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Stable Controller Design for Linear Systems

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Abstract

This paper is concerned with the problem of suboptimal stable mixed H_2/H_∞ control for linear time-invariant systems. The designed controllers are required to satisfy a prescribed H_∞ performance bound or a prescribed degree of stability. By reducing the stable controller synthesis problem to a multiobjective state feedback control problem for two different state models, sufficient conditions for the solvability of the considered problem are given in terms of solutions to algebraic Riccati equations and matrix inequalities. LMI-based iterative algorithms are developed to solve the stable controller synthesis problem. All of the proposed algorithms are shown to be convergent. An example is given to illustrate the proposed methods.

Keywords: Linear systems; dynamic output feedback; mixed H_2/H_∞ control; stable controller; LMI.

1 Introduction

The problem of designing a stable controller to stabilize a given plant with some performance specifications has been extensively investigated by a number of authors, see [2], [5], [6]-[8] [11], [13]-[18]. In [16], it has been shown that a necessary and sufficient condition for the existence of a stable stabilizing controller is the parity interlacing property. A plant P is said to satisfy the parity interlacing property if the number of poles of P between any pair of real right half-plane blocking zeros is even. Some procedures for constructing stable stabilizing controllers are given in [12, 16], which involve the construction of a unit in H_∞ satisfying certain interpolation conditions that may result in very large order controllers. For the stable H_∞ controller design, a method using a state space approach is proposed in [11]. In [6], sufficient conditions are also obtained for the synthesis of SISO finite dimensional suboptimal stable H_∞ controllers by converting the problem into a Nevanlinna-Pick interpolation problem. In [18], a sufficient condition for the existence of a stable suboptimal H_∞ controller is derived in terms of positive definite stabilizing solution to a certain algebraic Riccati equation. However, the order of the designed controller is two times that of the plant. A result proposing stable H_∞ controllers which have the same order as the plant is given in [17], and the H_∞ performance of the stable H_∞ controllers is investigated. More recently, a method of designing a stable H_∞ controller is also given by using the Riccati equation approach, where the designed controller is of the same order as that of the plant, and satisfies the same H_∞-norm bound as that of the resulting closed-loop system [2]. For the problem of designing stable H_2 controller, [5] presents an algorithm which requires the minimization of a nonlinear objective function with nonlinear inequality constraints. Based on the cost function modification, a sufficient condition for the stable H_2 control problem is given in [13]. The method is also extended to the design of a stable controller for mixed H_2/H_∞ problem [8].

This paper will be concerned with the problem of designing suboptimal stable mixed H_2/H_∞ controllers for linear time-invariant systems. The controller to be designed is required to satisfy a prescribed H_∞ performance bound or a prescribed degree of stability. The stable controller synthesis problem is reduced to a multiobjective state feedback control problem for two different state models. New sufficient conditions for the solvability of the problem are given in terms of solutions to Riccati equations and matrix inequalities by solving the multiobjective state feedback control problem. *Iterative* algorithms are developed to solve the stable controller synthesis problem. All of the proposed algorithms are shown to be convergent, and the numerical example illustrates the advantage of the proposed algorithms. The paper is organized as follows. Section 2 formulates the problem under consideration and gives some preliminaries. The suboptimal stable mixed

H_2/H_∞ control problem is addressed in Section 3. A numerical example is given in Section 4.

2 Problem statement and preliminaries

Consider a linear time-invariant system Σ described by equations

$$\Sigma: \quad \dot{x} = Ax + B_1 w + B_2 u \qquad (1)$$
$$z_0 = C_0 x + D_0 u \qquad (2)$$
$$z_1 = C_1 x + D_1 u \qquad (3)$$
$$y = C_2 x + D_2 w \qquad (4)$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, $y \in R^q$ is the measured output, $w \in R^r$ is the disturbance inputs, and $z_0 \in R^{r_0}$ and $z_1 \in R^{r_1}$ are the outputs to be regulated. The dynamic output feedback controller K is given by

$$K: \quad \dot{\xi} = A_K \xi + B_K y \qquad (5)$$
$$u = C_K \xi + D_K y \qquad (6)$$

where $\xi \in R^v$. The resulting closed-loop system Σ_c with the controller K is described as follows

$$\Sigma_c: \quad \dot{x}_e = A_e x_e + B_e w \qquad (7)$$
$$z_0 = C_{0e} x_e + J_{0e} w \qquad (8)$$
$$z_1 = C_{1e} x_e + J_{1e} w \qquad (9)$$

