Gale, Shapley (1962), there is a renewed interest in this kind of models which cover, of course, an important aspect of the economic activity. Without being complete we quote some of the papers: Shapley, Shubik (1971), Shapley, Scarf (1974), Crawford, Knoer (1981), Kaneko (1982), Kelso, Crawford (1982), Quinzii (1984), Roth (1984), Rochford (1984), Demange, Gale (1985), Crawford, Rochford (1986), and finally the excellent book of Roth and Sotomayor (1990).

Shapley and Shubik study an assignment market, viewed as a TU game, and its core. Kaneko generalizes their model without the TU assumption and establishes by means of balancedness the non-emptiness of the core. Indeed, most of the papers in the area (originating with the one by Gale and Shapley) consider core-like ideas of stability. The core, as Shapley and Shubik have remarked, does not always express the relative bargaining power of the players in an assignment game. An innovative paper ist the one by Rochford, which models bargaining between matched pairs in the TU game of Shapley and Shubik. A player bargains there with his/her partner using a threat point which is based on the outside opportunities given the current payoff to other pairs. Rochford defines a set of equilibria which is stable under rebargaining and shows that this set coincides with the intersection of the core and kernel of the respective TU game. We will use here a slight modification of Kaneko's NTU model, and the conceptual approach to bargaining devised by Rochford for

TU assignment games. The present paper has three main goals: First, for the class of NTU assignment games, we generalize the intersection of the core and prekernel, while preserving all the main properties of the old solution. Second, we generalize Rochford's results, without the TU assumption. Finally, the study of reduced games and consistency requirements relate the model and solution concept to other works and solution concepts in the area.

This paper is organized as follows: In Section 1 we present some notations and preliminary definitions. In Section 2 we present the model of NTU assignment games (NTU-AG) and study briefly their core. In Section 3 we study reduced games of NTU assignment games and establish some properties of their core which are basic for the following analysis. In Section 4 we define a set of stable bargained equilibria for NTU-AG, and prove its non-emptiness by showing convergence of a bargaining process to the set of equilibria. In Section 5 we characterize the set of equilibria by means of axioms and compare this to axiomatic and geometric characterizations of the intersection of the core and prekernel.

1 Notations and Preliminaries

Let U be a finite set of players. A coalition S is a non-empty set of U. A payoff vector for N is a function x: $N \Rightarrow \mathbb{R}$, thus $\mathbb{R}^N$ is the set of all payoff vectors. x^S denotes the restriction of x to members of the coalition S. $x(S)$ denotes the sum $\sum_{i \in S} x^i$. $\mathop{\mathrm{X}}_q A_q$ denotes the cartesian product of sets A_q.

0^S denotes the vector in $\mathbb{R}^S$ with all coordinates equal to zero. $|S|$ denotes the cardinality of coalition S. Let $x, y \in \mathbb{R}^N$. We write: $x \geq y$ if $x^i \geq y^i$ for all $i \in N$; $x > y$ if $x \geq y$ and $x \neq y$; $x >> y$ if $x^i > y^i$ for all $i \in N$.

Let $A \subseteq \mathbb{R}^k$. A is comprehensive if $x \in A$ and $x \geq y$ imply $y \in A$. The boundary of A is denoted by ∂A and the interior of A by A°. $\mathbb{R}^k_+$ is the restriction of $\mathbb{R}^k$ to vectors with non-negative coordinates.

Definition 1.1:

a) A TU game is a pair (N,v) where N is a coalition and v is a function which assigns to each coalition $S \subseteq N$ a real number $v(S)$. We assume $v(\emptyset) = 0$.
We denote

$$X(N,v) = \{ x \mid x \in \mathbb{R}^N \text{ and } x(N) \leq v(N) \} \tag{1.1}$$

b) Let (N,v) be a TU game. The core of (N,v), $C(N,v)$, is defined by:

$$C(N,v) = \{ x \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subseteq N \} \tag{1.2}$$

c) Let (N,v) be a TU game. We denote for $i,j \in N$, $i \neq j$ and $x \in \mathbb{R}^N$:

$$s_{ij}(x,N,v) = s_{ij}(x) = \max_{S \subseteq N} \{ v(S) - x(S) \mid i \in S \text{ and } j \notin S \} \tag{1.3}$$

The prekernel of (N,v), PreK(N,v), is defined by:

$$\text{PreK}(N,v) = \{x \mid x(N) = v(N) \text{ and } s_{ij}(x) = s_{ji}(x) \text{ for all } i,j \in N,\ i \neq j\} \tag{1.4}$$

d) Let $S \subseteq N$, a coalition, and let $x \in X(N,v)$. The reduced game with respect to S and x is the game (S,v_x) where:

$$v_x(T) = \begin{cases} 0, & \text{if } T \text{ is empty} \quad (1.5) \\ v(N) - x(N \setminus T), & \text{if } T = S \quad (1.6) \\ \max\{ v(T \cup Q) - x(Q) \mid Q \subseteq N \setminus S \}, & \text{otherwise} \quad (1.7) \end{cases}$$

e) A TU game (N,v) is super-additive if, for all coalitions $S,T \subseteq N$ with $S \cap T = \emptyset$, we have:

$$v(S \cup T) \geq v(S) + v(T) \tag{1.8}$$

■

Definition 1.2:

a) A NTU game is a pair (N,V) where N is a coalition and V is a function which assigns to each coalition $S \subseteq N$ a subset $V(S)$ of $\mathbb{R}^S$, such that

$V(S)$ is non-empty and comprehensive (1.9)

$V(S) \cap (x^S + \mathbb{R}^S_+)$ is bounded for every $x^S \in \mathbb{R}^S$ (1.10)

$V(S)$ is closed (1.11)

if $x^S, y^S \in \partial V(S)$ and $x^S \geq y^S$ then $x^S = y^S$ (1.12)

Let (N,V) be an NTU game and let $x \in V(N)$.

