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Stable central limit theorems for super Ornstein-Uhlenbeck processes*

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Abstract

In this paper, we study the asymptotic behavior of a supercritical (ξ, ψ)-superprocess $(X_t)_{t\geq 0}$ whose underlying spatial motion ξ is an Ornstein-Uhlenbeck process on $\mathbb{R}^d$ with generator $L = \frac{1}{2}\sigma^2\Delta - bx\cdot\nabla$ where $\sigma, b > 0$; and whose branching mechanism ψ satisfies Grey's condition and a perturbation condition which guarantees that, when $z \to 0$, $\psi(z) = -\alpha z + \eta z^{1+\beta}(1 + o(1))$ with $\alpha > 0$, $\eta > 0$ and $\beta \in (0, 1)$. Some law of large numbers and $(1+\beta)$-stable central limit theorems are established for $(X_t(f))_{t\geq 0}$, where the function f is assumed to be of polynomial growth. A phase transition arises for the central limit theorems in the sense that the forms of the central limit theorem are different in three different regimes corresponding to the branching rate being relatively small, large or critical at a balanced value.

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1 Introduction

1.1 Motivation

Let $d \in \mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{R}_+ := [0, \infty)$. Let $\xi = \{(\xi_t)_{t\geq 0}; (\Pi_x)_{x\in\mathbb{R}^d}\}$ be an $\mathbb{R}^d$-valued Ornstein-Uhlenbeck process (OU process) with generator

$$Lf(x) = \frac{1}{2}\sigma^2\Delta f(x) - bx\cdot\nabla f(x), \quad x \in \mathbb{R}^d, f \in C^2(\mathbb{R}^d),$$

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where $\sigma > 0$ and $b > 0$ are constants. Let ψ be a function on $\mathbb{R}_+$ of the form

$$\psi(z) = -\alpha z + \rho z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\, \pi(dy), \quad z \in \mathbb{R}_+, \tag{1.1}$$

where $\alpha > 0$, $\rho \geq 0$ and π is a measure on $(0,\infty)$ with $\int_{(0,\infty)}(y \wedge y^2)\, \pi(dy) < \infty$. ψ is referred to as a branching mechanism and π is referred to as the Lévy measure of ψ. Denote by $\mathcal{M}(\mathbb{R}^d)$ the space of all finite Borel measures on $\mathbb{R}^d$. For $f, g \in \mathcal{B}(\mathbb{R}^d, \mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$, write $\mu(f) = \int f(x)\mu(dx)$ and $\langle f, g\rangle = \int f(x)g(x)dx$ whenever the integrals make sense. We say a real-valued Borel function $f : (t,x) \mapsto f(t,x)$ on $\mathbb{R}_+ \times \mathbb{R}^d$ is *locally bounded* if, for each $t \in \mathbb{R}_+$, we have $\sup_{s\in[0,t],x\in\mathbb{R}^d} |f(s,x)| < \infty$. We say that an $\mathcal{M}(\mathbb{R}^d)$-valued Hunt process $X = \{(X_t)_{t\geq 0}; (\mathbb{P}_\mu)_{\mu\in\mathcal{M}(\mathbb{R}^d)}\}$ on $(\Omega, \mathscr{F})$ is a *super Ornstein-Uhlenbeck process (super-OU process)* with branching mechanism ψ, or a (ξ,ψ)-superprocess, if for each non-negative bounded Borel function f on $\mathbb{R}^d$, we have

$$\mathbb{P}_\mu[e^{-X_t(f)}] = e^{-\mu(V_t f)}, \quad t \geq 0, \mu \in \mathcal{M}(\mathbb{R}^d),$$

where $(t,x) \mapsto V_t f(x)$ is the unique locally bounded non-negative solution to the equation

$$V_t f(x) + \Pi_x\Big[\int_0^t \psi(V_{t-s} f(\xi_s))\, ds\Big] = \Pi_x[f(\xi_t)], \quad x \in \mathbb{R}^d, t \geq 0.$$

The existence of such super-OU process X is well known, see [13] for instance.

Recently, there have been quite a few papers on laws of large numbers for superdiffusions. In [15, 16, 17], some weak laws of large numbers (convergence in law or in probability) were established. The strong law of large numbers for superprocesses was first studied in [9], followed by [10, 11, 14, 26, 30, 44] under different settings. For a good survey on recent developments in laws of large numbers for branching Markov processes and superprocesses, see [14].

The strong law of large numbers for the super-OU process X above can be stated as follows: Under some conditions on ψ (these conditions are satisfied under our Assumptions 1 and 2 below), there exists an Ω_0 of $\mathbb{P}_\mu$-full probability for every $\mu \in \mathcal{M}(\mathbb{R}^d)$ such that on Ω_0, for every Lebesgue-a.e. continuous bounded non-negative function f on $\mathbb{R}^d$, we have $\lim_{t\to\infty} e^{-\alpha t} X_t(f) = H_\infty \langle f, \varphi\rangle$, where H_∞ is the limit of the martingale $e^{-\alpha t} X_t(1)$ and φ is the invariant density of the OU process ξ defined in (1.4) below. See [11, Theorem 2.13 & Example 8.1] and [14, Theorem 1.2 & Example 4.1].

In this paper, we will establish some spatial central limit theorems (CLTs) for the super-OU process X above. Our key assumption is that ψ satisfies Grey's condition and some perturbation condition which guarantees that, when $z \to 0$, $\psi(z) = -\alpha z + \eta z^{1+\beta}(1 + o(1))$ with $\alpha > 0$, $\eta > 0$ and $\beta \in (0,1)$. Our goal is to find $(F_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$ so that $(X_t(f) - G_t)/F_t$ converges weakly to some non-degenerate random variable as $t \to \infty$, for a large class of functions f. Note that, in the setting of this paper, $X_t(f)$ typically has infinite second moment.

There are many papers on CLTs for branching processes, branching diffusions and superprocesses, under the second moment condition. See [18, 20, 21] for supercritical Galton-Watson processes (GW processes), [24, 25] for supercritical multi-type GW processes, [4, 5, 6] for supercritical multi-type continuous time branching processes and [3] for general supercritical branching Markov processes under certain conditions. Some spatial CLTs for supercritical branching OU processes with binary branching mechanism were proved in [1], and some spatial CLTs for supercritical super-OU processes with branching mechanisms satisfying a fourth moment condition were proved in [33]. These two papers made connections between CLTs and branching rate regimes. Some spatial CLTs for supercritical super-OU processes with branching mechanisms satisfying only a second moment condition were established in [36]. Moreover, compared with the results of [1, 33], the limit distributions in [36] are non-degenerate. Since then, a series of spatial CLTs for a large class of general supercritical branching Markov processes and superprocesses with spatially dependent branching mechanisms were proved in [37, 38, 39]. Functional versions of the CLTs were established in [23] for supercritical multitype branching processes, and in [40] for supercritical superprocesses.

There are also many limit theorems for supercritical branching processes and branching Markov processes with branching mechanisms of infinite second moment. Heyde [19] established some CLTs for supercritical GW processes when the offspring distribution belongs to the domain of attraction of a stable law of index $\alpha \in (1, 2]$, and proved that the limit laws are stable laws. Similar results for supercritical multi-type GW processes and supercritical continuous time branching processes, under some p-th ($p \in (1, 2]$) moment condition on the offspring distribution, were given in Asmussen [2]. Recently, Marks and Miloś [31] considered the limit behavior of supercritical branching OU processes with a special stable offspring distribution. They established some spatial CLTs in the small and critical branching rate regimes, but they did not prove any CLT type result in the large branching rate regime. We also mention here that very recently [22] considered stable fluctuations of Biggins' martingales in the context of branching random walks, and [35] considered the asymptotic behavior of a class of critical superprocesses with spatially dependent stable branching mechanism.

As far as we know, this paper is the first to study spatial CLTs for supercritical superprocesses without the second moment condition.

1.2 Main results

We will always assume that the following assumption holds.

Assumption 1.1. *The branching mechanism ψ satisfies Grey's condition, i.e., there exists $z' > 0$ such that $\psi(z) > 0$ for all $z > z'$ and $\int_{z'}^{\infty} \psi(z)^{-1} dz < \infty$.*

For $\mu \in \mathcal{M}(\mathbb{R}^d)$, write $\|\mu\| = \mu(1)$. It is known (see [27, Theorems 12.5 & 12.7] for example) that, under Assumption 1.1, the *extinction event* $D := \{\exists t \geq 0, \text{ s.t. } \|X_t\| = 0\}$ has positive probability with respect to $\mathbb{P}_\mu$ for each $\mu \in \mathcal{M}(\mathbb{R}^d)$. In fact, $\mathbb{P}_\mu(D) = e^{-\bar{v}\|\mu\|}$ where $\bar{v} := \sup\{\lambda \geq 0 : \psi(\lambda) = 0\} \in (0, \infty)$ is the largest root of ψ.

Denote by Γ the gamma function. For any σ-finite signed measure μ, we use $|\mu|$ to denote the total variation measure of μ. In this paper, we will also assume the following:

Assumption 1.2. *There exist constants $\eta > 0$ and $\beta \in (0, 1)$ such that*

$$\int_{(1,\infty)} y^{1+\beta+\delta} \left| \pi(dy) - \frac{\eta \, dy}{\Gamma(-1-\beta) y^{2+\beta}} \right| < \infty \tag{1.2}$$

for some $\delta > 0$.

We will show in Subsection 2.1 that if Assumption 1.2 holds, then η and β are uniquely determined by the Lévy measure π. In the remainder of the paper, we will always use η and β to denote the constants in Assumption 1.2. Note that δ is not uniquely determined by π. In fact, if $\delta > 0$ is a constant such that (1.2) holds, then replacing δ by any smaller positive number, (1.2) still holds. Therefore, Assumption 1.2 is equivalent to the following statement: There exist constants $\eta > 0$ and $\beta \in (0, 1)$ such that, for all small enough $\delta > 0$, (1.2) holds.

Remark 1.3. Roughly speaking, Assumption 1.2 says that ψ is "not too far away" from

$\widetilde{\psi}(z) := -\alpha z + \eta z^{1+\beta}$ near 0. In fact, if we consider their difference

$$\begin{aligned}\psi_1(z) &:= \psi(z) - \widetilde{\psi}(z) \\ &= \rho z^2 + \int_{(0,\infty)} (e^{-yz} - 1 + yz)\Big(\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta) y^{2+\beta}}\Big), \quad z \geq 0,\end{aligned} \tag{1.3}$$

then it can be verified that (see Lemma 2.1 below) $\psi_1(z)/z^{1+\beta} \xrightarrow[z\to 0]{} 0$. Therefore, we can write $\psi(z) = -\alpha z + z^{1+\beta}(\eta + o(1))$ as $z \to 0$. One can further write that $\psi(z) = -\alpha z + z^{1+\beta} l(z)$ where l is a function on $[0,\infty)$ which is slowly varying at 0.

Remark 1.4. It will be proved in Lemma 2.3 that, under Assumption 1.2, ψ satisfies the $L\log L$ condition, i.e., $\int_{(1,\infty)} y \log y\, \pi(dy) < \infty$. This guarantees that H_∞, the limit of the non-negative martingale $(e^{-\alpha t}\|X_t\|)_{t\geq 0}$, is non-degenerate.

Let us introduce some notation in order to give the precise formulation of our main result. Denote by $\mathcal{B}(\mathbb{R}^d, \mathbb{R})$ the space of all $\mathbb{R}$-valued Borel functions on $\mathbb{R}^d$. Denote by $\mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$ the space of all $\mathbb{R}_+$-valued Borel functions on $\mathbb{R}^d$. We use $(P_t)_{t\geq 0}$ to denote the transition semigroup of ξ. Define $P_t^\alpha f(x) := e^{\alpha t} P_t f(x) = \Pi_x[e^{\alpha t} f(\xi_t)]$ for each $x \in \mathbb{R}^d$, $t \geq 0$ and $f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$. It is known that, see [28, Proposition 2.27] for example, $(P_t^\alpha)_{t\geq 0}$ is the *mean semigroup* of X in the sense that $\mathbb{P}_\mu[X_t(f)] = \mu(P_t^\alpha f)$ for all $\mu \in \mathcal{M}(\mathbb{R}^d)$, $t \geq 0$ and $f \in \mathcal{B}(\mathbb{R}^d, \mathbb{R}_+)$.

The limit behavior of X is closely related to the spectral property of the OU semigroup $(P_t)_{t\geq 0}$ which we now recall (See [32] for more details). It is known that the OU process ξ has an invariant probability on $\mathbb{R}^d$

$$\varphi(x)dx := \Big(\frac{b}{\pi\sigma^2}\Big)^{d/2} \exp\Big(-\frac{b}{\sigma^2}|x|^2\Big) dx \tag{1.4}$$

which is a symmetric multivariate Gaussian distribution. Let $L^2(\varphi)$ be the Hilbert space with inner product

$$\langle f_1, f_2\rangle_\varphi := \int_{\mathbb{R}^d} f_1(x) f_2(x) \varphi(x) dx, \quad f_1, f_2 \in L^2(\varphi).$$

Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For each $p = (p_k)_{k=1}^d \in \mathbb{Z}_+^d$, write $|p| := \sum_{k=1}^d p_k$, $p! := \prod_{k=1}^d p_k!$ and $\partial_p := \prod_{k=1}^d (\partial^{p_k}/\partial x_k^{p_k})$. The *Hermite polynomials* are defined by

$$H_p(x) := (-1)^{|p|} \exp(|x|^2) \partial_p \exp(-|x|^2), \quad x \in \mathbb{R}^d, p \in \mathbb{Z}_+^d.$$

It is known that $(P_t)_{t\geq 0}$ is a strongly continuous semigroup in $L^2(\varphi)$ and its generator L has discrete spectrum $\sigma(L) = \{-bk : k \in \mathbb{Z}_+\}$. For $k \in \mathbb{Z}_+$, denote by $\mathcal{A}_k$ the eigenspace corresponding to the eigenvalue $-bk$, then $\mathcal{A}_k = \mathrm{Span}\{\phi_p : p \in \mathbb{Z}_+^d, |p| = k\}$ where

$$\phi_p(x) := \frac{1}{\sqrt{p! 2^{|p|}}} H_p\Big(\frac{\sqrt{b}}{\sigma} x\Big), \quad x \in \mathbb{R}^d, p \in \mathbb{Z}_+^d.$$

In other words, $P_t \phi_p(x) = e^{-b|p|t} \phi_p(x)$ for all $t \geq 0$, $x \in \mathbb{R}^d$ and $p \in \mathbb{Z}_+^d$. Moreover, $\{\phi_p : p \in \mathbb{Z}_+^d\}$ forms a complete orthonormal basis of $L^2(\varphi)$. Thus for each $f \in L^2(\varphi)$, we have

$$f = \sum_{k=0}^\infty \sum_{p\in\mathbb{Z}_+^d : |p|=k} \langle f, \phi_p\rangle_\varphi \phi_p, \quad \text{in } L^2(\varphi). \tag{1.5}$$

For each function $f \in L^2(\varphi)$, define *the order of f* as

$$\kappa_f := \inf\big\{k \geq 0 : \exists\, p \in \mathbb{Z}_+^d, \text{ s.t. } |p| = k \text{ and } \langle f, \phi_p\rangle_\varphi \neq 0\big\}$$

which is the lowest non-zero frequency in the eigen-expansion (1.5). Note that $\kappa_f \geq 0$ and that, if $f \in L^2(\varphi)$ is non-zero, then $\kappa_f < \infty$. In particular, the order of any non-zero constant function is zero.

Denote by $\mathcal{M}_c(\mathbb{R}^d)$ the space of all finite Borel measures of compact support on $\mathbb{R}^d$. For $p \in \mathbb{Z}_+^d$, define $H_t^p := e^{-(\alpha-|p|b)t}X_t(\phi_p)$ for all $t \geq 0$. If $\alpha\tilde{\beta} > |p|b, \tilde{\beta} := \beta/(1+\beta)$, then for all $\gamma \in (0,\beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, we will prove in Lemma 3.2 that $(H_t^p)_{t\geq 0}$ is a $\mathbb{P}_\mu$-martingale bounded in $L^{1+\gamma}(\mathbb{P}_\mu)$. Thus the limit $H_\infty^p := \lim_{t\to\infty} H_t^p$ exists $\mathbb{P}_\mu$-almost surely and in $L^{1+\gamma}(\mathbb{P}_\mu)$.

We first present a law of large numbers for our model which extends the strong laws of large numbers of [11, 14] in which the first order asymptotic ($\kappa_f = 0$) was identified. Denote by $\mathcal{P}$ the class of functions of polynomial growth on $\mathbb{R}^d$, i.e.,

$$\mathcal{P} := \{f \in \mathcal{B}(\mathbb{R}^d,\mathbb{R}) : \exists C > 0, n \in \mathbb{Z}_+ \text{ s.t. } \forall x \in \mathbb{R}^d, |f(x)| \leq C(1+|x|)^n\}. \tag{1.6}$$

It is clear that $\mathcal{P} \subset L^2(\varphi)$.

Theorem 1.5. *If $f \in \mathcal{P}$ satisfies $\alpha\tilde{\beta} > \kappa_f b$, then for all $\gamma \in (0,\beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$,*

$$e^{-(\alpha-\kappa_f b)t}X_t(f) \xrightarrow[t\to\infty]{} \sum_{p\in\mathbb{Z}_+^d:|p|=\kappa_f} \langle f,\phi_p\rangle_\varphi H_\infty^p \quad \text{in } L^{1+\gamma}(\mathbb{P}_\mu).$$

Moreover, if f is twice differentiable and all its second order partial derivatives are in $\mathcal{P}$, then we also have almost sure convergence.

If $f \in \mathcal{B}(\mathbb{R}^d,\mathbb{R}_+)$ is non-zero and bounded, then $\kappa_f = 0$. Hence, Theorem 1.5 says that for any $\gamma \in (0,\beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, as $t \to \infty$, $e^{-\alpha t}X_t(f) \to \langle f,\varphi\rangle H_\infty$ in $L^{1+\gamma}(\mathbb{P}_\mu)$. Moreover, if f is twice differentiable and all its second order partial derivatives are in $\mathcal{P}$, then we also have a.s. convergence. However, to get a.s. convergence for bounded non-negative Lebesgue-a.e. continuous functions f, we do not need f to be twice differentiable. See [11, Theorem 2.13 & Example 8.1] and [14, Theorem 1.2 & Example 4.1].

For the rest of this subsection, we focus on the CLTs of $X_t(f)$ for a large collection of $f \in \mathcal{P}\setminus\{0\}$. Write $\tilde{u} = \frac{u}{1+u}$ for each $u \neq -1$. It turns out that there is a phase transition in the sense that the results are different in the following three regimes:

1. the small branching rate regime where f satisfies $\alpha\tilde{\beta} < \kappa_f b$;
2. the critical branching rate regime where f satisfies $\alpha\tilde{\beta} = \kappa_f b$; and
3. the large branching rate regime where f satisfies $\alpha\tilde{\beta} > \kappa_f b$.

Here, small (resp. large) branching rate means that the branching rate α is small (resp. large) compared to κ_f; and critical branching rate means that the branching rate α is at a critical balanced value compared to κ_f. To present our result, we define a family of operators $(T_t)_{t\geq 0}$ on $\mathcal{P}$ by

$$T_t f := \sum_{p\in\mathbb{Z}_+^d} e^{-\left||p|b-\alpha\tilde{\beta}\right|t}\langle f,\phi_p\rangle_\varphi \phi_p, \quad t \geq 0, f \in \mathcal{P}, \tag{1.7}$$

and a family of $\mathbb{C}$-valued functionals $(m_t)_{0\leq t<\infty}$ on $\mathcal{P}$ by

$$m_t[f] := \eta\int_0^t du \int_{\mathbb{R}^d} (-iT_u f(x))^{1+\beta}\varphi(x)\, dx, \quad 0 \leq t < \infty, f \in \mathcal{P}. \tag{1.8}$$

Define $\mathcal{C}_s := \mathcal{P} \cap \overline{\mathrm{Span}}\{\phi_p : \alpha\tilde{\beta} < |p|b\}$, $\mathcal{C}_c := \mathcal{P} \cap \mathrm{Span}\{\phi_p : \alpha\tilde{\beta} = |p|b\}$ and $\mathcal{C}_l := \mathcal{P} \cap \mathrm{Span}\{\phi_p : \alpha\tilde{\beta} > |p|b\}$. Note that $\mathcal{C}_s$ is an infinite dimensional space, $\mathcal{C}_l$ and $\mathcal{C}_c$ are finite dimensional spaces, and $\mathcal{C}_c$ might be empty. For $f \in \mathcal{P} \setminus \{0\}$, in Lemma 2.6 and Proposition 2.7 below, we will show that

$$m[f] := \begin{cases} \lim_{t\to\infty} m_t[f], & f \in \mathcal{C}_s \oplus \mathcal{C}_l, \\ \lim_{t\to\infty} \frac{1}{t} m_t[f], & f \in \mathcal{P} \setminus \mathcal{C}_s \oplus \mathcal{C}_l, \end{cases} \tag{1.9}$$

is well defined, and moreover, there exists a $(1+\beta)$-stable random variable ζ^f with characteristic function $\theta \mapsto e^{m[\theta f]}$. The main result of this paper is as follows.

Theorem 1.6. *If $\mu \in \mathcal{M}_c(\mathbb{R}^d) \setminus \{0\}$, then under $\mathbb{P}_\mu(\cdot|D^c)$, the following hold:*

1. *if $f \in \mathcal{C}_s \setminus \{0\}$, then $\|X_t\|^{-\frac{1}{1+\beta}} X_t(f) \xrightarrow[t\to\infty]{d} \zeta^f$;*
2. *if $f \in \mathcal{C}_c \setminus \{0\}$, then $\|tX_t\|^{-\frac{1}{1+\beta}} X_t(f) \xrightarrow[t\to\infty]{d} \zeta^f$;*
3. *if $f \in \mathcal{C}_l \setminus \{0\}$, then*

$$\|X_t\|^{-\frac{1}{1+\beta}} \Big(X_t(f) - \sum_{p\in\mathbb{Z}_+^d : \alpha\tilde{\beta} > |p|b} \langle f, \phi_p\rangle_\varphi e^{(\alpha-|p|b)t} H_\infty^p \Big) \xrightarrow[t\to\infty]{d} \zeta^{-f}.$$

At this point, we should mention that the theorem above does not cover all $f \in \mathcal{P}$. Theorem 1.6.(1) can be rephrased as if $f \in \mathcal{P}\setminus\{0\}$ satisfies $\alpha\tilde{\beta} < \kappa_f b$, then under $\mathbb{P}_\mu(\cdot|D^c)$, $\|X_t\|^{-\frac{1}{1+\beta}} X_t(f) \xrightarrow[t\to\infty]{d} \zeta^f$. Combining the first two parts of Theorem 1.6, one can easily get that if $f \in \mathcal{P}$ satisfies $\alpha\tilde{\beta} = \kappa_f b$, then under $\mathbb{P}_\mu(\cdot|D^c)$, $\|tX_t\|^{-\frac{1}{1+\beta}} X_t(f) \xrightarrow[t\to\infty]{d} \zeta^f$. A general $f \in \mathcal{P}$ can be decomposed as $f_s + f_c + f_l$ with $f_s \in \mathcal{C}_s$, $f_c \in \mathcal{C}_c$ and $f_l \in \mathcal{C}_l$. For $f \in \mathcal{P}$ satisfying $\alpha\tilde{\beta} > \kappa_f b$, f_s and f_c maybe non-zero. In this case, we do not have a CLT yet. We conjecture that the limit random variables in Theorem 1.6 for $f \in \mathcal{C}_s$, $f \in \mathcal{C}_c$ and $f \in \mathcal{C}_l$ are independent. If this is valid, we can get a CLT for $X_t(f)$ for all $f \in \mathcal{P}$. This independence is valid under the second moment condition, see [38]. We leave the question of the independence of the limit stable random variables to a future project.

We now give some intuitive explanation of the branching rate regimes and the phase transition. Similar explanation has been given in the context of branching-OU processes, see [31]. First we mention that a super-OU process arises as the "high intensity" limit of a sequence of branching-OU processes, see [28] for example. A superprocess can be thought of as a cloud of infinitesimal branching "particles" moving in space. The phase transition is due to an interplay of two competing effects in the system: coarsening and smoothing. The coarsening effect corresponds to the increase of the spatial inequality, and is a consequence of the branching: simply an area with more particles will produce more offspring. The smoothing effect corresponds to the decrease of the spatial inequality and is a consequence of the mixing property of the OU processes: each OU "particle" will "forget" its initial position exponentially fast.

Let us consider $X_t(\phi_p)$ as an example and discuss how the parameters α, β, b and $|p|$ influence those two effects:

- The branching rate α captures the mean intensity of the branching in the system. Therefore, the lager the branching rate α, the stronger the coarsening effect.
- The tail index β describes the heaviness of the tail of the offspring distribution which belongs to the domain of attraction of some $(1+\beta)$-stable random variable. When β is smaller i.e. the tail is heavier, then it is more likely that one particle can suddenly have a large amount of offspring. In other words, the larger the tail index β, the smaller the fluctuation of offspring number, and then the stronger the coarsening effect.

- The drift parameter b is related to the level of the mixing property of the OU particles. The larger the drift parameter b, the faster the OU-particles forgetting their initial position, and therefore the stronger the smoothing effect.
- The order $|p|$ is related to the capability of ϕ_p capturing the mixing property of the OU particles. In particular, in the case that $|p| = 0$, no mixing property can be captured by $\phi_p \equiv 1$ since we are only considering the total mass $\|X_t\|$. In general, the higher the order $|p|$, the more mixing property can be captured by ϕ_p, and therefore the stronger the smoothing effect.

Here we discuss the role of the other parameters ρ, η and σ in our model:

- The coefficient ρ does not influence the result since ρz^2 in the branching mechanism ψ is a part of the small perturbation ψ_1 (see Remark 1.3).
- The coefficients η and σ are hidden in the definition of the functional $m[f]$, and therefore influence the actual distribution of the limiting $(1+\beta)$-stable random variable ξ^f. Their role in the coarsening and smoothing effects are negligible compared to the four parameters α, β, b and $|p|$ mentioned above.

