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STABLE DENSITIES UNDER CHANGE OF SCALE AND TOTAL VARIATION INEQUALITIES

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In this paper it is shown that if q is the density of a symmetric stable density, then for $c \in (0, 1) \cup (1, \infty)$, the graph of $q(x)$ intersects the graph of $cq(cx)$ at only two points. The argument proceeds by introducing a new characterization of unimodality for densities and involves a representation for symmetric stable random variables that is also useful for simulating such random variables. Finally our results are applied to prove some inequalities concerning the total variation norm of the difference of two symmetric stable densities.

1. Introduction. Some of the basic properties of stable densities are still unknown (or at least unproved). In this paper we show that the picture that is usually drawn to illustrate how centered normal densities behave under change of scale (see Figure 1) is valid for any symmetric stable density $q(x, \alpha)$ of fixed index $\alpha \in (0, 2]$. More precisely, we show that for any $c \in (0, 1) \cup (1, \infty)$ the graph of $q(x, \alpha)$ crosses the graph of $cq(cx, \alpha)$ at exactly two points symmetric around the origin. We proceed by first proving that what we shall call "the single intersection property" holds for the density of positive stable random variables. Namely, if $p(x, \alpha)$ is the density of a positive stable random variable of index $\alpha \in (0, 1)$, then for $c \in (0, 1) \cup (1, \infty)$ the graph of $p(x, \alpha)$ crosses the graph of $cp(cx, \alpha)$ exactly once for $x > 0$. The "single intersection property" for a positive random variable Z turns out to be equivalent to the unimodality of $\log Z$. If Z is a positive stable random variable of index $\alpha \in (0, 1)$ we use a result of Chernin and Ibragimov [1] to prove that $\log Z$ is unimodal. This result of Chernin and Ibragimov enables us to write positive stable random variables in terms of uniform and exponential random variables. This representation is of use in another context, i.e. in direct Monte-Carlo simulation of stable random variables.

Our paper is organized as follows. Section 2 contains certain facts we shall need from the theory of totally positive kernels as exposited in Karlin as well as a quick introduction to stable densities. Section 3 contains our treatment of the single intersection property for densities as well as some new characterizations of unimodality. This section will also contain a number of theorems about the behavior of the total variation norm $||\cdot||$ on densities. (If $p_1(x)$ and $p_2(x)$ are densities on $(-\infty, +\infty)$ then $||p_1 - p_2|| \equiv \int_{-\infty}^{+\infty} |p_1(x) - p_2(x)|\, dx$.) Section 4

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contains our proof of the fact that the single intersection property holds for $p(x, \alpha)$ and for $q(|x|, \alpha)$. Section 5 ends with a picture which illustrates the geometry of how symmetric unimodal densities $q(x)$ cross under change of scale when $q(|x|)$ has the single intersection property. This section also contains some counterexamples and an argument that shows that neither $p(x, \alpha)/cp(cx, \alpha)$ nor $q(x, \alpha)/cq(cx, \alpha)$ are monotone as x varies in $(0, \infty)$, for $c \in (0, 1) \cup (1, \infty)$ and α near 2. This makes us feel that the regularity properties we prove here for stable densities are about as much as we can hope for.

2. Preliminary facts about stable and totally positive densities. For $0 < \alpha \leqq 2$ we shall let $q(x, \alpha)$ denote the symmetric density function on $(-\infty, +\infty)$ with characteristic function $\exp(-|t|^\alpha)$ for t real. (We let $q_c(x, \alpha)$ denote the density $cq(cx, \alpha)$.) For $0 < \alpha < 1$ we let $p(x, \alpha)$ denote the density function on $(0, \infty)$ with Laplace transform $\exp(-s^\alpha)$ for $s > 0$. (We let $p_c(x, \alpha)$ denote the density $cp(cx, \alpha)$.) The proof of the existence of these densities can be found in Feller ([2], page 424 and page 540).

If $X_1, \cdots, X_n$ are independent random variables all with density $q_c(x, \alpha)$, then it is clear that $n^{-(1/\alpha)}(X_1 + \cdots + X_n)$ also has density $q_c(x, \alpha)$. (A similar statement is true about $p_c(x, \alpha)$.) For this reason we say that the densities $q_c(x, \alpha)$ and $p_c(x, \alpha)$ are strictly stable of index α. The densities $q_c(x, \alpha)$ and $p_c(x, \alpha)$ are the only stable densities we will study in this paper, though there are many others, not necessarily symmetric or giving mass 1 to $(0, \infty)$.

We now give a quick summary of what we shall need to know about totally positive kernels and densities. The following definitions are taken from Karlin [5].

DEFINITION 2.1. Let $f(t)$ be a function with values in $[-\infty, +\infty]$, defined on I, a subset of the real line. Let $S^-(f) = \sup S^-(f(t_1), \cdots, f(t_n))$ where the sup is taken over all sets $t_1 < \cdots < t_n$ with t_i in I and n is arbitrary but finite. Here $S^-(x_1, \cdots, x_n)$ is the number of sign changes of the indicated sequence, zero terms discarded.