where $x_e = [x^T \ \ \xi^T]^T$,

$$A_e = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}$$
$$B_e = \begin{bmatrix} B_1 + B_2 D_K D_2 \\ B_K D_2 \end{bmatrix}$$
$$C_{0e} = [\, C_0 + D_0 D_K C_2 \quad D_0 C_K \,], \qquad J_{0e} = D_0 D_K D_2$$
$$C_{1e} = [\, C_1 + D_1 D_K C_2 \quad D_1 C_K \,], \qquad J_{1e} = D_1 D_K D_2$$

By [9], the mixed H_2/H_∞ performance measure $J(\Sigma, K, \gamma)$ of the stable system Σ_c is defined as follows

$$J(\Sigma, K, \gamma) = \begin{cases} \infty, & \text{if } J_{0e} \neq 0 \\ \mathrm{Trace}(C_{0e} Y C_{0e}^T), & \text{otherwise} \end{cases} \qquad (10)$$

where $Y \geq 0$ is the stabilizing solution to the following algebraic Riccati equation

$$A_e Y + Y A_e^T + (Y C_{1e}^T + B_e J_{1e}^T)(\gamma^2 - J_{1e} J_{1e}^T)^{-1} \times (Y C_{1e}^T + B_e J_{1e}^T)^T + B_e B_e^T = 0 \qquad (11)$$

Definition 2.1: A symmetric matrix X_0 is said to be a stabilizing solution to the Riccati equation $A^T X + XA - XMX + N = 0$ if it satisfies the Riccati equation and the matrix $A - MX_0$ is stable.

Then the problem under consideration is as follows.

Suboptimal β-stable mixed H_2/H_∞ control problem: Given constants $\gamma > 0$, $\alpha > 0$ and $\beta > 0$, find a stable controller K with $\|K(s)\|_\infty < \beta$ such that the closed-loop system Σ_c is internally stable and the mixed H_2/H_∞ performance measure $J(\Sigma, K, \gamma)$ of the closed-loop system Σ_c satisfies $J(\Sigma, K, \gamma) < \alpha$.

By imposing the constraint of an H_∞ performance bound on the designed controller, the robust stability of the closed-loop system may be guaranteed. The problem of designing a stable H_∞ controller to satisfy an H_∞ performance was considered in [17], and [2] for the special case $\gamma = \beta$, respectively. When β is sufficiently large, the above suboptimal β-stable mixed H_2/H_∞ control problem becomes one of designing a stable H_2/H_∞ controller. In the sequel, the more general problem of designing an H_2/H_∞ controller with the prescribed degree of stability r will also be addressed. If there is no constraints of K being stable and $\|K\|_\infty < \beta$, then the problem is reduced to the mixed H_2/H_∞ control problem considered in [9]. In particular, [9] has shown that for a given $\gamma > 0$, the computation of the optimal mixed H_2/H_∞ performance $J_{opt}(\Sigma, \gamma)$ and the construction of a suboptimal compensator can be approached via convex optimization, where

$$J_{opt}(\Sigma, \gamma) = \inf_K \{ \, J(\Sigma, K, \gamma) : \ \|T_{z_1 w}(K)\|_\infty < \gamma \text{ and } A_e \text{ is stable} \} \qquad (12)$$

with the transfer function $T_{z_1 w}(K)$ being defined by

$$T_{z_1 w}(K) = J_{1e} + C_{1e}(sI - A_e)^{-1} B_e \qquad (13)$$

The following assumptions will be used in the sequel.
Assumption A1: The triple (C_2, A, B_2) is stabilizable and detectable.
Assumption A2: The pair (A, B_1) is stabilizable.
Assumption A3: $D_2 [B_1^T \ \ D_2^T] = [0 \ \ I]$.

3 Stable mixed H_2/H_∞ controller design

In this section, we will present new sufficient conditions for the solvability of the suboptimal stable mixed H_2/H_∞ control problem, and an algorithm will be proposed to minimize the mixed H_2/H_∞ performance $J(\Sigma, K, \gamma)$ under the constraints of K being stable and $\|K\|_\infty < \beta$. The proofs are omitted, see [15] for the details.