b) A coalition S can improve upon x if there exists $y^S \in V(S)$ such that $y^S >> x^S$. The core of (N,V), $C(N,V)$, is defined by:

$$C(N,V) = \{ x \mid x \in V(N) \text{ and no coalition can improve upon } x \}. \tag{1.13}$$

c) Let $S \subseteq N$, a coalition, and let $x \in V(N)$. The reduced game with respect to S and x is the game (S,V_x), where

$$V_x(S) = \{ y^S \mid (y^S, x^{N \setminus S}) \in V(N) \}, \tag{1.14}$$

$$V_x(T) = \bigcup_{Q \subseteq N \setminus S} \{ y^T \mid (y^T, x^Q) \in V(T \cup Q) \}, \text{ if } T \underset{\neq}{\subset} S,\ T \neq \emptyset \tag{1.15}$$

d) An NTU game (N,V) is super-additive if, for all coalitions $S,T \subseteq N$ with $S \cap T = \emptyset$, we have:

$$V(S \cup T) \supseteq V(S) \times V(T) \tag{1.16}$$

e) Let (N,V) be an NTU game. For $i \in N$ we denote:

$$v^i = \sup \{ x^i \mid x^i \in V(i) \} \tag{1.17}$$

In the same fashion, for a reduced game (T,V_x), we denote for $i \in T$:

$$v^i_{x,T} = \sup \{ x^i \mid x^i \in V_x(i) \} \tag{1.18}$$

Let (N,V) be an NTU game and let $x \in V(N)$. x is Pareto-optimal (PO) if there is no $y \in V(N)$ with $y > x$. x is individually rational (IR) if for all $i \in N$ we have $x^i \geq v^i$.

f) We denote by βA the north-east boundary of a compact, convex, comprehensive (relative to $\mathbb{R}^2_+$) and full dimensional set A in R^2_+, i.e.,

$$\beta A = \{ x \mid x \in A \text{ and } y > x \text{ imply } y \notin A \} \tag{1.19}$$

■

Definition 1.3: Let Γ be a class of NTU games.

a) A solution on Γ is a function σ which assigns to each game $(N,V) \in \Gamma$ a subset $\sigma(N,V)$ of $V(N)$.

b) A solution σ on Γ has the reduced game property (RGP) if it satisfies the following: If $(N,V) \in \Gamma$, $S \subseteq N$, $S \neq \emptyset$, and $x \in \sigma(N,V)$, then $(S,V_x) \in \Gamma$ and $x^S \in \sigma(S,V_x)$.

c) A solution on Γ has the converse reduced game property (CRGP) if it satisfies the following: If $(N,V) \in \Gamma$, $x \in V(N)$, and for every pair $S = \{i,j\}$ with $i,j \in N$, $i \neq j$ it is true that $(S,V_x) \in \Gamma$ and $x^S \in \sigma(S,V_x)$, then $x \in \sigma(N,V)$.

■

The definition of RGP and CRGP are similar for TU games. We refer the reader to Peleg (1985a, 1986) for these and other properties of solutions, and discussions.

2 NTU Assignment Games

We consider a society formed by two distinct groups, "men" and "women". To each player we associate a set which elements represent the possible incomes as a single. To each pair we will associate a set of feasible allocations for this pair, in case it's members decide to "marry". The basic idea is that a marriage may bring utility gains to both parts involved, relative to the utility as singles. For various economic interpretations see Kaneko (1982), Rochford (1984) or Demange, Gale (1985).

Definition 2.1: Let F be the set of female players, M the set of male players, both finite, non-empty sets, with $M \cap F = \emptyset$. We assume $|M| = |F|$, otherwise a set D of dummies is added to the smaller set. Let $N = F \cup M$ (including dummies). Let $|N| = 2n$ (after the dummies were added). Male players will be denoted by m_i, m_k, etc. or simply by m, and similarly for females (with f instead of m!).

To each coalition $S \subseteq N$ with $|S| \leq 2$ we associate a set $V'(S) \subset \mathbb{R}^S_+$ such that:

$$V'(S) \text{ is non-empty, compact, convex, comprehensive relative to } \mathbb{R}^S_+ \tag{2.1}$$

$$V'(S) = 0^S, \qquad \text{if } S \subseteq D \tag{2.2}$$

$$V'(S) \supseteq \underset{\substack{T \subseteq S \\ |T|=1}}{\mathrm{X}} V'(T) \text{ if } |S| = 2 \tag{2.3}$$

$$V'(S) = \underset{\substack{T \subseteq S \\ |T|=1}}{\mathrm{X}} V'(T), \text{ if } S \cap D \neq \emptyset \text{ or if } S \cap M = \emptyset \text{ or if } S \cap F = \emptyset \tag{2.4}$$

If $|S| = 2$ and $V'(S) \neq \underset{\substack{T \subseteq S \\ |T|=1}}{\mathrm{X}} V'(T)$ then the following hold:

There exists $x^S \in V'(S)$ with $(x^i,x^j) >> (v^i,v^j)$ where $i,j \in S$ and v^i, v^j are defined in 1.17 with respect to the sets $V'(i)$ and $V'(j)$.
If $x^S, y^S \in \beta V'(S)$ and $x^S \geq y^S$ then $x^S = y^S$ (2.6)

■

Condition 2.1 is clear. We assume that zero ist the worst possible outcome for a player and dummies get indeed zero (2.2). Condition 2.3 says that the players can achieve, in any pair, what they could get as singles (super-additivity).

Condition 2.4 says that being married with a dummy or with a member of one's own sex can not be more "productive" than staying single. Condition 2.5 says that marriage can not be productive to one side only. It is also complementary to condition 2.6 which says that the north-east boundary does not contain segments parallel to an axis (non-levelness).

We define now formally an NTU game on the society N. We look at all possible partitions of a coalition in sets consisting of singles or mixed pairs.

Thus, we keep in mind the idea of the original TU assignment game of Shapley and Shubik, but we do not allow free transfer of utility in coalitions of size bigger than two. This model is only slightly different from the "central assignment game" of Kaneko (1982).

Definition 2.2: Let S be a coalition in N. We denote:

$$t(S) = \{ T \subseteq S \mid |T| = 1 \text{ or } (|T| = 2 \text{ and } T \cap M \neq \emptyset \text{ and } T \cap F \neq \emptyset)\} \quad (2.7)$$

$p(S) = (T_1,T_2,\ldots,T_k)$ will be called a t-partition of S if it is a partition of S and $T_i \in t(S)$ for $1 \leq i \leq k$.

$P(S)$ will denote the set of all t-partitions of S.

An NTU assignment game (NTU-AG) is a pair (N,V) where N is a set of players like in Definition 2.1 and V assigns to each coalition S in N the set

$$V(S) = \bigcup \{ \underset{T \in p(S)}{\mathrm{X}} V'(T) \mid p(S) \in P(S) \} \quad (2.8)$$

■

Remark 2.3: In the future we will assume that $V'(S) = 0^S$ for S with $|S| = 1$ and thus we have also $V'(S) = 0^S$ for S like in 2.4. Given an NTU-AG we can always normalize it in this way, and we will require the solution concepts to be independent of this normalization. The more general definition is needed for a correct use of consistency properties in the future.