We let $S^+(f)$ be defined similarly in terms of $S^+(f(t_1), \cdots, f(t_n))$, where $S^+(x_1, \cdots, x_n)$ denotes the maximum number of sign changes of the indicated sequence, zero terms being given arbitrary sign.

DEFINITION 2.2. A real function (frequently called kernel) $K(x, y)$ of two variables x, y ranging over two subsets of the real line I and J is said to be totally positive of order r on $I \times J$, if for all $x_1 < \cdots < x_m$, $y_1 < \cdots < y_m$, $m \leqq r$, and $(x_i, y_j) \in I \times J$, we have $\det(K(x_i, y_j)) \geqq 0$. If strict inequality holds the kernel K is said to be strictly totally positive of order r on $I \times J$.

DEFINITION 2.3. A density $f(x)$ is said to be a (strict) Polya density if the kernel $f(x - y)$ is (strictly) totally positive of all orders.

The following theorem is crucial to our work.

THEOREM A. *Suppose $K(x, y)$ is Borel measurable and* (*strictly*) *totally positive of order r on $I \times J$. Let f be a real Borel function on J with $S^-(f) \leqq r - 1$. Define* *g on I by $g(x) = \int_J K(x, y)f(y)\,dy$, and assume this integral exists for all x in I. Then $S^-(g) \leqq S^-(f)(S^+(g) \leqq S^-(f)$.*

See Karlin [5, page 233] for a more general version of this theorem.

We note that by [5, page 18] any kernel of the form $K(x, y) = \exp(f(x)g(y))$ is strictly totally positive on $(0, \infty) \times (0, \infty)$ if both f and g are strictly increasing on $(0, \infty)$. From this it follows simply that $K(x, y) = h(x)k(y)\exp(f(x)g(y))$ is strictly totally positive on $(0, \infty) \times (0, \infty)$ if (1) h and k are both greater than 0 on $(0, \infty)$ and (2) f and g are both strictly increasing on $(0, \infty)$; since $\det(h(x_i)k(y_j)\exp(f(x_i)g(y_j))) = \prod_{i,j} h(x_i)k(y_j)\det(\exp(f(x_i)g(y_j)))$.

We now make some remarks about densities to explain our conventions. First we note that if a random variable X has a density function f then f is defined only up to sets of measure 0. We shall, of course, try to use as regular a version of the density of X as possible, if the density exists; however, we also note that it is convenient sometimes to assign the value $+\infty$ to the value at some point of the density of X. This happens, for instance, if X is unimodal, i.e. there exists some point m such that the distribution function of X is convex to the left of m and concave to the right of m. If X is unimodal and has a density, we choose a version $f(x)$ of its density that is increasing to the left of m and decreasing to the right of m. In some cases we then have that $\lim_{x\to m} f(x) = \infty$, which "forces" us to assign the value $+\infty$ to $f(m)$.

The following theorems concerning kernels totally positive of order two will also be useful to us.

THEOREM B. *Let $f(x)$ be a real valued density. Then $f(x - y)$ is a totally positive kernel of order 2* iff *$\log f(x)$ is concave on some interval I and $-\infty$ outside of I.* (See Karlin [5, page 159]).

THEOREM C. *Let X and Y be independent random variables. If X is unimodal and Y has a density f with $f(x - y)$ totally positive of order 2, then $X + Y$ is unimodal.* (See [3].)

3. The single intersection property; connection with unimodality and total variation. Let J be a finite or infinite interval in R. For any function f defined on J with values in $[-\infty, +\infty]$ let $I^-(f)$ denote the closure in R of $\{x \mid f(x) \leqq 0\}$ and $I^+(f)$ denote the closure in R of $\{x \mid f(x) \geqq 0\}$. If f_1 and f_2 are $[-\infty, +\infty]$ valued functions defined on J with $f_1 - f_2$ well defined (i.e. f_1 and f_2 do not assume the same value $+\infty$ or $-\infty$ at the same point y in J) we say that f_1 crosses f_2 at y_0 in J^0 (the interior of J) if $f_1(y) - f_2(y)$ has exactly one strict change in sign in any sufficiently small interval containing y_0. It can be seen that $S^+(f_1 - f_2) = 1$ if and only if $\{I^-(f_1 - f_2), I^+(f_1 - f_2)\} = \{U, U'\}$ where U and U' are intervals in R with $U \cap U' = \{a\}$ for some real a. In that case f_1 crosses f_2 at a. It can also be seen that $S^-(f_1 - f_2) = 1$ if and only if $\{I^-(f_1 - f_2), I^+(f_1 - f_2)\} = \{U, U'\}$ where U and U' are intervals in R with $U \cap U' = [a, b]$ and $\inf_{x\in J} x < a \leqq b < \sup_{x\in J} x$. In that case we shall call the set $[a, b]$ the crossing set for $f_1 - f_2$.