Suppose that the Riccati equation

$$AY + YA^T + Y(\frac{1}{\gamma^2} C_1^T C_1 - C_2^T C_2) Y + B_1 B_1^T = 0 \qquad (14)$$

has a stabilizing solution $Y \geq 0$ for $\gamma > 0$. Define the auxiliary systems Σ_Y and Σ_{KY} as follows:

$$\Sigma_Y: \quad \dot{x} = A_{11} x + B_{11} w + B_{22} u \qquad (15)$$
$$z_{Y0} = C_0 x + D_0 u \qquad (16)$$

$$\Sigma_{KY}: \quad \begin{aligned} z_{Y1} &= C_1 x + D_1 u && (17) \\ \dot{x} &= A_{12}x + B_{11}w + B_{22}u && (18) \\ z_{K1} &= u && (19) \end{aligned}$$

where C_0, D_0, C_1 and D_1 are the same as in (2) and (3), and

$$A_{11} = A + \frac{1}{\gamma^2} Y C_1^T C_1 \quad (20)$$

$$B_{11} = Y C_2^T \quad (21)$$

$$B_{22} = B_2 + \frac{1}{\gamma^2} Y C_1^T D_1 \quad (22)$$

$$A_{12} = A + \frac{1}{\gamma^2} Y C_1^T C_1 - Y C_2^T C_2 \quad (23)$$

Then we have the following lemma.

Lemma 3.1: If there exists a common state feedback gain C_K for the systems Σ_Y and Σ_{KY} such that both $A_{11} + B_{22}C_K$ and $A_{12} + B_{22}C_K$ are stable, $J(\Sigma_Y, C_K, \gamma) < \alpha - \text{Trace}(C_0 Y C_0^T)$, and $\|T_{KY}(s, C_K)\|_\infty < \beta$, then the dynamic output feedback controller K described by (5) and (6) with the C_K and

$$A_K = A_{12} + B_{22}C_K, \quad B_K = B_{11} = Y C_2^T, \quad D_K = 0 \quad (24)$$

solves the suboptimal β-stable mixed H_2/H_∞ control problem, where

$$T_{KY}(s, C_K) = C_K(sI - A_K)^{-1} B_K \quad (25)$$

The key idea here lies in the construction of the two auxiliary systems Σ_Y and Σ_{KY}, especially Σ_{KY}. The system Σ_Y is constructed in [9] for solving the suboptimal mixed H_2/H_∞ output feedback control problem, which reduces the original outut feedback control problem into a state feedback control problem for Σ_Y. Notice that the designed controller can be given by (24), the matrices A_{12}, B_{22} and B_{11} are defined by (21)-(23) and only one parameter matrix C_K is to be determined such that $\| K(s) \|_\infty < \beta$ ($K(s) = T_{KY}(s, C_K)$). So the auxiliary system Σ_{KY} is constructed to reduce the problem of finding a gain C_K such that $\| K(s) \|_\infty < \beta$ into a state feedback control problem for Σ_{KY}. Thus, the suboptimal β-stable mixed H_2/H_∞ control problem cab be reduced to the problem of finding a state feedback gain C_K such that both the mixed H_2/H_∞ state feedback control problem for Σ_Y and the H_∞ state feedback control problem for Σ_{KY} are solved by the same C_K. This implies that the stable controller synthesis problem can be reduced to the multiobjective state feedback control problem for the two different systems Σ_Y and Σ_{KY}. This approach differs from all previous existing approaches to the stable control design problem in [11], [6], [18], [17], [2], [13] and [8].

First, a sufficient condition for the solvability of the suboptimal β-stable mixed H_2/H_∞ control problem is given in the following theorem.

Theorem 3.2: Consider the system Σ described by equations (1)-(4). Let $\gamma > 0$, $\alpha > 0$ and $\beta > 0$ be given constants. Suppose the following
(i) Assumptions A1-A3 hold;
(ii) The Riccati equation (14) has a stabilizing solution $Y \geq 0$;
(iii) There exist matrices $X > 0$, $Q > 0$ and W such that the following inequalities hold

$$\begin{bmatrix} M_a & XC_1^T + W^T D_1^T \\ C_1 X + D_1 W & -\gamma^2 I \end{bmatrix} < 0 \quad (26)$$

$$\begin{bmatrix} -Q & C_0 X + D_0 W \\ XC_0^T + W^T D_0^T & -X \end{bmatrix} < 0 \quad (27)$$