1.3 An outline of the methodology

Let us give some intuitive explanation of the methodology used in this paper. For any $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and any random variable Y with finite mean, we define $\mathcal{I}_s^t Y := \mathcal{I}_s^t[Y,\mu] := \mathbb{P}_\mu[Y|\mathscr{F}_t] - \mathbb{P}_\mu[Y|\mathscr{F}_s]$ where $0 \le s \le t < \infty$. We will use the shorter notation $\mathcal{I}_s^t Y$ when there is no danger of confusion. For $f \in \mathcal{P}$, consider the following decomposition over the time interval $[0,t]$:

$$X_t(f) := \sum_{k=0}^{\lfloor t\rfloor -1} \mathcal{I}_{t-k-1}^{t-k} X_t(f) + \mathcal{I}_0^{t-\lfloor t\rfloor} X_t(f) + X_0(P_t^\alpha f), \quad t \ge 0.$$

To find the fluctuation of $X_t(f)$, we will investigate the fluctuation of each term on the right hand side above. The second term and third term are negligible after rescaling, and for the first term, we will establish a multi-variate unit interval CLT, which says that

$$\Big(\|X_t\|^{-\frac{1}{1+\beta}} \mathcal{I}_{t-k-1}^{t-k} X_t(f)\Big)_{k=0}^n \xrightarrow[t\to\infty]{d} (\zeta_k^f)_{k=0}^n,$$

where $(\zeta_k^f)_{k\in\mathbb{N}}$ are some independent $(1+\beta)$-stable random variables. If $f \in \mathcal{C}_s \setminus \{0\}$, then it can be argued that $\sum_{k=0}^{\lfloor t\rfloor} \zeta_k^f \xrightarrow[t\to\infty]{d} \zeta^f$ and then intuitively we have $\|X_t\|^{-\frac{1}{1+\beta}} X_t(f) \xrightarrow[t\to\infty]{d} \zeta^f$. If $f \in \mathcal{C}_c \setminus \{0\}$, then it can be argued that $t^{-\frac{1}{1+\beta}} \sum_{k=0}^{\lfloor t\rfloor} \zeta_k \xrightarrow[t\to\infty]{d} \zeta^f$ and then intuitively we have $\|tX_t\|^{-\frac{1}{1+\beta}} X_t(f) \xrightarrow[t\to\infty]{d} \zeta^f$. If $f \in \mathcal{C}_l$, the general idea is almost the same, except that we need to consider the decomposition over the time interval $[t,\infty)$.

This paper is our first attempt on stable CLTs for superprocesses. There are still many open questions. Ren, Song and Zhang have established some spatial CLTs in [38] for a class of superprocesses with general spatial motions under the assumption that the branching mechanisms satisfy a second moment condition. We hope to prove spatial CLTs for superprocesses with general motions without the second moment assumption on the branching mechanism in a future project.

Recall that our Assumption 1.2 says that the branching mechanism ψ is $-\alpha z + \eta z^{1+\beta}$ plus a small perturbation $\psi_1(z)$ which satisfies (1.2) with some $\delta > 0$. It would be interesting to consider more general branching mechanisms.

The following correspondence between (sub)critical branching mechanisms and Bernstein functions is well known, see, for instance, [7, Theorem VII.4(ii)] and [8, Proposition 7]. Suppose that $f, g : (0,\infty) \to [0,\infty)$ are related by $f(x) = xg(x)$. Then f is a (sub)critical branching mechanism with $\lim_{x\to 0} f(x) = 0$ iff g is a Bernstein function with a decreasing Lévy density. We now use this correspondence to give some examples of branching mechanisms satisfying Assumptions 1.1 and 1.2. If h is a complete Bernstein function which is regularly varying at 0 with index $\beta_1 \in (\beta, 1)$, then

$$\psi(z) := -\alpha z + \rho z^2 + \eta z^{1+\beta} + zh(z), \qquad z > 0,$$

satisfies Assumptions 1.1 and 1.2. If $\beta_1 \in (\beta, 1)$, $c_1 \in (0, \eta/\Gamma(-1-\beta))$ and $c_2 \geq 1$, then

$$\psi(z) := -\alpha z + \rho z^2 + \eta z^{1+\beta} - \int_{c_2}^{\infty} (e^{-yz} - 1 + yz) \frac{c_1 dy}{y^{1+\beta_1}}, \qquad z \in \mathbb{R}_+,$$

satisfies Assumptions 1.1 and 1.2.

The rest of the paper is organized as follows: In Subsection 2.1 we will give some preliminary results for the branching mechanism ψ. In Subsections 2.2 and 2.3 we will give some estimates for some operators related to the super-OU process X. In Subsection 2.4 we will give the definitions of the $(1+\beta)$-stable random variables involved in this paper. In Subsection 2.5 we will give some refined estimates for the OU semigroup. In Subsection 2.6 we will give some estimates for the small value probability of continuous state branching processes. In Subsection 2.7 we will give upper bounds for the $(1+\gamma)$-moments for our superprocesses. These estimates and upper bounds will be crucial in the proofs of our main results. In Subsection 3.1, we will give the proof of Theorem 1.5. In Subsections 3.2–3.5, we will give the proof of Theorem 1.6. In the Appendix, we consider a general superprocess $(X_t)_{t\geq 0}$ and we prove that the characteristic exponent of $X_t(f)$ satisfies a complex-valued non-linear integral equation. This fact will be used at several places in this paper, and we think it is of independent interest.

2 Preliminaries

2.1 Branching mechanism

Let ψ be the branching mechanism given in (1.1). Suppose that Assumptions 1.1 and 1.2 hold. In this subsection, we give some preliminary results on ψ. Recall that η and β are the constants in Assumption 1.2. Let $\mathbb{C}_+ := \{x + iy : x \in \mathbb{R}_+, y \in \mathbb{R}\}$ and $\mathbb{C}_+^0 := \{x + iy : x \in (0,\infty), y \in \mathbb{R}\}$.

Lemma 2.1. *The function ψ_1 given by (1.3) can be uniquely extended as a complex-valued continuous function on $\mathbb{C}_+$ which is holomorphic on $\mathbb{C}_+^0$. Moreover, for all $\delta > 0$ small enough, there exists $C > 0$ such that for all $z \in \mathbb{C}_+$, we have $|\psi_1(z)| \leq C|z|^{1+\beta+\delta} + C|z|^2$.*

Proof. According to Lemma A.2 below and the uniqueness of holomorphic extensions, we know that ψ_1 can be uniquely extended as a complex-valued continuous function on $\mathbb{C}_+$ which is holomorphic on $\mathbb{C}_+^0$. The extended ψ_1 has the following form:

$$\psi_1(z) = \rho z^2 + \int_{(0,\infty)} (e^{-yz} - 1 + yz) \Big(\pi(dy) - \frac{\eta \, dy}{\Gamma(-1-\beta) y^{2+\beta}}\Big), \quad z \in \mathbb{C}_+.$$

Now, according to Assumption 1.2, for all small enough $\delta > 0$, we have

$$\begin{aligned}|\psi_1(z)| &\le \rho|z|^2 + \int_{(0,\infty)} (|yz| \wedge |yz|^2)\Big|\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\Big| \\ &\le |z|^2\Big(\rho + \int_{(0,1)} y^2\Big|\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\Big|\Big) \\ &\quad + |z|^{1+\beta+\delta}\int_{(1,\infty)} y^{1+\beta+\delta}\Big|\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\Big|, \quad z \in \mathbb{C}_+,\end{aligned}$$

as desired. □

The following lemma says that the constants η, β in Assumption 1.2 are uniquely determined by the Lévy measure π.

Lemma 2.2. *Suppose Assumption 1.2 holds. Suppose that there are $\eta', \delta' > 0$ and $\beta' \in (0,1)$ such that*

$$\int_{(1,\infty)} y^{1+\beta'+\delta'}\,\Big|\pi(dy) - \frac{\eta'\, dy}{\Gamma(-1-\beta)y^{2+\beta'}}\Big| < \infty.$$

Then $\eta' = \eta$ and $\beta' = \beta$.

Proof. Without loss of generality, we assume that $\beta + \delta \le \beta' + \delta'$. Using the fact that $y^{1+\beta+\delta} \le y^{1+\beta'+\delta'}$ for $y \ge 1$, we get

$$\int_{(1,\infty)} y^{1+\beta+\delta}\Big|\pi(dy) - \frac{\eta'\, dy}{\Gamma(-1-\beta)y^{2+\beta'}}\Big| < \infty.$$

Comparing this with Assumption 1.2, we get

$$\int_{(1,\infty)} y^{1+\beta+\delta}\Big|\frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}} - \frac{\eta'\, dy}{\Gamma(-1-\beta)y^{2+\beta'}}\Big| < \infty.$$

In other words, if we denote by $\widetilde{\pi}(dy)$ the measure $\eta'\Gamma(-1-\beta)^{-1}y^{-2-\beta'}dy$, then $\widetilde{\pi}$ is a Lévy measure which satisfies Assumption 1.2. Applying Lemma 2.1 to $\widetilde{\pi}$, we have that there exists $c > 0$ such that

$$|\eta z^{1+\beta} - \eta' z^{1+\beta'}| \le cz^{1+\beta+\delta} + cz^2, \quad z \in \mathbb{R}_+.$$

Dividing both sides by $z^{1+\beta}$ we have $|\eta - \eta' z^{\beta'-\beta}| \le cz^\delta + cz^{1-\beta}, z \in \mathbb{R}_+$. This implies that $\eta' z^{\beta'-\beta} \xrightarrow[\mathbb{R}^+ \ni z \to 0]{} \eta > 0$. So we must have $\beta' = \beta$ and $\eta' = \eta$. □

Lemma 2.3. *If ψ satisfies Assumption 1.2, then ψ satisfies the $L\log L$ condition, i.e., $\int_{(1,\infty)} y\log y\ \pi(dy) < \infty$.*

Proof. Using Assumption 1.2 and the fact that $y \log y \le y^{1+\beta+\delta}$ for y large enough, we get

$$\int_{(1,\infty)} y\log y\,\Big|\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\Big| < \infty.$$

Therefore we have

$$\int_{(1,\infty)} y\log y\,\Big(\pi(dy) - \frac{\eta\, dy}{\Gamma(-1-\beta)y^{2+\beta}}\Big) < \infty.$$

Combining this with $\int_{(1,\infty)} \frac{\eta \log y\ dy}{\Gamma(-1-\beta)y^{1+\beta}} < \infty$, we immediately get the desired result. □

2.2 Definition of controller

Denote by $\mathcal{B}(\mathbb{R}^d,\mathbb{C})$ the space of all $\mathbb{C}$-valued Borel functions on $\mathbb{R}^d$. Recall that $\mathcal{P}$ is given in (1.6). Define $\mathcal{P}^+ := \mathcal{P}\cap\mathcal{B}(\mathbb{R}^d,\mathbb{R}_+)$ and $\mathcal{P}^* := \{f\in\mathcal{B}(\mathbb{R}^d,\mathbb{C}) : |f|\in\mathcal{P}^+\}$.

In this paper, we say R is a *monotone operator on* $\mathcal{P}^+$ if $R:\mathcal{P}^+\to\mathcal{P}^+$ satisfies that $Rf\leq Rg$ for all $f\leq g$ in $\mathcal{P}^+$. For a function $h:[0,\infty)\to[0,\infty)$, we say R is an *h-controller* if R is a monotone operator on $\mathcal{P}^+$ and that $R(\theta f)\leq h(\theta)Rf$ for all $f\in\mathcal{P}^+$ and $\theta\in[0,\infty)$. For subsets $\mathcal{D},\mathcal{I}\subset\mathcal{P}^*$ and an operator R on $\mathcal{P}^+$, we say an operator A is *controlled by R from $\mathcal{D}$ to $\mathcal{I}$* if $A:\mathcal{D}\to\mathcal{I}$ and that $|Af|\leq R|f|$ for all $f\in\mathcal{D}$; we say a family of operators $\mathscr{O}$ is *uniformly controlled by R from $\mathcal{D}$ to $\mathcal{I}$* if each operator $A\in\mathscr{O}$ is controlled by R from $\mathcal{D}$ to $\mathcal{I}$. For subsets $\mathcal{D},\mathcal{I}\subset\mathcal{P}^*$ and a function $h:[0,\infty)\to[0,\infty)$, we say an operator A (resp. a family of operators $\mathscr{O}$) is *h-controllable* (resp. *uniformly h-controllable*) from $\mathcal{D}$ to $\mathcal{I}$ if there exists an h-controller R such that A (resp. $\mathscr{O}$) is controlled (resp. uniformly controlled) by R from $\mathcal{D}$ to $\mathcal{I}$.

For two operators $A:\mathcal{D}_A\subset\mathcal{P}^*\to\mathcal{P}^*$ and $B:\mathcal{D}_B\subset\mathcal{P}^*\to\mathcal{P}^*$, define $(A\times B)f(x) := Af(x)\times Bf(x)$ for all $f\in\mathcal{D}_A\cap\mathcal{D}_B$ and $x\in\mathbb{R}^d$. For any $a\in\mathbb{R}$ and any operator $A:\mathcal{D}_A\to\mathcal{B}(\mathbb{R}^d,\mathbb{C}\setminus(-\infty,0])$, define $A^{\times a}f(x) := (Af(x))^a$ for all $f\in\mathcal{D}_A$ and $x\in\mathbb{R}^d$.

The following lemma is easy to verify.

Lemma 2.4. *For each $i\in\{0,1\}$, let $\mathscr{O}_i$ be a family of operators which is uniformly controlled by an h_i-controller R_i from $\mathcal{D}_i\subset\mathcal{P}^*$ to $\mathcal{I}_i\subset\mathcal{P}^*$. Then the following statements hold:*

1. *If $\mathcal{I}_0\subset\mathcal{D}_1$, then $\{A_1A_0 : A_i\in\mathscr{O}_i, i=0,1\}$ is uniformly controlled by the $(h_1\circ h_0)$-controller R_1R_0 from $\mathcal{D}_0$ to $\mathcal{I}_1$.*
2. *$\{A_1\times A_0 : A_i\in\mathscr{O}_i, i=0,1\}$ is uniformly controlled by the $(h_1\times h_0)$-controller $R_1\times R_0$ from $\mathcal{D}_0\cap\mathcal{D}_1$ to $\mathcal{P}^*$.*
3. *$\{A_1+A_0 : A_i\in\mathscr{O}_i, i=0,1\}$ is uniformly controlled by the $(h_1\vee h_0)$-controller R_1+R_0 from $\mathcal{D}_0\cap\mathcal{D}_1$ to $\mathcal{P}^*$.*
4. *If $\mathcal{I}_0\subset\mathcal{B}(\mathbb{R}^d,\mathbb{C}\setminus(\infty,0])$ and $a>0$, then $\{A^{\times a} : A\in\mathscr{O}_0\}$ is uniformly controlled by the (h_0^a)-controller $R_0^{\times a}$ from $\mathcal{D}_0$ to $\mathcal{P}^*$.*
5. *Suppose that $\mathscr{O}_0=\{A_\theta : \theta\in\Theta\}$ where Θ is an index set. Further suppose that $(\Theta,\mathcal{J})$ is a measurable space and that $(\theta,x)\mapsto A_\theta f(x)$ is $\mathcal{J}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable for each $f\in\mathcal{D}$. Then the following space of operators*
$$\Big\{f\mapsto\int_\Theta A_\theta f\,\nu(d\theta) : \nu \text{ is a probability measure on } (\Theta,\mathcal{J})\Big\}$$
is uniformly controlled by R_0 from $\mathcal{D}_0$ to $\mathcal{P}^*$.

2.3 Controllers for the super-OU processes

Let X be our super-OU process with branching mechanism ψ satisfying Assumptions 1.1 and 1.2. In this subsection, we will define several operators and study some of their properties that will be used in this paper.

Define $\psi_0(z)=\psi(z)+\alpha z$ for $z\in\mathbb{R}_+$. According to Lemma 2.1, ψ,ψ_1 and ψ_0 can all be uniquely extended as complex-valued continuous functions on $\mathbb{C}_+$ which are also holomorphic on $\mathbb{C}_+^0$. For all $f\in\mathcal{B}(\mathbb{R}^d,\mathbb{C}_+)$ and $x\in\mathbb{R}^d$, define $\Psi f(x)=\psi\circ f(x)$, $\Psi_0 f(x)=\psi_0\circ f(x)$ and $\Psi_1 f(x)=\psi_1\circ f(x)$.

For all $t\in[0,\infty), x\in\mathbb{R}^d$ and $f\in\mathcal{P}$, let $U_tf(x) := \operatorname{Log}\mathbb{P}_{\delta_x}[e^{i\theta X_t(f)}]|_{\theta=1}$ be the value of the characteristic exponent of the infinitely divisible random variable $X_t(f)$ (See the paragraph after Lemma A.3). It follows from (A.8) that $-U_tf(x)$ takes values in $\mathbb{C}_+$. Furthermore, we know from Proposition A.6 that

$$U_tf(x)-\int_0^t P^\alpha_{t-s}\Psi_0(-U_sf)(x)ds = iP^\alpha_tf(x),\quad t\in[0,\infty), x\in\mathbb{R}^d, f\in\mathcal{P}. \tag{2.1}$$

For all $t \geq 0$ and $f \in \mathcal{P}$, we define

$$Z_t f := \int_0^t P^\alpha_{t-s}\big(\eta(-iP^\alpha_s f)^{1+\beta}\big)ds, \qquad Z'_t f := \int_0^t P^\alpha_{t-s}\big(\eta(-U_s f)^{1+\beta}\big)ds,$$
$$Z''_t f := \int_0^t P^\alpha_{t-s}\Psi_1(-U_s f)ds, \qquad Z'''_t f := (Z'_t - Z_t + Z''_t)f.$$

Then we have that

$$U_t - iP^\alpha_t = Z'_t + Z''_t = Z_t + Z'''_t, \quad t \geq 0. \tag{2.2}$$

For all $\kappa \in \mathbb{Z}_+$ and $f \in \mathcal{P}$, define

$$Q_\kappa f := \sup_{t\geq 0} e^{\kappa b t}|P_t f|, \qquad Qf := Q_{\kappa_f} f. \tag{2.3}$$

Then according to [31, Fact 1.2], Q is an operator from $\mathcal{P}$ to $\mathcal{P}$.

Lemma 2.5. *Under Assumptions 1.1 and 1.2, the following statements are true:*

(1) $(-U_t)_{0\leq t\leq 1}$ *is uniformly* θ*-controllable from* $\mathcal{P}$ *to* $\mathcal{P}^* \cap \mathcal{B}(\mathbb{R}^d, \mathbb{C}_+)$.

(2) $(P^\alpha_t)_{0\leq t\leq 1}$ *is uniformly* θ*-controllable on* $\mathcal{P}^*$.

(3) Ψ_0 *is* $(\theta^2 \vee \theta^{1+\beta})$*-controllable from* $\mathcal{P}^* \cap \mathcal{B}(\mathbb{R}^d, \mathbb{C}_+)$ *to* $\mathcal{P}^*$.

(4) $(U_t - iP^\alpha_t)_{0\leq t\leq 1}$ *is uniformly* $(\theta^2 \vee \theta^{1+\beta})$*-controllable from* $\mathcal{P}$ *to* $\mathcal{P}^*$.

(5) $(Z'_t - Z_t)_{0\leq t\leq 1}$ *is uniformly* $(\theta^{2+\beta} \vee \theta^{1+2\beta})$*-controllable from* $\mathcal{P}$ *to* $\mathcal{P}^*$.

(6) For all $\delta > 0$ *small enough, we have that* $(Z''_t)_{0\leq t\leq 1}$ *is uniformly* $(\theta^2 \vee \theta^{1+\beta+\delta})$*-controllable from* $\mathcal{P}$ *to* $\mathcal{P}^*$.

(7) For all $\delta > 0$ *small enough, we have that* $(Z'''_t)_{0\leq t\leq 1}$ *is uniformly* $(\theta^{2+\beta} \vee \theta^{1+\beta+\delta})$*-controllable from* $\mathcal{P}$ *to* $\mathcal{P}^*$.

Proof. (1). According to (A.8), $-U_t$ is an operator from $\mathcal{P}$ to $\mathcal{B}(\mathbb{R}^d, \mathbb{C}_+)$. It follows from (A.9) that for all $g \in \mathcal{P}$, $0 \leq t \leq 1$ and $x \in \mathbb{R}^d$, we have $|U_t g(x)| \leq \sup_{0\leq u\leq 1} P^\alpha_u |g|(x)$. We claim that $f \mapsto \sup_{0\leq u\leq 1} P^\alpha_u f$ is a map from $\mathcal{P}^+$ to $\mathcal{P}^+$. In fact, if $f \in \mathcal{P}^+$, there exists constant $c > 0$ such that

$$0 \leq \sup_{0\leq u\leq 1} P^\alpha_u f \leq \sup_{0\leq u\leq 1} P_u(e^{\alpha u}e^{-\kappa_f b u}e^{\kappa_f b u} f) \leq c \sup_{0\leq u\leq 1}(e^{\kappa_f b u}P_u f) \leq cQf \in \mathcal{P}.$$

It is clear that $f \mapsto \sup_{0\leq u\leq 1} P^\alpha_u f$ is a θ-controller.

(2). Similar to the proof of (1).

(3). By Lemma 2.1, there exist $C, \delta > 0$ satisfying $\beta + \delta < 1$ such that for all $f \in \mathcal{P}^* \cap \mathcal{B}(\mathbb{R}^d, \mathbb{C}_+)$, it holds that $|\Psi_0 f| \leq \eta|f|^{1+\beta} + |\Psi_1 f| \leq \eta|f|^{1+\beta} + C|f|^2 + C|f|^{1+\beta+\delta}$. Note that the operator

$$f \mapsto \eta f^{1+\beta} + Cf^2 + Cf^{1+\beta+\delta}, \quad f \in \mathcal{P}^+,$$

is a $(\theta^2 \vee \theta^{1+\beta})$-controller.

(4). From (1)–(3) above and Lemma 2.4.(1), we know that the operators

$$f \mapsto P^\alpha_{t-s}\Psi_0(-U_s f), \quad 0 \leq s \leq t \leq 1,$$

are uniformly $(\theta^2 \vee \theta^{1+\beta})$-controllable. Combining this with (2.1) and Lemma 2.4.(5), we get the desired result.

(5). Notice that from Lemma A.3,

$$|(-U_t f)^{1+\beta} - (-iP^\alpha_t f)^{1+\beta}| \leq (1+\beta)|U_t f - iP^\alpha_t f|(|U_t f|^\beta + |iP^\alpha_t f|^\beta).$$

Now using (1), (2) and (4) above, and Lemma 2.4, we get that the operators

$$f \mapsto (-U_t f)^{1+\beta} - (-iP_t^\alpha f)^{1+\beta}, \quad 0 \le t \le 1,$$

are uniformly $(\theta^{2+\beta} \vee \theta^{1+2\beta})$-controllable. Combining with Lemma 2.4, and

$$(Z'_t - Z_t)f = \int_0^t P_{t-s}^\alpha \Big(\eta((-U_s f)^{1+\beta} - (-iP_s^\alpha f)^{1+\beta})\Big) ds, \quad 0 \le t \le 1, f \in \mathcal{P},$$

we get the desired result.

(6). By Lemma 2.1, for all $\delta > 0$ small enough, there exists $C > 0$ such that

$$|\Psi_1(f)| \le C(|f|^2 + |f|^{1+\beta+\delta}), \quad f \in \mathcal{P}^* \cap \mathcal{B}(\mathbb{R}^d, \mathbb{C}_+).$$

Note that, for all $\delta, C > 0$,

$$f \mapsto C(f^2 + f^{1+\beta+\delta}), \quad f \in \mathcal{P}^+$$

is a $(\theta^2 \vee \theta^{1+\beta+\delta})$-controller. Therefore, for all $\delta > 0$ small enough, we have that Ψ_1 is a $(\theta^2 \vee \theta^{1+\beta+\delta})$-controllable operator from $\mathcal{P}^* \cap \mathcal{B}(\mathbb{R}^d, \mathbb{C}_+)$ to $\mathcal{P}^*$. Combining this with (1)–(2) above, and Lemma 2.4, we get that, for all $\delta > 0$ small enough, the operators

$$f \mapsto Z''_t f = \int_0^t P_{t-s}^\alpha \Psi_1(-U_s f) ds, \quad 0 \le t \le 1,$$

are uniformly $(\theta^2 \vee \theta^{1+\beta+\delta})$-controllable from $\mathcal{P}$ to $\mathcal{P}^*$.

(7). Since $Z'''_t = (Z'_t - Z_t) + Z''_t$, the desired result follows from (5)–(6) above and Lemma 2.4.(3). ☐

2.4 Stable distributions

Recall that the operators $(T_t)_{t\ge0}$ are defined by (1.7), and the functionals $(m_t)_{0\le t<\infty}$ and m are given by (1.8) and (1.9) respectively.

Lemma 2.6. *$(T_t)_{t\ge0}$, $(m_t)_{0\le t<\infty}$ and m are well defined.*

Proof. Step 1. We will show that for each $f \in \mathcal{P}$, there exists $h \in \mathcal{P}$ such that $|T_t f| \le e^{-\delta t} h$ for each $t \ge 0$, where

$$\delta := \inf\left\{ \left|\tilde\beta\alpha - |p|b\right| : p \in \mathbb{Z}_+^d, \langle f, \phi_p\rangle_\varphi \neq 0 \right\} \ge 0. \tag{2.4}$$

From this upper bound, it can be verified that $(T_t)_{t\ge0}$ and $(m_t)_{0\le t<\infty}$ are well defined. In fact, we can write $f = f_0 + f_1$ with $f_0 \in \mathcal{C}_s \oplus \mathcal{C}_c$ and $f_1 \in \mathcal{C}_l$. According to [31, Lemma 2.7], there exists $h_0 \in \mathcal{P}$ such that for each $t \ge 0$,

$$|T_t f_0| = \Big| \sum_{p\in\mathbb{Z}_+^d : \tilde\beta\alpha \le |p|b} e^{-(|p|b - \tilde\beta\alpha)t} \langle f, \phi_p\rangle_\varphi \phi_p \Big| = e^{\tilde\beta\alpha t} |P_t f_0| \le e^{-(\kappa_{(f_0)} b - \tilde\beta\alpha)t} h_0 \le e^{-\delta t} h_0.$$

On the other hand,

$$|T_t f_1| \le e^{-\delta t} \sum_{p\in\mathbb{Z}_+^d : \tilde\beta\alpha > |p|b} |\langle f, \phi_p\rangle_\varphi \phi_p| =: e^{-\delta t} h_1, \quad t \ge 0.$$

So the desired result in this step follows with $h := h_0 + h_1$.