The following lemma gives a new characterization of unimodality which we shall need. The notation $T_d \cdot f$ stands for the d translate of a function f, i.e. $(T_d \cdot f)(x) = f(x + d)$.

LEMMA 3.1. *Let X be a random variable with density. Then X is unimodal if and only if there is some version f of the density of X which is finite at all points except possibly one and such that $S^-(f - T_d \cdot f) = 1$ for all $d \in (-\infty, 0) \cup (0, +\infty)$.*

PROOF. We prove one direction of the theorem by remarking that X unimodal implies that there is some number m and some version f of the density of X such that f is finite, increasing on $(-\infty, m)$ and such that f is finite, decreasing on (m, ∞). If $d > 0$ then $T_d \cdot f \geqq f$ on $(-\infty, m - d]$, $f \geqq T_d \cdot f$ on $[m, \infty)$, and $T_d \cdot f - f$ is decreasing on $[m - d, m]$. It follows that $S^-(f - T_d \cdot f) = 1$, since $I^-(f - T_d \cdot f) = (-\infty, b_d]$ and $I^+(f - T_d \cdot f) = [a_d, \infty)$ where $a_d \leqq b_d$. (For further reference we note here that $[a_d, b_d] \subset [m - d, m]$.) The argument for $d < 0$ is similar.

We now argue the reverse implication. We are given that $S^-(f - T_d \cdot f) = 1$ for all $d > 0$. Let $F(x)$ denote the distribution function of X. We note that

$$\Delta_d^2 F(x) = \int_{x-d}^{x} (T_d \cdot f - f)\, dx \tag{3.1}$$

where $\Delta_d^2 F(x) = F(x + d) - 2F(x) + F(x - d)$. We can rewrite the right hand side of (3.1) as $((T_d \cdot f - f) * I_{[-d,0]})(x)$; we then apply Theorems A and B to conclude that $S^-(\Delta_d^2 F) = 1$ for all $d > 0$. This suffices to prove that F is unimodal by the following argument.

We let F_ε stand for $F * N_\varepsilon$ where N_ε stands for the normal distribution with mean zero and variance ε. Since the density of N_ε is a strict Polya density by [5, page 19] we conclude in particular that $S^+(\Delta_d^2 F_\varepsilon) = 1$. This implies that for some real number $c = c(d, \varepsilon)$ we have $\{x \mid \Delta_d^2 F_\varepsilon(x) > 0\} = (-\infty, c)$ and $\{x \mid \Delta_d^2 F_\varepsilon(x) < 0\} = (c, +\infty)$. Now let d tend to zero along some subsequence d_n such that $c(d_n, \varepsilon) \to r = r(\varepsilon)$. As is well known, F_ε is infinitely differentiable, and since $(d^2/dx^2)F_\varepsilon(x) = \lim_{n\to\infty} \Delta_d^2 F_\varepsilon(x)/d_n^2$ we conclude that $(d^2/dx^2)F_\varepsilon$ is positive on $(-\infty, r)$ and negative on $(r, +\infty)$ hence F_ε is unimodal. Now $F_\varepsilon \to F$ in distribution as $\varepsilon \to 0$, hence F is unimodal as well by [4, page 66].

REMARK 3.1. It is worthwhile to remark that by suitably rearranging the above proof we get an even more general criterion for X to be unimodal, namely X is unimodal if and only if $S^-(\Delta_d^2 F) = 1$ for all $d > 0$, where F is the distribution function for X. (Here we do not assume the existence of a density for X).

REMARK 3.2. Using the argument in the first half of Lemma 3.1 it is easy to see that if X has a density f with a unique mode m such that f is strictly monotone on either side of m, then $S^+(f - T_d \cdot f) = 1$ for all $d \in (-\infty, 0) \cup (0, \infty)$. Conversely if f is not strictly monotone on some side of m and $S^+(f - T_d \cdot f) = 1$ for all $d > 0$ then f must be constant on some interval (since f in any case will be monotone on either side of m); this implies $S^+(f - T_d \cdot f) = \infty$ which is a contradiction. We shall call a density f with $S^+(f - T_d \cdot f) = 1$ for all $d \in (-\infty, 0) \cup (0, +\infty)$, a *strictly unimodal density*.

We now introduce the single intersection property. It will play a crucial role in our geometric approach to the study of stable densities. The notation f_c stands for the change of scale by c, i.e. $f_c(x) = cf(cx)$. If X is a random variable with density f then X/c has density f_c.

DEFINITION 3.1. Let J be a subinterval of R. Let f be a $[-\infty, +\infty]$ valued function defined on J. We say f has the *single intersection property on* J if $f - f_c$ is well defined for all $c \in (0, 1) \cup (1, \infty)$ and if $S^-(f - f_c) = 1$ for all such c. If $S^+(f - f_c) = 1$ for all $c \in (0, 1) \cup (1, \infty)$ we say that f has the *strict single intersection property on* J.