Because N has an even number of players (after we added the dummies) and because of the normalization and condition 2.3, we can omit from the union which forms $V(N)$ those t-partitions of N which contain singles. Their contribution will be covered by partitions consisting only of coalitions of size two. We will, in the future, consider only such partitions of N, denoted simply by "p" instead of $p(N)$.

An NTU-AG is, strictly speaking, neither an NTU game in the sense of Definition 1.2 – a, nor "central assignment game" in the sense of Kaneko. However, the technical differences have, in our context, no effect, and we will use some of the results in Kaneko (1982) and Peleg (1985). ■

Theorem 2.4: (Kaneko) Let (N,V) be an NTU assignment game. The core, $C(N,V)$, of this game is not empty.

Due the special structure of an NTU-AG, only coalitions of size two are significant when considering the core concept:

Lemma 2.5: Let (N,V) be an NTU-AG, let $x \in V(N)$. Then $x \notin C(V,N)$ if and only if there exists a coalition $S \subseteq N$ with $|S| = 2$, $S \cap M \neq \emptyset$, $S \cap F \neq \emptyset$, $S \cap D = \emptyset$ and $y^S \in V'(S)$ with $y^S >> x^S$.

Proof: It is clear that if such an S exists, the we have $V(S) = V'(S)$ and $x \notin C(N,V)$. For the converse, assume $x \notin C(N,V)$. Then there exists a coalition $Q \subseteq N$ and $y^Q \in V(Q)$ with $y^Q >> x^Q$. $x \in V(N) \subseteq \mathbb{R}^N_+$, so $y^Q >> 0^Q$ and this implies that Q is the required coalition if $|Q| \leq 2$. (Remark that we used the zero normalization). If $|Q| > 2$, then $y^Q \in \underset{T \in p(Q)}{\mathrm{X}} V'(T)$ for a $p(Q) \in P(Q)$. Because $y^Q >> 0^Q$, there exists a $T \in p(Q)$ with $V'(T) \neq 0^T$ and T must have the required form. $y^T \in V'(T)$ and $y^T >> x^T$. ■

In the TU assignment game of Shapley-Shubik the core is a convex (thus connected) set and there is, generically, only one matching compatible with allocations in the core (see Rochford (1984)). In any case, each vector in the core is compatible with any matching for which allocations in the core can be found. The structure of the core of an NTU-AG may be very different, and we need the following:

Definition 2.6: Let (N,V) be an NTU-AG and let $x \in V(N)$. Then

a) $$P(x) = \{ p \mid p \in P(N) \text{ and } x \in \underset{T \in p}{\mathrm{X}} V'(T) \} \tag{2.9}$$

b) For $p \in P(N)$,
$$C_p(N,V) = C(N,V) \cap \underset{T \in p}{\mathrm{X}} V'(T) \tag{2.10}$$
■

Thus, $P(x)$ is the set of t-partitions of N for which $x \in V(N)$ is feasible, and $C_p(N,V)$ is the set of vectors in the core which are feasible for a t-partition of N.

The core of an NTU-AG may consist *generically* of several non-empty, non-convex and even non-connected sets of the form $C_p(N,V)$. This is the main source of technical difficulties. (For details in a related model see Roth, Sotomayor (1990)).

Remark 2.7: The conditions in Definition 2.1 (and the zero normalization) have the effect that, for a two person coalition T with $V'(T) \neq 0^T$, the north-east boundary $\beta V'(T)$ can be represented by the graph of a continuous, concave, decreasing function. The proof of these well known facts is straightforward and will be omitted. Now let $x \in C(N,V)$, let $p \in P(x)$, so that $x \in C_p(N,V)$, and let T be a two person coalition with $V'(T) \neq 0^T$. If $T \in p$, then $x^T \in V'(T)$ and, because x can not be improved upon, we get $x^T \in \beta V'(T)$. If $T \notin p$ then, from the same reason we must have $x^T \in \beta V'(T)$ *or* $x^T \notin V'(T)$. ■

3 Reduced Games of NTU Assignment Games

We have two main reasons for studying reduced games:
First, we can avoid the notion of "excess" and the usual definition of the kernel (or prekernel), and we will be able to generalize naturally the intersection of the core and kernel (or prekernel) for NTU assignment games, using axiomatic and geometric characterizations of the TU case.

Second, reduced games and related consistency properties capture here important features of bargaining like the role of outside options or stability under rebargaining.

Lemma 3.1: Let (N,V) be an NTU-AG, let $x \in V(N)$ and let $S \subseteq N$, $S = \{m,f\}$. Then, in the game, (S,V_x), the following hold:

$$\text{a) } V_x(m) = \bigcup_{f_k \neq f} \{ y^m \mid (y^m, x^{f_k}) \in V'(m,f_k) \} \cup \{0^m\} \tag{3.1}$$

$$\text{b) } V_x(f) = \bigcup_{m_k \neq m} \{ y^f \mid (x^{m_k}, y^f) \in V'(m_k,f) \cup \{0^f\} \tag{3.2}$$

Proof: Remark that the unions can be taken only over non-dummy players. To prove a), we first note that, by the definition of reduced games, the union is contained in $V_x(m)$. For the converse inclusion, let $y^m \in V_x(m)$.

If $y^m = 0^m$ the statement is clear. Otherwise we have $(y^m, x^Q) \in \underset{T \in p^*}{\mathrm{X}} V'(T)$ where Q is a coalition in $N \setminus S$ and p^* is a t-partition of $Q \cup \{m\}$. If m would be a single or matched with a dummy in p^* we would have $y^m = 0^m$, so the partner of m in p^* is an f_i with $f_i \in F \setminus \{f\}$, $f_i \notin D$ and $(y^m, x^{f_i}) \in V'(m,f_i)$.
The proof of b) is similar. ■

Lemma 3.2: Let (N,V) be an NTU-AG, let $x \in C(N,V)$ and let $p \in P(x)$. Let $S \in p$, $S = \{m,f\}$. Then, in the game (S,V_x), the following hold:

a) $x^S \in C(S,V_x)$, in particular $C(S,V_x) \neq \emptyset$.

b) $x^m \geq v^m_{x,S}$ and $x^f \geq v^f_{x,S}$.

$$\text{c) } V_x(S) = V'(S) \tag{3.3}$$

Proof: a) This is just an instance of the reduced game property (RGP) of the core of NTU games (see Peleg (1985) – Lemma 4.5).

b) Follows immediately from a) by remarking that a vector in the core is in particular individually rational (IR).

c) Suppose first that $y^S \in V'(S)$. We have $x \in \underset{T \in p}{\mathrm{X}} V'(T)$ because $p \in P(x)$. From $S \in p$ and $y^S \in V'(S)$ we derive that $(y^S, x^{N \setminus S}) \in \underset{T \in p}{\mathrm{X}} V'(T)$, and by Definition 2.2 of an NTU-AG we get $(y^S, x^{N \setminus S}) \in V(N)$. Finally, we conclude that $y^S \in V_x(S)$ by using the definition of reduced games. Remark that we used only $x \in V(N)$.