Step 2. We will show that if $f \in \mathcal{C}_s \oplus \mathcal{C}_l$, then $m[f]$ is well defined. In fact, let δ be given by (2.4), then in this case $\delta > 0$. Now, according to Step 1 there exists $h \in \mathcal{P}$ such that $|T_t f| \le e^{-\delta t} h$ for each $t \ge 0$. This exponential decay implies the desired result in this step.

Step 3. We will show that if $f \in \mathcal{P} \setminus (\mathcal{C}_s \oplus \mathcal{C}_l)$, then $m[f]$ is also well defined. In fact, f can be decomposed as $f = f_c + f_{sl}$ where $f \in \mathcal{C}_c \setminus \{0\}$ and $f_{sl} \in \mathcal{C}_s \oplus \mathcal{C}_l$. Note that $T_t f_c = f_c$ for each $t \geq 0$. Also note that in Step 2, we already have shown that there exist $\delta > 0$ and $h \in \mathcal{P}^+$ such that for each $t \geq 0$, we have $|T_t f_{sl}| \leq e^{-\delta t} h$. Therefore, using Lemma A.3 we have

$$\begin{aligned}&|(-iT_t f)^{1+\beta} - (-if_c)^{1+\beta}| \leq (1+\beta)(|T_t f|^\beta + |f_c|^\beta)|T_t f_{sl}|\\&\leq (1+\beta)(|f_c + T_t f_{sl}|^\beta + |f_c|^\beta) e^{-\delta t} h \leq (1+\beta)((|f_c| + |h|)^\beta + |f_c|^\beta) e^{-\delta t} h =: e^{-\delta t} g,\end{aligned}$$

where $g \in \mathcal{P}^+$. Therefore

$$\begin{aligned}&\Big|\frac{1}{t} m_t[f] - \langle (-if_c)^{1+\beta}, \varphi\rangle\Big| = \Big|\frac{1}{t}\cdot t \int_0^1 \big\langle (-iT_{rt} f)^{1+\beta} - (-if_c)^{1+\beta}, \varphi\big\rangle \, dr\Big| \qquad (2.5)\\&\leq \int_0^1 \big\langle |(-iT_{rt} f)^{1+\beta} - (-if_c)^{1+\beta}|, \varphi\big\rangle \, dr \leq \langle g, \varphi\rangle \int_0^1 e^{-\delta r t} \, dr \xrightarrow[t\to\infty]{} 0. \qquad \square\end{aligned}$$

Proposition 2.7. *For each $f \in \mathcal{P} \setminus \{0\}$, there exists a non-degenerate $(1+\beta)$-stable random variable ζ^f such that $E[e^{i\theta\zeta^f}] = e^{m[\theta f]}$ for all $\theta \in \mathbb{R}$.*

The proof of the above proposition relies on the following lemma:

Lemma 2.8. *Let q be a measure on $\mathbb{R}^d \setminus \{0\}$ with $\int_{\mathbb{R}^d\setminus\{0\}} |x|^{1+\beta} q(dx) \in (0, \infty)$. Then*

$$\theta \mapsto \exp\Big\{\int_{\mathbb{R}^d\setminus\{0\}} (i\theta\cdot x)^{1+\beta} q(dx)\Big\}, \quad \theta \in \mathbb{R}^d,$$

is the characteristic function of an $\mathbb{R}^d$-valued $(1+\beta)$-stable random variable.

Proof. It follows from disintegration that there exist a measure λ on $S := \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ and a kernel $k(\xi, dt)$ from S to $\mathbb{R}_+$ such that

$$\int_{\mathbb{R}^d\setminus\{0\}} f(x) q(dx) = \int_S \lambda(d\xi) \int_{\mathbb{R}_+} f(t\xi) k(\xi, dt), \quad f \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}, \mathbb{R}_+).$$

We define another measure λ_0 on S by

$$\lambda_0(d\xi) := \frac{1}{\Gamma(-1-\beta)} \int_0^\infty t^{1+\beta} k(\xi, dt) \lambda(d\xi),$$

where Γ is the Gamma function. Then λ_0 is a non-zero finite measure, since

$$\begin{aligned}\lambda_0(S) =& \frac{1}{\Gamma(-1-\beta)} \int_S \lambda(d\xi) \int_0^\infty |t\xi|^{1+\beta} k(\xi, dt)\\=& \frac{1}{\Gamma(-1-\beta)} \int_{\mathbb{R}^d\setminus\{0\}} |x|^{1+\beta} q(dx) \in (0, \infty).\end{aligned}$$

Define a measure ν on $\mathbb{R}^d \setminus \{0\}$ by

$$\int_{\mathbb{R}^d\setminus\{0\}} f(x)\nu(dx) = \int_S \lambda_0(d\xi) \int_0^\infty f(r\xi) \frac{dr}{r^{2+\beta}}, \quad f \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}, \mathbb{R}_+).$$

Then, according to [41, Remark 14.4], ν is the Lévy measure of a $(1+\beta)$-stable distribution on $\mathbb{R}^d$, say μ, whose characteristic function is

$$\hat{\mu}(\theta) = \exp\Big\{\int_{\mathbb{R}^d\setminus\{0\}} (e^{-i\theta\cdot y} - 1 + i\theta\cdot y)\nu(dy)\Big\}, \quad \theta \in \mathbb{R}.$$

Finally, according to (A.2), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus\{0\}}(e^{-i\theta\cdot y}-1+i\theta\cdot y)\nu(dy)=\int_S \lambda_0(d\xi)\int_0^\infty (e^{-ir\theta\cdot\xi}-1+ir\theta\cdot\xi)\frac{dr}{r^{2+\beta}}\\
&=\int_S \lambda(d\xi)\int_0^\infty (e^{-ir\theta\cdot\xi}-1+ir\theta\cdot\xi)\frac{dr}{\Gamma(-1-\beta)r^{2+\beta}}\int_0^\infty t^{1+\beta}k(\xi,dt)\\
&=\int_S \lambda(d\xi)\int_0^\infty (i\theta\cdot\xi)^{1+\beta}t^{1+\beta}k(\xi,dt)=\int_S \lambda(d\xi)\int_0^\infty (i\theta\cdot t\xi)^{1+\beta}k(\xi,dt)\\
&=\int_{\mathbb{R}^d}(i\theta\cdot x)^{1+\beta}q(dx).
\end{aligned}$$
□

Proof of Proposition 2.7. Suppose that $f\in\mathcal{C}_s\oplus\mathcal{C}_l$. Note that $m[\theta f]$ can be written as

$$m[\theta f]=\eta\int_0^\infty ds\int_{\mathbb{R}^d}(-i\theta T_s f(x))^{1+\beta}\varphi(x)\,dx,\quad \theta\in\mathbb{R}. \tag{2.6}$$

Therefore, according to Lemma 2.8, in order to show that ζ^f is a $(1+\beta)$-stable random variable we only need to show that

$$\int_0^\infty ds\int_{\mathbb{R}^d}|T_s f(x)|^{1+\beta}\varphi(x)\,dx\in(0,\infty). \tag{2.7}$$

According to the Step 1 in the proof of Lemma 2.6, we know that there exist $\delta>0$ and $h\in\mathcal{P}$ such that $|T_s f|\le e^{-\delta s}h$ for each $s\ge 0$. The claim (2.7) then follows.

If $f\in\mathcal{P}\setminus(\mathcal{C}_s\oplus\mathcal{C}_l)$, then f can be written by $f=f_c+(f-f_c)$ where $f_c\in\mathcal{C}_c\setminus\{0\}$ and $f-f_c\in\mathcal{C}_s\oplus\mathcal{C}_l$. In this case, according to (2.5), $m[\theta f]$ has an integral representation:

$$m[\theta f]=\int_{\mathbb{R}^d}(-i\theta f_c(x))^{1+\beta}\varphi(x)\,dx,\quad \theta\in\mathbb{R}.$$

Finally, according to Lemma 2.8 and the fact that $\int_{\mathbb{R}^d}|f_c(x)|^{1+\beta}\varphi(x)\,dx\in(0,\infty)$, we have that ζ^f is a non-degenerate $(1+\beta)$-stable random variable. □

2.5 A refined estimate for the OU semigroup

It turns out that our proof of the CLT relies on the following refined estimate for the OU semigroup.

Lemma 2.9. *Suppose that $g\in\mathcal{P}$, then there exists $h\in\mathcal{P}^+$ such that for all $f\in\mathcal{P}_g:=\{\theta T_n g: n\in\mathbb{Z}_+,\theta\in[-1,1]\}$ and $t\ge 0$, we have $|P_t(Z_1 f-\langle Z_1 f,\varphi\rangle)|\le e^{-bt}h$.*

Proof. Fix $g\in\mathcal{P}$. We write $g=g_0+g_1$ with $g_0\in\mathcal{C}_s\oplus\mathcal{C}_c$ and $g_1\in\mathcal{C}_l$, and $q_f:=Z_1 f-\langle Z_1 f,\varphi\rangle\in\mathcal{P}^*$ for each $f\in\mathcal{P}$.

Step 1. We claim that we only need to prove the result for all $f\in\widetilde{\mathcal{P}}_g:=\{T_{n+1}g:n\in\mathbb{Z}_+\}$. In fact, both $\operatorname{Re} q_g$ and $\operatorname{Im} q_g$ are functions in $\mathcal{P}$ of order ≥ 1. The result is valid for $f=T_0 g=g$ according to [31, Fact 1.2]. Also, note that if the result is valid for some $f\in\mathcal{P}$, it is also valid for any θf with $\theta\in[-1,1]$.

Step 2. We show that $\{T_s g: s>0\}\subset C_\infty(\mathbb{R}^d)\cap\mathcal{P}$. In fact, for each $s>0$,

$$T_s g=T_s(g_0+g_1)=e^{\alpha\tilde\beta s}P_s g_0+\sum_{p\in\mathbb{Z}_+^d:\alpha\tilde\beta>|p|b}\langle g_1,\phi_p\rangle_\varphi e^{-(\alpha\tilde\beta-|p|b)s}\phi_p.$$

Notice that the second term is in $C_\infty(\mathbb{R}^d)\cap\mathcal{P}$ since it is a finite sum of polynomials, and the first term is also in $C_\infty(\mathbb{R}^d)\cap\mathcal{P}$ according to [31, Fact 1.1].

Step 3. We show that there exists $h_3 \in \mathcal{P}^+$ such that for all $j \in \{1, \dots, d\}$ and $f \in \widetilde{\mathcal{P}}_g$, it holds that $|\partial_j f| \le h_3$. In fact, it is known that (see [32] for example)

$$P_t f(x) = \int_{\mathbb{R}^d} f\big(xe^{-bt} + y\sqrt{1-e^{-2bt}}\big)\varphi(y)\,dy, \quad t \ge 0, x \in \mathbb{R}^d, f \in \mathcal{P}. \tag{2.8}$$

For $f \in C_\infty(\mathbb{R}^d) \cap \mathcal{P}$ it can be verified from above that

$$\partial_j P_t f = e^{-bt} P_t \partial_j f, \quad t \ge 0, j \in \{1, \dots, d\}. \tag{2.9}$$

Thanks to Step 2, $T_1 g_0 \in C_\infty(\mathbb{R}^d) \cap \mathcal{P}$. According to [31, Fact 1.3] and the fact that $\alpha\tilde\beta \le \kappa_{g_0} b$, we have for each $j \in \{1, \dots, d\}$,

$$\kappa_{(\partial_j T_1 g_0)} \ge \kappa_{(T_1 g_0)} - 1 = \kappa_{g_0} - 1 \ge \frac{\alpha\tilde\beta}{b} - 1.$$

Therefore, there exists $h_3' \in \mathcal{P}^+$ such that for all $n \in \mathbb{Z}_+$ and $j \in \{1, \dots, d\}$,

$$\begin{aligned}&|\partial_j T_{n+1} g_0| = |\partial_j e^{\alpha\tilde\beta n} P_n T_1 g_0| = e^{\alpha\tilde\beta n - bn}|P_n \partial_j T_1 g_0|\\ &\le e^{\alpha\tilde\beta n - bn}\exp\{-\kappa_{(\partial_j T_1 g_0)} bn\} Q\partial_j T_1 g_0 \le Q\partial_j T_1 g_0 \le h_3'.\end{aligned}$$

On the other hand, there exists $h_3'' \in \mathcal{P}^+$ such that for all $n \in \mathbb{Z}_+$ and $j \in \{1, \dots, d\}$,

$$\begin{aligned}&|\partial_j T_{n+1} g_1| = \Big|\sum_{p \in \mathbb{Z}_+^d : \alpha\tilde\beta > |p|b} e^{-(\alpha\tilde\beta - |p|b)(n+1)}\langle g_1, \phi_p\rangle_\varphi \partial_j \phi_p\Big|\\ &\le \sum_{p \in \mathbb{Z}_+^d : \alpha\tilde\beta > |p|b} |\langle g_1, \phi_p\rangle_\varphi \partial_j \phi_p| \le h_3''.\end{aligned}$$

Then the desired result in this step follows.

Step 4. We show that there exists $h_4 \in \mathcal{P}^+$ such that for all $j \in \{1, \dots, d\}, u \in [0,1]$ and $f \in \widetilde{\mathcal{P}}_g$, it holds that $|\partial_j P^\alpha_{1-u}(-iP^\alpha_u f)^{1+\beta}| \le h_4$. In fact, thanks to Step 2 and (2.9), for all $j \in \{1, \dots, d\}, u \in [0,1]$ and $f \in \widetilde{\mathcal{P}}_g$, we have

$$\begin{aligned}&\partial_j P^\alpha_{1-u}(-iP^\alpha_u f)^{1+\beta} = e^{-(1-u)b} P^\alpha_{1-u} \partial_j (-iP^\alpha_u f)^{1+\beta}\\ &= (1+\beta)e^{-(1-u)b} P^\alpha_{1-u}[(-iP^\alpha_u f)^\beta \partial_j(-iP^\alpha_u f)]\\ &= -i(1+\beta)e^{-(1-u)b} P^\alpha_{1-u}[(-iP^\alpha_u f)^\beta e^{-ub} P^\alpha_u \partial_j f]\\ &= -i(1+\beta)e^{-b} e^{(1-u)\alpha} e^{u\alpha(1+\beta)} P_{1-u}[(-iP_u f)^\beta P_u \partial_j f].\end{aligned}$$

Recall from Step 1 in the proof of Lemma 2.6 there exists $h_4' \in \mathcal{P}^+$ such that for each $f \in \{T_s g : s \ge 0\}$ it holds that $|f| \le h_4'$. Therefore, using Step 3, we have for all $j \in \{1, \dots, d\}, u \in [0,1]$ and $f \in \widetilde{\mathcal{P}}_g$,

$$\begin{aligned}&|\partial_j P^\alpha_{1-u}(-iP^\alpha_u f)^{1+\beta}| \le (1+\beta)e^{\alpha(1+\beta)} P_{1-u}[(P_u|f|)^\beta P_u|\partial_j f|]\\ &\le (1+\beta)e^{\alpha(1+\beta)} P_{1-u}[(P_u h_4')^\beta P_u h_3] \le (1+\beta)e^{\alpha(1+\beta)} Q_0[(Q_0 h_4')^\beta Q_0 h_3],\end{aligned}$$

where Q_0 is defined by (2.3). This implies the desired result in this step.

Step 5. We show that there exists $h_5 \in \mathcal{P}^+$ such that for each $f \in \widetilde{\mathcal{P}}_g$, we have $|\nabla(Z_1 f)| \le h_5$. In fact, according to Step 4, for all $j \in \{1, \dots, d\}$, $f \in \widetilde{\mathcal{P}}_g$ and compact $A \subset \mathbb{R}^d$, we have

$$\int_0^1 \sup_{x \in A} |(\partial_j P^\alpha_{1-u}(-iP^\alpha_u f)^{1+\beta})(x)|\,du \le \sup_{x \in A} h_4(x) < \infty.$$

Using this and [12, Theorem A.5.2], for all $j \in \{1, \dots, d\}$, $f \in \widetilde{\mathcal{P}}_g$ and $x \in \mathbb{R}^d$, it holds that

$$|\partial_j Z_1 f(x)| = \Big| \int_0^1 (\partial_j P^\alpha_{1-u}(-iP^\alpha_u f)^{1+\beta})(x)\, du \Big| \le h_4(x).$$

Now, the desired result for this step is valid.

Step 6. Let h_5 be the function in Step 5. There are $c_0, n_0 > 0$ such that for all $x \in \mathbb{R}^d$, $h_5(x) \le c_0(1+|x|)^{n_0}$. Note that for all $x, y \in \mathbb{R}^d$, $1 + |x| + |y| \le (1+|x|)(1+|y|)$; and that for all $\theta \in [0,1]$, $|\sqrt{1-\theta} - 1| \le \theta$. Write $D_{x,y} = \{ax + by : a, b \in [0,1]\}$ fo $x, y \in \mathbb{R}^d$. Using (2.8) and Step 5, there exists $h_6 \in \mathcal{P}^+$ such that for all $t \ge 0$, $f \in \widetilde{\mathcal{P}}_g$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
|P_t q_f(x)| &= \Big| \int_{\mathbb{R}^d} ((Z_1 f)(xe^{-bt} + y\sqrt{1-e^{-2bt}}) - Z_1 f(y))\varphi(y)\, dy \Big| \\
&\le \int_{\mathbb{R}^d} \Big(\sup_{z \in D_{x,y}} |\nabla (Z_1 f)(z)| \Big) |xe^{-bt} + y\sqrt{1-e^{-2bt}} - y| \varphi(y)\, dy \\
&\le e^{-bt} \int_{\mathbb{R}^d} c_0 (1 + |x| + |y|)^{n_0} (|x| + |y|) \varphi(y)\, dy \\
&\le c_0 e^{-bt} (1+|x|)^{n_0} \Big(|x| \int_{\mathbb{R}^d} (1+|y|)^{n_0} \varphi(y)\, dy + \int_{\mathbb{R}^d} (1+|y|)^{n_0} |y| \varphi(y)\, dy \Big) \\
&\le e^{-bt} h_6(x).
\end{aligned}$$

□

2.6 Small value probability

In this subsection, we digress briefly from our super-OU process and consider a (supercritical) *continuous-state branching process (CSBP)* $\{(Y_t)_{t\ge 0}; \mathbf{P}_x\}$ with branching mechanism ψ given by (1.1). Such a process $\{(Y_t)_{t\ge 0}; \mathbf{P}_x\}$ is defined as an $\mathbb{R}^+$-valued Hunt process satisfying

$$\mathbf{P}_x[e^{-\lambda Y_t}] = e^{-x v_t(\lambda)}, \quad x \in \mathbb{R}^+, t \ge 0, \lambda \in \mathbb{R}^+,$$

where for each $\lambda \ge 0$, $t \mapsto v_t(\lambda)$ is the unique positive solution to the equation

$$v_t(\lambda) - \int_0^t \psi(v_s(\lambda))\, ds = \lambda, \quad t \ge 0. \tag{2.10}$$

It can be verified that for each $\mu \in \mathcal{M}(\mathbb{R}^d)$ with $x = \|\mu\|$, we have $\{(\|X_t\|)_{t\ge 0}; \mathbb{P}_\mu\} \overset{\text{law}}{=} \{(Y_t)_{t\ge 0}; \mathbf{P}_x\}$.

Our goal in this subsection is to determine how fast the probability $\mathbf{P}_x(0 < e^{-\alpha t} Y_t \le k_t)$ converges to 0 when $t \mapsto k_t$ is a strictly positive function on $[0, \infty)$ such that $k_t \to 0$ and $k_t e^{\alpha t} \to \infty$ as $t \to \infty$. Suppose that Grey's condition is satisfied i.e., there exists $z' > 0$ such that $\psi(z) > 0$ for all $z > z'$, and that $\int_{z'}^\infty \psi(z)^{-1} dz < \infty$. Also suppose that the $L \log L$ condition is satisfied i.e., $\int_1^\infty y \log y\, \pi(dr) < \infty$. We write $W_t = e^{-\alpha t} Y_t$ for each $t \ge 0$.

Proposition 2.10. *Suppose that $t \mapsto k_t$ is a strictly positive function on $[0, \infty)$ such that $k_t \to 0$ and $k_t e^{\alpha t} \to \infty$ as $t \to \infty$. Then, for each $x \ge 0$, there exist $C, \delta > 0$ such that*

$$\mathbf{P}_x(0 < W_t \le k_t) \le C(k_t^\delta + e^{-\delta t}), \quad t \ge 0.$$

Proof. Step 1. We recall some known facts about the CSBP (Y_t). For each $\lambda \ge 0$, we denote by $t \mapsto v_t(\lambda)$ the unique positive solution of (2.10). Letting $\lambda \to \infty$ in (2.10), we have by monotonicity that $\bar{v}_t := \lim_{\lambda \to \infty} v_t(\lambda)$ exists in $(0, \infty]$ for all $t \ge 0$, and that

$$\mathbf{P}_x(Y_t = 0) = e^{-x\bar{v}_t}, \quad t \ge 0, x \ge 0. \tag{2.11}$$

It is known, see [28, Theorems 3.5–3.8] for example, that under Grey's condition $\bar{v} := \lim_{t\to\infty} \bar{v}_t \in [0,\infty)$ exists and is the largest root of ψ on $[0,\infty)$. Letting $t \to \infty$ in (2.11), we have by monotonicity that

$$\mathbf{P}_x(\exists t \geq 0, Y_t = 0) = e^{-x\bar{v}}, \quad x \geq 0.$$

Note the derivative of ψ, i.e.,

$$\psi'(z) = -\alpha + 2\rho z + \int_{(0,\infty)} (1 - e^{-zy}) y\pi(dy), \quad z \geq 0,$$

is non-decreasing. This says that ψ is a convex function. Also notice that $\psi'(0+) = -\alpha < 0$ and that there exists $z > 0$ such that $\psi(z) > 0$. Therefore we have (i) $\bar{v} > 0$; (ii) $\psi(z) < 0$ on $z \in (0, \bar{v})$; and (iii) $\psi(z) > 0$ on $z \in (\bar{v}, \infty)$. It is also known, see [28, Proposition 3.3] for example, that (i) if $\lambda \in (0, \bar{v})$, then $0 < \lambda \leq v_t(\lambda) < \bar{v}$; (ii) if $\lambda \in (\bar{v}, \infty)$, then $\bar{v} < v_t(\lambda) \leq \lambda < \infty$; and (iii) for each $\lambda \in (0,\infty) \setminus \{\bar{v}\}$ and $t \geq 0$, we always have $\int_{v_t(\lambda)}^{\lambda} \psi(z)^{-1} dz = t$. Taking $\lambda \to \infty$ and using the monotone convergence theorem, we have that

$$\int_{\bar{v}_t}^{\infty} \frac{dz}{\psi(z)} = t, \quad t \geq 0. \tag{2.12}$$

Step 2. We will show that, for each $x \geq 0$ there exists a constant $c_1 > 0$ such that

$$\mathbf{P}_x(0 < W_t \leq k_t) \leq c_1 \big(|\bar{v} - v_t(k_t^{-1} e^{-\alpha t})| + |\bar{v}_t - \bar{v}| \big), \quad t \geq 0.$$

In fact, for all $x \geq 0$ and $t \geq 0$, we have

$$\begin{aligned}
&\mathbf{P}_x(0 < W_t \leq k_t) = \mathbf{P}_x(e^{-k_t^{-1} W_t} \geq e^{-1}, W_t > 0) \\
&\leq e\mathbf{P}_x[e^{-k_t^{-1} W_t}; W_t > 0] = e\big(\mathbf{P}_x[e^{-k_t^{-1} W_t}] - \mathbf{P}_x(W_t = 0)\big) \\
&= e\big(e^{-x v_t(k_t^{-1} e^{-\alpha t})} - e^{-x\bar{v}_t}\big) \leq ex\big(|\bar{v} - v_t(k_t^{-1} e^{-\alpha t})| + |\bar{v}_t - \bar{v}|\big),
\end{aligned}$$

as desired in this step.

Step 3. We will show that there exist $c_2, \delta_1, t_0 > 0$ such that

$$|\bar{v}_t - \bar{v}| \leq c_2 e^{-\delta_1 t}, \quad t \geq t_0.$$

In fact, since ψ is a convex function, we must have $\tau := \psi'(\bar{v}) > 0$ and that $\psi(z) \geq (z - \bar{v})\tau$ for each $z \geq \bar{v}$. According to Grey's condition, we can find $z_0 > \bar{v}$ such that $t_0 := \int_{z_0}^{\infty} \psi(z)^{-1} dz < \infty$. For each $t > t_0$, according to (2.12), we have

$$\begin{aligned}
t - t_0 &= \int_{\bar{v}_t}^{\infty} \frac{dz}{\psi(z)} - \int_{z_0}^{\infty} \frac{dz}{\psi(z)} = \int_{\bar{v}_t}^{z_0} \frac{dz}{\psi(z)} \\
&\leq \int_{\bar{v}_t}^{z_0} \frac{dz}{(z - \bar{v})\tau} = \frac{1}{\tau}\big(\log(z_0 - \bar{v}) - \log(\bar{v}_t - \bar{v})\big).
\end{aligned}$$

Rearranging, we get $\bar{v}_t - \bar{v} \leq (z_0 - \bar{v}) e^{-\tau(t - t_0)}$, for all $t \geq t_0$. This implies the desired result in this step.