THEOREM 3.1. *Let X be a positive random variable with density. Then the following are equivalent*:

(i) *There is some version p of the density of X with the single intersection property on* $(0, \infty)$.
(ii) *The random variable* $\log X$ *is unimodal.*

PROOF. Assume (i) first. Since p has the single intersection property on $(0, \infty)$ we must in particular have $p - p_c$ well defined for all $c \in (0, 1) \cup (1, \infty)$. This forces p to be finite everywhere on $(0, \infty)$ except possibly at one point where the value $+\infty$ is assumed. The density f of $\log X$ is calculated to be $f(t) = e^t p(e^t)$ defined on $(-\infty, +\infty)$ with the exception of at most one point where the value $+\infty$ is assumed.

Now $p(x) > p_c(x)$ is equivalent to $f(t) > (T_d \cdot f)(t)$ for $x > 0$, where $t = \log x$ and $d = \log c$. We conclude that $S^-(p - p_c) = 1$ for all $c \in (0, 1) \cup (1, \infty)$ if and only if $S^-(f - T_d \cdot f) = 1$ for all $d \in (-\infty, 0) \cup (0, +\infty)$. By Lemma 3.1 we conclude that $\log X$ is unimodal. The argument to show (ii) implies (i) proceeds exactly as above, but in reverse order.

COROLLARY 3.1. *Let X be a symmetric random variable with density. Then there exists a version q of the density of X with $q - q_c$ well defined on $(-\infty, \infty)$ for all $c \in (0, 1) \cup (1, \infty)$ and with $S^-(q - q_c) = 2$ if and only if* $\log |X|$ *is unimodal.*

PROOF. Apply Theorem 3.1 separately to the positive part of X and the negative part of X.

We end this section with some inequalities concerning the total variation norm $\| \;\|$ on densities.

THEOREM 3.2. *Let $g(x)$ be a unimodal density on $(-\infty, \infty)$. Then for $d_1 < d_2 < d_3$ we have* $\|T_{d_1} \cdot g - T_{d_2} \cdot g\| \leqq \|T_{d_1} \cdot g - T_{d_3} \cdot g\|$.

PROOF. Without loss of generality we may suppose $d_1 = 0$. Referring back to the notation of Lemma 3.1, we define a_i, b_i by setting $(I^-(g - T_{d_1} \cdot g, I^+(g - T_{p_1} \cdot g)) \equiv ((-\infty, b_i], [a_i, \infty))$ for $i = 2, 3$. We note that the unimodality of g implies that $[a_2, b_2] \subset [m - d_2, m]$ where m is any mode for g (as we have seen in the proof of Lemma 3.1) we conclude that $T_{d_2} \cdot g$ is decreasing on (a_2, ∞) hence $T_{d_2} \cdot g \geqq T_{d_3} \cdot g$ on $[a_2, \infty)$. Since $g \geqq T_{d_2} \cdot g$ on $[a_2, \infty)$ by definition we conclude

$$(3.2) \qquad \int_{a_2}^{\infty} |g(x) - g(x + d_2)|\, dx \leqq \int_{a_2}^{\infty} |g(x) - g(x + d_3)|\, dx .$$

Now let $d = d_3 - d_2$. For $x \leqq a_2$ we have $T_{d_2} \cdot g \geqq g \geqq T_{-d} \cdot g$, where the last inequality follows since g is decreasing for $x \leqq a_2$. We get

$$\begin{aligned} \int_{-\infty}^{a_2} |g(x) - g(x + d_2)|\, dx &\leqq \int_{-\infty}^{a_2} |g(x - d) - g(x - d + d_3)|\, dx \\ &= \int_{-\infty}^{a_2 - d} |g(x) - g(x + d_3)|\, dx . \end{aligned}$$

We conclude that

$$(3.3) \qquad \int_{-\infty}^{a_2} |g(x) - g(x + d_2)|\, dx \leqq \int_{-\infty}^{a_2} |g(x) - g(x + d_3)|\, dx .$$

We conclude from (3.2) and (3.3) together that

$$\int_{-\infty}^{+\infty} |g(x) - g(x + d_2)|\, dx \leqq \int_{-\infty}^{+\infty} |g(x) - g(x + d_3)|\, dx .$$

COROLLARY 3.2. *Let X be a positive random variable with density $p(x)$ which has the single intersection property on $(0, \infty)$. Then for $0 < c_1 < c_2 < c_3$ we have* $||p_{c_1} - p_{c_2}|| \leqq ||p_{c_1} - p_{c_3}||$.

PROOF. Let g be the density of $\log X$. By Theorem 3.2 and Lemma 3.1 we know that

$$||T_{\log c_1} \cdot g - T_{\log c_2} \cdot g|| \leqq ||T_{\log c_1} \cdot g - T_{\log c_3} \cdot g|| .$$

However for any $c, c' \in R$ we have

$$||p_c - p_{c'}|| = 2 \sup_A (P[X \in cA] - P[X \in c'A])$$

where the sup is taken over all Borel subsets of $(0, \infty)$. Using the fact that $\log x$ is a bimeasurable 1–1 map from $(0, \infty)$ to $(-\infty, \infty)$, we can rewrite the last expression as

$$2 \sup_B (P[\log X \in \log c + B] - P[\log X \in \log c' + B])$$

where the sup is taken over all Borel subsets of $(-\infty, +\infty)$. This in turn equals $||T_{\log c} \cdot g - T_{\log c'} g||$ and the conclusion of this corollary is now evident.