For the converse inclusion, let $y^S \in V_x(S)$, which means that $(y^S, x^{N \setminus S}) \in V(N)$. Thus, $(y^S, x^{N \setminus S}) \in \underset{T \in p'}{\mathrm{X}} V'(T)$ for a certain $p' \in P(N)$. If $S \in p'$ the result is clear. Otherwise, let f_k be the partner of m in p'. Then $(y^m, x^{f_k}) \in V'(m,f_k)$ and by Lemma 3.1 we get $y^m \leq v^m_{x,S}$. Combined with b) we obtain $y^m \leq x^m$, and using a similar argument for f we have $y^S \leq x^S$. $x^S \in V'(S)$ and by comprehensiveness we conclude that $y^S \in V'(S)$. ■

For x and S like in the previous Lemma we obtained a full picture of the reduced game (S,V_x) : It is easily seen that $V_x(m)$ and $V_x(f)$ are closed, bounded intervals containing the origin. Their upper limits, $v^m_{x,S}$ *and* $v^f_{x,S}$ respectively, represent the best opportunities of m and f outside their present "marriage", given the current payoff $x^{N \setminus S}$ to the other players. $V_x(S)$ is compact, convex, comprehensive relative to $\mathbb{R}^S_+$, and with non-leveled north-east boundary. By Lemma 3.2-b and comprehensiveness we have also $(v^m_{x,S}, v^f_{x,S}) \in V_x(S)$. We will use these facts when we look at (S,V_x) as a well defined bargaining problem.

When defining solution concepts for assignment games we would like to have the following property : A payoff vector x is an "equilibrium" for a matching p if each pair in p is in "equilibrium". We show next how this is translated, in our context, for the core. The reader might compare this with the "converse reduced game property" (CRGP) of the core of general NTU games (Peleg 1985a).

Lemma 3.3: Let (N,V) be an NTU-AG, let $x \in V(N)$ and $p \in P(x)$. If for each $T \in p$ it is true that $x^T \in C(T, V_x)$, then $x \in C(N,V)$.

Proof: If $x \notin C(N,V)$, then we have by Lemma 2.5 a coalition $S = \{m,f\}$ in N with $S \cap D = \emptyset$ and $y^S \in V'(S)$ with $y^S \gg x^S$. If $S \in p$, then we have $V'(S) \subseteq V_x(S)$ (see the first part of the proof of Lemma 3.2-c). Then $y^S \in V_x(S)$ and we get a contradiction to $x^S \in C(S,V_x)$. If $S \notin p$, let $T = \{m,f_k\}$, $T \in p$. By comprehensiveness $(y^m, x^f) \in V'(S)$ and this implies that $y^m \leq v^m_{x,T}$ (remark that $f_k \neq f$). x^T is IR in (T, V_x), so $x^m \geq v^m_{x,T}$, and we get a contradiction to $y^S \gg x^S$. ■

Let (N,V) be an NTU-AG, let $x \in C(N,V)$, let $p \in P(x)$ and $T \in p$. Suppose that we keep $x^{N \setminus T}$ fixed. The following question arises: What is the "bargaining range" of the pair T, if we look at the core as a "window" in the set $V(N)$? In other words : which are those y^T such that $(y^T, x^{N \setminus T}) \in C(N,V)$? An answer to this for general TU games is found in Maschler, Peleg, Shapley (1979) of in Aumann, Dreze (1974), where reduced games are more explicitely used. We show in Moldovanu (1989) that the result of Aumann and Dreze (Theorem 5, there) does not hold for general NTU games. However, for NTU assignment games we have the following important result:

Lemma 3.4: Let (N,V) be an NTU-AG, let $x \in C(N,V)$, $p \in P(x)$ and $T \in p$ with $T = \{m,f\}$. Then $(y^T, x^{N\setminus T}) \in C(N,V)$ if and only if $y^T \in C(T,V_x)$.

Proof: Remark first that $y^T \in C(T,V_x)$ implies $y^T \in V_x(T)$ which implies by the definition of reduced games that $(y^T, x^{N\setminus T}) \in V(N)$.

Let now $y^T \in C(T,V_x)$ and suppose that $w = (y^T, x^{N\setminus T}) \notin C(N,V)$. Then, by Lemma 2.5 we have an $S \subseteq N$ with $|S| = 2$, $S \cap M \neq \emptyset$ and $z^S \in V'(S)$ with with $z^S >> w^S$. If $S \subseteq N\setminus T$ we get a contradiction to $x \in C(N,V)$, and if $S = T$ we get a contradiction to $y^T \in C(T,V_x)$, because $V'(T) = V_x(T)$ by Lemma 3.2.

Assume then that S contains a member of T and a member of $N\setminus T$. We assume w.l.o.g that $S = \{m,f_k\}$, with $f_k \neq f$. By comprehensiveness $(z^m, x^{f_k}) \in V'(S)$ and this implies $z^m \leq v^m_{x,T}$. y^T is IR in (T,V_x) so $y^m \geq v^m_{x,T}$ and by the last two inequalities we get a contradiction to $z^m > y^m$.

The converse part follows immediately from the reduced game property of the core, by noting that the values of (T,V_x) depend *only* on $x^{N\setminus T}$. ■

The last Lemma shows also that, given $x \in C(N,V)$, the "bargaining range" of a pair matched in $p \in P(x)$ is a connected, closed set, homeomorphic to an interval. This is so because this "range" coincides, as shown, with the coe of a two-person (reduced) game. Finally, the reader may ask why we do not consider "bargaining ranges" for members not matched in p. The next Lemma shows that a "bargaining range" for such players does not provide any flexibility.