$$\text{Trace}(Q) < \alpha - \text{Trace}(C_0 Y C_0^T) \quad (28)$$

$$\begin{bmatrix} M_b & W^T \\ W & -\beta^2 I \end{bmatrix} < 0 \quad (29)$$

where A_{11}, B_{11}, B_{22} and A_{12} are as defined in (20)-(23), $M_a = A_{11}X + XA_{11}^T + B_{22}W + W^T B_{22}^T + B_{11}B_{11}^T$ and $M_b = A_{12}X + XA_{12}^T + B_{22}W + W^T B_{22}^T + B_{11}B_{11}^T$. Then the dynamic output feedback controller K described by (5) and (6) with (24), and

$$C_K = W X^{-1} \quad (30)$$

solves the suboptimal β-stable mixed H_2/H_∞ control problem.

Note that (26)-(28) are the conditions for solving the suboptimal mixed H_2/H_∞ control problem, and (29) for $\|T_{KY}(s, C_K)\|_\infty < \beta$ (hence $\|K(s)\|_\infty < \beta$). For the case of designing a stable H_2/H_∞ controller with the prescribed degree of stability r (but without $\|T_{KY}(s, C_K)\|_\infty < \beta$, i.e., $\|K(s)\|_\infty < \beta$), we have the following

Corollary 3.3: With all assumptions as in Theorem 3.2 except for (29) being replaced by

$$(A_{12}+rI)X + X(A_{12}+rI)^T + B_{22}W + W^T B_{22}^T < 0 \quad (31)$$

where $r \geq 0$ is a constant. Then the controller K given by (5) and (6) with (30) and (24) is with the prescribed degree of stability r, and stabilizes the system Σ with $J(\Sigma, K, \gamma) < \alpha$.

Remark 3.4: Theorem 3.2 presents a sufficient condition for the suboptimal β-stable mixed H_2/H_∞ control problem in terms of solutions to Riccati equation (14) and matrix inequalities (26)-(29). For the fixed $Y \geq 0$ satisfying (14), (26)-(29) are LMIs, which can be solved by using the LMI Toolbox [4]. When there is no constraints of $K(s)$ being stable and $\|K(s)\|_\infty < \beta$ (i.e., remove (29)), Theorem 3.2 is reduced to the necessary and sufficient condition for the suboptimal mixed

H_2/H_∞ control problem in [9]. Corollary 3.3 presents a sufficient condition under which the designed mixed H_2/H_∞ controller is with a prescribed degree of stability r.

It should be noted that the conditions (26)-(29) are given based on the assumption of existence of a *common* Lyapunov function for the two closed-loop systems from Σ_Y and Σ_{KY} with $u = C_K x$, which may result in some conservativeness. In the following, we will present another sufficient condition without this conservativeness. Denote

$$\Delta_Y(P_1, P_{10}, C_K, \gamma) = \begin{bmatrix} \Delta_{Y11}(P_1, P_{10}) & P_1 B_{11} & C_K^T + (P_1 B_{22} + C_1^T D_1) R_1^{-1} \\ B_{11}^T P_1 & -\gamma^2 I & 0 \\ C_K + R_1^{-1}(B_{22}^T P_1 + D_1^T C_1) & 0 & -R_1^{-1} \end{bmatrix} \quad (32)$$

$$\Delta_{KY}(P_2, P_{20}, C_K, \beta) = \begin{bmatrix} \Delta_{KY11}(P_2, P_{20}) & P_2 B_{11} & C_K^T + P_2 B_{22} \\ B_{11}^T P_2 & -\beta^2 I & 0 \\ C_K + B_{22}^T P_2 & 0 & -I \end{bmatrix} \quad (33)$$

where

$$R_1 = D_1^T D_1 > 0 \quad (34)$$

$$\begin{aligned} \Delta_{Y11}(P_1, P_{10}) = & P_1(A_{11} - B_{22} R_1^{-1} D_1^T C_1) \\ & + (A_{11} - B_{22} R_1^{-1} D_1^T C_1)^T P_1 \quad (35) \\ & + C_1^T (I - D_1 R_1^{-1} D_1^T) C_1 \\ & - P_1 B_{22} R_1^{-1} B_{22}^T P_{10} \quad (36) \\ & - P_{10} B_{22} R_1^{-1} B_{22}^T P_1 \\ & + P_{10} B_{22} R_1^{-1} B_{22}^T P_{10} \quad (37) \end{aligned}$$

$$\begin{aligned} \Delta_{KY11}(P_2, P_{20}) = & P_2 A_{12} + A_{12}^T P_2 - P_2 B_{22} B_{22}^T P_{20} \\ & - P_{20} B_{22} B_{22}^T P_2 + P_{20} B_{22} B_{22}^T P_{20} \quad (38) \end{aligned}$$