COROLLARY 3.3. *Let X be a random variable with density $q(x)$. Suppose both $q(x)$ and $q(-x)$ have the strict single intersection property on $(0, \infty)$. Then for $0 < c_1 < c_2 < c_3$ we have*

$$||q_{c_1} - q_{c_2}|| \leqq ||q_{c_1} - q_{c_3}|| .$$

PROOF. Apply the last corollary to each of $q(x)$ and $q(-x)$ separately.

4. Application to stable densities. The following result is stated in [1] and can be gotten by imitating their calculations earlier in this paper. (We have corrected a slight misprint in the formula presented there.)

LEMMA 4.1. *Let $\alpha \in (0, 1)$ and let $p(x, \alpha)$ be as in Section 2. Then for $x \geqq 0$*

$$(4.1) \qquad p(x, \alpha) = \frac{1}{\pi}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{x}\right)^{(1-\alpha)^{-1}} \int_0^\pi a(\varphi) \exp\left(-\left(\frac{1}{x}\right)^{\alpha/(1-\alpha)} a(\varphi)\right) d\varphi$$

where

$$a(\varphi) = \left(\frac{\sin(\alpha\varphi)}{\sin(\varphi)}\right)^{(1-\alpha)^{-1}} \left(\frac{\sin((1-\alpha)\varphi)}{\sin(\alpha\varphi)}\right).$$

COROLLARY 4.1. *Let L and U be independent random variables where L is exponential with density* $\exp(-x)$ *for* $x \geqq 0$, *and U is uniform on* $[0, \pi]$. *Then for* $\alpha \in (0, 1)$ *we claim that* $p(x, \alpha)$ *is the density of* $(a(U)/L)^{(1-\alpha)/\alpha}$.

PROOF. Let $h(x) = \pi^{-1} \int_0^\pi a(\varphi) \exp(-xa(\varphi))\, d\varphi$. Clearly $h(x)$ is the density of $L/a(U)$. By a trivial change of variable we conclude that the density of $(a(U)/L)^{(1-\alpha)/\alpha}$ is $p(x, \alpha)$.

REMARK. It is convenient to rewrite $a(\varphi)$ as $(b(\varphi))^{(1-\alpha)^{-1}}$ where $b(\varphi) = (\sin(\alpha\varphi)/\sin(\varphi))^\alpha(\sin((1-\alpha)\varphi)/\sin(\varphi))^{1-\alpha}$. This form is used in [6].

LEMMA 4.2. *Let h be a differentiable strictly increasing function defined on* $(0, 1)$. *Let U be a random variable uniformly distributed on* $(0, 1)$. *Then the density* $p(y)$ *of the random variable* $h(U)$ *is finite everywhere. Also*

(1) *If* $h'(s)$ *is increasing then* $h(U)$ *is unimodal.*

(2) *If* $d/ds \log h(s)$ *is increasing then* $\log h(U)$ *is unimodal.*

PROOF. The function h is strictly increasing hence the density of $h(U)$ at the point $y = h(x)$ must be $\lim_{\Delta y \to 0} (h^{-1}(y + \Delta y) - h^{-1}(y))/\Delta y = 1/h'(h^{-1}(y))$. By continuity of h, the range of h is some bounded or unbounded interval I. It is clear that $p(y) = 0$ for $y \notin I$, and $p(y) > 0$ for $y \in I$.

To show that p is unimodal it suffices to show that $p(y)$ decreases on I, which is clear. This proves (1). To prove (2), just apply (1) to $\log h(x)$.

REMARK. Under the hypotheses of Lemma 2 one can show that $d^2/dy^2 \log(p(y)) < 0$ iff $h'h''' - 2(h'')^2 \geqq 0$. In other words the condition that $h'h''' - 2(h'')^2 \geqq 0$ suffices to conclude that $p(y)$ is a totally positive density of order 2. (See Theorem B.) We do not prove this in detail because when $\alpha = \frac{1}{2}$, if we let $h(\varphi) = \log(b(\varphi))$ (where $b(\varphi) = (\sin(\alpha\varphi)/\sin(\varphi))^\alpha(\sin((1-\alpha)\varphi)/\sin\varphi)^{1-\alpha}$), it turns out that $h'h''' - 2(h'')^2 < 0$. Hence, the following theorem is the most we can prove about the random variable $\log Z$, where Z is positive stable.