Lemma 3.5: Let (N,V) be an NTU-AG, let $x \in C(N,V)$, $p \in P(x)$ and let $T = \{m,f\}$, $T \notin p$. Then $V_x(T) = V_x(m) \times V_x(f)$, and $C(T,V_x) = \{x^T\}$.

Proof: By RGP we know that $x^T \in C(T,V_x)$. x^T is IR in (T,V_x) so we obtain $(x^m, x^f) \geq (v^m_{x,T}, v^f_{x,T})$. Let f_k be the partner of m in p and let $S = \{m,f_k\}$. $x^S \in V'(S)$ and this implies $x^m \leq v^m_{x,T}$, so we obtain $x^m = v^m_{x,T}$. A similar argument for f and comprehensiveness establish $V_x(T) \supseteq V_x(m) \times V_x(f)$.

For the converse inclusion, let $y^T \in V_x(T)$ and assume $(y^m, y^f) > (v^m_{x,T}, v^f_{x,T})$, so $y^T > x^T$. Assume also w.l.o.g that $y^m > v^m_{x,T}$. By the definition of reduced games we have $z = (y^T, x^{N\setminus T}) \in V(N)$, and let $p' \in P(z)$. If $T \notin p'$ then, for a certain $f_j \neq f$, we have $(y^m, x^{f_j}) \in V'(m,f_j)$. This implies that $y^m \leq v^m_{x,T}$ and we get a contradiction. If $T \in p'$ then $y^T \in V'(T)$. By comprehensiveness $x^T \in V'(T)$ and, because $T \notin p$, we get a contradiction to $x \in C(N,V)$ by Remark 2.7 and non-levelness. It is now clear that x^T is the only point in $C(T,V_x)$.

■

The same result applies also for a coalition T of two men or two women. The proof is easy and is left to the reader.

4 Bargaining in NTU Assignment Games

After some preparations, we are now ready to describe the bilateral bargaining process and the solution concept. We translate for the NTU case ideas of Rochford, by making a strong use of reduced games and consistency requirements.

Let (N,V) be an NTU-AG, let $x \in V(N)$, $p \in P(x)$, $T \in p$, $T = \{m,f\}$. The members of T observe their reduced game (T,V_x). As remarked before, the upper limits of the intervals $V_x(m)$ and $V_x(f)$ are, respectively, the best opportunities of m and f outside T. These values are, in a sense, a kind of threats : if m does not get at least $v^m_{x,T}$ he has an incentive to break the matching, and similarly for f. Loosely speaking, we can say that m and f face a bargaining problem represented by (T,V_x). Thus, assuming that $x^{N\setminus T}$ is fixed, the members of T bargain over $V_x(T)$ taking in account their outside options.

If $x \in C(N,V)$, then Lemma 3.4 shows that any choice in $C(T,V_x)$ preserves the stability implied by the core.

We assume that matched pairs solve their bargaining problem using solutions which obey some "standard" axioms : symmetry (SYM), independence of equivalent utility representations (IEUR), Pareto-optimality (PO), individual rationality (IR). For the relevant definitions see Roth (1979).

Remark that we do not assume that all pairs choose an agreement point using the same solution concept! We assume only that, once a pair has agreed on a solution concept, it sticks to this concept.

Definition 4.1: A solution to a bargaining problem will be called standard if it obeys the mentioned axioms: SYM, IEUR, PO, IR. The standard solution adopted by a pair T will be called the T-standard solution. ■

The Nash, Raiffa-Kalai-Smorodinski and Maschler-Perles solution concepts are, among others, instances of standard solutions in our context. In relation to TU games the term was proposed by Aumann, Maschler (1985) and the connection will be made clear in Section 5.

We define now the solution concept for NTU assignment games:

Definition 4.2: A stable bargained solution to the NTU assignment game (N,V) is a set $SB_p(N,V)$, where $p \in P(N)$ and

$$SB_p(N,V) = \{ x \mid x \in \underset{T\in p}{\mathrm{X}} V'(T) \text{ and, for each } T \in p,\ x^T \text{ is the } T\text{-standard solution to } (T,V_x) \} \tag{4.1}$$

We define also : $$SB(N,V) = \bigcup_{p\in P(N)} SB_p(N,V) \tag{4.2}$$
■

Explanation to the Definition: Remark first that, by the properties IR and PO of a standard solution, x^T should be in $C(T,V_x)$ for each $T \in p$. Then, by Lemma 3.3, we have $SB_p(N,V) \subseteq C_p(N,V)$ and hence $SB(N,V) \subseteq C(N,V)$. Imagine now that an allocation $x \in \underset{T\in p}{\mathrm{X}} V'(T)$ is proposed:

As explained before, each pair in T in p bargains over the problem (T,V_x). The result (if existing!) is an allocation y, but the outside opportunities for the members of a pair T changed, because $x^{N\setminus T}$ changed to $y^{N\setminus T}$. As a consequence of this a member of T may wish to rebargain and this process may never end. If the proposed x belongs to $SB_p(N,V)$, the choice of each $T \in p$ is exactly x^T, so after bargaining we fall back on x and the process stops.

The consistency properties of this solution are obvious. ■

We set now to the task of proving the non-emptiness of a set $SB_p(N,V)$. This result will be proved by showing convergence of a bargaining process to the set of equilibria. We generalize also a result of Rochford which showed convergence when starting from two special points ("best" for women, respectively men).

First, we make the additional assumption that the T-standard solutions are continuous with respect to the threat point (TPC). Remark that all above mentioned instances of standard solutions have indeed this property.

We need also the following:

Definition 4.3: Let (N,V) be an NTU-AG, let $p = \{ T_1,T_2,...,T_n \} \in P(N)$ with $C_p(N,V) \neq \emptyset$ and let $x \in C_p(N,V)$.

a) We define the following chain of maps:
$x = (x^{T_1}, x^{T_2},...,x^{T_n}) \to (y^{T_1}, x^{T_2},...,x^{T_n}) \to (y^{T_1}, y^{T_2}, x^{T_3},...,x^{T_n}) \to ... \to (y^{T_1}, y^{T_2},...,y^{T_n}) = y$, where y^{T_1} is the T_1-standard solution with TPC to (T_1,V_x) and similarly, for $1 < i \leq n$, y^{T_i} is the T_i-standard solution with TPC to the game $(T_i, V_{(y^{T_1},..., y^{T_{i-1}}, x^{T_i},..., x^{T_n})})$.

b) We define an operator B on $C_p(N,V)$, $B(x) = y$, where y is defined by the precedent chain. (We prove below that B is well defined!)

c) Let $z,w \in C_p(N,V)$ and let $T_i \in p$. Then $z^{T_i}, w^{T_i} \in \beta V'(T_i)$ and we denote by $< z^{T_i}, w^{T_i}>$ the closed path in $\beta V'(T_i)$ between these points.

