# STABLE CONSTANT MEAN CURVATURE TORI AND THE ISOPERIMETRIC PROBLEM IN THREE SPACE FORMS

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## Introduction

Let $\psi : M \to N$ be an immersion of an orientable surface into a three dimensional oriented Riemannian manifold. Then $\psi$ has constant mean curvature if and only if it is a critical point of the area functional for any compactly supported variation that preserves the volume *enclosed* by the surface. In this context we say that the constant mean curvature immersion $\psi$ is *stable* if the second variation formula of the area, which we call henceforth the *index form* of $\psi$, is non negative for all variations of the above type. Otherwise, $\psi$ is stable if for any $f \in C^\infty(M)$ with compact support such that $\int_M f\,dA = 0$, we have

$$I(f,f) = \int_M \{|\nabla f|^2 - (\mathrm{Ric}(\xi) + |\sigma|^2) f^2\}\,dA \geq 0, \tag{1}$$

where $\nabla f$ is the gradient of $f$, $\xi$ is the unit normal vector field of the immersion, Ric is the Ricci curvature of the ambient manifold $N$, and $|\sigma|^2$ is the square of the norm of the second fundamental form $\sigma$ of $\psi$. For more details see [3].

We are interested in studying stability when the metric induced on $M$ is complete. If the ambient manifold is a simply connected complete space form $N(c)$ of constant sectional curvature $c$, then it has been proved by Barbosa and do Carmo [2], Barbosa, do Carmo and Eschenburg [3], El Soufi and Ilias [8], and Heintze [13] that the only compact with no boundary stable surface is the umbilical sphere. In fact this result has been proved for arbitrary dimension.

Let $\psi : M \to N$ be a stable constant mean curvature immersion of a complete orientable surface into a three dimensional oriented Riemannian manifold. Then the following interesting, although partial results, are known

**Theorem 1.** *If $N$ is compact and has positive Ricci curvature, then $M$ is compact and connected, and genus(M)$\leq 3$.*

**Theorem 2.** *If $N$ is complete, has non negative Ricci curvature, positive injectivity radius and its sectional curvature is bounded from above, then either*

(i) *$\psi$ is totally geodesic, or*

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(ii) *M is compact and connected and genus(M)*$\leq 5$.

The compactness in theorem 1 follows from the results of Fischer–Colbrie [9] and López and Ros [20, theorem 4], and in theorem 2 from the sharper results of Frensel [11], and da Silveira [25, theorem 1.6], see also [4]. The connectedness of $M$ is easy to prove, in both theorems, because otherwise we could construct a non identically zero locally constant function with vanishing mean value contradicting the inequality (1). Connectedness is relevant in stability problems in two different ways: first, the connected components of an unstable surface could be stable and second, the most interesting stable surfaces appear as solutions to the isoperimetric problem (see below) and in general these surfaces are not connected.

The genus estimate in theorems 1 and 2 is obtained by El Soufi and Ilias [8], Frensel [11] and Yau [28]. If the genus of the surface is 4 or 5 we conclude easily from their proof that $M$ must be a minimal surface.

Theorem 1 is applied to elliptic space forms and theorem 2 to the flat ones. In particular taking $N$ equal to the Euclidean 3–space or to the unit 3–sphere, we obtain the results of López and Ros [20], Palmer [22], and da Silveira [25].

These results give a classification of complete stable constant mean curvature immersions in $\mathbb{R}^3$ and in $S^3(1)$. When the ambient manifold $N(c)$, $c \geq 0$, is not simply connected the problem is unsolved. We can easily construct flat stable constant mean curvature tori in some quotients or $\mathbb{R}^3$ and $S^3(1)$. More interesting examples have been found in flat three dimensional tori by Ross [24], who has proved that the classical Schwarz minimal surfaces of genus three are stable. Also, from the informal argument exposed by Frankel at the end of [10], it is natural to hope that the Poincaré dodecahedral space, an elliptic space form which is covered 120 times by the three sphere, contains a stable embedded constant mean curvature, in fact minimal, surface of genus two.