THEOREM 4.1. *Let Z be a random variable with density* $p(y, \alpha)$ *for* $\alpha \in (0, 1)$. *Then for* $c \in (0, 1) \cup (1, \infty)$ *then densities* $p(y, \alpha)$ *and* $p_c(y, \alpha)$ *on* $(0, \infty)$ *are equal exactly once (where they cross).*

PROOF. Write Z as $b(U)^{1/\alpha}L^{(\alpha-1)/\alpha}$ as in Corollary 4.1 where U is uniform, on $(0, \pi)$ and L is exponential. We have $\log Z = \alpha^{-1} \log b(U) + (\alpha - 1)/\alpha \log L$. By Lemma 4.2, to check that $\log b(U)$ is unimodal we need only to prove

$\log b(\varphi)$ is strictly increasing and $d/d\varphi \log b(\varphi)$ is increasing on $(0, \pi)$. However $d/d\varphi \log b(\varphi) = \alpha(d/d\varphi) \log (\sin(\alpha\varphi)/\sin\varphi) + (1-\alpha)\, d/d\varphi \log ((\sin(1-\alpha)\varphi)/\sin\varphi)$. Now $d/d\varphi \log (\sin(\alpha\varphi)/\sin\varphi) = \alpha \cot(\alpha\varphi) - \cot\varphi$. This expression is positive for $0 < \alpha < 1$, as we can check by differentiation with respect to α. Similarly $d/d\varphi \log (\sin(1-\alpha)\varphi/\sin\varphi)$ is positive, and we conclude $\log b(\varphi)$ is strictly increasing in $(0, \pi)$. We differentiate again to get $d^2/d\varphi^2 \log (\sin(\alpha\varphi)/\sin\varphi) = -\alpha^2/\sin^2(\alpha\varphi) + 1/\sin^2(\varphi)$. This expression in positive for $0 < \alpha < 1$ because $(\sin(\alpha\varphi)/\sin\varphi) > \alpha$ as we can check by differentiating with respect to α. So we conclude as before that $d/d\varphi \log b(\varphi)$ is increasing and hence that $\alpha^{-1} \log b(U)$ is unimodal. By Theorem 3.1 we see that $S^-(r - r_c) = 1$ where r is the density of $b(U)^{1/\alpha}$.

Now for $\gamma > 0$ the density of $L^{-\gamma}$ is $\gamma^{-1}(x^{-1})^{(\gamma+1)/\gamma} \exp\{-(x^{-1})^\gamma\}$ for $x \geqq 0$.

We conclude that for $y \geqq 0$

$$p(x, \alpha) = \gamma^{-1} \int_0^\infty \frac{1}{y} \left(\frac{y}{x}\right)^{(\gamma+1)/\gamma} e^{-(y/x)^\gamma} r(y)\, dy$$

where $\gamma = (1-\alpha)/\alpha$. The density $p_c(x, \alpha)$ has the same representation with r_c replacing r. Now the kernel $K(x, y) = 1/y(y/x)^{(\gamma+1)/\gamma} \exp\{-(y/x)^\gamma\}$ is strictly totally positive by [5, page 18], hence we conclude that $S^+(p(\cdot, \alpha) - p_c(\cdot, \alpha)) \leqq 1$. This means that $p_c(x, \alpha)$ and $p(x, \alpha)$ are equal no more than once. By [4, page 48] both p and p_c are continuous; it follows that they are equal exactly once (where they cross) since otherwise one would dominate the other. □

REMARK. Let $\gamma > 0$ and let Z have density $p(x, \alpha)$. The above proof shows that the density of Z^γ is equal to the density of cZ^γ exactly once (where they actually cross).

THEOREM 4.2. *Let* $\alpha \in (0, 2]$. *Let* $c \in (0, 1) \cup (1, \infty)$. *Let* $q_c(x, a) = cq(cx, a)$. *Then* $q(x, a)$ *and* $q_c(x, a)$ *are equal only twice, at points symmetric about* 0 (*where they actually cross.*)

PROOF. Since $(q(x, \alpha)$ and $q_c(x, \alpha)$ are symmetric, it suffices to show that $q(x, \alpha)$ equals $q_c(x, \alpha)$ only once for $x \geqq 0$. Let Z have density $p(x, \alpha/2)$ and let X be normal with mean 0 and variance 1. If Z and X are independent, then some multiple of $Z^{\frac{1}{2}}X$ has density $q(x, \alpha)$ by Feller [2, page 562]. (In fact $2^{\frac{1}{2}}$ is the correct multiple, but we do not need this information.)

For $x \geqq 0$ we conclude that for some $k > 0$

$$q(x, \alpha) = \frac{k^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \frac{1}{y} e^{-\frac{1}{2}k(x/y)^2} h(y)\, dy\,,$$

where h is the density of $Z^{\frac{1}{2}}$. A similar expression holds for $q_c(x, \alpha)$, except that h_c, the density of $c^{-1}Z^{\frac{1}{2}}$, is substituted for h. Now $S^-(h - h_c) = 1$ and the kernel $y^{-1}e^{-\frac{1}{2}k(x/y)^2}$ is strictly totally positive on $(0, \infty) \times (0, \infty)$ by [5, page 18]. It follows that $S^+(q - q_c) \leqq 1$ where $q = q(x, \alpha)$ defined for $x > 0$ and $q_c = q_c(x, \alpha)$ is also defined for $x > 0$. We argue as in Theorem 4.1 to show that $q = q_c$

exactly once on $(0, \infty)$ and that the point of equality is a cross-over point. (Note that we need not worry about equality at 0 since $q(0) = \int_{-\infty}^{+\infty} (2\pi)^{-1}e^{-|\mu|^\alpha}\, d\mu > 0$; hence $q_c(0) = cq(0) \neq q(0)$ for $c \in (0, 1) \cup (1, \infty)$.) □