We define an equivalence relation "$\sim$" on $C_p(N,V)$ as follows : $z \sim w$ if and only if for each $T_i \in p$ and for each $u^{T_i} \in \, < z^{T_i}, w^{T_i}>$ there exists $u^{N\setminus T_i}$ such that $u = (u^{T_i}, u^{N\setminus T_i}) \in C_p(N,V)$. The reader may check that this is indeed an equivalence relation. The equivalence class of x will be denoted by $[x]$. ■

Thus, the coordinates of $B(x)$ are determined, two at a time, by bargaining which takes in consideration the outside opportunities available at that time. The reader may compare this formulation with the function $a(x)$ in Rochford (1984). It is not a-priori clear that the operator B is well defined, nor what its range is because the y^{T_i} are chosen from cores (of reduced games) which may be empty.

Lemma 4.4: The operator B of Definition 4.3 is well defined and $B(x) = y \in C_p(N,V)$. Moreover, B is continuous and $B(x) \sim x$.

Proof: Let $x = (x^{T_1}, x^{N \setminus T_1}) \in C_p(N,V)$ and observe the first step in the chain defining B. By Lemma 3.2–a we know that $C(T_1,V_x)$ is not empty and by the properties IR and PO of the T_1-standard solution we know that $y^{T_1} \in C(T_1,V_x)$. By Lemma 3.4 we obtain that $(y^{T_1}, x^{N \setminus T_1})$, which is the result after the first step in the chain, belongs to $C(N,V)$, and obviously also to $C_p(N,V)$. Further, we know that x^{T_1}, $y^{T_1} \in C(T_1,V_x)$ and, by Lemma 3.4, we have $(u^{T_1}, x^{N \setminus T_1}) \in C_p(N,V)$ for each $u^{T_1} \in C(T_1,V_x)$. $C(T_1,V_x)$ is a connected closed path in $\beta(V'/T_1)$ and we obtain that $x = (x^{T_1}, x^{N \setminus T_1}) \sim (y^{T_1}, x^{N \setminus T_1})$. This first step in the chain is also continuous because, for $T_1 = \{m,f\}$, the values $v^m_{x,T}$ *and* $v^f_{x,T}$ depend continuously on $x^{N \setminus T_1}$ (see Lemma 3.1 and Remark 2.7), and because the T_1-standard solution has the TPC property.

With a similar argument for each step in the chain we obtain the desired result. ■

The bargaining process which we use is formed by repetitions of the procedure which defines the operator B in Definition 4.3.

The more complicated structure of the core of an NTU-AG enables convergence when starting from several initial allocations. We observe first a "lattice" property of the core of an NTU-AG (compare with Shapley, Shubik (1971)–Theorem 3.)

Lemma 4.5: Let (N,V) be an NTU-AG, let $p \in P(N)$ with $C_p(N,V) \neq \emptyset$ and let $x,y \in C_p(N,V)$. We define two vectors z,u in $\mathbb{R}^N$ by:

$$z^{m_i} = \min(x^{m_i}, y^{m_i}) \; ; \; z^{f_j} = \max(x^{f_j}, y^{f_j}) \qquad \text{for } 1 \leq i,j \leq n. \tag{4.3}$$

$$u^{m_i} = \max(x^{m_i}, y^{m_i}) \; ; \; u^{f_j} = \min(x^{f_j}, y^{f_j}) \qquad \text{for } 1 \leq i,j \leq n. \tag{4.4}$$

Then the vectors z,u belong to $C_p(N,V)$.

Proof: Let $S = \{m_k, f_h\} \in p$. If $x^{m_k} = y^{m_k}$ then $x^S = y^S$ because $x^S, y^S \in \beta V'(S)$ and this set is non-leveled. In this case we have $z^S = x^S = y^S$ and $z^S \in V'(S)$. Assume w.l.o.g that $x^{m_k} < y^{m_k}$. This implies $x^{f_h} > y^{f_h}$ and by definition we have $z^S = x^S$ and $z^S \in V'(S)$. The same kind of argument for each pair shows that
$z \in \mathop{X}\limits_{T \in p} V'(T)$ and hence that $z \in V(N)$.

If $z \notin C_p(N,V)$ then, by Lemma 2.5, we have a coalition Q in N with $|Q| = 2$, $Q \cap M \neq \emptyset$, $Q \cap F \neq \emptyset$ and $w^Q \in V'(Q)$ with $w^Q \gg z^Q$. Let $Q = \{m,f\}$ and assume that $z^m = x^m$. Then we have $w^m > x^m$, and $w^f > x^f$ because $z^f \geq x^f$. This is a contradiction to $x \in C(N,V)$. If $z^m = y^m$ we get a contradiction to $y \in C(N,V)$.

The same kind of argument proves also that $u \in C_p(N,V)$. ■

Lemma 4.6: Let (N,V) be an NTU-AG, let $p \in P(N)$ with $C_p(N,V) \neq \emptyset$ and let $[x]$ be an equivalence class in $C_p(N,V)$ (see Definition 4.3.–c). There exist payoff vectors $w_{M,[x]}$, $w_{F,[x]} \in [x]$ such that, for all $y \in [x]$, we have

$$w_{M,[x]} <_p y <_p w_{F,[x]} \tag{4.5}$$

where "$<_p$" is an order relation defined on $C_p(N,V)$ by : $x <_p y$ if $x,y \in C_p(N,V)$ and $x^{f_j} \leq y^{f_j}$ for $1 \leq j \leq n$.

Proof: If $x,y \in C_p(N,V)$ and if $y \in [x]$, then it is easy to see that $z,u \in [x]$ where z,u were defined in the previous Lemma. Thus $[x]$ is a complete lattice with respect to the defined order relation, $[x]$ is compact and therefore we have maximal and minimal elements. (Remark also that, if $y \in [x]$, also the path-connected component of y in $C_p(N,V)$ is included in $[x]$.) ■

Remark how interests of all members of one side of the game are polarised in the minimal of maximal element. We denote by $\mathrm{Pol}_p(N,V)$ the union of these points over all classes $[x]$ in $C_p(N,V)$.