In this paper we first prove that

*"stable constant mean curvaturetori in three space forms are flat".*

In the flat case these tori are quotients of a plane or of the flat circular cylinder in $\mathbb{R}^3$. In the elliptic one the tori must be a quotient of a Clifford torus $S^1(a)\times S^1(b) \subset S^3(1)$, with $a^2 + b^2 = 1$. In the hyperbolic case, the tori are quotients of horospheres or of tubes about geodesics.

Our next result gives the classification of complete stable constant mean curvaturesurfaces in the three dimensional real projective space with its standard metric. More precisely

> *"an orientable complete stable constant mean curvaturesurface in* $\mathbb{RP}^3(1)$ *is a geodesic sphere, a twofold covering of a real projective plane or an embedded flat tube of radius r, with* $\pi/6 \leq r \leq \pi/3$*, about a geodesic".*

We also prove that any stable constant mean curvaturesurface of genus 2 or 3 in an elliptic space form covered 3 or 4 times by the three sphere $S^3(1)$ must be embedded.

The isoperimetric problem is one of the great global problems in Riemannian geometry. A part of it can be stated as follows: if $N$ is a 3–dimensional compact Riemannian manifold, given a positive number $V$, with $0 < V <$ volume$(N)$, find the embedded compact surfaces of least area which enclose a domain of volume $V$. In this form the problem has always a smooth compact solution because of the existence result of Almgrem [1] and the regularity theorem of Taylor [27]; this solution has constant mean curvatureand is stable. The same result holds if $N$ is a complete non compact flat three manifold. The solution of this problem may be condensed in a function called the *isoperimetric profile* of $N$ which associates to each positive number $V$ the area $A$ of the surface of least area enclosing a volume $V$. For simply connected complete space forms the problem was solved by Schmidt [26]. In the metric product $S^2(1) \times \mathbb{R}$ it was solved by Pedrosa [23], see also [4]. For complete flat three manifolds the problem is reduced to classical differential geometry and is, essentially, unsolved. The case of flat three dimensional tori seems to be specially complicate and attractive.

As an application of our results, we solve the isoperimetric problem for the real projective space, a problem proposed by do Carmo [4], and for oriented complete flat three manifolds with cyclic fundamental group. In both cases the surfaces of least area enclosing a given volume are geodesic spheres or flat tubes about geodesics.

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## 1. Stability of constant mean curvaturetori

Let $(M, ds^2)$ be a compact orientable Riemannian surface and $N(c)$ a complete oriented three dimensional Riemannian manifold with constant sectional curvature $c$. If $\psi : M \to N(c)$ is an isometric immersion with constant mean curvature$H$ and second fundamental form $\sigma$, we can consider on $M$ the associated holomorphic quadratic differential $\sigma^{2,0}$ such that, if $z = x + i\,y$ is a local conformal coordinate in $M$, then

$$\sigma^{2,0} = 4\,\sigma(\partial_z, \partial_z)\,dz^2 = \{\sigma(\partial_x, \partial_x) - \sigma(\partial_y, \partial_y) - 2i\,\sigma(\partial_x, \partial_y)\}\;dz^2.$$

It follows that $\sigma^{2,0}(p) = 0$ if and only if $p$ is an umbilic point of the immersion and that either all the points are umbilic or the umbilic points are isolated. Suppose that $b^2 = 4(c + H^2) > 0$. Then at an umbilic point $p$ of $M$ the Gauss curvature is given by $K(p) = b^2/4 > 0$ and, so, the totally umbilical case corresponds to genus$(M)$=0. If $M$ is a surface of genus $g > 0$ the number of umbilic points, counted with their multiplicity, is $4g-4$ and, in particular, constant mean curvaturetori have no umbilic points. If we denote by $P$ the finite set of umbilic points of $\psi$, we can consider on $M \setminus P$ the flat metric $ds_0^2 = b\,|\sigma^{2,0}|$, which is conformal to $ds^2$. If $w$ is the function on $M \setminus P$ defined by $ds^2 = (\exp(2w)/b^2)\; ds_0^2$, then

$$\Delta_0 w + \sinh(w)\cosh(w) = 0 \tag{2}$$

where $\Delta_0$ is the Laplacian of the flat metric $ds_0^2$. Moreover, if we denote by $k_i$, $i = 1, 2$, the principal curvatures of the immersion $\psi$ and by $K$ the Gauss curvature of the metric $ds^2$, then

$$\begin{gathered} k_i = H \pm (b/2)\exp(-2w),\ \ i = 1, 2,\ \text{ and} \\ K = (b^2/4)(1 - \exp(-4w)). \end{gathered} \tag{3}$$

so $K \equiv 0$ if and only if $w \equiv 0$. For more details see [7], [14] and [18].