Step 4. We will show that there exist $c_3, \delta_2, t_1 > 0$ such that

$$|\bar{v} - v_t(k_t^{-1} e^{-\alpha t})| \leq c_3 k_t^{\delta_2}, \quad t \geq t_1.$$

Define $\rho_t := 1 + (\log k_t)/(t\alpha)$ for all $t \geq 0$. By the fact that $k_t^{-1} e^{-\alpha t} = e^{-\alpha \rho_t t}$ for all $t \geq 0$ and the condition that $k_t e^{\alpha t} \xrightarrow[t\to\infty]{} \infty$, we have $\rho_t t \xrightarrow[t\to\infty]{} \infty$. Since the $L \log L$ condition is satisfied, we have (see [29] for example), $W_t \xrightarrow[t\to\infty]{a.s.} W_\infty$, where the martingale limit W_∞ is a non-degenerate positive random variable. This implies that

$$v_t(e^{-\alpha t}) = -\log \mathbf{P}_1[e^{-W_t}] \xrightarrow[t\to\infty]{} -\log \mathbf{P}_1[e^{-W_\infty}] =: z^* \in (0,\infty).$$

The $L\log L$ condition also guarantees that (see again [29] for example) $\{W_\infty = 0\} = \{\exists t \geq 0, X_t = 0\}$ a.s. in $\mathbf{P}_1$. This and the non-degeneracy of W_∞ imply that

$$z^* = -\log \mathbf{P}_1[e^{-W_\infty}] < -\log \mathbf{P}_1(W_\infty = 0) = \bar{v}.$$

Fix an arbitrary $\epsilon \in (0,\tau)$. According to the fact that $\tau = \psi'(\bar{v}) > 0$, there exists $z_0 \in (0,\bar{v})$ such that for all $z \in (z_0,\bar{v})$, we have $-\psi(z) \geq (\bar{v}-z)(\tau-\epsilon)$. Fix this z_0. For t large enough, we have $0 < k_t^{-1}e^{-\alpha t} < v_t(k_t^{-1}e^{-\alpha t}) < \bar{v}$. Then we have for $t > 0$ large enough,

$$\begin{aligned} t &= \int_{k_t^{-1}e^{-\alpha t}}^{v_t(k_t^{-1}e^{-\alpha t})} \frac{dz}{-\psi(z)} = \Big(\int_{e^{-\alpha\rho_t t}}^{v_{\rho_t t}(e^{-\alpha\rho_t t})} + \int_{v_{\rho_t t}(e^{-\alpha\rho_t t})}^{z_0} + \int_{z_0}^{v_t(k_t^{-1}e^{-\alpha t})} \Big) \frac{dz}{-\psi(z)} \\ &= \rho_t t + O(1) + \int_{z_0}^{v_t(k_t^{-1}e^{-\alpha t})} \frac{dz}{-\psi(z)}, \end{aligned}$$

where we used the fact that

$$\int_{v_{\rho_t t}(e^{-\alpha\rho_t t})}^{z_0} \frac{dz}{-\psi(z)} \xrightarrow[t\to\infty]{} \int_{z^*}^{z_0} \frac{dz}{-\psi(z)}.$$

Now we have, for t large enough,

$$\begin{aligned} t &\leq \rho_t t + O(1) + \int_{z_0}^{v_t(k_t^{-1}e^{-\alpha t})} \frac{dz}{(\bar{v}-z)(\tau-\epsilon)} \\ &= \rho_t t + O(1) - \frac{1}{\tau-\epsilon}\Big(\log\big(\bar{v} - v_t(e^{-\alpha\rho_t t})\big) - \log(\bar{v}-z_0)\Big). \end{aligned}$$

Rearranging, we get, for t large enough,

$$e^{-t(\tau-\epsilon)} \geq e^{-\rho_t t(\tau-\epsilon)+O(1)}(\bar{v} - v_t(e^{-\alpha\rho_t t})).$$

Therefore, there exist $c_3 > 0$ and $t_1 > 0$ such that for all $t \geq t_1$,

$$\bar{v} - v_t(k_t^{-1}e^{-\alpha t}) \leq e^{-t(\tau-\epsilon)+(1+\frac{\log k_t}{t\alpha})t(\tau-\epsilon)+O(1)} \leq c_3 k_t^{\frac{\tau-\epsilon}{\alpha}}.$$

This implies the desired result in this step.

Finally, by Steps 2-4, we have for each $x \geq 0$, there exist $c_4, \delta_3, t_2 > 0$ such that

$$\mathbf{P}_x(0 < W_t \leq k_t) \leq c_4(k_t^{\delta_3} + e^{-\delta_3 t}), \quad t \geq t_2.$$

Note that the left hand side is always bounded from above by 1, so we can take $t_2 = 0$ in the above statement. □

2.7 Moments for super-OU processes

In this subsection, we want to find some upper bound for the $(1+\gamma)$-th moment of $X_t(g)$, where $\gamma \in (0,\beta)$ and $g \in \mathcal{P}$.

Lemma 2.11. *There is a* $(\theta^2 \vee \theta^{1+\beta})$*-controller* R *such that for all* $0 \leq t \leq 1$, $g \in \mathcal{P}$, $\lambda > 0$ *and* $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, *we have*

$$\mathbb{P}_\mu(|\mathcal{I}_0^t X_t(g)| > \lambda) \leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda} \mu(R|\theta g|)d\theta.$$

Proof. It is elementary calculus (see the proof of [12, Theorem 3.3.6] for example) that for $u > 0$ and $x \neq 0$,

$$\frac{1}{u}\int_{-u}^{u}(1-e^{i\theta x})\,d\theta = 2-\frac{2\sin ux}{ux} \geq \mathbf{1}_{ux>2}.$$

Denote by R the $(\theta^2 \vee \theta^{1+\beta})$-controller in Lemma 2.5.(4). Then, using Lemma A.1 we get

$$\begin{aligned}
&|\mathbb{P}_\mu(|\mathcal{I}_0^t X_t(g)| > \lambda)| \leq \Big|\frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}(1-\mathbb{P}_\mu[e^{i\theta \mathcal{I}_0^t X_t(g)}])d\theta\Big| \\
&\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}|1-e^{\mu(U_t(\theta g)-iP_t^\alpha(\theta g))}|d\theta \leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\mu(|U_t(\theta g)-iP_t^\alpha(\theta g)|)d\theta \\
&\leq \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\mu(R|\theta g|)d\theta.
\end{aligned}$$

□

Lemma 2.12. *For all $h \in \mathcal{P}^+$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, there exists $C > 0$ such that for all $\kappa \in \mathbb{Z}_+$, $\lambda > 0$ and $0 \leq r \leq s \leq t < \infty$ with $s - r \leq 1$, we have*

$$\sup_{g\in\mathcal{P}:Q_\kappa g\leq h}\mathbb{P}_\mu(|\mathcal{I}_r^s X_t(g)| > \lambda) \leq Ce^{\alpha r}\Big(\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{1+\beta}+\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{2}\Big).$$

Proof. Denote by R the $(\theta^2 \vee \theta^{1+\beta})$-controller in Lemma 2.11. Fix $h \in \mathcal{P}^+$, $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ $\kappa \in \mathbb{Z}_+$ and $0 \leq r \leq s \leq t < \infty$ with $s - r \leq 1$. Suppose that $g \in \mathcal{P}$ satisfies $Q_\kappa g \leq h$. Using the Markov property of X, we get

$$\begin{aligned}
&\mathbb{P}_\mu(|\mathcal{I}_r^s X_t(g)| > \lambda) = \mathbb{P}_\mu\Big[\mathbb{P}_\mu[|X_s(P_{t-s}^\alpha g)-X_r(P_{t-r}^\alpha g)| > \lambda|\mathscr{F}_r]\Big] \\
&= \mathbb{P}_\mu\big[\mathbb{P}_{X_r}(|X_{s-r}(P_{t-s}^\alpha g)-X_0(P_{t-r}^\alpha g)| > \lambda)\big] \\
&= \mathbb{P}_\mu\big[\mathbb{P}_{X_r}(|\mathcal{I}_0^{s-r}X_{s-r}(P_{t-s}^\alpha g)| > \lambda)\big] \leq \mathbb{P}_\mu\Big[\frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}X_r(R|\theta P_{t-s}^\alpha g|)d\theta\Big] \\
&\leq \mathbb{P}_\mu\Big[\frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}X_r(R|\theta e^{(t-s)(\alpha-\kappa b)}h|)d\theta\Big] \\
&\leq \mathbb{P}_\mu[X_r(Rh)]\frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}(|\theta e^{(t-s)(\alpha-\kappa b)}|^{1+\beta}+|\theta e^{(t-s)(\alpha-\kappa b)}|^2)d\theta \\
&= \mu(P_r^\alpha Rh)\Big(\frac{2^{2+\beta}}{2+\beta}\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{1+\beta}+\frac{2^3}{3}\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{2}\Big) \\
&\leq Ce^{\alpha r}\Big(\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{1+\beta}+\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{2}\Big),
\end{aligned}$$

where $C := \Big(\frac{2^{2+\beta}}{2+\beta}+\frac{2^3}{3}\Big)\mu(Q_0 Rh) > 0$. □

For each random variable $\{Y;\mathbb{P}\}$ and $p \in [1,\infty)$, we write $\|Y\|_{\mathbb{P};p} := \mathbb{P}[|Y|^p]^{1/p}$. Recall that we write $\tilde{u} = \frac{u}{1+u}$ for each $u \neq -1$.

Lemma 2.13. *For all $h \in \mathcal{P}$, $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and $\gamma \in (0,\beta)$, there exists $C > 0$ such that for all $\kappa \in \mathbb{Z}_+$ and $0 \leq r \leq s \leq t < \infty$ with $s - r \leq 1$, we have*

$$\sup_{g\in\mathcal{P}:Q_\kappa g\leq h}\|\mathcal{I}_r^s X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \leq Ce^{t\alpha(1-\tilde{\gamma})+(t-s)(\alpha\tilde{\gamma}-\kappa b)}.$$

Proof. Fix $h \in \mathcal{P}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. Let C_0 be the constant in the Lemma 2.12. For all $\kappa \in \mathbb{Z}_+$, $0 \le r \le s \le t$ with $s - r \le 1$, $g \in \mathcal{P}$ with $Q_\kappa g \le h$, and $c > 0$, we have

$$\begin{aligned}
&\mathbb{P}_\mu[|\mathcal{I}_r^s X_t(g)|^{1+\gamma}] = (1+\gamma)\int_0^\infty \lambda^\gamma \mathbb{P}_\mu(|\mathcal{I}_r^s X_t(g)| > \lambda) d\lambda \\
&\le (1+\gamma)\int_0^c \lambda^\gamma d\lambda + (1+\gamma)\int_c^\infty \lambda^\gamma \mathbb{P}_\mu(|\mathcal{I}_r^s X_t(g)| > \lambda) d\lambda \\
&\le c^{1+\gamma} + C_0 e^{\alpha r}(1+\gamma)\int_c^\infty \Big(\Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^{1+\beta} + \Big(\frac{e^{(t-s)(\alpha-\kappa b)}}{\lambda}\Big)^2\Big)\lambda^\gamma d\lambda \\
&\le c^{1+\gamma} e^{\alpha r} + C_0 e^{\alpha r}(1+\gamma)\Big(\frac{e^{(1+\beta)(t-s)(\alpha-\kappa b)}}{(\beta-\gamma)c^{\beta-\gamma}} + \frac{e^{2(t-s)(\alpha-\kappa b)}}{(1-\gamma)c^{1-\gamma}}\Big).
\end{aligned}$$

Taking $c = e^{(t-s)(\alpha-\kappa b)}$, we get

$$\mathbb{P}_\mu\big[|\mathcal{I}_r^s X_t(g)|^{1+\gamma}\big] \le e^{(1+\gamma)(t-s)(\alpha-\kappa b)} e^{\alpha r}\Big(1 + C_0\frac{1+\gamma}{\beta-\gamma} + C_0\frac{1+\gamma}{1-\gamma}\Big).$$

Note that

$$\begin{aligned}
&(1+\gamma)(t-s)(\alpha-\kappa b) + \alpha r = (t-s)\alpha + (t-s)(\gamma\alpha - (1+\gamma)\kappa b) + \alpha r \\
&\le t\alpha + (t-s)(\gamma\alpha - (1+\gamma)\kappa b).
\end{aligned}$$

So the desired result is true. □

Lemma 2.14. *For all $h \in \mathcal{P}$, $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $\gamma \in (0,\beta)$ and $\kappa \in \mathbb{Z}_+$, there exists a constant $C > 0$ such that for all $t \ge 0$, we have*

1. $\sup_{g\in\mathcal{P}:Q_\kappa g \le h} \|X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \le Ce^{(\alpha-\kappa b)t}$ *provided $\alpha\tilde{\gamma} > \kappa b$;*
2. $\sup_{g\in\mathcal{P}:Q_\kappa g \le h} \|X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \le Cte^{\frac{\alpha}{1+\gamma}t}$ *provided $\alpha\tilde{\gamma} = \kappa b$;*
3. $\sup_{g\in\mathcal{P}:Q_\kappa g \le h} \|X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \le Ce^{\frac{\alpha}{1+\gamma}t}$ *provided $\alpha\tilde{\gamma} < \kappa b$.*

Proof. Fix $\gamma \in (0,\beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. Let C be the constant in Lemma 2.13. Using the triangle inequality, for all $\kappa \in \mathbb{Z}_+$, $g \in \mathcal{P}$ with $Q_\kappa g \le h$ and $t \ge 0$, we have

$$\begin{aligned}
&\|X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \le \sum_{l=0}^{\lfloor t\rfloor -1} \big\|\mathcal{I}_{t-l-1}^{t-l} X_t(g)\big\|_{\mathbb{P}_\mu;1+\gamma} + \big\|\mathcal{I}_0^{t-\lfloor t\rfloor} X_t(g)\big\|_{\mathbb{P}_\mu;1+\gamma} + |\mu(P_t^\alpha g)| \\
&\le C^{\frac{1}{1+\gamma}} e^{\frac{\alpha}{1+\gamma}t} \sum_{l=0}^{\lfloor t\rfloor} e^{\frac{\gamma\alpha-\kappa(1+\gamma)b}{1+\gamma}l} + e^{(\alpha-\kappa b)t}\mu(h).
\end{aligned}$$

By calculating the sum on the right, we get the desired result. □

3 Proofs of main results

In this section, we will prove the main results of this paper. For simplicity, we will write $\widetilde{\mathbb{P}}_\mu = \mathbb{P}_\mu(\cdot|D^c)$ in this section.

3.1 Law of large numbers

In this subsection, we prove Theorem 1.5. For this purpose, we first prove the almost sure and $L^{1+\gamma}(\mathbb{P}_\mu)$ convergence of a family of martingales for $\gamma \in (0,\beta)$. Recall that L is the infinitesimal generator of the OU-process. For $f \in \mathcal{P} \cap C^2(\mathbb{R}^d)$ and $a \in \mathbb{R}$, we define

$$M_t^{f,a} := e^{-(\alpha-ab)t} X_t(f) - \int_0^t e^{-(\alpha-ab)s} X_s((L+ab)f)\, ds. \tag{3.1}$$

Let $(\mathscr{F}_t)_{t\ge0}$ be the natural filtration of X. The following lemma says that $\{M_t^{f,a} : t \ge 0\}$ is a martingale with respect to $(\mathscr{F}_t)_{t\ge0}$.

Lemma 3.1. *For all $f \in \mathcal{P} \cap C^2(\mathbb{R}^d)$, $a \in \mathbb{R}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, the process $(M_t^{f,a})_{t\geq 0}$ is a $\mathbb{P}_\mu$-martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$.*

Proof. Put $\bar{f} := (L + ab)f$. It follows easily from Ito's formula that

$$P_t^{ab} f(x) = f(x) + \int_0^t P_s^{ab} \bar{f}(x)\, ds, \quad t \geq 0, x \in \mathbb{R}^d, \tag{3.2}$$

where $P_t^{ab} := e^{abt} P_t$. For $0 \leq s \leq t$, we have

$$\mathbb{P}_\mu[M_t^{f,a} | \mathscr{F}_s] = e^{-(\alpha - ab)t} \mathbb{P}_\mu \left[X_t(f) | \mathscr{F}_s\right] - \mathbb{P}_\mu\Big[\int_0^t e^{-(\alpha - ab)u} X_u(\bar{f})\, du \Big| \mathscr{F}_s\Big] \tag{3.3}$$
$$= e^{-(\alpha - ab)t} X_s(P_{t-s}^\alpha f) - \int_0^s e^{-(\alpha - ab)u} X_u(\bar{f})\, du - \int_s^t e^{-(\alpha - ab)u} X_s(P_{u-s}^\alpha \bar{f})\, du.$$

Using (3.2) and Fubini's theorem, we have

$$\begin{aligned}
&\int_s^t e^{-(\alpha - ab)u} X_s(P_{u-s}^\alpha \bar{f})\, du = e^{-(\alpha - ab)s} \int_s^t X_s(P_{u-s}^{ab} \bar{f})\, du \\
&= e^{-(\alpha - ab)s} X_s \left(\int_0^{t-s} P_u^{ab} \bar{f}\, du\right) = e^{-(\alpha - ab)s} \left(X_s(P_{t-s}^{ab} f) - X_s(f)\right) \\
&= e^{-(\alpha - ab)t} X_s(P_{t-s}^\alpha f) - e^{-(\alpha - ab)s} X_s(f).
\end{aligned}$$

Using this and (3.3), we get the desired result. ☐

Recall that, for $p \in \mathbb{Z}_+^d$, ϕ_p is an eigenfunction of L corresponding to the eigenvalue $-|p|b$ and $H_t^p = e^{-(\alpha - |p|b)t} X_t(\phi_p)$ for each $t \geq 0$.

Lemma 3.2. *For all $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and $p \in \mathbb{Z}_+^d$, $(H_t^p)_{t\geq 0}$ is a $\mathbb{P}_\mu$-martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$. Moreover if $\alpha\tilde{\beta} > |p|b$, the martingale is bounded in $L^{1+\gamma}(\mathbb{P}_\mu)$ for each $\gamma \in (0, \beta)$. Thus the limit $H_\infty^p := \lim_{t\to\infty} H_t^p$ exists $\mathbb{P}_\mu$-a.s. and in $L^{1+\gamma}(\mathbb{P}_\mu)$ for each $\gamma \in (0, \beta)$.*

Proof. Fix a $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and a $p \in \mathbb{Z}_+^d$. It follows from Lemma 3.1 that $(H_t^p)_{t\geq 0}$ is a $\mathbb{P}_\mu$-martingale. Further suppose that $\alpha\tilde{\beta} > |p|b$. Then there exists a $\gamma_0 \in (0, \beta)$ which is close enough to β so that $\alpha\tilde{\gamma} > |p|b$ for each $\gamma \in [\gamma_0, \beta)$. Using Lemma 2.14 and the fact $\kappa_{\phi_p} = |p|$, we get that, for each $\gamma \in [\gamma_0, \beta)$, there exists a constant $C > 0$ such that

$$\|H_t^p\|_{\mathbb{P}_\mu; 1+\gamma} \leq C e^{-(\alpha - |p|b)t} e^{(\alpha - |p|b)t} = C, \quad t \geq 0.$$

For each $\gamma \in (0, \gamma_0)$ there exists a constant $C' > 0$ such that

$$\|H_t^p\|_{\mathbb{P}_\mu; 1+\gamma} \leq \|H_t^p\|_{\mathbb{P}_\mu; 1+\gamma_0} \leq C', \quad t \geq 0.$$

Therefore, for each $\gamma \in (0, \beta)$, the martingale $(H_t^p)_{t\geq 0}$ is bounded in $L^{1+\gamma}(\mathbb{P}_\mu)$. ☐

Lemma 3.3. *Suppose that $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and that $p \in \mathbb{Z}_+^d$ satisfies $\alpha\tilde{\beta} > |p|b$. Then for each $\gamma \in (0, \beta)$ satisfying $\alpha\tilde{\gamma} > |p|b$, there exists a constant $C > 0$ such that,*

$$\|H_t^p - H_s^p\|_{\mathbb{P}_\mu; 1+\gamma} \leq C e^{-(\alpha\tilde{\gamma} - |p|b)s}, \quad 0 \leq s < t \leq \infty.$$

Proof. Thanks to Lemma 3.2, we only need to prove the inequality when $0 \leq s < t < \infty$. Suppose $p \in \mathbb{Z}_+^d$, $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and $\gamma \in (0, \beta)$ with $\alpha\tilde{\gamma} > |p|b$ are fixed. Using Lemma 2.13 with $g = \phi_p$ and $k = |p|$, we know that there exists a constant $C_1 > 0$ such that for all $0 \leq r \leq s$ with $s - r \leq 1$,

$$\|H_s^p - H_r^p\|_{\mathbb{P}_\mu; 1+\gamma} \leq C_1 e^{-(\alpha\tilde{\gamma} - |p|b)s}.$$

Thus there exists $C_2 > 0$ such that for all $0 \le s < t$,

$$\begin{aligned}&\|H_t^p - H_s^p\|_{\mathbb{P}_\mu;1+\gamma}\\ &\le \|H_{\lfloor s\rfloor+1}^p - H_s^p\|_{\mathbb{P}_\mu;1+\gamma} + \sum_{k=\lfloor s\rfloor+1}^{\lfloor t\rfloor} \|H_{k+1}^p - H_k^p\|_{\mathbb{P}_\mu;1+\gamma} + \|H_t^p - H_{\lfloor t\rfloor+1}^p\|_{\mathbb{P}_\mu;1+\gamma}\\ &\le C_1\Big(e^{-(\alpha\tilde{\gamma}-|p|b)s} + \sum_{k=\lfloor s\rfloor+1}^{\lfloor t\rfloor} e^{-(\alpha\tilde{\gamma}-|p|b)k} + e^{-(\alpha\tilde{\gamma}-|p|b)t}\Big) \le C_2 e^{-(\alpha\tilde{\gamma}-|p|b)s}.\end{aligned}$$

□

Proof of Theorem 1.5. Fix $f \in \mathcal{P}$ such that $\alpha\beta > \kappa_f b(1+\beta)$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$. Write

$$f = \sum_{p\in\mathbb{Z}_+^d:|p|\ge\kappa_f} \langle f, \phi_p\rangle_\varphi \phi_p =: \sum_{p\in\mathbb{Z}_+^d:|p|=\kappa_f} \langle f, \phi_p\rangle_\varphi \phi_p + \widetilde{f}.$$

Then

$$e^{-(\alpha-\kappa_f b)t} X_t(f) = \sum_{p\in\mathbb{Z}_+^d:|p|=\kappa_f} \langle f, \phi_p\rangle_\varphi H_t^p + e^{-(\alpha-\kappa_f b)t} X_t(\widetilde{f}), \quad t \ge 0.$$

According to Lemma 3.2, we have

$$\sum_{p\in\mathbb{Z}_+^d:|p|=\kappa_f} \langle f, \phi_p\rangle_\varphi H_t^p \xrightarrow[t\to\infty]{} \sum_{p\in\mathbb{Z}_+^d:|p|=\kappa_f} \langle f, \phi_p\rangle_\varphi H_\infty^p,$$

$\mathbb{P}_\mu$-a.s. and in $L^{1+\gamma}(\mathbb{P}_\mu)$ for each $\gamma \in (0,\beta)$. Therefore, it suffices to show that

$$J_t := e^{-(\alpha-\kappa_f b)t} X_t(\widetilde{f}), \quad t \ge 0,$$

converges to 0 in $L^{1+\gamma}(\mathbb{P}_\mu)$ for all $\gamma \in (0,\beta)$, and converges almost surely provided f is twice differentiable and all its second order partial derivatives are in $\mathcal{P}$.

Step 1. Let $g \in \mathcal{P}$. Let $\kappa > 0$ be such that $\kappa < \kappa_g$ and $\kappa b < \alpha\tilde{\beta}$. We will show that for each $\gamma \in (0,\beta)$ there exist $C_1, \delta_1 > 0$ such that

$$\|e^{-(\alpha-\kappa b)t} X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \le C_1 e^{-\delta_1 t}, \quad t \ge 0.$$

In order to do this, we choose a $\gamma_0 \in (0,\beta)$ close enough to β such that $\kappa b < \alpha\tilde{\gamma}$ for each $\gamma \in [\gamma_0,\beta)$. According to Lemma 2.14, we have for each $\gamma \in (0,\beta)$,

1. if $\gamma \in [\gamma_0,\beta)$ and $\alpha\tilde{\gamma} > \kappa_g b$, then there exists $C_2 > 0$ such that
$$\|e^{-(\alpha-\kappa b)t} X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \le C_2 e^{-(\alpha-\kappa b)t} e^{(\alpha-\kappa_g b)t} \le C_2 e^{-(\kappa_g-\kappa)bt}, \quad t \ge 0;$$
2. if $\gamma \in [\gamma_0,\beta)$ and $\alpha\tilde{\gamma} = \kappa_g b$, then there exists $C_3 > 0$ such that
$$\|e^{-(\alpha-\kappa b)t} X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \le C_3 t e^{-(\alpha-\kappa b)t} e^{\frac{\alpha}{1+\gamma}t} = C_3 t e^{-(\alpha\tilde{\gamma}-\kappa b)t}, \quad t \ge 0;$$
3. if $\gamma \in [\gamma_0,\beta)$ and $\alpha\tilde{\gamma} < \kappa_g b$, then there exists $C_4 > 0$ such that
$$\|e^{-(\alpha-\kappa b)t} X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \le C_4 e^{-(\alpha-\kappa b)t} e^{\frac{\alpha}{1+\gamma}t} = C_4 e^{-(\alpha\tilde{\gamma}-\kappa b)t}, \quad t \ge 0;$$
4. if $\gamma \in (0,\gamma_0)$, then thanks to (1)–(3) above and the fact that
$$\|e^{-(\alpha-\kappa b)t} X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \le \|e^{-(\alpha-\kappa b)t} X_t(g)\|_{\mathbb{P}_\mu;1+\gamma_0},$$
there exist $C_5, \delta_2 > 0$ such that
$$\|e^{-(\alpha-\kappa b)t} X_t(g)\|_{\mathbb{P}_\mu;1+\gamma} \le C_5 e^{-\delta_2 t}, \quad t \ge 0.$$

Thus, the desired conclusion in this step is valid. In particular, by taking $g = \widetilde{f}$ and $\kappa = \kappa_f$, we get that J_t converges to 0 in $L^{1+\gamma}(\mathbb{P}_\mu)$ for any $\gamma \in (0, \beta)$.