Theorem 3.5: Consider the system Σ described by equations (1)-(4). Let $\gamma > 0$, $\alpha > 0$ and $\beta > 0$ be given constants. Suppose the following
(i) Assumptions A1-A3 and (34) hold;
(ii) The Riccati equation (14) has a stabilizing solution $Y \geq 0$;
(iii) There exist matrices $P_1 > 0$, $P_{10} > 0$, $P_2 > 0$, $P_{20} > 0$, $Q > 0$ and C_K such that (28) and the following inequalities hold

$$\Delta_Y(P_1, P_{10}, C_K, \gamma) < 0 \quad (39)$$

$$\Delta_{KY}(P_2, P_{20}, C_K, \beta) < 0 \quad (40)$$

$$\begin{bmatrix} -Q & C_0 + D_0 C_K \\ C_0^T + C_K^T D_0^T & -\frac{1}{\gamma^2} P_1 \end{bmatrix} < 0 \quad (41)$$

Then the dynamic output feedback controller K described by (5) and (6) with the C_K and (24) solves the suboptimal β-stable mixed H_2/H_∞ control problem.

Corollary 3.6: With all assumptions as in Theorem 3.5, except that inequality (40) is replaced by

$$\Delta_{KYr}(P_2, P_{20}, C_K, r) = \begin{bmatrix} 2rP_2 + \Delta_{KY11}(P_2, P_{20}) & C_K^T + P_2 B_{22} \\ C_K + B_{22}^T P_2 & -I \end{bmatrix} < 0 \quad (42)$$

where $r \geq 0$ is a constant. Then the controller K given by (5) and (6) with the above C_K and (24) is with the prescribed degree of stability r, and stabilizes the system Σ and $J(\Sigma, K, \gamma) < \alpha$.

Remark 3.7: The sufficient condition of Theorem 3.5 is weaker than that of Theorem 3.2, and Theorem 3.5 contains no conservativeness from the assumption on the existence of a *common* Lyapunov function for Σ_Y and Σ_{KY} with $u = C_K x$. However, it should be mentioned that Theorem 3.5 is still a sufficient condition for the stable controller design problem. There might be no solution if the order of the controller is restricted to that of the plant. In comparison with the condition given in terms of LMIs in Theorem 3.2, the inequalities (28), and (39)-(41) normally are not LMIs, which cannot be solved directly. This class of matrix inequality conditions was used in [10] for the simultaneous linear-quadratic optimal control via static output feedback. But, when P_{10} and P_{20} are given, then (28) and (39)-(41) are LMIs with respect to the variables P_1, P_2, C_K, Q and β^2. This property can be used to form the following convergent iterative algorithms.

First, by combining Theorem 3.2 and Theorem 3.5, we have the following *iterative* algorithm to minimize $J(\Sigma, K, \gamma)$ for given $\gamma > 0$ and $\beta > 0$.

Algorithm 3.8: Let $\gamma > 0$ and $\beta > 0$ be given constants.
Step 1. Solve the Riccati equation (14) to obtain the stabilizing solution $Y \geq 0$.
Step 2. Minimize Trace(Q) subject to the LMI constraints (26), (27) and (29), and denote $C_{Kopt} = W_{opt} X_{opt}^{-1}$.
Step 3. Minimize β_0^2 subject to $X_0 > 0$ and the LMI constraint (45) below; and minimize Trace(Q_0) subject to LMI constraints (43) and (44).

$$\begin{bmatrix} M_c & X(C_1^T + C_{Kopt}^T D_1^T) \\ (C_1 + D_1 C_{Kopt}) X & -\gamma^2 I \end{bmatrix} < 0 \quad (43)$$

$$\begin{bmatrix} -Q_0 & (C_0 + D_0 C_{Kopt}) X \\ X(C_0 + D_0 C_{Kopt})^T & -X \end{bmatrix} < 0 \quad (44)$$

$$\begin{bmatrix} M_d & X_0 C_{Kopt}^T \\ C_{Kopt} X_0 & -\beta_0^2 I \end{bmatrix} < 0 \quad (45)$$

where $M_c = (A_{11} + B_{22} C_{Kopt}) X + X(A_{11} + B_{22} C_{Kopt})^T + B_{11} B_{11}^T$, and $M_d = (A_{12} + B_{22} C_{Kopt}) X_0 + X_0 (A_{12} + B_{22} C_{Kopt})^T + B_{11} B_{11}^T$. Denote $P_{10}^0 = \gamma^2 X_{opt}^{-1}$ and $P_{20}^0 = \beta_{0opt}^2 X_{0opt}^{-1}$.