When the ambient space is $N(c)$ then, using the Gauss equation, the index form (1) is given by

$$I(f, f) = \int_M \{|\nabla f|^2 - (b^2 - 2K)\, f^2\}\, dA, \tag{4}$$

for all $f \in C^\infty(M)$.

As $w$ satisfies the sinh–Gordon equation (2) and at umbilic points $w$ goes to $+\infty$, if $w \not\equiv 0$ its zeros are given by a finite set of $C^1$–immersed circles in $M$, see [6]. Now we will prove that stability implies a restriction on the nodal sets of $w$.

**Theorem 3.** *Let $\psi : M \to N(c)$ be an immersion with constant mean curvature $H$ from a compact orientable surface of positive genus into a oriented three space form $N(c)$. Let $K$ be the Gauss curvature of $M$. Suppose that $b^2 = 4(c + H^2) > 0$.*
*If $K \not\equiv 0$ and $\psi$ is stable, then the open set $\{p \in M|\ K(p) < 0\}$ is connected and each connected component of $\{p \in M|\ K(p) > 0\}$ must contain one umbilic point at least.*

*Proof.* If $f$ is a function compactly supported in $M \setminus P$ then, as outside of the set of umbilic points we have from (3) that $b^2 - 2K = b^2\cosh(2w)\exp(-2w) = b^2\,(\cosh^2 w + \sinh^2 w)\exp(-2w)$, the index form (4) can be written

$$I(f, f) = \int_M \{|\nabla_0 f|^2 - (\cosh^2 w + \sinh^2 w)\, f^2\}\, dA_0, \tag{5}$$

where the length of the gradient of $f$ and the measure are taken with respect to the flat metric $ds_0^2$.

Let $\Omega$ be a connected component of $\{p \in M|\ K(p) \neq 0\}$ without umbilic points. As sign($K$)=sign($w$), the function $f$ on $M$ defined by

$$f = \begin{cases} \sinh(w), & \text{on } \Omega \\ 0, & \text{on } M \setminus \Omega, \end{cases}$$

is in the Sobolev space $H^1(M)$. Moreover, umbilic points cannot lie at the boundary of $\Omega$ because the Gauss curvature is positive for the first kind of points and zero for the second one. So $f$ has compact support in $M \setminus P$. Hence integrating by parts and using (2) we obtain

$$\begin{aligned}
\int_M |\nabla_0 f|^2 \, dA_0 &= \int_M \langle \nabla_0 f, \nabla_0 \sinh(w) \rangle \, dA_0 \\
&= -\int_M f \, \Delta_0 \sinh(w) \, dA_0 \\
&= -\int_\Omega \sinh(w) \, \Delta_0 \sinh(w) \, dA_0 \\
&= -\int_\Omega \Big\{ \sinh(w) \Big\{ \sinh(w) |\nabla_0 w|^2 + \cosh(w) \Delta_0 w \Big\} \Big\} \, dA_0 \\
&= \int_\Omega \sinh^2(w) \Big\{ \cosh^2 w - |\nabla_0 w|^2 \Big\} \, dA_0 .
\end{aligned}$$

From this last equality and (5) we conclude that

$$\begin{aligned}
I(f,f) &= \int_\Omega \Big\{ \sinh^2 w (\cosh^2 w - |\nabla_0 w|^2) \\
&\quad - (\cosh^2 w + \sinh^2 w) \sinh^2 w \Big\} \, dA_0 = \\
&= -\int_\Omega \sinh^2 w (\sinh^2 w + |\nabla_0 w|^2) \, dA_0 < 0.
\end{aligned}$$

If we could find two connected components of $\{p \in M | \; K(p) \neq 0\}$ without umbilic points, we would get two functions in $H^1(M)$, $f_1$, $f_2$ whose supports are disjoint and such that $I(f_i, f_i) < 0$, i=1,2. So, a certain linear combination of both functions will give another function $f$ with $\int_M f \, dA = 0$ and $I(f,f) < 0$, which is impossible by the stability of $\psi$. At this point the theorem follows directly because the set $\{p \in M | \; K(p) < 0\}$ is nonvoid, by the Gauss–Bonnet theorem, and does not contain umbilic points. □

If genus($M$)=1, we obtain the following stronger conclusion, *without restrictions* on the values of $c$ and $H$.