Step 2. We further assume that $f \in C^2(\mathbb{R}^d)$ and $D^2 f \in \mathcal{P}$. We will show that J_t converges to 0 almost surely. For $a \geq 0$, $t \geq 0$, and $g \in \mathcal{P} \cap C^2(\mathbb{R}^d)$ satisfying $D^2 g \in \mathcal{P}$, we define

$$L_t^{g,a} := \int_0^t e^{-(\alpha - ab)s} X_s((L + ab)g)ds, \quad Y_t^{g,a} := \int_0^t e^{-(\alpha - ab)s} |X_s((L + ab)g)|ds.$$

Now choose $a_0 \in (\kappa_f, \kappa_f + 1)$ close enough to κ_f so that $a_0 b < \alpha\tilde{\beta}$. According to (3.1),

$$J_t = e^{-(a_0 - \kappa_f)bt}(M_t^{\widetilde{f},a_0} + L_t^{\widetilde{f},a_0}), \quad t \geq 0.$$

So we only need to show that

$$e^{-(a_0-\kappa_f)bt} M_t^{\widetilde{f},a_0} \xrightarrow[t\to\infty]{} 0, \quad e^{-(a_0-\kappa_f)bt} L_t^{\widetilde{f},a_0} \xrightarrow[t\to\infty]{} 0, \quad \mathbb{P}_\mu\text{-a.s.}$$

Notice that $\kappa_{(L+a_0 b)\widetilde{f}} \geq \kappa_{\widetilde{f}} \geq \kappa_f + 1 > a_0$. By Step 1, for any fixed $\gamma \in (0, \beta)$, there exist $C_6, \delta_3 > 0$ such that for each $t \geq 0$,

$$\|e^{-(\alpha - a_0 b)t} X_t(\widetilde{f})\|_{\mathbb{P}_\mu;1+\gamma} \leq C_6 e^{-\delta_3 t}, \quad \|e^{-(\alpha - a_0 b)t} X_t(L\widetilde{f} + a_0 b\widetilde{f})\|_{\mathbb{P}_\mu;1+\gamma} \leq C_6 e^{-\delta_3 t}.$$

Now, by the triangle inequality, for each $t \geq 0$,

$$\begin{aligned}
&\|L_t^{\widetilde{f},a_0}\|_{\mathbb{P}_\mu;1+\gamma} \leq \|Y_t^{\widetilde{f},a_0}\|_{\mathbb{P}_\mu;1+\gamma} \\
&\leq \int_0^t \|e^{-(\alpha - a_0 b)s} X_s(L\widetilde{f} + a_0 b\widetilde{f})\|_{\mathbb{P}_\mu;1+\gamma} ds \leq C_6 \int_0^t e^{-\delta_3 s} ds \leq \frac{C_6}{\delta_3}.
\end{aligned}$$

Since $Y_t^{\widetilde{f},a_0}$ is increasing in t, it converges to some finite random variable $Y_\infty^{\widetilde{f},a_0}$ almost surely and in $L^{1+\gamma}(\mathbb{P}_\mu)$. Consequently, we have

$$\lim_{t\to\infty} e^{-(a_0-\kappa_f)bt} |L_t^{\widetilde{f},a_0}| \leq \lim_{t\to\infty} e^{-(a_0-\kappa_f)bt} |Y_t^{\widetilde{f},a_0}| = 0, \quad \mathbb{P}_\mu\text{-a.s.}$$

On the other hand, the martingale $M_t^{\widetilde{f},a_0}$ satisfies

$$\|M_t^{\widetilde{f},a_0}\|_{\mathbb{P}_\mu;1+\gamma} \leq \|e^{-(\alpha - a_0 b)t} X_t(\widetilde{f})\|_{\mathbb{P}_\mu;1+\gamma} + \|L_t^{\widetilde{f},a_0}\|_{\mathbb{P}_\mu;1+\gamma} \leq C_6(e^{-\delta_3 t} + \frac{1}{\delta_3}), \quad t \geq 0.$$

This implies that the martingale converges almost surely. Consequently,

$$\lim_{t\to\infty} e^{-(a_0-\kappa_f)bt} M_t^{\widetilde{f},a_0} = 0, \quad \mathbb{P}_\mu\text{-a.s.}. \qquad \square$$

3.2 Central limit theorems for unit time intervals

In this subsection, we will establish the following CLT.

Theorem 3.4. *If $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and $f \in \mathcal{P} \setminus \{0\}$, then under $\mathbb{P}_\mu(\cdot|D^c)$, we have*

$$\Upsilon_t^f := \frac{X_{t+1}(f) - X_t(P_1^\alpha f)}{\|X_t\|^{1-\tilde{\beta}}} \xrightarrow[t\to\infty]{d} \zeta_0^f,$$

where ζ_0^f is a $(1+\beta)$-stable random variable with characteristic function $\theta \mapsto e^{\langle Z_1(\theta f), \varphi\rangle}$.

In fact, we prove a stronger result:

Proposition 3.5. *For all $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and $g \in \mathcal{P} \setminus \{0\}$, there exist $C, \delta > 0$ such that for all $t \geq 1$ and $f \in \mathcal{P}_g := \{\theta T_n g : n \in \mathbb{Z}_+, \theta \in [-1,1]\}$, we have*

$$\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\Upsilon_t^f} - e^{\langle Z_1 f, \varphi\rangle}; D^c|\mathscr{F}_t]\big|\Big] \leq Ce^{-\delta t}.$$

Proof. Fix $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and $g \in \mathcal{P} \setminus \{0\}$.

Step 1. Write $A_t(\epsilon) := \{\|X_t\| \geq e^{(\alpha-\epsilon)t}\}$ for $t \geq 0$ and $\epsilon > 0$. We will show that for all $f \in \mathcal{P} \setminus \{0\}$, $\epsilon > 0$ and $t \geq 0$, it holds that

$$\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\Upsilon_t^f} - e^{\langle Z_1(f), \varphi\rangle}; D^c|\mathscr{F}_t]\big|\Big] \leq J_1^f(t,\epsilon) + J_2^f(t,\epsilon) + J_3^f(t,\epsilon),$$

where

$$J_1^f(t,\epsilon) := \mathbb{P}_\mu[|X_t(Z_1'''(\theta_t f))|; A_t(\epsilon)], \quad J_2^f(t,\epsilon) := \mathbb{P}_\mu[|X_t(Z_1(\theta_t f)) - \langle Z_1 f, \varphi\rangle|; A_t(\epsilon)],$$
$$J_3(t,\epsilon) := 2\mathbb{P}_\mu(A_t(\epsilon)\Delta D^c), \quad \theta_t := \|X_t\|^{-(1-\tilde{\beta})}.$$

In fact, it follows from (2.2), the definitions of U_1, Z_1''' and Z_1, that for all $t \geq 0$,

$$\begin{aligned}\mathbb{P}_\mu[e^{i\Upsilon_t^f}|\mathscr{F}_t] &= \mathbb{P}_\mu[\exp\{i\theta_t X_{t+1}(f) - i\theta_t X_t(P_1^\alpha f)\}|\mathscr{F}_t] \\ &= \exp\{X_t((U_1 - iP_1^\alpha)(\theta_t f))\} = \exp\{X_t((Z_1 + Z_1''')(\theta_t f))\}.\end{aligned} \tag{3.4}$$

From Lemma 2.8, we get that $\theta \mapsto \langle Z_1(\theta f), \varphi\rangle$ is the characteristic function of some $(1+\beta)$-stable random variable, and then $\mathrm{Re}\langle Z_1 f, \varphi\rangle \leq 0$. Using this, (3.4), (A.8) and the fact $|e^{-x} - e^{-y}| \leq |x-y|$ for all $x, y \in \mathbb{C}_+$, we get for each $t \geq 0$ and $\epsilon > 0$,

$$\begin{aligned}&\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\Upsilon_t^f} - e^{\langle Z_1 f, \varphi\rangle}; D^c|\mathscr{F}_t]\big|\Big] \\ &\leq \mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\Upsilon_t^f} - e^{\langle Z_1 f, \varphi\rangle}; A_t(\epsilon)|\mathscr{F}_t]\big| + 2\mathbb{P}_\mu(A_t(\epsilon)\Delta D^c|\mathscr{F}_t)\Big] \\ &= \mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\Upsilon_t^f}|\mathscr{F}_t] - e^{\langle Z_1 f, \varphi\rangle}\big|; A_t(\epsilon)\Big] + J_3(t,\epsilon) \\ &\leq \mathbb{P}_\mu\Big[\big|e^{X_t((Z_1+Z_1''')(\theta_t f))} - e^{\langle Z_1 f, \varphi\rangle}\big|; A_t(\epsilon)\Big] + J_3(t,\epsilon) \\ &\leq \mathbb{P}_\mu\Big[\big|X_t((Z_1 + Z_1''')(\theta_t f)) - \langle Z_1 f, \varphi\rangle\big|; A_t(\epsilon)\Big] + J_3(t,\epsilon) \\ &\leq J_1^f(t,\epsilon) + J_2^f(t,\epsilon) + J_3(t,\epsilon).\end{aligned}$$

Step 2. We will show that for $\epsilon > 0$ small enough, there exist $C_2, \delta_2 > 0$ such that for all $t \geq 1$ and $f \in \mathcal{P}_g$, we have $J_1^f(t,\epsilon) \leq C_2 e^{-\delta_2 t}$.

In fact, let $\delta_0 > 0$ be the constant in Lemma 2.5.(7) and let R be the corresponding $(\theta^{2+\beta} \vee \theta^{1+\beta+\delta_0})$-controller. According to Step 1 in the proof of Lemma 2.6, there exists $h_2 \in \mathcal{P}^+$ such that for each $f \in \mathcal{P}_g$ it holds that $|f| \leq h_2$. Then, we have for all $t \geq 0$, $\epsilon > 0$ and $f \in \mathcal{P}_g$,

$$|Z_1'''(\theta_t f)|\mathbf{1}_{A_t(\epsilon)} \leq R(|\theta_t f|)\mathbf{1}_{A_t(\epsilon)} \leq R\Big(\frac{h_2}{e^{(\alpha-\epsilon)t(1-\tilde{\beta})}}\Big) \leq \sum_{\rho \in \{\delta_0, 1\}} e^{-\frac{1+\beta+\rho}{1+\beta}(\alpha-\epsilon)t} Rh_2.$$

Thus for all $t \geq 0$, $\epsilon > 0$ and $f \in \mathcal{P}_g$,

$$J_1^f(t,\epsilon) \leq \sum_{\rho \in \{\delta_0,1\}} e^{-\frac{1+\beta+\rho}{1+\beta}(\alpha-\epsilon)t}\mathbb{P}_\mu[X_t(Rh_2)] \leq \sum_{\rho \in \{\delta_0,1\}} \mu(Q_0 Rh_2) e^{-(\alpha\frac{\rho}{1+\beta} - \epsilon\frac{1+\beta+\rho}{1+\beta})t},$$

where Q_0 is defined by (2.3). By taking $\epsilon > 0$ small enough, we get the desired result in this step.

Step 3. We will show that for $\epsilon > 0$ small enough there exist $C_3, \delta_3 > 0$ such that for all $t \geq 0$ and $f \in \mathcal{P}_g$, we have $J_2^f(t,\epsilon) \leq C_3 e^{-\delta_3 t}$. In fact, for all $t \geq 0$, and $f \in \mathcal{P}_g$,

$$X_t(Z_1(\theta_t f)) - \langle Z_1 f, \varphi\rangle = \theta_t^{1+\beta} X_t(Z_1 f) - \langle Z_1 f, \varphi\rangle = \frac{1}{\|X_t\|} X_t(Z_1 f - \langle Z_1 f, \varphi\rangle),$$

and therefore,

$$J_2^f(t,\epsilon) = \mathbb{P}_\mu\Big[\Big|\frac{1}{\|X_t\|} X_t(Z_1 f - \langle Z_1 f, \varphi\rangle)\Big|; A_t(\epsilon)\Big] \leq e^{-(\alpha-\epsilon)t}\mathbb{P}_\mu[|X_t(q_f)|],$$

where $q_f = Z_1 f - \langle Z_1 f, \varphi\rangle \in \mathcal{P}^*$. It follows from Lemma 2.9 that there exists $h_3 \in \mathcal{P}$ such that for each $f \in \mathcal{P}_g$, we have $Q_1(\operatorname{Re} q_f) \leq h_3$ and $Q_1(\operatorname{Im} q_f) \leq h_3$, where Q_1 is given by (2.3) with $\kappa = 1$. In the rest of this step, we fix a $\gamma \in (0,\beta)$ small enough such that $\alpha\gamma < b < (1+\gamma)b$. According to Lemma 2.14.(3) (with $\kappa = 1$), there exists $C_3 > 0$ such that for all $t \geq 0$ and $f \in \mathcal{P}_g$,

$$\begin{aligned}&\mathbb{P}_\mu\left[|X_t(q_f)|\right] \leq \|X_t(\operatorname{Re} q_f)\|_{\mathbb{P}_\mu,1+\gamma} + \|X_t(\operatorname{Im} q_f)\|_{\mathbb{P}_\mu,1+\gamma}\\ &\leq 2 \sup_{q\in\mathcal{P}:Q_1 q \leq h_3} \|X_t(q)\|_{\mathbb{P}_\mu;1+\gamma} \leq C_3 e^{\frac{\alpha t}{1+\gamma}}.\end{aligned}$$

Therefore, for all $t \geq 0, \epsilon > 0$ and $f \in \mathcal{P}_g$, we have

$$J_2^f(t,\epsilon) \leq C_3 e^{-(\alpha-\epsilon)t} e^{\frac{\alpha t}{1+\gamma}} \leq C_3 e^{-(\alpha\tilde{\gamma}-\epsilon)t}.$$

By taking $\epsilon > 0$ small enough, we get the required result in this step.

Step 4. We will show that, for each $\epsilon \in (0,\alpha)$, there exist $C_4, \delta_4 > 0$ such that for all $t \geq 1$, $J_3(t,\epsilon) \leq C_4 e^{-\delta_4 t}$. In fact, we have for all $t \geq 0, \epsilon > 0$,

$$\mathbb{P}_\mu(A_t(\epsilon), D) = \mathbb{P}_\mu[\mathbb{P}_\mu(D|\mathscr{F}_t); A_t(\epsilon)] = \mathbb{P}_\mu[e^{-\bar{v}\|X_t\|}; A_t(\epsilon)] \leq \exp(-\bar{v}\|\mu\| e^{(\alpha-\epsilon)t}).$$

On the other hand, by Proposition 2.10, for each $\epsilon \in (0,\alpha)$, there exist $C_4, \delta_4 > 0$ such that for all $t \geq 0$,

$$\mathbb{P}_\mu(A_t(\epsilon)^c, D^c) \leq \mathbb{P}_\mu(0 < e^{-\alpha t}\|X_t\| \leq e^{-\epsilon t}) \leq C_4(e^{-\epsilon\delta_4 t} + e^{-\delta_4 t}).$$

Combining these results, we get the desired result in this step.

Step 5. Combining the results in Steps 1–4, we immediately get the desired result. □

The following corollary will be used later in the proof of Theorem 1.6.

Corollary 3.6. *If $g \in \mathcal{P}\setminus\{0\}$ and $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, then there exist $C, \delta > 0$ such that for all $l \leq n$ in $\mathbb{Z}_+$ and $(f_j)_{j=l}^n \subset \mathcal{P}_g$,*

$$\Big|\widetilde{\mathbb{P}}_\mu\Big[\prod_{k=l}^n e^{i\Upsilon_k^{f_k}} - \prod_{k=l}^n e^{\langle Z_1 f_k, \varphi\rangle}\Big]\Big| \leq C e^{-\delta l}. \tag{3.5}$$

Proof. For $l \leq n$ in $\mathbb{Z}_+$, $k \in \{l,\dots,n\}$ and $(f_j)_{j=l}^n \subset \mathcal{P}_g$, define

$$a_k := \widetilde{\mathbb{P}}_\mu\Big[\prod_{j=l}^k e^{i\Upsilon_j^{f_j}}\Big] \times \Big(\prod_{j=k+1}^n e^{\langle Z_1 f_j, \varphi\rangle}\Big).$$

Then for all $l \leq n$ in $\mathbb{Z}_+$, $k \in \{l,\dots,n\}$ and $(f_j)_{j=l}^n \subset \mathcal{P}_g$, we have

$$\begin{aligned}a_k - a_{k-1} &= \mathbb{P}_\mu(D^c)^{-1}\mathbb{P}_\mu\Big[(e^{i\Upsilon_k^{f_k}} - e^{\langle Z_1 f_k,\varphi\rangle})\prod_{j=l}^{k-1} e^{i\Upsilon_j^{f_j}}; D^c\Big]\Big(\prod_{j=k+1}^n e^{\langle Z_1 f_j,\varphi\rangle}\Big)\\ &= \mathbb{P}_\mu(D^c)^{-1}\mathbb{P}_\mu\Big[\mathbb{P}_\mu[e^{i\Upsilon_k^{f_k}} - e^{\langle Z_1 f_k,\varphi\rangle}; D^c|\mathscr{F}_k]\prod_{j=l}^{k-1} e^{i\Upsilon_j^{f_j}}\Big]\Big(\prod_{j=k+1}^n e^{\langle Z_1 f_j,\varphi\rangle}\Big).\end{aligned}$$

By Lemma 3.5, there exist $C_0, \delta_0 > 0$ such that for all $l \le n$ in $\mathbb{Z}_+$, $k \in \{l, \dots, n\}$, and $(f_j)_{j=l}^n \subset \mathcal{P}_g$, we have

$$|a_k - a_{k-1}| \le \mathbb{P}_\mu(D^c)^{-1}\mathbb{P}_\mu\Big[\big|\mathbb{P}_\mu[e^{i\Upsilon_k^{f_k}} - e^{\langle Z_1 f_k, \varphi\rangle}; D^c|\mathscr{F}_k]\big|\Big] \le C_0 e^{-\delta_0 k}.$$

Therefore, there exist $C, \delta > 0$ such that for all $l \le n$ in $\mathbb{Z}_+$ and each $(f_j)_{j=l}^n \subset \mathcal{P}_g$, we have

$$\text{LHS of (3.5)} = |a_n - a_{l-1}| \le \sum_{k=l}^n |a_k - a_{k-1}| \le \sum_{k=l}^n C_0 e^{-\delta_0 k} \le Ce^{-\delta l}. \qquad \square$$

3.3 Central limit theorem for $f \in \mathcal{C}_s$

Proof of Theorem 1.6.(1). Fix $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $f \in \mathcal{C}_s$ and $t_0 > 1$ large enough so that $\lceil t - \ln t\rceil \le \lfloor t \rfloor - 1$ for all $t \ge t_0$. For each $t \ge t_0$, in this proof we write $\theta_t = \|X_t\|^{\tilde\beta - 1}$,

$$\begin{aligned}\theta_t X_t(f) = I_1^f(t) + I_2^f(t) + I_3^f(t) := \Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor} \theta_t \mathcal{I}_{t-k-1}^{t-k} X_t(f)\Big) \\ + \Big(\theta_t \mathcal{I}_0^{t-\lfloor t\rfloor} X_t(f) + \sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor - 1} \theta_t \mathcal{I}_{t-k-1}^{t-k} X_t(f)\Big) + (\theta_t X_0(P_t^\alpha f)), \quad (3.6)\end{aligned}$$

and $I_0^f(t) := \sum_{k=0}^{\lfloor t-\ln t\rfloor} \Upsilon_{t-k-1}^{T_k \tilde f}$, where $\tilde f := e^{\alpha(\tilde\beta - 1)} f$.

Step 1. We show that $I_0^f(t) \xrightarrow[t\to\infty]{d} \zeta^f$. In fact, for each $k \in \mathbb{Z}_+$, we have $T_k \tilde f \in \mathcal{P}_{\tilde f} := \{\theta T_n \tilde f : n \in \mathbb{Z}_+, \theta \in [-1, 1]\}$. Therefore from Corollary 3.6 we get that there exist $C_1, \delta_1 > 0$ such that

$$\Big|\widetilde{\mathbb{P}}_\mu[e^{iI_0^f(t)}] - \exp\Big(\sum_{k=0}^{\lfloor t-\ln t\rfloor} \langle Z_1 T_k \tilde f, \varphi\rangle\Big)\Big| \le C_1 e^{-\delta_1(t - \lfloor t-\ln t\rfloor)}, \quad t \ge t_0.$$

On the other hand, using (2.6) and the fact that $\varphi(x)dx$ is the invariant probability of the semigroup $(P_t)_{t\ge 0}$, we have

$$\begin{aligned}\sum_{k=0}^\infty \langle Z_1 T_k \tilde f, \varphi\rangle &= \sum_{k=0}^\infty \int_0^1 \langle P_u^\alpha((-iP_{1-u}^\alpha T_k \tilde f)^{1+\beta}), \varphi\rangle \, du \qquad (3.7)\\ &= \sum_{k=0}^\infty \int_0^1 e^{\alpha u} \langle (-iP_{1-u}^\alpha T_k \tilde f)^{1+\beta}, \varphi\rangle \, du \\ &= \sum_{k=0}^\infty \int_0^1 \langle (-iT_{k+1-u} f)^{1+\beta}, \varphi\rangle \, du = \int_0^\infty \langle (-iT_u f)^{1+\beta}, \varphi\rangle \, du = m[f].\end{aligned}$$

Therefore, we have $\widetilde{\mathbb{P}}_\mu[e^{iI_0^f(t)}] \xrightarrow[t\to\infty]{} e^{m[f]}$. Since $I_0^f(t)$ is linear in f, we can replace f with θf, $\theta \in \mathbb{R}$, and then the desired result in this step follows.

Step 2. We show that $I_1^f(t) - I_0^f(t) \xrightarrow[t\to\infty]{d} 0$. In fact, by [12, Lemma 3.4.3] we have that for each $t \ge t_0$,

$$|\widetilde{\mathbb{P}}_\mu[e^{i(I_1^f(t) - I_0^f(t))}] - 1| \le \sum_{k=0}^{\lfloor t-\ln t\rfloor} \widetilde{\mathbb{P}}_\mu\big[|Y_{t,k}|\big], \qquad (3.8)$$

where $Y_{t,k} := \exp(i\Upsilon_{t-k-1}^{T_k \tilde f} - i\theta_t \mathcal{I}_{t-k-1}^{t-k} X_t(f)) - 1$. We claim that there exist $C_2, \delta_2 > 0$ such that $\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|] \le C_2 e^{-\delta_2(t-k-1)}$ for all $k \in \mathbb{Z}_+$ and $t \ge k+1$. Then there exists $C_2' > 0$

such that for each $t \geq t_0$, $|\widetilde{\mathbb{P}}_\mu[e^{i(I_1^f(t)-I_0^f(t))}] - 1| \leq C_2' t^{-\delta_1}$ which, combined with the fact that $I_1^f(t) - I_0^f(t)$ is linear in f, completes this step.

We will show the claim above in the following substeps 2.1 and 2.2. First we choose $\gamma \in (0, \beta)$ close enough to β so that there exist $\eta, \eta' > 0$ with $\alpha\tilde{\gamma} > \eta > \eta - 3\eta' > \alpha\tilde{\beta} - \alpha\tilde{\gamma} > 0$; and define for $k \in \mathbb{Z}_+$ and $t \geq k+1$,

$$\mathcal{D}_{t,k} := \{|H_t - H_{t-k-1}| \leq e^{-\eta(t-k-1)}, H_{t-k-1} > 2e^{-\eta'(t-k-1)}\},$$

where $H_t := e^{-\alpha t}\|X_t\|$.

Substep 2.1. We show that there exist $C_{2.1}, \delta_{2.1} > 0$ such that for all $k \in \mathbb{Z}_+$ and $t \geq k+1$, $\widetilde{\mathbb{P}}_\mu\big[|Y_{t,k}|; \mathcal{D}_{t,k}^c\big] \leq C_{2.1} e^{-\delta_{2.1}(t-k)}$. In fact, it follows from Proposition 2.10, Lemma 3.3 with $|p| = 0$ and Chebyshev's inequality that there exist $C_{2.1}', \delta_{2.1}' > 0$ such that for all $k \geq 0$ and $t \geq k+1$,

$$\begin{aligned}
&\widetilde{\mathbb{P}}_\mu(\mathcal{D}_{t,k}^c) \leq \widetilde{\mathbb{P}}_\mu(|H_t - H_{t-k-1}| > e^{-\eta(t-k-1)}) + \widetilde{\mathbb{P}}_\mu(H_{t-k-1} \leq 2e^{-\eta'(t-k-1)}) \\
&\leq \mathbb{P}_\mu(D^c)^{-1} e^{\eta(t-k-1)} \mathbb{P}_\mu[|H_t - H_{t-k-1}|] + \mathbb{P}_\mu(D^c)^{-1} \mathbb{P}_\mu(H_{t-k-1} \leq 2e^{-\eta'(t-k-1)}; D^c) \\
&\leq \mathbb{P}_\mu(D^c)^{-1} e^{\eta(t-k-1)} \|H_t - H_{t-k-1}\|_{\mathbb{P}_\mu;1+\gamma} + \mathbb{P}_\mu(D^c)^{-1} \mathbb{P}_\mu(0 < H_{t-k-1} \leq 2e^{-\eta'(t-k-1)}) \\
&\leq C_{2.1}' e^{-(\alpha\tilde{\gamma}-\eta)(t-k-1)} + C_{2.1}' e^{-\delta_{2.1}'(t-k-1)}.
\end{aligned}$$

This implies the desired result in this substep, since $|Y_{t,k}| \leq 2$ a.s..