Step 4. Minimize Trace(Q^j) subject to $P_2^j > 0$ and the LMI constraints

$$\Delta_Y(P_1^j, P_{10}^j, C_K^j, \gamma) < 0, \quad \Delta_{KY}(P_2^j, P_{20}^j, C_K^j, \beta) < 0$$

$$\begin{bmatrix} -Q^j & C_0 + D_0 C_K^j \\ (C_0 + D_0 C_K^j)^T & -\frac{1}{\gamma^2} P_1^j \end{bmatrix} < 0$$

where $P_{10}^j = P_{1opt}^{j-1}$, $P_{20}^j = P_{2opt}^{j-1}$, $j = 1, 2, \cdots$, and P_{1opt}^{j-1} and P_{2opt}^{j-1} are the solutions of the $(j-1)$th optimization. When Trace(Q_{opt}^{j-1}) − Trace(Q_{opt}^j) $< \epsilon$ for some $\epsilon > 0$, stop.

Remark 3.9: In Algorithm 3.8, Step 2 and Step 3 provide initial solutions P_{10}^0 and P_{20}^0 for the iterative computation in Step 4. It is easy to see that Trace(Q_{opt}^j) $\leq$ Trace(Q_{opt}^{j-1}), $j = 1, 2, \cdots$, so the sequence $\{\text{Trace}(Q_{opt}^j)\}_{j=1}^{\infty}$ is convergent, which implies that for any $\epsilon > 0$, the inequality Trace(Q_{opt}^{j-1}) − Trace(Q_{opt}^j) $< \epsilon$ will be satisfied for some large enough j. So the algorithm is convergent with respect to the optimization objective Trace(Q^j). The algorithm is based on the combination of Theorem 3.2 and Theorem 3.5, the iterative part of it (Step 4 in Algorithm 3.8) from Theorem 3.5 will improve the trade-off between α, γ and β, which will be illustrated in Section 5 by an example.

Similarly, combining Corollary 3.3 and Corollary 3.6, we have the following convergent algorithm to minimize $J(\Sigma, K, \gamma)$ under the constraints of $\|T_{z_1 w}(K)\|_\infty < \gamma$ and $K(s)$ with the prescribed degree of stability r.

Algorithm 3.10: Let $\gamma > 0$ and $r > 0$ be given constants.
Step 1. Solve the Riccati equation (14) to obtain the stabilizing solution $Y \geq 0$.
Step 2. Minimize Trace(Q) subject to the LMI constraints (26), (27) and (31). Denote $C_{Kopt} = W_{opt} X_{opt}^{-1}$.
Step 3. Minimize Trace(Q_0) subject to the LMI constraints (43), (44); and minimize Trace(P_2) subject to $P_2 > 0$ and the LMI constraint

$$P_2(A_{12} + rI + B_{22} C_{Kopt}) + (A_{12} + rI + B_{22} C_{Kopt})^T P_2 + C_{Kopt}^T C_{Kopt} < 0 \quad (46)$$

Denote $P_{10}^0 = \gamma^2 X_{opt}^{-1}$ and $P_{20}^0 = P_{2opt}$.
Step 4. Minimize Trace(Q^j) subject to $P_2^j > 0$ and the LMI constraints

$$\Delta_Y(P_1^j, P_{10}^j, C_K^j, \gamma) < 0, \quad \Delta_{KYr}(P_2^j, P_{20}^j, C_K^j, r) < 0$$

$$\begin{bmatrix} -Q^j & C_0 + D_0 C_K^j \\ (C_0 + D_0 C_K^j)^T & -\frac{1}{\gamma^2} P_1^j \end{bmatrix} < 0$$

where $P_{10}^j = P_{1opt}^{j-1}$, $P_{20}^j = P_{2opt}^{j-1}$, $j = 1, 2, \cdots$, and P_{1opt}^{j-1} and P_{2opt}^{j-1} are the solutions of the $(j-1)$th optimization. When Trace(Q_{opt}^{j-1}) − Trace(Q_{opt}^j) $< \epsilon$ for some $\epsilon > 0$, stop.