**Corollary 4.** *Let $\psi : (M, ds^2) \to N(c)$ be an isometric immersion with constant mean curvature$H$. If $M$ is a torus and $\psi$ is stable, then the metric $ds^2$ is flat.*

*Proof.* Suppose first that $b^2 = 4(c + H^2) > 0$. As $M$ has no umbilic points, we conclude using theorem 3 that the subset $\{p \in M | \; K(p) > 0\}$ is empty. So the result follows from the Gauss–Bonnet theorem.

If $c + H^2 \leq 0$, then it follows from the Gauss equation that the curvature of $M$ is less than or equal to zero and, from the Gauss–Bonnet theorem, must be identically zero. □

It is an elementary fact that a flat constant mean curvaturesurface in $N(0)$ is locally congruent to a plane or to a right circular cylinder in $\mathbb{R}^3$, in $N(1)$ to a Clifford torus, that is, to a product of circles on $S^3(1)$, and in $N(-1)$ to a horosphere or to a tube about a geodesic.

## 2. Stability of constant mean curvaturesurfaces in elliptic space forms

Every orientable three dimensional space form $N(1)$ with positive sectional curvature $c = 1$ determines a finite Riemannian covering $\Pi : S^3(1) \to N(1)$ from the standard unit sphere $S^3(1)$ over $N(1)$. In particular the real projective space $\mathbb{RP}^3(1)$ determines a two sheeted covering. In this section we shall obtain some information about compact stable constant mean curvaturesurfaces in $N(1)$ when the number of sheets of $\Pi$ is small.

We shall need the following result

**Theorem 5.** *If $\psi : M \to S^3(1)$ is an immersion of a compact orientable surface, then*

$$\int_M (1 + H^2)\, dA \geq 4\pi, \tag{6}$$

*and the equality holds if and only if $M$ is an umbilic sphere.*

*Moreover, if*

$$\int_M (1 + H^2)\, dA \leq 8\pi, \tag{7}$$

*then the immersion $\psi$ is an embedding.*

For the first part of the theorem see [5]. The second statement is proved in [19] if $\int_M (1 + H^2)\, dA < 8\pi$. When $\int_M (1 + H^2)\, dA = 8\pi$ the result is essentially due to Kusner [17]. As it is not explicitely stated in [17], we explain briefly the proof of this result. If $\psi$ is not an embedding, then there exists a point $p$ in $S^3(1)$ such that $\psi^{-1}(p)$ contains exactly two points, see [19]. Now, taking stereographic projection from the point $p$, we transform the immersion $\psi$ into a complete minimal surface with finite total curvature in the Euclidean space $\mathbb{R}^3$ and two embedded flat ends, see [17]. But it follows from [17] that such a surface must be embedded, and so its flat ends must be parallel. Hence we see that the linear function orthogonal to the ends is a harmonic bounded function, and then it must be constant. The result follows from this contradiction.

A compact surface of genus zero immersed with constant mean curvaturein a spherical three space form is totally umbilical and hence is stable. If the genus of the surface is one the stability problem is solved by theorem 3. From theorem 1 it remains to study the stability problem when the genus is 2 or 3. Some of our arguments in the following result are similar to those used in [8], [19], [21] and [28].

**Theorem 6.** *Let $\psi : M \to N(1)$ be an immersion with constant mean curvature$H$ from a compact orientable surface of genus greater than one into an oriented elliptic 3–space form $N(1)$. Suppose that the universal covering $\Pi : S^3(1) \to N(1)$ has $k$ sheets and that $\psi$ is stable. Then*

(i) $(1 + H^2)\, Area(M) \leq 2\pi$.

(ii) $k \geq 3$.