For proving the convergence of the bargaining process we need the following observation: It is natural in our context to assume that an improvement / disimprovement in the outside opportunities of an agent should not disadvantage / advantage this agent in his present bargaining. The outside opportunities are represented here by the "threat" point in the bargaining problem obtained as a reduced game, so we are lead to consider the following property:

Definition 4.7: Let $(K;e)$ and $(K;d)$ be bargaining problems of players i,j and let Ψ be a solution (see Roth (1979)). We say that Ψ has individual threat point monotonicity (ITPM) if it satisfies the following: If $e^i \geq d^i$ and $e^j \leq d^j$ then $\Psi^i(K;e) \geq \Psi^i(K;d)$ and $\Psi^j(K;e) \leq \Psi^j(K;d)$. ■

Indeed, all the mentioned instances of standard solutions have ITPM. The proofs are not difficult. See also the related "power" axiom R1 in Livne (1986).

We are now ready to prove the main result of this paper:

Theorem 4.8: Let (N,V) be an NTU-AG and let $p \in P(N)$ with $C_p(N,V) \neq \emptyset$. Let $x \in \mathrm{Pol}_p(N,V)$ and assume that, for each $T \in p$, the T-standard solution has TPC and ITPM. Let B be the operator of Definition 4.3 and let B^k be its powers defined in the usual way for any natural number k.

Then $y = \lim_{k \to \infty} B^k(x)$ exists, and y belongs to the corresponding (to choices of T-standard solutions) set $SB_p(N,V)$. In particular, $SB_p(N,V) \neq \emptyset$.

Proof: (Remark that there may be indeed different sets $SB_p(N,V)$ because of the may possible choices of T-standard solutions.)

We write x_k for $B^k(x)$ and x_0 for $x \in \mathrm{Pol}_p(N,V)$. We assume w.l.o.g that $x_o = w_{M,[x]}$ and we know by Lemma 4.4. that $x_k \sim x_{k+1}$.

We prove now by induction that, for $m_i \in M, f_j \in F, 1 \leq i,j \leq n$, we have $x^{m_i}_{k+1} \leq x^{m_i}_k$ and $x^{f_j}_{k+1} \geq x^{f_j}_k$. The assertion is true for $k = 0$ because $x_0 = w_{M,[x]}$.

Assume that the assertion is true for $k = r$ and let $T_1 \in p$, w.l.o.g $T_1 = \{m_1,f_1\}$. We easily obtain that $v^{m_1}_{x_{r+1},T_1} \leq v^{m_1}_{x_r,T_1}$ and $v^{f_1}_{x_{r+1},T_1} \geq v^{f_1}_{x_r,T_1}$.

In the same manner, for all $T \in p$, the threat point in their bargaining problem moves from stage r to stage $r+1$ in a fashion described in Definition 4.7. Because the T-standard solutions have ITPM we get the desired result for $k = r+1$.

Thus, for all $m_i \in M$, $x^{m_i}_k$ is a monotone non-increasing sequence and, because $[x]$ is compact, the limit of this sequence exists. The same holds for the non-decreasing sequences $x^{f_j}_k$.

Finally, because $x_k \in [x]$ for all k and because $[x]$ is closed, we obtain that $y = \lim_{k\to\infty} x_k = \lim_{k\to\infty} B^k(x) \in [x]$. By the continuity of B we have $B(y) = y$, and obviously $y \in SB_p(N,V)$. ■

Corollary 4.9: Let (N,V) be an NTU-AG. Then $SB(N,V) \neq \emptyset$.

Proof: Obvious from $C(N,V) \neq \emptyset$ (Theorem 2.4) and the definition of $SB(N,V)$ together with the Theorem 4.8. ■

Remark 4.10: In the proof of the main Theorem we used only the properties IR and PO of a standard solution. Thus, the existence of equilibria extends to a much wider class of bargaining procedures. One may even consider a "non-cooperative" method satisfying IR and PO (and, of course, TPC and ITPM).

The other properties of a standard solution (IEUR, SYM) are requested just for the connection to the TU case and the generalization of the intersection of the core and kernel which we have in mind. ■

The dynamical proof which we used for showing the existence of equilibria is not entirely constructive because it uses the non-emptiness of the core, which is proved in Kaneko (1982) by applying Scarf's Theorem on balanced games.

This is different from the TU case, where the non-emptiness of the core is established by solving a linear program (Shapley, Shubik (1971)).

5 Axiomatics and TU Games

In this section we characterize axiomatically the set of stable bargained equilibria $SB(N,V)$ for NTU assignment games, and we compare this with the intersection of the core and the kernel of TU assignment games. We will conclude that we generalized in a natural way this intersection and Rochford's work.

Peleg (1989) considers the following axioms: NE – non-emptiness; ETP – equal treatment property; COV – covariance; IR – individual rationality; WRGP – weak reduced game property; CRGP – converse reduced game property.

COV is the suitable form of IEUR for the TU case; ETP coincides with SYM for solutions consisting of an unique point; WRGP is just RGP applied to reduced games with maximum two players.

Peleg shows that, on the class of market games, the intersection of the core and pre-kernel is the unique solution satisfying NE, COV, ETP, IR, WRGP, CRGP.

It is well known (see for example Aumann, Maschler (1985)) that for two person TU games there is a unique point-valued solution satisfying NE, COV, SYM, PO. For a TU game $(\{1,2\}, v)$ this solution, which is called "standard", is given by

$$x^i = \frac{v(12)-v(1)-v(2)}{2} + v(i)\,,\ i = 1,2 \tag{5.1}$$

This makes sense also for a game which is not super-additive, and the solution is not IR there.

Remark that, for a super-additive game, this solution is exactly the middle point of the segment with endpoints $(v(1),v(12)-v(1))$ and $(v(12)-v(2),v(2))$. This segment is exactly the core of the game. For such a game, the prekernel and all standard solutions in the sense of Definition 4.1 pick the vector given by 5.1.

Let TRGP be the reduced game property applied only to reduced games of two players. The reader will not find it difficult to prove that, alternatively, one can characterize the intersection of the core and prekernel, on the same class, by: NE, CRGP, TRGP, and for two person games: SYM, COV, PO. (IR is also implied!)