Substep 2.2. We will show that there exist $C_{2.2}, \delta_{2.2} > 0$ such that for all $k \in \mathbb{Z}_+$ and $t \geq k+1$, it holds that $\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|; \mathcal{D}_{t,k}] \leq C_{2.2} e^{-\delta_{2.2}(t-k)}$. In fact, noticing that for $f \in \mathcal{C}_s$ and $k \in \mathbb{Z}_+$, we have $T_k f = e^{\alpha(\tilde{\beta}-1)k} P_k^\alpha f$; and therefore for all $k \in \mathbb{Z}_+$ and $t \geq k+1$,

$$\Upsilon_{t-k-1}^{T_k \tilde{f}} = \frac{X_{t-k}(T_k \tilde{f}) - X_{t-k-1}(P_1^\alpha T_k \tilde{f})}{\|X_{t-k-1}\|^{1-\tilde{\beta}}} = \frac{\mathcal{I}_{t-k-1}^{t-k} X_t(f)}{\|e^{\alpha(k+1)} X_{t-k-1}\|^{1-\tilde{\beta}}}.$$

Since $|e^{ix} - e^{iy}| \leq |x - y|$ for all $x, y \in \mathbb{R}$, we have for all $k \in \mathbb{Z}_+$ and $t \geq k+1$,

$$\begin{aligned}
\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|; \mathcal{D}_{t,k}] &\leq \widetilde{\mathbb{P}}_\mu\Big[|\mathcal{I}_{t-k-1}^{t-k} X_t(f)| \cdot \Big|\|e^{\alpha(k+1)} X_{t-k-1}\|^{\tilde{\beta}-1} - \|X_t\|^{\tilde{\beta}-1}\Big|; \mathcal{D}_{t,k}\Big] \\
&\leq e^{\alpha(\tilde{\beta}-1)t} \widetilde{\mathbb{P}}_\mu\big[|\mathcal{I}_{t-k-1}^{t-k} X_t(f)| \cdot K_{t,k}\big],
\end{aligned}$$

where

$$K_{t,k} := \Big|\frac{H_t^{1-\tilde{\beta}} - H_{t-k-1}^{1-\tilde{\beta}}}{H_t^{1-\tilde{\beta}} H_{t-k-1}^{1-\tilde{\beta}}}\Big| \mathbf{1}_{\mathcal{D}_{t,k}}.$$

Note that, since $\eta' < \eta$, we have almost surely on $\mathcal{D}_{t,k}$,

$$H_t \geq H_{t-k-1} - e^{-\eta(t-k-1)} \geq 2e^{-\eta'(t-k-1)} - e^{-\eta(t-k-1)} \geq e^{-\eta'(t-k-1)}.$$

Therefore, for all $k \in \mathbb{Z}_+$ and $t \geq k+1$, almost surely on $\mathcal{D}_{t,k}$,

$$\begin{aligned}
&\Big|H_t^{1-\tilde{\beta}} - H_{t-k-1}^{1-\tilde{\beta}}\Big| \leq (1-\tilde{\beta}) \max\{H_t^{-\tilde{\beta}}, H_{t-k-1}^{-\tilde{\beta}}\} |H_t - H_{t-k-1}| \\
&\leq (1-\tilde{\beta}) \max\{e^{\eta'(t-k-1)}, \frac{1}{2} e^{\eta'(t-k-1)}\}^{\tilde{\beta}} e^{-\eta(t-k-1)} \leq (1-\tilde{\beta}) e^{-(\eta-\eta')(t-k-1)}
\end{aligned}$$

and $|H_t^{1-\tilde{\beta}} H_{t-k-1}^{1-\tilde{\beta}}| \geq 2^{\frac{1}{1+\beta}} e^{-2\eta'(t-k-1)}$. Thus, there exists $C_{2.2}' > 0$ such that for all $k \geq 0, t \geq k+1$, almost surely

$$K_{t,k} \leq C_{2.2}' e^{-(\eta-3\eta')(t-k-1)}.$$

Now, by Lemma 2.13, there exists $C''_{2.2} > 0$ such that for all $k \geq 0$ and $t \geq k+1$,

$$\begin{aligned}
&\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|; \mathcal{D}_{t,k}] \leq C'_{2.2} e^{\alpha(\tilde{\beta}-1)t} \widetilde{\mathbb{P}}_\mu[|\mathcal{I}^{t-k}_{t-k-1} X_t(f)|] e^{-(\eta-3\eta')(t-k-1)} \\
&\leq \frac{C'_{2.2}}{\mathbb{P}_\mu(D^c)} e^{\alpha(\tilde{\beta}-1)t} \|\mathcal{I}^{t-k}_{t-k-1} X_t(f)\|_{\mathbb{P}_\mu;1+\gamma} e^{-(\eta-3\eta')(t-k-1)} \\
&\leq C''_{2.2} e^{\alpha(\tilde{\beta}-\tilde{\gamma})t} e^{(\alpha\tilde{\gamma}-\kappa_f b)k} e^{-(\eta-3\eta')(t-k)} \leq C''_{2.2} e^{\alpha(\tilde{\beta}-\tilde{\gamma})(t-k)} e^{-(\eta-3\eta')(t-k)},
\end{aligned}$$

as desired in this step. In the last inequality, we used the fact that $f \in \mathcal{C}_s$ and therefore $\alpha\tilde{\beta} < \kappa_f b$.

Step 3. We show that $I^f_2(t) \xrightarrow[t\to\infty]{d} 0$. First fix a $\gamma \in (0,\beta)$ in this step. From the fact that $\kappa_f b - \alpha\tilde{\gamma} > \alpha(\tilde{\beta}-\tilde{\gamma})$, we can choose $\epsilon > 0$ small enough so that $q := \kappa_f b - \alpha\tilde{\gamma} > \alpha(\tilde{\beta}-\tilde{\gamma}) + 2\epsilon(1-\tilde{\beta})$. Now writing $\mathcal{E}_t := \{\|X_t\| > e^{(\alpha-\epsilon)t}\}$, according to Proposition 2.10, there exist $C_3, \delta_3 > 0$ such that

$$\widetilde{\mathbb{P}}_\mu(\mathcal{E}^c_t) \leq \frac{1}{\mathbb{P}_\mu(D^c)} \mathbb{P}_\mu(0 < e^{-\alpha t}\|X_t\| \leq e^{-\epsilon t}) \leq C_3 e^{-\delta_3 t}, \quad t \geq 0.$$

Therefore,

$$|\widetilde{\mathbb{P}}_\mu[e^{iI^f_2(t)} - 1; \mathcal{E}^c_t]| \leq 2\widetilde{\mathbb{P}}_\mu(\mathcal{E}^c_t) \leq 2C_3 e^{-\delta_3 t}, \quad t \geq t_0. \tag{3.9}$$

According to Lemma 2.13, there exist $C'_3, C''_3, C'''_3 > 0$ such that for each $t \geq t_0 > 1$,

$$\begin{aligned}
&|\widetilde{\mathbb{P}}_\mu[(e^{iI^f_2(t)} - 1); \mathcal{E}_t]| \leq \widetilde{\mathbb{P}}_\mu[|I^f_2(t)|; \mathcal{E}_t] \\
&\leq (e^{(\alpha-\epsilon)t})^{\tilde{\beta}-1} \Big(\sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor -1} \widetilde{\mathbb{P}}_\mu[|\mathcal{I}^{t-k}_{t-k-1} X_t(f)|] + \widetilde{\mathbb{P}}_\mu[|\mathcal{I}^{t-\lfloor t\rfloor}_0 X_t(f)|] \Big) \\
&\leq (e^{(\alpha-\epsilon)t})^{\tilde{\beta}-1} \Big(\sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor -1} \|\mathcal{I}^{t-k}_{t-k-1} X_t(f)\|_{\mathbb{P}_\mu;1+\gamma} + \|\mathcal{I}^{t-\lfloor t\rfloor}_0 X_t(f)\|_{\mathbb{P}_\mu;1+\gamma} \Big) \\
&\leq C'_3 e^{\alpha(\tilde{\beta}-\tilde{\gamma})t} e^{\epsilon(1-\tilde{\beta})t} \sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor} e^{(\alpha\tilde{\gamma}-\kappa_f b)k} \leq C'_3 e^{qt} e^{-\epsilon(1-\tilde{\beta})t} \sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor} e^{-qk} \\
&\leq C''_3 e^{q(t-\lceil t-\ln t\rceil)} e^{-\epsilon(1-\tilde{\beta})t} \leq C''_3 t^q e^{-\epsilon(1-\tilde{\beta})t}.
\end{aligned}$$

From this and (3.9), we get that $\widetilde{\mathbb{P}}_\mu[e^{iI^f_2(t)}] \xrightarrow[t\to\infty]{} 1$. Note that $I^f_2(t)$ is linear in f so we can replace f with θf for $\theta \in \mathbb{R}$ and get the desired result in this step.

Step 4. We will show that $I^f_3(t) \xrightarrow[t\to\infty]{\widetilde{\mathbb{P}}_\mu\text{-}a.s.} 0$. In fact, we have

$$|I^f_3(t)| \leq \frac{X_0(|P^\alpha_t f|)}{\|X_t\|^{1-\tilde{\beta}}} \leq \frac{e^{\alpha t-\kappa_f bt} X_0(Qf)}{(e^{\alpha t} H_t)^{1-\tilde{\beta}}} = e^{(\alpha\tilde{\beta}-k_f b)t} H^{\tilde{\beta}-1}_t X_0(Qf) \xrightarrow[t\to\infty]{\widetilde{\mathbb{P}}_\mu\text{-}a.s.} 0.$$

Step 5. Combining Steps 1–4, we complete the proof of Theorem 1.6.(1). □

3.4 Central limit theorem for $f \in \mathcal{C}_c$

Proof of Theorem 1.6.(2). Fix $\mu \in \mathcal{M}_c(\mathbb{R}^d)$, $f \in \mathcal{C}_c$ and $t_0 > 1$ large enough so that $\lceil t - \ln t\rceil \leq \lfloor t\rfloor - 1$ for each $t \geq t_0$. For each $t \geq t_0$, in this proof we write $\theta_t = \|tX_t\|^{\tilde{\beta}-1}$, define $I^f_i(t)$ using (3.6) for $i = 1,2,3$, and set $I^f_0(t) := t^{\tilde{\beta}-1} \sum_{k=0}^{\lfloor t-\ln t\rfloor} \Upsilon^{T_k \tilde{f}}_{t-k-1}$, where $\tilde{f} = e^{\alpha(\tilde{\beta}-1)} f$.

Step 1. We show that $I^f_0(t) \xrightarrow[t\to\infty]{d} \zeta^f$. In fact, for each $t \geq t_0(>1)$ we have $t^{\tilde{\beta}-1} < 1$; and therefore, for each $k \in \mathbb{Z}_+$, we have $t^{\tilde{\beta}-1} T_{k+1} f \in \mathcal{P}_f := \{\theta T_n f : n \in \mathbb{Z}_+, \theta \in [-1,1]\}$.

Therefore from Proposition 3.6 and that $\tilde{\beta}-1=-\frac{1}{1+\beta}$ we get that there exist $C_1,\delta_1>0$ such that

$$\Big|\widetilde{\mathbb{P}}_\mu[e^{iI_0^f(t)}]-\exp\Big(\frac{1}{t}\sum_{k=0}^{\lfloor t-\ln t\rfloor}\langle Z_1T_k\tilde{f},\varphi\rangle\Big)\Big|\le C_1e^{-\delta_1(t-\lfloor t-\ln t\rfloor)}\le\frac{C_1}{t^{\delta_1}},\quad t\ge t_0.$$

Since $f\in\mathcal{C}_c\setminus\{0\}$, we have $T_k\tilde{f}=\tilde{f}$ for each $k\in\mathbb{Z}_+$. Similar to the argument in (3.7) we have

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{\lfloor t-\ln t\rfloor}\langle Z_1T_k\tilde{f},\varphi\rangle=\langle Z_1\tilde{f},\varphi\rangle=\langle(-if)^{1+\beta},\varphi\rangle=m[f].$$

Therefore $\widetilde{\mathbb{P}}_\mu[e^{iI_0^f(t)}]\xrightarrow[t\to\infty]{}e^{m[f]}$. The desired result in this step follows.

Step 2. We show that $I_1^f(t)-I_0^f(t)\xrightarrow[t\to\infty]{d}0$. In fact, similar to Step 2 in the proof of Theorem 1.6.(1), we have (3.8) is valid with $Y_{t,k}:=\exp(it^{\tilde{\beta}-1}\Upsilon_{t-k-1}^{T_k\tilde{f}}-i\theta_t\mathcal{I}_{t-k-1}^{t-k}X_t(f))-1$. Similarly, we claim that there exist $C_2,\delta_2>0$ such that $\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|]\le C_2e^{-\delta_2(t-k-1)}$ for all $k\in\mathbb{N}$ and $t\ge k+1$, and then the desired result in this step follows.

We will show the claim above in the following substeps 2.1 and 2.2. First we choose $\gamma\in(0,\beta)$ close enough to β so that there exist $\eta,\eta'>0$ with $\alpha\tilde{\gamma}>\eta>\eta-3\eta'>\alpha\tilde{\beta}-\alpha\tilde{\gamma}>0$; and define, for $k\in\mathbb{N}$ and $t\ge k+1$, $\mathcal{D}_{t,k}:=\{|H_t-H_{t-k-1}|\le e^{-\eta(t-k-1)},H_{t-k-1}>2e^{-\eta'(t-k-1)}\}$.

Substep 2.1. Similar to Substep 2.1 in the proof of Theorem 1.6.(1), there exist $C_{2.1},\delta_{2.1}>0$ such that for all $k\in\mathbb{N}$ and $t\ge k+1$, $\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|;\mathcal{D}_{t,k}^c]\le C_{2.1}e^{-\delta_{2.1}(t-k)}$. We omit the details.

Substep 2.2. We will show that there exist $C_{2.2},\delta_{2.2}>0$ such that for all $k\in\mathbb{N}$ and $t\ge k+1$, $\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|;\mathcal{D}_{t,k}]\le C_{2.2}e^{-\delta_{2.2}(t-k)}$. In fact, noticing that for $f\in\mathcal{C}_c$ and $k\in\mathbb{Z}_+$, we have $T_kf=e^{\alpha(\tilde{\beta}-1)k}P_k^\alpha$; and therefore for all $k\in\mathbb{Z}_+$ and $t\ge k+1$,

$$t^{\tilde{\beta}-1}\Upsilon_{t-k-1}^{T_k\tilde{f}}=\frac{X_{t-k}(T_k\tilde{f})-X_{t-k-1}(P_1^\alpha T_k\tilde{f})}{\|tX_{t-k-1}\|^{1-\tilde{\beta}}}=\frac{\mathcal{I}_{t-k-1}^{t-k}X_t(f)}{\|te^{\alpha(k+1)}X_{t-k-1}\|^{1-\tilde{\beta}}}.$$

The rest is similar to Substep 2.2 in the proof of Theorem 1.6.(2). We omit the details.

Step 3. We show that $I_2^f(t)\xrightarrow[t\to\infty]{d}0$. In fact, writing $\mathcal{E}_t:=\{\|X_t\|>t^{-1/2}e^{\alpha t}\}$, according to Proposition 2.10, there exist $C_3,\delta_3>0$ such that

$$\widetilde{\mathbb{P}}_\mu(\mathcal{E}_t^c)\le\frac{1}{\mathbb{P}_\mu(D^c)}\mathbb{P}_\mu(0<e^{-\alpha t}\|X_t\|\le t^{-1/2})\le C_3(t^{-\delta_3}+e^{-\delta_3t}),\quad t\ge0.$$

Therefore,

$$|\widetilde{\mathbb{P}}_\mu[e^{iI_2^f(t)}-1;\mathcal{E}_t^c]|\le2\widetilde{\mathbb{P}}_\mu(\mathcal{E}_t^c)\le C_3(t^{-\delta_3}+e^{-\delta_3t}),\quad t\ge t_0. \tag{3.10}$$

Choose a $\gamma\in(0,\beta)$ close enough to β so that $\alpha(\tilde{\beta}-\tilde{\gamma})\le\frac{1}{2}(1-\tilde{\beta})$. According to Lemma

2.13, there exist $C_3', C_3'', C_3''' > 0$ such that for each $t \geq t_0 (> 1)$,

$$\begin{aligned}
&|\widetilde{\mathbb{P}}_\mu[(e^{iI_2^f(t)} - 1)\mathbf{1}_{\mathcal{E}_t}]| \leq \widetilde{\mathbb{P}}_\mu[|I_2^f(t)|\mathbf{1}_{\mathcal{E}_t}] \\
&\leq (t^{\frac{1}{2}} e^{\alpha t})^{\tilde{\beta}-1} \Big(\sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor -1} \widetilde{\mathbb{P}}_\mu[|\mathcal{I}_{t-k-1}^{t-k} X_t(f)|] + \widetilde{\mathbb{P}}_\mu[|\mathcal{I}_0^{t-\lfloor t\rfloor} X_t(f)|] \Big) \\
&\leq C_3' t^{\frac{1}{2}(\tilde{\beta}-1)} e^{\alpha(\tilde{\beta}-1)t} \Big(\sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor -1} \|\mathcal{I}_{t-k-1}^{t-k} X_t(f)\|_{\mathbb{P}_\mu;1+\gamma} + \|\mathcal{I}_0^{t-\lfloor t\rfloor} X_t(f)\|_{\mathbb{P}_\mu;1+\gamma} \Big) \\
&\leq C_3' t^{\frac{1}{2}(\tilde{\beta}-1)} e^{\alpha(\tilde{\beta}-\tilde{\gamma})t} \sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor} e^{(\alpha\tilde{\gamma}-\kappa_f b)k} = C_3' t^{\frac{1}{2}(\tilde{\beta}-1)} e^{\alpha(\tilde{\beta}-\tilde{\gamma})t} \sum_{k=\lceil t-\ln t\rceil}^{\lfloor t\rfloor} e^{-\alpha(\tilde{\beta}-\tilde{\gamma})k} \\
&\leq C_3'' t^{\frac{1}{2}(\tilde{\beta}-1)} e^{\alpha(\tilde{\beta}-\tilde{\gamma})(t-\lceil t-\ln t\rceil)} \leq C_3''' t^{\frac{1}{2}(\tilde{\beta}-1)} t^{\alpha(\tilde{\beta}-\tilde{\gamma})}.
\end{aligned}$$

From this and (3.10), we get the desired result in this step.

Step 4. Similar to Step 4 in the proof of Theorem 1.6.(1), we can verify that $I_3(t) \xrightarrow[t\to\infty]{\widetilde{\mathbb{P}}_\mu\text{-a.s.}} 0$. We omit the details.

Step 5. Combining Steps 1–4, we complete the proof of Theorem 1.6.(2). □

3.5 Central limit theorem for $f \in \mathcal{C}_l$

Proof of Theorem 1.6.(3). Fix $\mu \in \mathcal{M}_c(\mathbb{R}^d)$ and $f \in \mathcal{C}_l$. Define $\mathcal{N} := \{p \in \mathbb{Z}_+^d : \alpha\tilde{\beta} > |p|b\}$. In this proof we write for each $t \geq 0$,

$$\begin{aligned}
&\frac{X_t(f) - \sum_{p\in\mathbb{Z}_+^d:\alpha\tilde{\beta}\geq|p|b} \langle f, \phi_p\rangle_\varphi e^{(\alpha-|p|b)t} H_\infty^p}{\|X_t\|^{1-\tilde{\beta}}} = \sum_{p\in\mathcal{N}} \frac{\langle f,\phi_p\rangle_\varphi [X_t(\phi_p) - e^{(\alpha-|p|b)t} H_\infty^p]}{\|X_t\|^{1-\tilde{\beta}}} \\
&= \sum_{p\in\mathcal{N}} \frac{\langle f,\phi_p\rangle_\varphi e^{(\alpha-|p|b)t}(H_t^p - H_\infty^p)}{\|X_t\|^{1-\tilde{\beta}}} = \sum_{k=0}^\infty \sum_{p\in\mathcal{N}} \langle f,\phi_p\rangle_\varphi e^{(\alpha-|p|b)t} \frac{H_{t+k}^p - H_{t+k+1}^p}{\|X_t\|^{1-\tilde{\beta}}} \\
&=: \sum_{k=0}^\infty \widetilde{\Upsilon}_{t,k} = \Big(\sum_{k=0}^{\lfloor t^2\rfloor} \widetilde{\Upsilon}_{t,k} \Big) + \Big(\sum_{k=\lceil t^2\rceil}^\infty \widetilde{\Upsilon}_{t,k} \Big) =: I_1^f(t) + I_2^f(t),
\end{aligned}$$

and $I_0^f(t) := \sum_{k=0}^{\lfloor t^2\rfloor} \Upsilon_{t+k}^{-T_k\tilde{f}}$ where $\tilde{f} := \sum_{p\in\mathcal{N}} e^{-(\alpha-|p|b)} \langle f,\phi_p\rangle_\varphi \phi_p$.

Step 1. We show that $I_0^f(t) \xrightarrow[t\to\infty]{d} \zeta^{-f}$. In fact, since $T_k\tilde{f} \in \mathcal{P}_{\tilde{f}}$ for each $k \in \mathbb{Z}_+$, from Corollary 3.6 we have $\widetilde{\mathbb{P}}_\mu[e^{iI_0^f(t)}] \xrightarrow[t\to\infty]{} \exp\{\sum_{k=0}^\infty \langle Z_1 T_k(-\tilde{f}), \varphi\rangle\}$. Using (2.6) and the fact that $\varphi(x)dx$ is the invariant probability of the semigroup $(P_t)_{t\geq 0}$ we have

$$\begin{aligned}
\sum_{k=0}^\infty \langle Z_1 T_k(-\tilde{f}), \varphi\rangle &= \sum_{k=0}^\infty \int_0^1 \langle P_u^\alpha((iP_{1-u}^\alpha T_k\tilde{f})^{1+\beta}), \varphi\rangle \, du \\
&= \sum_{k=0}^\infty \int_0^1 e^{\alpha u} \langle (iP_{1-u}^\alpha T_k\tilde{f})^{1+\beta}, \varphi\rangle \, du \\
&= \sum_{k=0}^\infty \int_0^1 \langle (iT_{k+u}f)^{1+\beta}, \varphi\rangle \, du = \int_0^\infty \langle (iT_u f)^{1+\beta}, \varphi\rangle \, du = m[-f].
\end{aligned}$$

The result in this step follows.

Step 2. We show that $I_1^f(t) - I_0^f(t) \xrightarrow[t\to\infty]{d} 0$. In fact, by [12, Lemma 3.4.3] we have, for each $t \geq 0$, that $|\widetilde{\mathbb{P}}_\mu[e^{i(I_1^f(t)-I_0^f(t))} - 1]| \leq \sum_{k=0}^{\lfloor t^2\rfloor} \widetilde{\mathbb{P}}_\mu[|Y_{t,k}|]$ where $Y_{t,k} := e^{i(\widetilde{\Upsilon}_{t,k} - \Upsilon_{t+k}^{-T_k\tilde{f}})} - 1$.

We claim that there exist $C_2,\delta_2>0$ such that $\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|]\le C_2e^{-\delta_2 t}$ for all $t\ge 0$ and $k\in\mathbb{Z}_+$. Then $|\widetilde{\mathbb{P}}_\mu[e^{i(I_1^f(t)-I_0^f(t))}-1]|\le (t^2+1)C_2e^{-\delta_2 t}$ which completes this step.

We will show the claim above in the following substeps 2.1 and 2.2. First we choose $\gamma\in(0,\beta)$ close enough to β so that $\alpha\tilde\gamma>|p|b$ for each $p\in\mathcal{N}$; and even closer so that there exist $\eta,\eta'>0$ satisfying $\alpha\tilde\gamma>\eta>\eta-3\eta'>\alpha(\tilde\beta-\tilde\gamma)>0$. We also define $\mathcal{D}_{t,k}:=\{|H_t-H_{t+k}|\le e^{-\eta t}, H_t>2e^{-\eta' t}\}$.

Substep 2.1. Similar to Substep 2.1 in the proof of Theorem 1.6.(1), we have that there exist $C_{2.1},\delta_{2.1}>0$ such that for all $k\in\mathbb{Z}_+$ and $t\ge0$, $\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|;\mathcal{D}_{t,k}^c]\le C_{2.1}e^{-\delta_{2.1}t}$. We omit the details.