(iii) *If $k$ =3 or 4, the immersion $\psi$ is an embedding and the induced morphism between the fundamental groups $\psi_* : \pi_1(M) \to \pi_1(N(1))$ is surjective.*

*Proof.* Let $g$ be the genus of $M$. It is a known fact (see [12, p. 261]) that there exists a non constant meromorphic map $\phi : M \to S^2(1) \subset \mathbb{R}^3$ such that

$$\text{degree}(\phi) \leq 1 + \left[\frac{g+1}{2}\right], \tag{8}$$

where $[x]$ is the greatest integer less than or equal to $x$. Composing $\phi$ with a conformal diffeomorphism of the Riemann sphere $S^2$, we can suppose, see [19], that

$$\int_M \phi \, dA = 0.$$

Using this vectorial function as a test function in the index form (4) we obtain

$$\begin{aligned} 0 \leq I(\phi, \phi) &= \int_M \{|\nabla \phi|^2 - (b^2 - 2K)\}\, dA = \\ &= 8\pi\, \text{degree}(\phi) - 4 \int_M (1 + H^2)\, dA + 8\pi\,(1 - g), \end{aligned}$$

where in the last equality we have used that $\phi$ is a conformal map and the Gauss–Bonnet theorem. Estimating the degree of $\phi$ by (8) we obtain

$$\int_M (1 + H^2)\, dA \leq 2\pi \left(2 - g + \left[\frac{g+1}{2}\right]\right). \tag{9}$$

If $g \geq 4$ the right hand side of this inequality is nonpositive and this contradiction gives us the statement (ii) of theorem 1. If $g$=2 or 3 we obtain statement (i) in our theorem.

Let $\bar{M}$ be the pullback of $M$ via the covering map $\Pi : S^3(1) \to N(1)$. Note that $\bar{M}$ is not necessarily connected. Associated to this surface we have naturally defined an isometric immersion $\bar{\psi} : \bar{M} \to S^3(1)$ and a $k$–sheeted Riemannian covering $\bar{M} \to M$. As $\bar{\psi}$ is locally congruent to $\psi$ the new immersion has also constant mean curvature$H$. Then, using the statement (i) above, we have

$$\int_{\bar{M}} (1 + H^2)\, d\bar{A} = k \int_M (1 + H^2)\, dA \leq 2k\pi. \tag{10}$$

If $k \leq 2$, $\bar{\psi}$ should verify the equality in (6). But clearly genus$(\bar{M}) \geq g > 1$, and this contradiction proves (ii).

Now we prove (iii). Suppose that $k \le 4$. From (6) we conclude that

$$4\pi\,(\text{number of components of } \bar{M}) \le \int_{\bar{M}} (1+H^2)\, d\bar{A} \le 2k\pi \le 8\pi. \tag{11}$$

So it follows that $\bar{M}$ is connected: otherwise $\bar{M}$ would have precisely two connected components and, from theorem 5, each of these components should be an umbilic sphere in $S^3(1)$, and this is impossible because genus$(M) > 0$. This fact is equivalent to the surjectivity of $\psi_* : \pi_1(M) \to \pi_1(N(1))$. From (7) and (11) we obtain that $\bar{\psi}$ is an embedding and finally using the fact that $\psi_*$ is surjective, we conclude that also the immersion $\psi$ is an embedding. □

As a consequence of the above results we can now state a complete solution of the global stability problem in the three dimensional real projective space. This space form is the only two sheeted quotient of the sphere $S^3(1)$.

**Corollary 7.** *Let $\psi : M \to \mathbb{RP}^3(1)$ be a complete orientable constant mean curvaturesurface immersed into the real projective space. If the immersion is stable, then either*

(i) *$M$ is a compact surface with genus($M$)=0 and $\psi$ is an embedded geodesic sphere or a twofold covering of a totally geodesic projective plane, or*
(ii) *$M$ is a compact surface of genus 1 and $\psi$ is an embedded flat tube of radius $r$, with $\pi/6 \le r \le \pi/3$, about a geodesic.*

*Proof.* From theorem 1 we have that $M$ is a compact and connected surface with genus($M$)$\le$3. From theorem 6 we see that the cases genus($M$)=2 or 3 cannot hold.