5. The monotone likelihood ratio property, some counterexamples and a picture. A density $p(x)$ on $(0, \infty)$ is said to have the monotone likelihood property if $p(x)/p_c(x)$ is a monotone function of x for all $c \in (0, 1) \cup (1, \infty)$. It is seen (after rewriting $p(x)/p_c(x)$ as $p(e^y)/p(e^{y+d})$ for $y = \log x$, $d = \log c$) that this property is equivalent to the condition that $\log (Z)$ have a totally positive density of order 2. We consider now the densities $q(x, \alpha)$ for which we have shown that $q(x, \alpha)/q_c(x, \alpha)$ crosses 1 only once for $x \in (0, \infty)$. We now show that we cannot strengthen this result to show that $q(x, \alpha)/q_c(x, \alpha)$ is monotone on $(0, \infty)$. We first need the fact that for $\alpha \in (0, 2)$, the density $q(x, \alpha)$ is proportional to $x^{-1-\alpha}$ for x large; which follows by applying L'Hopital's rule to the fact shown in Feller [4, page 547] that $\int_x^\infty q(y, \alpha)\, dy$ is proportional to $x^{-\alpha}$ for x large and $\alpha \in (0, 2)$. From this we conclude that $\lim_{x\to\infty} q_c(x, \alpha)/q(x, \alpha) = c^{-\alpha}$ for $\alpha \in (0, 2)$. If $\alpha = 2$ then we compute $q_c(x, 2)/q(x, 2) = \exp\{-(c^2 - 1)x^2\}$. From this we conclude first that $q_c(x, 2)/q(x, 2)$ is monotone on $(0, \infty)$ (hence $\log |Y|$ has a totally positive density of order 2) and secondly that $\lim_{x\to\infty} q_c(x, 2)/q(x, 2) = 0$ or ∞ if $c > 1$ or $c < 1$ respectively. We use these two facts to show that $q_c(x, \alpha)/q(x, \alpha)$ is not monotone on $(0, \infty)$ for all $\alpha \in (0, 2)$.

By way of contradiction suppose that $c > 1$ and that $q_c(x, \alpha)/q(x, \alpha)$ is monotone on $(0, \infty)$; it must then be decreasing since $q_c(0, \alpha) > q(0, \alpha)$. We conclude

$$\lim_{x\to\infty} q_c(x, \alpha)/q(x, \alpha) \leqq q_c(y, \alpha)/q(y, \alpha) \tag{5.1}$$

for any $y > 0$. Let $\alpha \to 2$ in (5.1); we get

$$\lim_{\alpha\to 2} \lim_{x\to\infty} q_c(x, \alpha)/q(x, \alpha) \leqq \lim_{\alpha\to 2} q_c(y, \alpha)/q(y, \alpha)\, . \tag{5.2}$$

Now let $y \to \infty$ in (5.2), and relabel y by x, getting:

$$\lim_{\alpha\to 2} \lim_{x\to\infty} q_c(x, \alpha)/q(x, \alpha) \leqq \lim_{x\to\infty} \lim_{\alpha\to 2} q_c(x, \alpha)/q(x, \alpha)\, . \tag{5.3}$$

Since the right-hand side is 0 and the left-hand side is $\lim_{\alpha\to 2} c^{-\alpha}$ by the previous remarks, we see that (5.3) is a contradiction.

Hence we see that $cq(cx, \alpha)q(x, \alpha)$ cannot be decreasing in x for any infinite set of α containing 2 as a limit point.

It follows that $\log |X|$ does not have a totally positive density of order 2 for such a set of α. Let now Z have density $p(x, \alpha/2)$ and let Y, symmetric Gaussian with $E(Y^2) = 2$, be independent of Z. We know $YZ^{\frac{1}{2}}$ has density $q(x, \alpha)$, so if $\log (|Y|Z^{\frac{1}{2}})$ does not have a totally positive density of order two, then neither does $\log (|Z|)$ by Ibragimov's theorem in [3]. (Remember that $\log (|Y|)$ has a totally positive density of order 2 by the previous remarks).

COUNTEREXAMPLE 5.1. It might be conjectured that if $q(x)$ is a real valued unimodal density function then $S^-(q - q_c) = 2$ for all $c \in (0, 1) \cup (1, \infty)$. This conjecture is false, as the following argument shows.