Substep 2.2. We show that there exist $C_{2.2},\delta_{2.2}>0$ such that for all $k\in\mathbb{Z}_+$ and $t\ge0$, we have $\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|;\mathcal{D}_{t,k}]\le C_{2.2}e^{-\delta_{2,2}t}$. In fact, it can be verified that for all $k\in\mathbb{Z}_+$ and $t\ge0$,

$$\begin{aligned}\Upsilon_{t+k}^{-T_k\tilde f}&=\frac{X_{t+k}(P_1^\alpha T_k\tilde f)-X_{t+k+1}(T_k\tilde f)}{\|X_{t+k}\|^{1-\tilde\beta}}\\&=\sum_{p\in\mathcal{N}}\langle\tilde f,\phi_p\rangle_\varphi e^{-(\alpha\tilde\beta-|pb|)k}\frac{X_{t+k}(P_1^\alpha\phi_p)-X_{t+k+1}(\phi_p)}{\|X_{t+k}\|^{1-\tilde\beta}}\\&=\sum_{p\in\mathcal{N}}\langle f,\phi_p\rangle_\varphi e^{(\alpha-|p|b)t}\frac{H_{t+k}^p-H_{t+k+1}^p}{\|e^{-\alpha k}X_{t+k}\|^{1-\tilde\beta}}.\end{aligned}$$

Therefore for all $k\in\mathbb{Z}_+$ and $t\ge0$,

$$\begin{aligned}&|Y_{t,k}|\mathbf{1}_{\mathcal{D}_{t,k}}\le\Big(\sum_{p\in\mathcal{N}}|\langle f,\phi_p\rangle_\varphi|e^{(\alpha-|p|b)t}|H_{t+k}^p-H_{t+k+1}^p|\Big)\Big(\frac{1}{\|X_t\|^{1-\tilde\beta}}-\frac{1}{\|e^{-\alpha k}X_{t+k}\|^{1-\tilde\beta}}\Big)\mathbf{1}_{\mathcal{D}_{t,k}}\\&=\Big(\sum_{p\in\mathcal{N}}|\langle f,\phi_p\rangle_\varphi|e^{(\alpha-|p|b)t}|H_{t+k}^p-H_{t+k+1}^p|\Big)e^{\alpha(\tilde\beta-1)t}K_{t,k}\\&=\Big(\sum_{p\in\mathcal{N}}|\langle f,\phi_p\rangle_\varphi|e^{(\alpha\tilde\beta-|p|b)t}|H_{t+k}^p-H_{t+k+1}^p|\Big)K_{t,k},\end{aligned}$$

where

$$K_{t,k}:=\Big|\frac{H_t^{1-\tilde\beta}-H_{t+k}^{1-\tilde\beta}}{H_t^{1-\tilde\beta}H_{t+k}^{1-\tilde\beta}}\Big|\mathbf{1}_{\mathcal{D}_{t,k}}.$$

Similar to Substep 2.2 in the proof of Theorem 1.6.(1), we can verify that for all $k\in\mathbb{Z}_+$ and $t\ge0$, almost surely $K_{t,k}\le C_{2.2}''e^{-(\eta-3\eta')t}$. From this and Lemma 3.3 we know that there exists $C_{2.2}'''$ such that for all $k\in\mathbb{Z}_+$ and $t\ge0$,

$$\begin{aligned}&\widetilde{\mathbb{P}}_\mu[|Y_{t,k}|;\mathcal{D}_{t,k}]\le\mathbb{P}_\mu(D)^{-1}\mathbb{P}_\mu[|Y_{t,k}|;\mathcal{D}_{t,k}]\\&\le\mathbb{P}_\mu(D)^{-1}C_{2.2}''e^{-(\eta-3\eta')t}\sum_{p\in\mathcal{N}}|\langle f,\phi_p\rangle_\varphi|e^{(\alpha\tilde\beta-|p|b)t}\mathbb{P}_\mu[|H_{t+k}^p-H_{t+k+1}^p|]\\&\le\mathbb{P}_\mu(D)^{-1}C_{2.2}''e^{-(\eta-3\eta')t}\sum_{p\in\mathcal{N}}|\langle f,\phi_p\rangle_\varphi|e^{(\alpha\tilde\beta-|p|b)t}\|H_{t+k}^p-H_{t+k+1}^p\|_{\mathbb{P}_\mu;1+\gamma}\\&\le\mathbb{P}_\mu(D)^{-1}C_{2.2}''e^{-(\eta-3\eta')t}\sum_{p\in\mathcal{N}}|\langle f,\phi_p\rangle_\varphi|e^{(\alpha\tilde\beta-|p|b)t}e^{-(\alpha\tilde\gamma-|p|b)(t+k)}\\&\le C_{2.2}'''e^{-(\eta-3\eta')t}e^{(\alpha\tilde\beta-\alpha\tilde\gamma)t},\end{aligned}$$

as desired in this substep.

Step 3. We show that $I_2^f(t)\xrightarrow[t\to\infty]{d}0$. In order to do this, choose an $\epsilon\in(0,\alpha)$ and a $\gamma\in(0,\beta)$ close enough to β so that for each $p\in\mathcal{N}$, it holds that $\alpha\tilde\gamma>|p|b$. Define

$\mathcal{E}_t := \{\|X_t\| > e^{(\alpha-\epsilon)t}\}$. According to Proposition 2.10, there exist $C_3, \delta_3 > 0$ such that for each $t \geq 0$, $|\widetilde{\mathbb{P}}_\mu[e^{iI_2^f(t)} - 1; \mathcal{E}_t^c]| \leq 2\widetilde{\mathbb{P}}_\mu(\mathcal{E}_t^c) \leq C_3 e^{-\delta_3 t}$. On the other hand, according to Lemma 3.3, we know that there exist $C_3', C_3'' > 0$ and $\delta_3' > 0$ such that

$$\begin{aligned}
&|\widetilde{\mathbb{P}}_\mu[e^{iI_2^f(t)} - 1; \mathcal{E}_t]| \leq \widetilde{\mathbb{P}}_\mu[|I_2^f(t)|; \mathcal{E}_t] \leq \sum_{k=\lceil t^2 \rceil}^{\infty} \widetilde{\mathbb{P}}_\mu[|\widetilde{\Upsilon}_{t,k}|; \mathcal{E}_t] \\
&\leq \mathbb{P}_\mu(D^c)^{-1} \sum_{k=\lceil t^2 \rceil}^{\infty} \sum_{p\in\mathcal{N}} |\langle f, \phi_p\rangle_\varphi| e^{(\alpha-|p|b)t} \mathbb{P}_\mu\Big[\frac{|H^p_{t+k} - H^p_{t+k+1}|}{\|X_t\|^{1-\tilde{\beta}}}; \mathcal{E}_t\Big] \\
&\leq \mathbb{P}_\mu(D^c)^{-1} e^{(\alpha-\epsilon)(\tilde{\beta}-1)t} \sum_{k=\lceil t^2 \rceil}^{\infty} \sum_{p\in\mathcal{N}} |\langle f, \phi_p\rangle_\varphi| e^{(\alpha-|p|b)t} \|H^p_{t+k} - H^p_{t+k+1}\|_{\mathbb{P}_\mu;1+\gamma} \\
&\leq C_3' e^{(\alpha-\epsilon)(\tilde{\beta}-1)t} \sum_{k=\lceil t^2 \rceil}^{\infty} \sum_{p\in\mathcal{N}} |\langle f, \phi_p\rangle_\varphi| e^{(\alpha-|p|b)t} e^{-(\alpha\tilde{\gamma}-|p|b)(t+k)} \\
&= C_3'' e^{\alpha(\tilde{\beta}-\tilde{\gamma})t} e^{\epsilon(1-\tilde{\beta})t} e^{-\delta_3' t^2}.
\end{aligned}$$

To sum up we have that $\widetilde{\mathbb{P}}_\mu[e^{iI_2^f(t)}] \xrightarrow[t\to\infty]{} 1$, which completes this step.

Step 4. Combining Steps 1–3, we complete the proof of Theorem 1.6.(3). □

A Appedix

A.1 Analytic facts

In this subsection, we collect some useful analytic facts.

Lemma A.1. *For $z \in \mathbb{C}_+$, we have*

$$\Big|e^{-z} - \sum_{k=0}^{n} \frac{(-z)^k}{k!}\Big| \leq \frac{|z|^{n+1}}{(n+1)!} \wedge \frac{2|z|^n}{n!}, \quad n \in \mathbb{Z}_+. \tag{A.1}$$

Proof. Notice that $|e^{-z}| = e^{-\operatorname{Re} z} \leq 1$. Therefore, $|e^{-z} - 1| = |\int_0^1 e^{-\theta z} z d\theta| \leq |z|$. Also, notice that $|e^{-z} - 1| \leq |e^{-z}| + 1 \leq 2$. Thus (A.1) is true when $n = 0$. Now, suppose that (A.1) is true when $n = m$ for some $m \in \mathbb{Z}_+$. Then

$$\begin{aligned}
&\Big|e^{-z} - \sum_{k=0}^{m+1} \frac{(-z)^k}{k!}\Big| = \Big|\int_0^1 \Big(e^{-\theta z} - \sum_{k=0}^{m} \frac{(-\theta z)^k}{k!}\Big) z d\theta\Big| \\
&\quad \leq \Big(\int_0^1 \frac{|\theta z|^{m+1}}{(m+1)!}|z| d\theta\Big) \wedge \Big(\int_0^1 \frac{2|\theta z|^m}{m!}|z| d\theta\Big) = \frac{|z|^{m+2}}{(m+2)!} \wedge \frac{2|z|^{m+1}}{(m+1)!},
\end{aligned}$$

which says that (A.1) is true for $n = m + 1$. □

Lemma A.2. *Suppose that π is a measure on $(0,\infty)$ with $\int_{(0,\infty)} (y \wedge y^2)\pi(dy) < \infty$. Then the functions*

$$\begin{aligned}
h(z) &= \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(dy), \quad z \in \mathbb{C}_+, \\
h'(z) &= \int_{(0,\infty)} (1 - e^{-zy}) y \pi(dy), \quad z \in \mathbb{C}_+
\end{aligned}$$

are well defined, continuous on $\mathbb{C}_+$ and holomorphic on $\mathbb{C}_+^0$. Moreover,

$$\frac{h(z) - h(z_0)}{z - z_0} \xrightarrow[\mathbb{C}_+ \ni z \to z_0]{} h'(z_0), \quad z_0 \in \mathbb{C}_+.$$

Proof. It follows from Lemma A.1 that h and h' are well defined on $\mathbb{C}_+$. According to [42, Theorems 3.2. & Proposition 3.6], h' is continuous on $\mathbb{C}_+$ and holomorphic on $\mathbb{C}_+^0$.

It follows from Lemma A.1 that, for each $z_0 \in \mathbb{C}_+$, there exists $C > 0$ such that for $z \in \mathbb{C}_+$ close enough to z_0 and any $y > 0$,

$$\begin{aligned}&\Big|\frac{e^{-zy} - e^{-z_0 y} + (z - z_0)y}{z - z_0}\Big| = \frac{1}{|z - z_0|}\Big|\int_0^1 (-ye^{-(\theta z + (1-\theta)z_0)y} + y)(z - z_0)d\theta\Big|\\ &\le y\int_0^1 |1 - e^{-(\theta z+(1-\theta)z_0)y}|d\theta \le (2y) \wedge \Big(y^2\int_0^1 |\theta z + (1-\theta)z_0|d\theta\Big) \le C(y \wedge y^2).\end{aligned}$$

Using this and the dominated convergence theorem, we have

$$\begin{aligned}\frac{h(z) - h(z_0)}{z - z_0} &= \int_{(0,\infty)} \frac{e^{-zy} + zy - (e^{-z_0 y} + z_0 y)}{z - z_0}\pi(dy)\\ &\xrightarrow[\mathbb{C}_+\ni z\to z_0]{} \int_{(0,\infty)} (1 - e^{-z_0 y})y\pi(dy) = h'(z_0),\end{aligned}$$

which says that h is continuous on $\mathbb{C}_+$ and holomorphic on $\mathbb{C}_+^0$. ☐

For each $z \in \mathbb{C} \setminus (-\infty, 0]$, we define $\log z := \log|z| + i\arg z$ where $\arg z \in (-\pi, \pi)$ is uniquely determined by $z = |z|e^{i\arg z}$. For all $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\gamma \in \mathbb{C}$, we define $z^\gamma := e^{\gamma \log z}$. Then it is known, see [43, Theorem 6.1] for example, that $z \mapsto \log z$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. Therefore, for each $\gamma \in \mathbb{C}$, $z \mapsto z^\gamma$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. (We use the convention that $0^\gamma := \mathbf{1}_{\gamma=0}$.) Using the definition above we can easily show that $(z_1 z_0)^\gamma = z_1^\gamma z_0^\gamma$ provided $\arg(z_1 z_0) = \arg(z_1) + \arg(z_0)$.

It is known, see, for instance, [43, Theorem 6.1.3] and the remark following it, that the Gamma function Γ has an unique analytic extension in $\mathbb{C} \setminus \{0, -1, -2, \dots\}$ and that

$$\Gamma(z + 1) = z\Gamma(z), \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

Using this recursively, one gets that

$$\Gamma(x) := \int_0^\infty t^{x-1}\Big(e^{-t} - \sum_{k=0}^{n-1} \frac{(-t)^k}{k!}\Big)dt, \quad -n < x < -n + 1, n \in \mathbb{N}.$$

Fix a $\beta \in (0, 1)$. Using the uniqueness of holomorphic extension and Lemma A.2, we get that

$$z^\beta = \int_0^\infty (e^{-zy} - 1)\frac{dy}{\Gamma(-\beta)y^{1+\beta}}, \quad z \in \mathbb{C}_+,$$

and similarly that

$$z^{1+\beta} = \int_0^\infty (e^{-zy} - 1 + zy)\frac{dy}{\Gamma(-1-\beta)y^{2+\beta}}, \quad z \in \mathbb{C}_+. \tag{A.2}$$

Lemma A.2 also says that the derivative of $z^{1+\beta}$ is $(1 + \beta)z^\beta$ on $\mathbb{C}_+^0$.

Lemma A.3. *For all $z_0, z_1 \in \mathbb{C}_+$, we have*

$$|z_0^{1+\beta} - z_1^{1+\beta}| \le (1 + \beta)(|z_0|^\beta + |z_1|^\beta)|z_0 - z_1|.$$

Proof. Since $z^{1+\beta}$ is continuous on $\mathbb{C}_+$, we only need to prove the lemma assuming $z_0, z_1 \in \mathbb{C}_+^0$. Notice that

$$|z^\beta| = |e^{\beta\log|z| + i\beta\arg z}| = e^{\beta\log|z|} = |z|^\beta, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

Define a path $\gamma : [0,1] \to \mathbb{C}_+^0$ such that

$$\gamma(\theta) = z_0(1-\theta) + \theta z_1, \quad \theta \in [0,1].$$

Then, we have

$$\begin{aligned}|z_0^{1+\beta} - z_1^{1+\beta}| &\le (1+\beta)\int_0^1 |\gamma(\theta)^\beta| \cdot |\gamma'(\theta)| d\theta \le (1+\beta)\sup_{\theta\in[0,1]} |\gamma(\theta)|^\beta \cdot |z_1 - z_0| \\ &\le (1+\beta)(|z_1|^\beta + |z_0|^\beta)|z_1 - z_0|.\end{aligned}$$

□

Suppose that $\varphi(\theta)$ is a continuous function from $\mathbb{R}$ into $\mathbb{C}$ such that $\varphi(0) = 1$ and $\varphi(\theta) \neq 0$ for all $\theta \in \mathbb{R}$. Then according to [41, Lemma 7.6], there is a unique continuous function $f(\theta)$ from $\mathbb{R}$ into $\mathbb{C}$ such that $f(0) = 0$ and $e^{f(\theta)} = \varphi(\theta)$. Such a function f is called the distinguished logarithm of the function φ and is denoted as $\operatorname{Log}\varphi(\theta)$. In particular, when φ is the characteristic function of an infinitely divisible random variable Y, $\operatorname{Log}\varphi(\theta)$ is called the Lévy exponent of Y. This distinguished logarithm should not be confused with the $\log$ function defined on $\mathbb{C} \setminus (-\infty, 0]$. See the paragraph immediately after [41, Lemma 7.6].

A.2 Feynman-Kac formula with complex valued functions

In this subsection we give a version of the Feynman-Kac formula with complex valued functions. Suppose that $\{(\xi_t)_{t\in[r,\infty)}; (\Pi_{r,x})_{r\in[0,\infty),x\in E}\}$ is a (possibly non-homogeneous) Hunt process in a locally compact separable metric space E. We write

$$H_{(s,t)}^{(h)} := \exp\Big\{\int_s^t h(u,\xi_u)du\Big\}, \quad 0 \le s \le t, h \in \mathcal{B}_b([0,t]\times E, \mathbb{C}).$$

Lemma A.4. *Let $t \ge 0$. Suppose that $\rho_1, \rho_2 \in \mathcal{B}_b([0,t]\times E, \mathbb{C})$ and $f \in \mathcal{B}_b(E, \mathbb{C})$. Then*

$$g(r,x) := \Pi_{r,x}[H_{(r,t)}^{(\rho_1+\rho_2)} f(\xi_t)], \quad r \in [0,t], x \in E, \tag{A.3}$$

is the unique locally bounded solution to the equation

$$g(r,x) = \Pi_{r,x}[H_{(r,t)}^{(\rho_1)} f(\xi_t)] + \Pi_{r,x}\Big[\int_r^t H_{(r,s)}^{(\rho_1)} \rho_2(s,\xi_s) g(s,\xi_s)\, ds\Big], \quad r \in [0,t], x \in E.$$

Proof. The proof is similar to that of [13, Lemma A.1.5]. We include it here for the sake of completeness. We first verify that (A.3) is a solution. Notice that

$$\Pi_{r,x}\Big[\int_r^t |H_{(r,t)}^{(\rho_1)} \rho_2(s,\xi_s) H_{(s,t)}^{(\rho_2)} f(\xi_t)|\, ds\Big] \le \int_r^t e^{(t-r)\|\rho_1\|_\infty} e^{(t-s)\|\rho_2\|_\infty} \|\rho_2\|_\infty \|f\|_\infty\, ds < \infty.$$

Also notice that

$$\frac{\partial}{\partial s} H_{(s,t)}^{(\rho_2)} = -H_{(s,t)}^{(\rho_2)} \rho_2(s,\xi_s), \quad s \in (0,t).$$

Therefore, from the Markov property of ξ and Fubini's theorem we get that

$$\begin{aligned}&\Pi_{r,x}\Big[\int_r^t H_{(r,s)}^{(\rho_1)}\ (\rho_2 g)(s,\xi_s)\, ds\Big] = \Pi_{r,x}\Big[\int_r^t H_{(r,s)}^{(\rho_1)} \rho_2(s,\xi_s) \Pi_{s,\xi_s}[H_{(s,t)}^{(\rho_1+\rho_2)} f(\xi_t)]\, ds\Big] \\ &= \Pi_{r,x}\Big[\int_r^t H_{(r,t)}^{(\rho_1)} \rho_2(s,\xi_s) H_{(s,t)}^{(\rho_2)} f(\xi_t)\, ds\Big] = \Pi_{r,x}[H_{(r,t)}^{(\rho_1)} f(\xi_t)(H_{(r,t)}^{(\rho_2)} - 1)] \\ &= g(r,x) - \Pi_{r,x}[H_{(r,t)}^{(\rho_2)} f(\xi_t)].\end{aligned}$$

For uniqueness, suppose $\widetilde{g}$ is another solution. Put $h(r) = \sup_{x\in E} |g(r,x) - \widetilde{g}(r,x)|$. Then

$$h(r) \le e^{t\|\rho_1\|_\infty} \|\rho_2\|_\infty \int_r^t h(s) ds, \quad r \le t.$$

Applying Gronwall's inequality, we get that $h(r) = 0$ for $r \in [0,t]$. □

A.3 Superprocesses

In this subsection, we will give the definition of a general superprocess. Let E be locally compact separable metric space. Denote by $\mathcal{M}(E)$ the collection of all the finite measures on E equipped with the topology of weak convergence. For each function $F(x,z)$ on $E\times\mathbb{R}_+$ and each $\mathbb{R}_+$-valued function f on E, we use the convention: $F(x,f) := F(x,f(x)), x\in E$. A process $X=\{(X_t)_{t\geq 0};(\mathbf{P}_\mu)_{\mu\in\mathcal{M}(E)}\}$ is said to be a (ξ,ψ)-superprocess if

- the spatial motion $\xi=\{(\xi_t)_{t\geq 0};(\Pi_x)_{x\in E}\}$ is an E-valued Hunt process with its lifetime denoted by ζ;
- the branching mechanism $\psi : E\times[0,\infty)\to\mathbb{R}$ is given by

$$\psi(x,z)=-\rho_1(x)z+\rho_2(x)z^2+\int_{(0,\infty)}(e^{-zy}-1+zy)\pi(x,dy). \tag{A.4}$$

  where $\rho_1\in\mathcal{B}_b(E)$, $\rho_2\in\mathcal{B}_b(E,\mathbb{R}_+)$ and $\pi(x,dy)$ is a kernel from E to $(0,\infty)$ such that $\sup_{x\in E}\int_{(0,\infty)}(y\wedge y^2)\pi(x,dy)<\infty$;
- $X=\{(X_t)_{t\geq 0};(\mathbf{P}_\mu)_{\mu\in\mathcal{M}(E)}\}$ is an $\mathcal{M}(E)$-valued Hunt process with transition probability determined by

$$\mathbf{P}_\mu[e^{-X_t(f)}]=e^{-\mu(V_tf)},\quad t\geq 0,\mu\in\mathcal{M}(E),f\in\mathcal{B}_b^+(E),$$

  where for each $f\in\mathcal{B}_b(E)$, the function $(t,x)\mapsto V_tf(x)$ on $[0,\infty)\times E$ is the unique locally bounded non-negative solution to the equation

$$V_tf(x)+\Pi_x\Big[\int_0^{t\wedge\zeta}\psi(\xi_s,V_{t-s}f)ds\Big]=\Pi_x[f(\xi_t)\mathbf{1}_{t<\zeta}],\quad t\geq 0,x\in E.$$

We refer our readers to [28] for more discussions about the definition and the existence of superprocesses. To avoid triviality, we assume that $\psi(x,z)$ is not identically equal to $-\rho_1(x)z$.

Notice that the branching mechanism ψ can be extended into a map from $E\times\mathbb{C}_+$ to $\mathbb{C}$ using (A.4). Define

$$\psi'(x,z):=-\rho_1(x)+2\rho_2(x)z+\int_{(0,\infty)}(1-e^{-zy})y\pi(x,dy),\quad x\in E,z\in\mathbb{C}_+.$$

Then according to Lemma A.2, for each $x\in E$, $z\mapsto\psi(x,z)$ is a holomorphic function on $\mathbb{C}_+^0$ with derivative $z\mapsto\psi'(x,z)$. Define $\psi_0(x,z):=\psi(x,z)+\rho_1(x)z$ and $\psi_0'(x,z):=\psi'(x,z)+\rho_1(x)$.

Denote by $\mathbb{W}$ the space of $\mathcal{M}(E)$-valued càdlàg paths with its canonical path denoted by $(W_t)_{t\geq 0}$. We say X is *non-persistent* if $\mathbf{P}_{\delta_x}(\|X_t\|=0)>0$ for all $x\in E$ and $t>0$. Suppose that $(X_t)_{t\geq 0}$ is non-persistent, then according to [28, Section 8.4], there is a unique family of measures $(\mathbb{N}_x)_{x\in E}$ on $\mathbb{W}$ such that (i) $\mathbb{N}_x(\forall t>0,\|W_t\|=0)=0$; (ii) $\mathbb{N}_x(\|W_0\|\neq 0)=0$; and (iii) if $\mathcal{N}$ is a Poisson random measure defined on some probability space with intensity $\mathbb{N}_\mu(\cdot):=\int_E\mathbb{N}_x(\cdot)\mu(dx)$, then the superprocess $\{X;\mathbf{P}_\mu\}$ can be realized by $\widetilde{X}_0:=\mu$ and $\widetilde{X}_t(\cdot):=\mathcal{N}[W_t(\cdot)]$ for each $t>0$. We refer to $(\mathbb{N}_x)_{x\in E}$ as the *Kuznetsov measures* of X.