Maschler, Peleg, Shapley (1979) give a geometric characterization of the intersection of the core and kernel, which always coincides with the intersection of the core and prekernel. They show that an outcome in this intersection is always the midpoint of a certain bargaining range for *each* pair of players. By applying results of Aumann, Dreze (1974), where sections of the core are studied (see also our Lemma 3.4), one can easily show that the "bargaining range" in the geometric characterization is no other than the core of the reduced game for each pair, and thus the midpoint of this range is exactly the standard solution to this (reduced) game. In this way we obtain an alternative definition of the intersection of the core and prekernel (or kernel) for a general TU game with non-empty core:

$$\mathrm{PreK}(N,v) \cap C(N,v) = \{x \mid x \in C(N,v), \text{ and, for each pair } \{i,j\} \text{ where } i,j \in N \text{ and } i \neq j,\ (x^i,x^j) \text{ is the standard solution to the game } (\{i,j\}, v_x)\} \tag{5.2}$$

(Remark that we could require only that $x(N) = v(N)$, and that the standard solution is explicitely IR, like in our Definition 4.1. $x \in C(N,v)$ follows from CRGP.)

Remark 5.1: Let (N,V) be a general NTU game with non-empty core. We would like to define the intersection of the core and a "kernel" and we have seen how the notion of "excess" can be avoided by using reduced games. This also threw more light on the nature of such a solution. We define in a similar way:

$$SB(N,V) = \{x \mid x \in C(N,V) \text{ and, for } \textit{each} \text{ pair } \{i,j\} \text{ where } i,j \in N \text{ and } i \neq j, (x^i, x^j) \text{ is the Nash solution to the game } (\{i,j\}, V_x)\} \qquad (5.3)$$

The Nash solution is just a possible choice and may be replaced, of course, by other solution concept.

Unfortunately, such a set may be empty, even for very simple games. We borrow the following example from Maschler, Owen (1989): $N = \{1,2,3\}$ and

$$V(1,2) = \{(x^1,x^2) \mid 2x^1 + 3x^2 \leq 180\};\ V(N) = \{x^1,x^2,x^3) \mid x^1+x^2+x^3 \leq 120\};$$
$$V(T) = 0^T - \mathbb{R}^T_+ \text{ for all } T \subseteq N,\ T \neq N, \{1,2\}.$$

The reader may check that the equations leading to $x \in SB(N,V)$ do not have a solution. Maschler and Owen point this out looking for a consistent value, but this is the same thing as the definition of $SB(N,V)$ for this game.

For NTU assignment games we showed in Section 4 that we can ensure the existence of outcomes where each pair in a matching compatible with allocations in the core is in "equilibrium" (gets, say, the Nash solution to its reduced game). Actually, in such an outcome, *all* pairs are in equilibrium! Indeed, let (N,V) be an NTU-AG, let $p \in P(N)$ with $C_p(N,V) \neq \emptyset$, and let $x \in SB_p(N,V) \subseteq C_p(N,V)$. By IR and PO, a standard solution (Definition 4.1) must lie in the core of a two-person game. For a pair $T \notin p$ we know, by Lemma 3.5 and the observation after it, that the core of the reduced game (T,V_x) consists of the unique point x^T, thus such a pair gets also the Nash solution to its reduced game.

We are thus confronted with the same phenomena as with the bargaining set: a "straightforward" generalization exists for NTU games of pairs, but fails for general NTU games (see Peleg (1963)). ■

We have also the following axiomatic characterization (if the Nash solution is universally accepted):

Theorem 5.2: Let $\mathfrak{A}$ be the class of NTU assignment games. The set of stable bargained equilibria $SB(N,V)$ is the unique solution on $\mathfrak{A}$ which satisfies: NE, TRGP, CRGP and for two-person games: IIA, PO, IR, SYM, IEUR.

Proof: The proofs showing that SB has these properties should be clear by our treatement in Sections 3,4. For the uniqueness, let σ be a solution on $\mathfrak{A}$ which satisfies the mentioned axioms. It is clear that σ picks exactly the Nash solution for a two-person game in $\mathfrak{A}$. (IR is only needed to ensure this fact for (reduced) games where the threat point is already Pareto-optimal. It may be replaced, for example, by TPC). Let $(N,V) \in \mathfrak{A}$, let $x \in \sigma(N,V)$ and let $p \in P(x)$. By TRGP, $(T,V_x) \in \mathfrak{A}$ for $T \in p$ and therefore (T,V_x) is a super-additive two person game and its Nash solution is well defined. Again by TRGP, we have $x^T \in \sigma(T,V_x)$ and we obtain that x^T is actually the Nash solution to (T,V_x). This is true for any pair T in p, and we obtain that $x \in SB_p(N,V) \subseteq SB(N,V)$.

For the converse inclusion, let $x \in SB(N,V)$ and let $p \in P(x)$. Then $x \in SB_p(N,V)$ and, by definition, x^T is the Nash solution to (T,V_x) for all $T \in p$. We obtain that $x^T = \sigma(T,V_x)$ for all $T \in p$, and, by Remark 5.1, the same is true for any pair. Because σ has CRGP we conclude $x \in \sigma(N,V)$. ■

The reader may compare this with the already mentioned axiomatization of the intersection of the core and prekernel of TU games. The IIA can be, of course, replaced by other axioms yielding standard solutions.

The solution concept presented here for NTU assignment games does not coincide with Kalai's Kernel (neither it is included in it) which arises from using the excess functions g_S (see Kalai (1975), Example 1,2). In general, Kalai's Kernel and Nucleolus for a two person NTU, super-additive game consist of the Pareto-optimal point (x^1,x^2) with $x^1 - v^1 = x^2 - v^2$ (see Definition 1.2–e for v^i). This solution is not standard because it does not satisfy IEUR. For example, take $V(1) = V(2) = 0$ and $V(1,2) = \mathrm{conv}\{(0,0),(1,0),(0,2)\}$. Then Kalai's solution picks the point (2/3, 2/3) while any standard solution in our sense picks (1/2, 1). Remark that this is, of course, a very simple NTU-AG.

On the other hand, if there are pairs which choose this or other kind of proportional solution, the set of equilibria $SB(N,V)$ can be still defined and it is non-empty (see Remark 4.7). Then, it may be interesting to compare this solution with the egalitarian solution for NTU games which is characterized by proportionality for two person games and consistency (with another definition of reduced games) in Hart, Mas-Colell (1989).

Finally, some words on the TU assignment games of Shapley, Shubik (1971): By technically "transforming" such games into NTU assignment games (similar to Kaneko (1982), p. 208), the TU assignment games form a sub-class where all our results can be applied. By the analysis in Sections 3,4 or by the discussion in this Section, it should be obvious that, on this sub-class, the solution concept presented here is no other than Rochford's set of "symmetrically pairwise bargained allocations" (Rochford (1984)).

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