If genus($M$)=0, then $\psi$ is a totally umbilic immersion and so we have (i).

If genus($M$)=1, then from corollary 4, $M$ is a flat torus. It is a standard and simple fact that an immersion of this kind must be a finite Riemannian covering of a tube of radius $r$, $0 < r < \pi/2$, about a geodesic in $\mathbb{RP}^3(1)$. Using (9) and (10) for $g = 1$ and $k = 2$, we deduce that $\psi$ is an embedding.

If $\bar{T}$ is a tube of radius $r$, $0 < r < \pi/2$, about a geodesic in $S^3(1)$, then $\bar{T}$ is congruent to the standard embedding of the Clifford torus $S^1(\cos(r)) \times S^1(\sin(r)) \subset S^3(1) \subset \mathbb{R}^4$, and the Jacobi operator of this surface is $\Delta + 1/\cos^2 r + 1/\sin^2 r$. We obtain easily that the corresponding embedded tube in $\mathbb{RP}^3(1)$ is given from a intrinsic point of view by $T = \mathbb{R}^2/\Gamma$, where $\Gamma$ is the lattice in $\mathbb{R}^2$ generated by the vectors $(2\pi\cos(r), 0)$ and $(\pi\cos(r), \pi\sin(r))$. As the eigenvalues of the Laplacian of $T$ are

$$\left\{ \frac{m^2}{\cos^2(r)} + \frac{(2n+m)^2}{\sin^2(r)};\ n, m \in \mathbb{Z} \right\},$$

the tube $T$ is stable if and only if

$$\cos(r), \sin(r) \ge 1/2,$$

or, equivalently,

$$\pi/6 \leq r \leq \pi/3.$$

□

## 3. The isoperimetric problem

Among all embedded compact surfaces enclosing a given volume in an orientable compact 3–manifold or in a complete flat 3–space form, there exists one of least area, see [1]. This surface is a $(M, \epsilon, \delta)$–minimal set. Hence Taylor's regularity theorem [27] applies and so the only possible singularities of this surface occur along curves where three sheets meet in an equiangular way or at some isolated points where four of the above curves meet themselves. Since our surface separates the three–manifold just in two regions (inside and outside), it is regular although not necessary connected. In this way we obtain an embedded compact constant mean curvaturesurface which is stable. In this section we solve completely the isoperimetric problem in the real projective space and in the flat manifold $\mathbb{R}^3/S_\theta$, where $S_\theta$ is the subgroup generated by a screw motion in $\mathbb{R}^3$.

Let $f(V)$ be the function which determines the isoperimetric profile of the sphere $S^3(1)$, that is, for any $V \in (0, 2\pi^2)$, $f(V)$ denotes the area of the geodesic sphere in $S^3(1)$ which encloses a volume $V$. Let us denote by $g(V)$ the function $2V^{1/2}(\pi^2-V)^{1/2}$. To solve the equation $f(V) = g(V)$ in $[\pi^2/4, \pi^2/2]$ is equivalent to finding the fixed points of the explicitely computable map $f^{-1} \circ g$ in the above interval. A direct computation shows that $|(f^{-1} \circ g)'| < 1$, so there exists at least one fixed point of $f^{-1} \circ g$ (because $f(\pi^2/4) < g(\pi^2/4)$ and $f(\pi^2/2) > g(\pi^2/2)$); then $f^{-1} \circ g([\pi^2/4, \pi^2/2]) \subset [\pi^2/4, \pi^2/2]$ and we can conclude, from Banach's fixed point theorem, that there exists exactly one solution to $f(V) = g(V)$ in $(\pi^2/4, \pi^2/2)$. We denote this solution by $\mu$. Approximately $\mu \cong 4.1432835$. Now we state our first result

**Theorem 8.** *The isoperimetric profile of* $\mathbb{RP}^3(1)$ *is given by*

$$A(V) = \begin{cases} f(V), & \text{if } 0 < V \leq \mu \\ 2V^{1/2}(\pi^2 - V)^{1/2}, & \text{if } \mu \leq V \leq \pi^2 - \mu \\ f(\pi^2 + V), & \text{if } \pi^2 - \mu \leq V < \pi^2. \end{cases}$$