Let h, a, b be positive numbers. Define $q(x) = h$ for $0 \leqq x \leqq a$. Define $q(x) = (\frac{3}{8})h$ for $x = 2a$ and interpolate linearly between $x = a$ and $x = 2a$. We define $q(x) = 0$ for $x = 2a + b$ and interpolate linearly between $x = 2a$ and $x = 2a + b$. Let us now consider $q_2(x) = 2q(2x)$. We have $q_2(0) = 2h > h = q(0)$. Also $q_2(a) = 2q(2a) = 3h/4 < h = q(a)$, Now if we choose b sufficiently large it is clear that $q_2(2a) > q(2a)$ because $q_2(2a)/q_2(a) = b/(a + b)$ while $q(2a) = \frac{3}{8}h < \frac{3}{4}h = q_2(a)$.

We conclude that $q(x)$ crosses $2q(2x)$ at least four times. To finish, choose h so that $q(x)$ is a density.

COUNTEREXAMPLE 5.2. It might be conjectured that if $q(z)$ is a symmetric continuous density on $(-\infty, +\infty)$ with $S^+(q - q_c) = 2$ for all $c \in (0, 1) \cup (1, \infty)$ then $q(z)$ is unimodal. By Corollary 3.1 all we need to disprove this conjecture is an example of a symmetric non-unimodal random variable X with density, such that $\log(|X|)$ is unimodal. One such example follows.

Let a, b, h, k be positive numbers. Let $f(y) = ae^{(k+1)(y-b)}$ for $y \in (-\infty, b]$ and let $f(y) = ae^{-(h-1)(x-b)}$ for $y \in [b, +\infty)$. If $h > 1$ then a can be chosen to make $f(x)$ a density. $f(x)$ is clearly unimodal and corresponds to the density of $\log|X|$. If X is symmetric then its density $q(z)$ is equal to $ae^{-(k+1)b}|z|^k$ for $|z| < e^b$ and is equal to $ae^{(h-1)b}|z|^{-h}$ for $|z| > e^b$, hence X is not unimodal.

We now present a picture. It illustrates how q_{c_1}, q_{c_2}, and q_{c_3} intersect when $c_1 > c_2 > c_3$ and $q(x)$ is a unimodal symmetric continuous density function with $S^+(q - q_c) = 2$ for all $c \in (0, 1) \cup (1, \infty)$. One main goal of this paper was to prove this picture when $q(x) = q(x, \alpha)$, $\alpha \in (0, 2)$.

The picture needs a few words of justification to explain why $x_{12} < x_{13}$ and $x_{12} < x_{23}$, where $\pm x_{ij}$ denote the two points where q_{c_i} and q_{c_j} cross. To see this, we note that $q_{c_i}(0) = c_i q(0, \alpha) > 0$ so $q_{c_1}(0) > q_{c_2}(0) > q_{c_3}(0)$. We now let $c_1 \to +\infty$, fixing c_2 and c_3. In that case, $q_{c_1}(0) \to +\infty$, hence $q_{c_1}(\delta) \to 0$ for any $\delta > 0$ (Otherwise, $\int_{-\infty}^{+\infty} q_{c_1}(y)\, dy$ would exceed 1.) It follows that $x_{12} < x_{13}$ for c_1 large. Hence, $x_{12} < x_{23}$, for c_1 large, otherwise q_{c_2} would intersect q_{c_1} more than twice.

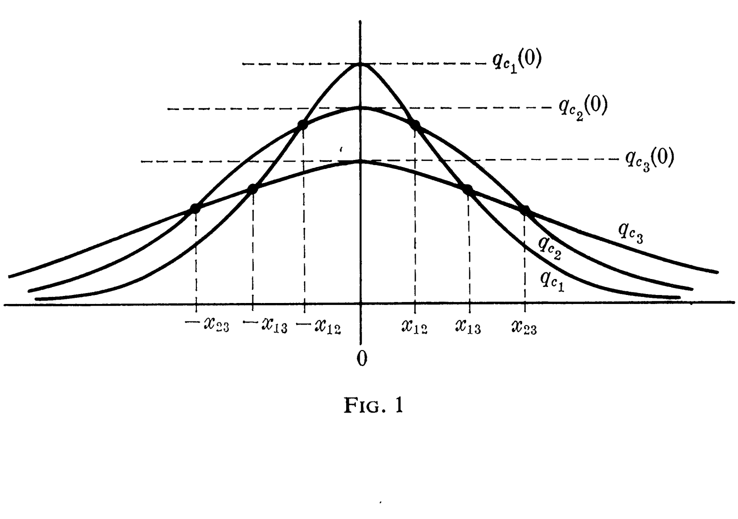

FIG. 1

Now let c_1 decrease. If for some $c_1 > c_2$ we have $x_{12} > x_{13}$ then by "continuity" there is some $c_1 \in (c_2, \infty)$ for which $x_{12} = x_{13}$. This implies $x_{23} = x_{12} = x_{13}$ which contradicts the fact that $\int_{-\infty}^{+\infty} q_{c_i}(y)\,dy = 1$ for $i = 1, 2, 3$. We conclude that $x_{12} < x_{13}$.

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