A.4 Semigroups for superprocesses

Let X be a non-persistent superprocess with its Kuznetsov measure denoted by $(\mathbb{N}_x)_{x\in E}$. We define the mean semigroup

$$P_t^{\rho_1}f(x):=\Pi_x[e^{\int_0^t\rho_1(\xi_s)ds}f(\xi_t)\mathbf{1}_{t<\zeta}],\quad t\geq 0,x\in E,f\in\mathcal{B}_b(E,\mathbb{R}_+).$$

It is known from [28, Proposition 2.27] and [27, Theorem 2.7] that for all $t > 0$, $\mu \in \mathcal{M}(E)$ and $f \in \mathcal{B}_b(E, \mathbb{R}_+)$,

$$\mathbb{N}_\mu[W_t(f)] = \mathbf{P}_\mu[X_t(f)] = \mu(P_t^{\rho_1} f). \tag{A.5}$$

Define

$$L_1(\xi) := \{f \in \mathcal{B}(E) : \forall x \in E, t \geq 0, \quad \Pi_x[|f(\xi_t)|] < \infty\},$$
$$L_2(\xi) := \{f \in \mathcal{B}(E) : |f|^2 \in L_1(\xi)\}.$$

Using monotonicity and linearity, we get from (A.5) that

$$\mathbb{N}_x[W_t(f)] = \mathbf{P}_{\delta_x}[X_t(f)] = P_t^{\rho_1} f(x) \in \mathbb{R}, \quad f \in L_1(\xi), t > 0, x \in E.$$

This says that the random variable $X_t(f)$ is well defined under probability $\mathbf{P}_{\delta_x}$ provided $f \in L_1(\xi)$. By the branching property of the superprocess, $X_t(f)$ is an infinitely divisible random variable. Therefore, we can write

$$U_t(\theta f)(x) := \operatorname{Log} \mathbf{P}_{\delta_x}[e^{i\theta X_t(f)}], \quad t \geq 0, f \in L_1(\xi), \theta \in \mathbb{R}, x \in E,$$

as its characteristic exponent. According to Campbell's formula, see [27, Theorem 2.7] for example, we have

$$\mathbf{P}_{\delta_x}[e^{i\theta X_t(f)}] = \exp(\mathbb{N}_x[e^{i\theta W_t(f)} - 1]), \quad t > 0, f \in L_1(\xi), \theta \in \mathbb{R}, x \in E.$$

Noticing that $\theta \mapsto \mathbb{N}_x[e^{i\theta W_t(f)} - 1]$ is a continuous function on $\mathbb{R}$ and that $\mathbb{N}_x[e^{i\theta W_t(f)} - 1] = 0$ if $\theta = 0$, according to [41, Lemma 7.6], we have

$$U_t(\theta f)(x) = \mathbb{N}_x[e^{iW_t(\theta f)} - 1], \quad t > 0, f \in L_1(\xi), \theta \in \mathbb{R}, x \in E. \tag{A.6}$$

Lemma A.5. *There exists a constant $C \geq 0$ such that for all $f \in L_1(\xi), x \in E$ and $t \geq 0$, we have*

$$|\psi(x, -U_t f)| \leq C P_t^{\rho_1}|f|(x) + C(P_t^{\rho_1}|f|(x))^2. \tag{A.7}$$

Proof. Noticing that $e^{\operatorname{Re} U_t f(x)} = |e^{U_t f(x)}| = |\mathbf{P}_{\delta_x}[e^{iX_t(f)}]| \leq 1$, we have

$$\operatorname{Re} U_t f(x) \leq 0. \tag{A.8}$$

Therefore, we can speak of $\psi(x, -U_t f)$ since $z \mapsto \psi(x, z)$ is well defined on $\mathbb{C}_+$. According to Lemma A.1, we have that

$$|U_t f(x)| \leq \mathbb{N}_x[|e^{iW_t(f)} - 1|] \leq \mathbb{N}_x[|iW_t(f)|] \leq (P_t^{\rho_1}|f|)(x). \tag{A.9}$$

Notice that, for any compact $K \subset \mathbb{R}$,

$$\mathbb{N}_x\Big[\sup_{\theta \in K}\Big|\frac{\partial}{\partial\theta}(e^{i\theta W_t(f)} - 1)\Big|\Big] \leq \mathbb{N}_x[|W_t(f)|] \sup_{\theta\in K}|\theta| \leq (P_t^{\rho_1}|f|)(x) \sup_{\theta \in K}|\theta| < \infty.$$

Therefore, according to [12, Theorem A.5.2] and (A.6), $U_t(\theta f)(x)$ is differentiable in $\theta \in \mathbb{R}$ with

$$\frac{\partial}{\partial\theta} U_t(\theta f)(x) = i\mathbb{N}_x[W_t(f)e^{i\theta W_t(f)}], \quad \theta \in \mathbb{R}.$$

Moreover, from the above, it is clear that

$$\sup_{\theta\in\mathbb{R}}\Big|\frac{\partial}{\partial\theta} U_t(\theta f)(x)\Big| \leq (P_t^{\rho_1}|f|)(x). \tag{A.10}$$

It follows from the dominated convergence theorem that $(\partial/\partial\theta)U_t(\theta f)(x)$ is continuous in θ. In other words, $\theta \mapsto -U_t(\theta f)(x)$ is a C^1 map from $\mathbb{R}$ to $\mathbb{C}_+$. Thus,

$$\psi(x, -U_t f) = -\int_0^1 \psi'(x, -U_t(\theta f))\frac{\partial}{\partial\theta}U_t(\theta f)(x)\, d\theta. \tag{A.11}$$

Notice that

$$\begin{aligned}
&|\psi'(x, -U_t f)| \\
&= \Big| -\rho_1(x) - 2\rho_2(x)U_t f(x) + \int_{(0,\infty)} y(1 - e^{yU_t f(x)})\pi(x, dy)\Big| \\
&= \Big| -\rho_1(x) - 2\rho_2(x)\mathbb{N}_x[e^{iW_t(f)} - 1] + \int_{(0,\infty)} y\mathbf{P}_{y\delta_x}[1 - e^{iX_t(f)}]\pi(x, dy)\Big| \\
&\le \|\rho_1\|_\infty + 2\rho_2(x)\mathbb{N}_x[W_t(|f|)] + \int_{(0,\infty)} y\mathbf{P}_{y\delta_x}[2 \wedge X_t(|f|)]\pi(x, dy) \\
&\le \|\rho_1\|_\infty + 2\|\rho_2\|_\infty P_t^{\rho_1}|f|(x) + \Big(\sup_{x\in E}\int_{(0,1]} y^2\pi(x, dy)\Big)\, P_t^{\rho_1}|f|(x) + 2\sup_{x\in E}\int_{(1,\infty)} y\pi(x, dy) \\
&=: C_1 + C_2(P_t^{\rho_1}|f|)(x),
\end{aligned} \tag{A.12}$$

where C_1, C_2 are constants independent of f, x and t. Now, combining the display above with (A.11) and (A.10) we get the desired result. □

This lemma also says that if $f \in L^2(\xi)$, then $\Pi_x\Big[\int_0^t \psi(\xi_s, -U_{t-s}f)ds\Big] \in \mathbb{C}, x \in E, t \ge 0$, is well defined. In fact, using Jensen's inequality and the Markov property, we have

$$\begin{aligned}
&\Pi_x\Big[\int_0^t |\psi(\xi_s, -U_{t-s}f)|ds\Big] \le \Pi_x\Big[\int_0^t (C_1 P_{t-s}^{\rho_1}|f|(\xi_s) + C_2 P_{t-s}^{\rho_1}|f|(\xi_s)^2)ds\Big] \\
&\le \int_0^t (C_1 e^{t\|\rho_1\|}\Pi_x[\Pi_{\xi_s}[|f(\xi_{t-s})|]] + C_2 e^{2t\|\rho_1\|}\Pi_x[\Pi_{\xi_s}[|f(\xi_{t-s})|]^2])\, ds \\
&\le \int_0^t (C_1 e^{t\|\rho_1\|}\Pi_x[|f(\xi_t)|] + C_2 e^{2t\|\rho_1\|}\Pi_x[|f(\xi_t)|^2])\, ds < \infty.
\end{aligned} \tag{A.13}$$

A.5 A complex-valued non-linear integral equation

Let X be a non-persistent superprocess. In this subsection, we will prove the following:

Proposition A.6. *If $f \in L_2(\xi)$, then for all $t \ge 0$ and $x \in E$,*

$$U_t f(x) - \Pi_x\Big[\int_0^t \psi(\xi_s, -U_{t-s}f)ds\Big] = i\Pi_x[f(\xi_t)]. \tag{A.14}$$

$$U_t f(x) - \int_0^t P_{t-s}^{\rho_1}\psi_0(\cdot, -U_s f)(x)\, ds = iP_t^{\rho_1} f(x). \tag{A.15}$$

To prove this, we will need the generalized spine decomposition theorem from [34]. Let $f \in \mathcal{B}_b(E, \mathbb{R}_+)$, $T > 0$ and $x \in E$. Suppose that $\mathbf{P}_{\delta_x}[X_T(f)] = \mathbb{N}_x[W_T(f)] = P_T^{\rho_1} f(x) \in (0, \infty)$, then we can define the following probability transforms:

$$d\mathbf{P}_{\delta_x}^{X_T(f)} := \frac{X_T(f)}{P_T^{\rho_1} f(x)}d\mathbf{P}_{\delta_x}; \quad d\mathbb{N}_x^{W_T(f)} := \frac{W_T(f)}{P_T^{\rho_1} f(x)}d\mathbb{N}_x.$$

Following the definition in [34], we say that $\{\xi, \mathbf{n}; \mathbf{Q}_x^{(f,T)}\}$ is a spine representation of $\mathbb{N}_x^{\langle W_T, f\rangle}$ if

- the spine process $\{(\xi_t)_{0\le t\le T};\mathbf{Q}_x^{(f,T)}\}$ is a copy of $\{(\xi_t)_{0\le t\le T};\Pi_x^{(f,T)}\}$, where

$$d\Pi_x^{(f,T)} := \frac{f(\xi_T)e^{\int_0^T\rho_1(\xi_s)ds}}{P_T^{\rho_1}f(x)}d\Pi_x;$$

- given $\{(\xi_t)_{0\le t\le T};\mathbf{Q}_x^{(f,T)}\}$, the immigration measure $\{\mathbf{n}(\xi,ds,dw);\mathbf{Q}_x^{(f,T)}[\cdot|(\xi_t)_{0\le t\le T}]\}$ is a Poisson random measure on $[0,T]\times\mathbb{W}$ with intensity

$$\mathbf{m}(\xi,ds,dw) := 2\rho_2(\xi_s)ds\cdot\mathbb{N}_{\xi_s}(dw) + ds\cdot\int_{y\in(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}(X\in dw)\pi(\xi_s,dy); \quad \text{(A.16)}$$

- $\{(Y_t)_{0\le t\le T};\mathbf{Q}_x^{(f,T)}\}$ is an $\mathcal{M}(E)$-valued process defined by

$$Y_t := \int_{(0,t]\times\mathbb{W}} w_{t-s}\mathbf{n}(\xi,ds,dw), \quad 0\le t\le T.$$

According to the spine decomposition theorem in [34], we have that

$$\{(X_s)_{s\ge 0};\mathbf{P}_{\delta_x}^{X_T(f)}\} \overset{f.d.d.}{=} \{(X_s+W_s)_{s\ge0};\mathbf{P}_{\delta_x}\otimes\mathbb{N}_x^{W_T(f)}\}, \quad \text{(A.17)}$$

$$\{(W_s)_{0\le s\le T};\mathbb{N}_x^{W_T(f)}\} \overset{f.d.d.}{=} \{(Y_s)_{s\ge0};\mathbf{Q}_x^{(f,T)}\}. \quad \text{(A.18)}$$

Proof of Proposition A.6. Assume that $f\in\mathcal{B}_b(E)$. Fix $t>0, r\in[0,t), x\in E$ and a strictly positive $g\in\mathcal{B}_b(E)$. Denote by $\{\xi,\mathbf{n};\mathbf{Q}_x^{(g,t)}\}$ the spine representation of $\mathbb{N}_x^{W_t(g)}$. Conditioned on $\{\xi;\mathbf{Q}_x^{(g,t)}\}$, denote by $\mathbf{m}(\xi,ds,dw)$ the conditional intensity of $\mathbf{n}$ in (A.16). Denote by $\Pi_{r,x}$ the probability of Hunt process $\{\xi;\Pi\}$ initiated at time r and position x. From Lemma A.1, we have $\mathbf{Q}_x^{(g,t)}$-almost surely

$$\begin{aligned}
&\int_{[0,t]\times\mathbb{W}}|e^{iw_{t-s}(f)}-1|\mathbf{m}(\xi,ds,dw) \le \int_{[0,t]\times\mathbb{W}}(|w_{t-s}(f)|\wedge 2)\mathbf{m}(\xi,ds,dw)\\
&\le \int_0^t\Big(2\rho_2(\xi_s)\mathbb{N}_{\xi_s}(W_{t-s}(|f|)) + \int_{(0,1]}y\mathbf{P}_{y\delta_{\xi_s}}[X_{t-s}(|f|)]\pi(\xi_s,dy) + 2\int_{(1,\infty)}y\pi(\xi_s,dy)\Big)ds\\
&\le \int_0^t(P_{t-s}^{\rho_1}|f|)(\xi_s)\Big(2\rho_2(\xi_s)+\int_{(0,1]}y^2\pi(\xi_s,dy)\Big)ds + 2t\sup_{x\in E}\int_{(1,\infty)}y\pi(x,dy)\\
&\le \Big(2\|\rho_2\|_\infty + \sup_{x\in E}\int_{(0,1]}y^2\pi(x,dy)\Big)te^{t\|\rho_1\|_\infty}\|f\|_\infty + 2t\sup_{x\in E}\int_{(1,\infty)}y\pi(x,dy)<\infty.
\end{aligned}$$

Using this, Fubini's theorem, (A.6) and (A.8) we have $\mathbf{Q}_x^{(g,t)}$-almost surely,

$$\begin{aligned}
&\int_{[0,t]\times\mathbb{N}}(e^{iw_{t-s}(f)}-1)\mathbf{m}(\xi,ds,dw)\\
&=\int_0^t\Big(2\rho_2(\xi_s)\mathbb{N}_{\xi_s}(e^{iW_{t-s}(f)}-1)+\int_{(0,\infty)}y\mathbf{P}_{y\delta_{\xi_s}}[e^{iX_{t-s}(f)}-1]\pi(\xi_s,dy)\Big)ds\\
&=\int_0^t\Big(2\rho_2(\xi_s)U_{t-s}f(\xi_s)+\int_{(0,\infty)}y(e^{yU_{t-s}f(\xi_s)}-1)\pi(\xi_s,dy)\Big)ds\\
&=-\int_0^t\psi_0'(\xi_s,-U_{t-s}f)ds.
\end{aligned}$$

Therefore, according to (A.18), Campbell's formula and above, we have that

$$\begin{aligned}
\mathbb{N}_x^{W_t(g)}[e^{iW_t(f)}] &= \mathbf{Q}_x^{(g,t)}\Big[\exp\Big\{\int_{[0,t]\times\mathbb{N}}(e^{iw_{t-s}(f)}-1)\mathbf{m}(\xi,ds,dw)\Big\}\Big] \quad \text{(A.19)}\\
&= \Pi_x^{(g,t)}[e^{-\int_0^t\psi_0'(\xi_s,-U_{t-s}f)ds}] = \frac{1}{P_t^{\rho_1}g(x)}\Pi_x[g(\xi_t)e^{-\int_0^t\psi'(\xi_s,-U_{t-s}f)ds}].
\end{aligned}$$

Let $\epsilon > 0$. Define $f^+ = (f \vee 0) + \epsilon$ and $f^- = (-f) \vee 0 + \epsilon$, then $f^\pm$ are strictly positive and $f = f^+ - f^-$. According to (A.17), we have that

$$\frac{\mathbf{P}_{\delta_x}[X_t(f^\pm)e^{iX_t(f)}]}{\mathbf{P}_{\delta_x}[X_t(f^\pm)]} = \mathbf{P}_{\delta_x}[e^{iX_t(f)}]\mathbb{N}_x^{W_t(f^\pm)}[e^{iX_t(f)}].$$

Using (A.19) and the above, we have

$$\begin{aligned}\frac{\mathbf{P}_{\delta_x}[X_t(f)e^{iX_t(f)}]}{\mathbf{P}_{\delta_x}[e^{iX_t(f)}]} &= \mathbf{P}_{\delta_x}[X_t(f^+)]\mathbb{N}_x^{W_t(f^+)}[e^{iX_t(f)}] - \mathbf{P}_{\delta_x}[X_t(f^-)]\mathbb{N}_x^{W_t(f^-)}[e^{iX_t(f)}] \\ &= \Pi_x[f(\xi_t)e^{-\int_0^t \psi'(\xi_s,-U_{t-s}f)ds}].\end{aligned}$$

Therefore, we have

$$\frac{\partial}{\partial\theta}U_t(\theta f)(x) = \frac{\mathbf{P}_{\delta_x}[iX_t(f)e^{iX_t(f)}]}{\mathbf{P}_{\delta_x}[e^{iX_t(f)}]} = \Pi_x[if(\xi_t)e^{-\int_0^t \psi'(\xi_s,-U_{t-s}(\theta f))ds}].$$

Since $\{(\xi_{r+t})_{t\geq 0}; \Pi_{r,x}\} \overset{d}{=} \{(\xi_t)_{t\geq 0}; \Pi_x\}$, we have

$$\begin{aligned}&\frac{\partial}{\partial\theta}U_{t-r}(\theta f)(x) = \Pi_x[if(\xi_{t-r})e^{-\int_0^{t-r} \psi'(\xi_s,-U_{t-r-s}(\theta f))ds}] \\ &= \Pi_{r,x}[if(\xi_t)e^{-\int_0^{t-r} \psi'(\xi_{r+s},-U_{t-r-s}(\theta f))ds}] = \Pi_{r,x}[if(\xi_t)e^{-\int_r^t \psi'(\xi_s,-U_{t-s}(\theta f))ds}].\end{aligned}$$

From (A.12), we know that for each $\theta \in \mathbb{R}$, $(t,x) \mapsto |\psi'(x, -U_t f(x))|$ is locally bounded (i.e. bounded on $[0,T] \times E$ for each $T \geq 0$). Therefore, we can apply Lemma A.4 and get that

$$\frac{\partial}{\partial\theta}U_{t-r}(\theta f)(x) + \Pi_{r,x}\Big[\int_r^t \psi'(\xi_s, -U_{t-s}(\theta f))\frac{\partial}{\partial\theta}U_{t-s}(\theta f)(\xi_s)\ ds\Big] = \Pi_{r,x}[if(\xi_t)]$$

and

$$\begin{aligned}&\frac{\partial}{\partial\theta}U_{t-r}(\theta f)(x) + \Pi_{r,x}\Big[\int_r^t e^{\int_r^s \rho_1(\xi_u)du}\psi_0'(\xi_s, -U_{t-s}(\theta f))\frac{\partial}{\partial\theta}U_{t-s}(\theta f)(\xi_s)\ ds\Big] \\ &= \Pi_{r,x}[ie^{\int_r^t \rho_1(\xi_s)ds}f(\xi_t)].\end{aligned}$$

Integrating the two displays above with respect to θ on [0,1], using Fubini's theorem, (A.10), (A.11) and (A.12), we get

$$U_{t-r}f(x) - \Pi_{r,x}\Big[\int_r^t \psi(\xi_s, -U_{t-s}f)\ ds\Big] = i\theta\Pi_{r,x}[f(\xi_t)]$$

and

$$U_{t-r}f(x) - \Pi_{r,x}\Big[\int_r^t e^{\int_r^s \rho_1(\xi_u)du}\psi_0(\xi_s, -U_{t-s}f)\ ds\Big] = i\Pi_{r,x}[e^{\int_r^t \rho_1(\xi_u)du}f(\xi_t)].$$

Taking $r = 0$, we get that (A.14) and (A.15) are true if $f \in \mathcal{B}_b(E)$.

The rest of the proof is to evaluate (A.14) and (A.15) for all $f \in L_2(\xi)$. We only do this for (A.14) since the argument for (A.15) is similar. Let $n \in \mathbb{N}$. Writing $f_n := (f^+ \wedge n) - (f^- \wedge n)$, then $f_n \xrightarrow[n\to\infty]{} f$ pointwise. From what we have proved, we have

$$U_t f_n(x) - \Pi_x\Big[\int_0^t \psi(\xi_s, -U_{t-s}f_n)\ ds\Big] = i\Pi_x[f_n(\xi_t)]. \tag{A.20}$$

Note that (i) $\Pi_x[f_n(\xi_t)] \xrightarrow[n\to\infty]{} \Pi_x[f(\xi_t)]$; (ii) by (A.6), the dominated convergence theorem and the fact that

$$|e^{iW_t(f_n)} - 1| \le W_t(|f|); \quad \mathbb{N}_x[W_t(|f|)] = (P_t^{\rho_1}|f|)(x) < \infty,$$

we have $U_t f_n(x) \xrightarrow[n\to\infty]{} U_t f(x)$, and (iii) by the dominated convergence theorem, (A.13) and the fact (see (A.7)) that

$$|\psi(\xi_s, -U_{t-s}f_n)| \le C_1 P_{t-s}^{\rho_1}|f|(\xi_s) + C_2 P_{t-s}^{\rho_1}|f|(\xi_s)^2,$$

we get that $\Pi_x[\int_0^t \psi(\xi_s, -U_{t-s}f_n)ds] \xrightarrow[n\to\infty]{} \Pi_x[\int_0^t \psi(\xi_s, -U_{t-s}f)ds]$. Using these, letting $n \to \infty$ in (A.20), we get the desired result. □

References

[1] R. Adamczak and P. Miłoś, *CLT for Ornstein-Uhlenbeck branching particle system*, Electron. J. Probab. **20** (2015), no. 42, 35 pp. MR-3339862

[2] S. Asmussen, *Convergence rates for branching processes*, Ann. Probab. **4** (1976), no. 1, 139–146. MR-0391286

[3] S. Asmussen and H. Hering, *Branching processes*, Progress in Probability and Statistics, 3. Birkhäuser Boston, Inc., Boston, MA, 1983. MR-0701538

[4] K. B. Athreya, *Limit theorems for multitype continuous time Markov branching processes. I. The case of an eigenvector linear functional*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **12** (1969), 320–332. MR-0254927

[5] K. B. Athreya, *Limit theorems for multitype continuous time Markov branching processes. II. The case of an arbitrary linear functional*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **13** (1969), 204–214. MR-0254928

[6] K. B. Athreya, *Some refinements in the theory of supercritical multitype Markov branching processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **20** (1971), 47–57. MR-0307367

[7] J. Bertoin, *Lévy processes*. Cambridge Tracts in Mathematics, 121. Cambridge University Press, 1996. MR-1406564

[8] J. Bertoin, B. Roynette and M. Yor, *Some connections between (sub)critical branching mechanisms and Bernstein functions*. arXiv:0412322v1.

[9] Z.-Q. Chen, Y.-X. Ren, and H. Wang, *An almost sure scaling limit theorem for Dawson-Watanabe superprocesses*, J. Funct. Anal. **254** (2008), no. 7, 1988–2019. MR-2397881

[10] Z.-Q. Chen, Y.-X. Ren, R. Song, and R. Zhang, *Strong law of large numbers for supercritical superprocesses under second moment condition*, Front. Math. China **10** (2015), no. 4, 807–838. MR-3352888

[11] Z.-Q. Chen, Y.-X. Ren, and T. Yang, *Skeleton decomposition and law of large numbers for supercritical superprocesses*, Acta Appl. Math. **159** (2019), 225–285. MR-3904489

[12] R. Durrett, *Probability: theory and examples*, Fourth edition. Cambridge Series in Statistical and Probabilistic Mathematics, 31. Cambridge University Press, Cambridge, 2010. MR-2722836

[13] E. B. Dynkin, *Superprocesses and partial differential equations*, Ann. Probab. **21** (1993), no. 3, 1185–1262. MR-1235414

[14] M. Eckhoff, A. E. Kyprianou, and M. Winkel, *Spines, skeletons and the strong law of large numbers for superdiffusions*, Ann. Probab. **43** (2015), no. 5, 2545–2610. MR-3395469

[15] J. Engländer, *Law of large numbers for superdiffusions: the non-ergodic case*, Ann. Inst. Henri Poincaré Probab. Stat. **45** (2009), no. 1, 1–6. MR-2500226

[16] J. Engländer and A. Winter, *Law of large numbers for a class of superdiffusions*, Ann. Inst. H. Poincaré Probab. Statist. **42** (2006), no. 2, 171–185. MR-2199796

[17] J. Engländer and D. Turaev, *A scaling limit theorem for a class of superdiffusions*, Ann. Probab. **30** (2002), no. 2, 683–722. MR-1905855

[18] C. C. Heyde, *A rate of convergence result for the super-critical Galton-Watson process*, J. Appl. Probability **7** (1970), 451–454. MR-0288859

[19] C. C. Heyde, *some central limit analogues for supercritical Galton-Watson processes*, J. Appl. Probability **8** (1971), 52–59. MR-0282422

[20] C. C. Heyde and B. M. Brown, *An invariance principle and some convergence rate results for branching processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, **20** (1971), 271–278. MR-0310987

[21] C. C. Heyde and J. R. Leslie, *Improved classical limit analogues for Galton-Watson processes with or without immigration*, Bull. Austral. Math. Soc. **5** (1971), 145–155. MR-0293731

[22] A. Iksanov, K. Kolesko, and M. Meiners, *Stable-like fluctuations of Bgins' martingales*, Stochastic Process. Appl. (2018). MR-4013869

[23] S. Janson, *Functional limit theorems for multitype branching processes and generalized Pólya urns*, Stochastic Process. Appl. **110** (2004), no. 2, 177–245. MR-2040966

[24] H. Kesten and B. P. Stigum, *Additional limit theorems for indecomposable multidimensional Galton-Watson processes*, Ann. Math. Statist. **37** (1966), 1463–1481. MR-0200979

[25] H. Kesten and B. P. Stigum, *A limit theorem for multidimensional Galton-Watson processes*, Ann. Math. Statist. **37** (1966), 1211–1223. MR-0198552

[26] M. A. Kouritzin, and Y.-X. Ren, *A strong law of large numbers for super-stable processes*, Stochastic Process. Appl. **121** (2014), no. 1, 505–521. MR-3131303

[27] A. E. Kyprianou, *Fluctuations of Lévy processes with applications*, Introductory lectures. Second edition. Universitext. Springer, Heidelberg, 2014. MR-3155252

[28] Z. Li, *Measure-valued branching Markov processes*, Probability and its Applications (New York). Springer, Heidelberg, 2011. MR-2760602

[29] R.-L. Liu, Y.-X. Ren, and R. Song, *$L\log L$ criterion for a class of superdiffusions*, J. Appl. Probab. **46** (2009), no. 2, 479–496. MR-2535827

[30] R.-L. Liu, Y.-X. Ren, and R. Song, *Strong law of large numbers for a class of superdiffusions*, Acta Appl. Math. **123** (2013), 73–97. MR-3010225

[31] R. Marks and P. Miłoś, *CLT for supercritical branching processes with heavy-tailed branching law*, arXiv:1803.05491v2.

[32] G. Metafune, D. Pallara, and E. Priola, *Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures*, J. Funct. Anal. **196** (2002), no. 1, 40–60. MR-1941990

[33] P. Miłoś, *Spatial central limit theorem for supercritical superprocesses*, J. Theoret. Probab. **31** (2018), no. 1, 1–40. MR-3769806

[34] Y.-X. Ren, R. Song, and Z. Sun, *Spine decompositions and limit theorems for a class of critical superprocesses*, Acta Appl. Math. (2019).

[35] Y.-X. Ren, R. Song, and Z. Sun, *Limit theorems for a class of critical superprocesses with stable branching*, arxiv:1807.02837v2.

[36] Y.-X. Ren, R. Song, and R. Zhang, *Central limit theorems for super Ornstein-Uhlenbeck processes*, Acta Appl. Math. **130** (2014), 9–49. MR-3180938

[37] Y.-X. Ren, R. Song, and R. Zhang, *Central limit theorems for supercritical branching Markov processes*, J. Funct. Anal. **266** (2014), no. 3, 1716–1756. MR-3146834

[38] Y.-X. Ren, R. Song, and R. Zhang, *Central limit theorems for supercritical superprocesses*, Stochastic Process. Appl. **125** (2015), no. 2, 428–457. MR-3293289

[39] Y.-X. Ren, R. Song, and R. Zhang, *Central limit theorems for supercritical branching nonsymmetric Markov processes*, Ann. Probab. **45** (2017), no. 1, 564–623. MR-3601657

[40] Y.-X. Ren, R. Song, and R. Zhang, *Functional central limit theorems for supercritical superprocesses*, Acta Appl. Math. **147** (2017), 137–175. MR-3592799

[41] K. Sato, *Lévy processes and infinitely divisible distributions*, Translated from the 1990 Japanese original. Revised by the author. Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge, 1999. MR-1739520

[42] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein functions.* Theory and applications. Second edition. De Gruyter Studies in Mathematics, 37. Walter de Gruyter & Co., Berlin, 2012. MR-2978140

[43] E. M. Stein and R. Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, 2. Princeton University Press, Princeton, NJ, 2003. MR-1976398

[44] L. Wang, *An almost sure limit theorem for super-Brownian motion*, J. Theoret. Probab. **23** (2010), no. 2, 401–416. MR-2644866

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