Remark 3.11: In [8], sufficient conditions for the stable mixed H_2/H_∞ control problem are given in terms of solutions to three coupled Riccati equations, which are difficult to solve. Comparing with the results in [8], Theorem 3.2 and Theorem 3.5 are given in terms of a Riccati equation and matrix inequalities by solving a multiobjective control problem, and convergent iterative algorithms are developed based on the two theorems. Moreover, the performances such as H_∞ norm and prescribed degree of stability of the designed controller are addressed, which are not covered in [8].

4 Example

In this section, we will present an example to illustrate the proposed algorithms.

Example 5.1: The example is to illustrate the use of Algorithm 3.10 to design a mixed H_2/H_∞ controller with a prescribed degree of stability r. The system model is as follows.

$$A = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = [1 \quad 0]$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$D_2 = [0 \quad 1], \quad C_0 = C_1, \quad D_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $\gamma = 0.5$, by using the result in [9] and LMI Control Toolbox [4], the optimal mixed H_2/H_∞ controller as shown in Table 1. The mixed H_2/H_∞ cost J_{opt} ($= J_d = J_a$) of the closed-loop system is 0.0842, and the stability degree of the controller is less than 0.4816.

Consider the problem of designing an H_2/H_∞ controller with the prescribed degree r of stability for $r = 1$. By using Algorithm 3.10 without Step 4 and with Step 4 (200 iterations), the computed results are shown in Table 1.

Table 1: Comparative Results for Example 1 with $\gamma = 0.5$

	Controller Parameters	J_d	J_a	Poles of Controller
Optimal H_2/H_∞	A_{Km}, B_{Km}, C_{Km}, D_{Km}	0.0842	0.0842	−0.4816, −1.5490
Algorithm 3.10N	A_{Km1}, B_{Km1}, C_{Km1}, D_{Km1}	0.1068	0.1004	$-1.2820 \pm 0.3694i$
Algorithm 3.10	A_{Km2}, B_{Km2}, C_{Km2}, D_{Km2}	0.0888	0.0888	−1.005, −1.1393

where Algorithm 3.10N represents Algorithm 3.10 without Step 4; J_d denotes the designed mixed H_2/H_∞

cost; and J_a denotes the actually achieved mixed H_2/H_∞ cost.

From Table 1, the results obtained via Algorithm 3.10 without Step 4 are conservative, where $J_a \neq J_d$ and the controller has much larger stability degree than the designed value 1. However, Algorithm 3.10 with Step 4 gives better results, where $J_a = J_d$ and the controller has only a slightly larger stability degree than the designed value 1. Comparing with the optimal mixed H_2/H_∞ cost J_{opt}, it is easy to see that the expense of increasing the stability degree of the controller (from -0.4816 to -1.005) is an increase of 5.34% (from 0.0842 to 0.0887) in the optimal mixed H_2/H_∞ cost. The controller parameters are given in the Appendix.

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Appendix: Controller Parameters

Controller parameters for Example 5.1:

$$A_{Km} = \begin{bmatrix} -0.8077 & 0.5000 \\ 0.4835 & -1.2229 \end{bmatrix}, \quad B_{Km} = \begin{bmatrix} 0.0641 \\ 0.1540 \end{bmatrix}$$

$$C_{Km} = [\,-0.4785 \quad -0.2229\,]$$
$$D_{Km} = 0$$

$$A_{Km1} = \begin{bmatrix} -0.8077 & 0.5000 \\ -0.7228 & -1.7563 \end{bmatrix}, \quad B_{Km1} = \begin{bmatrix} 0.0641 \\ 0.1540 \end{bmatrix}$$

$$C_{Km1} = [\,-1.6847 \quad -0.7563\,]$$
$$D_{Km1} = 0$$

$$A_{Km2} = \begin{bmatrix} -0.8077 & 0.5000 \\ -0.1313 & -1.3372 \end{bmatrix}, \quad B_{Km2} = \begin{bmatrix} 0.0641 \\ 0.1540 \end{bmatrix}$$

$$C_{Km2} = [\,-1.0932 \quad -0.3372\,]$$
$$D_{Km2} = 0$$