*Moreover, in the first case the solution of the isoperimetric problem is a geodesic ball, in the second one the solution is a tubular neighborhood of a geodesic and in the last one the solution of the problem is the outside of a geodesic ball.*

*Proof.* It is enough to compare area and volume among smooth domains in $\mathbb{RP}^3(1)$ bounded by stable constant mean curvaturesurfaces. By corollary 7 these surfaces are geodesic spheres or flat tubes of radius $r$, $\pi/6 \leq r \leq \pi/3$ about geodesics. The tubes have area $A = 2\pi\cos(r)\sin(r)$ and they separate $\mathbb{RP}^3(1)$ in two connected open domains with volumes $V_1 = \pi^2\cos^2(r)$ and $V_2 = \pi^2\sin^2(r)$, so that the relation between the area and the volume enclosed by the tubes is

$$A = 2V^{1/2}(\pi^2 - V)^{1/2}, \qquad \text{whenever } \frac{\pi^2}{4} \le V \le \frac{3\pi^2}{4}.$$

For geodesic balls in $\mathbb{RP}^3(1)$ the relation between area and volume is, of course, given by $A = f(V)$ and for the outside of a geodesic ball by $A = f(\pi^2 + V)$. The theorem follows directly from these relations. $\square$

We consider now the flat manifold $\mathbb{R}^3/S_\theta$, where $S_\theta$ is the subgroup of isometries of the Euclidean space generated by the screw motion

$$(x, y, x) \longmapsto (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta), z + \lambda),$$

with $\theta \in [0, 2\pi)$ and $\lambda > 0$. We wish to find the embeddings $\psi : M \to \mathbb{R}^3/S_\theta$ of compact surfaces which are stable as constant mean curvatureimmersions. As there are no compact totally geodesic surfaces in $\mathbb{R}^3/S_\theta$, by theorem 2, $M$ must be a compact and connected surface. On the other hand, if $\bar{M}$ is a connected pullback surface of $M$ in the universal covering $\mathbb{R}^3$ of $\mathbb{R}^3/S_\theta$, and $\bar{\psi} : \bar{M} \to \mathbb{R}^3$ is the corresponding proper constant mean curvatureembedding, then either $\bar{M}$ is compact and, from Alexandrov theorem [14], $\bar{\psi}$ is totally umbilical, or $\bar{\psi}$ is a simply periodic proper embedding inside a right circular cylinder in $\mathbb{R}^3$ and from the results of Korevaar, Kusner and Solomon [16, theorem 2.10], $\bar{M}$ is an embedded revolution surface in $\mathbb{R}^3$. Consequently $M$ must be a torus and, by corollary 4, it is a quotient of a right circular cylinder whose axis has to be the one of the screw motion. With this result and single computations we have proved the following theorem

**Theorem 9.** *The isoperimetric profile of $\mathbb{R}^3/S_\theta$ is given by*

$$A(V) = \begin{cases} (4\pi)^{1/3}(3V)^{2/3}, & \text{if } 0 < V \le \frac{4\pi\lambda^3}{3^4} \\ (4\pi\lambda V)^{1/2}, & \text{if } \frac{4\pi\lambda^3}{3^4} \le V. \end{cases}$$

*In the first case the domain of least area enclosing the volume $V$ is the Euclidean ball and in the second one the solution is the domain enclosed by the embedded torus $(S^1(a) \times \mathbb{R})/S_\theta$, for suitable values of $a$.*

*Remark.* The classical Schwarz minimal $\mathcal{P}$–surface of genus three in the cubic three–torus is a stable constant mean curvaturesurface, but the domain enclosed by this surface is not a solution of the isoperimetric problem, see [24]. By continuity, some of their constant mean curvaturecompanions discovered by Karcher [15] are also stable. In the same way it seems natural to hope that the Lawson constant mean curvaturesurface of genus two [18], and some of their constant mean curvaturecompanions, see [15], are stable in the space $T^2 \times \mathbb{R}$, where $T^2$ is the square flat two dimensional torus. However, the best candidates to solve the isoperimetric problem in these spaces are geodesic balls, tubular neighborhoods of closed geodesics and slices bounded by a pair of parallel planes.

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