arXiv:2202.07897v1 [math.PR] 16 Feb 2022

# STABLE FLUCTUATIONS OF ITERATED PERTURBED RANDOM WALKS IN INTERMEDIATE GENERATIONS OF A GENERAL BRANCHING PROCESS TREE

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ABSTRACT. Consider a general branching process, a.k.a. Crump-Mode-Jagers process, generated by a perturbed random walk $\eta_1$, $\xi_1+\eta_2$, $\xi_1+\xi_2+\eta_3,\ldots$. Here, $(\xi_1,\eta_1)$, $(\xi_2,\eta_2),\ldots$ are independent identically distributed random vectors with arbitrarily dependent positive components. Denote by $N_j(t)$ the number of the $j$th generation individuals with birth times $\leq t$. Assume that $j=j(t)\to\infty$ and $j(t)=o(t^a)$ as $t\to\infty$ for some explicitly given $a>0$ (to be specified in the paper). The corresponding $j$th generation belongs to the set of intermediate generations. We provide sufficient conditions under which finite-dimensional distributions of the process $(N_{\lfloor j(t)u\rfloor}(t))_{u>0}$, properly normalized and centered, converge weakly to those of an integral functional of a stable Lévy process with finite mean.

## 1. INTRODUCTION AND MAIN RESULT

1.1. **Definition and motivation.** Let $(\xi_1,\eta_1)$, $(\xi_2,\eta_2),\ldots$ be independent copies of an $\mathbb{R}^2$-valued random vector $(\xi,\eta)$ with arbitrarily dependent components. Denote by $(S_i)_{i\geq 0}$ the zero-delayed standard random walk with increments $\xi_i$ for $i\in\mathbb{N}$, that is, $S_0:=0$ and $S_i:=\xi_1+\cdots+\xi_i$ for $i\in\mathbb{N}$. Define

$$T_i:=S_{i-1}+\eta_i,\quad i\in\mathbb{N}.$$

The sequence $T:=(T_i)_{i\in\mathbb{N}}$ is called *perturbed random walk* (PRW). The so defined PRW is a non-trivial generalization of the standard random walk. Apart from being an interesting object of investigation, the PRW is known to be an important ingredient of perpetuities [9], the Bernoulli sieve [1, 8], $G/G/\infty$-queues [12], a perturbed branching random walk [2], to name but a few. A detailed exposition of various PRW's properties and its applications can be found in [11].

In what follows we assume that $\xi$ and $\eta$ are almost surely (a.s.) positive. Now we recall the construction of a general branching process generated by $T$. Imagine a population of individuals initiated at time 0 by one individual, the ancestor. An individual born at time $s\geq 0$ produces offspring whose birth times have the same distribution as $(s+T_i)_{i\in\mathbb{N}}$. All individuals act independently of each other. An individual resides in the $j$th generation if it has exactly $j$ ancestors. For $j\in\mathbb{N}$ and $t\geq 0$, denote by $T^{(j)}$ the collection of the birth times in the $j$th generation and by $N_j(t)$ the number of the $j$th generation individuals with birth times $\leq t$. We call the sequence $\mathscr{T}:=(T^{(j)})_{j\in\mathbb{N}}$ *iterated perturbed random walk on a general branching process tree*. Also, we call the $j$th generation *early*, *intermediate* or *late* depending on whether $j$ is fixed, $j=j(t)\to\infty$ and $j(t)=o(t)$ as $t\to\infty$, or $j=j(t)$ is of order $t$. According to Proposition 2.1 in [5], there exists a constant $a_0>0$ such that, for $j\geq at$, $a>a_0$ and large $t$, $N_j(t)=0$ a.s.

The sequence $\mathscr{T}$ has been introduced and used in [6] (see also [14]) as an auxiliary tool in the analysis of the nested occupancy scheme in random environment generated by stick-breaking. Later on, it was realised that $\mathscr{T}$ was an interesting mathematical object on its own. Of particular interest is the question on how the properties of $T$ transform when passing to the early, the intermediate and then the late generations.

2020 *Mathematics Subject Classification.* Primary: 60F05, 60J80; Secondary: 60K05.
*Key words and phrases.* general branching process; perturbed random walk; stable Lévy process; weak convergence.

Answering this question leads to a new generalization of renewal theory for the perturbed random walks, see [5] and [15] for the first results in this direction. Although the iterated perturbed random walk on a general branching process tree is a particular instance of a branching random walk, we believe that understanding its properties provides some insight into the behavior of branching random walks and general branching processes with arbitrary (but admissible) inputs.

1.2. **Main result.** Throughout the paper we write $\Longrightarrow$, $\overset{\mathrm{d}}{\longrightarrow}$ and $\overset{\mathrm{f.d.d.}}{\longrightarrow}$ to denote weak convergence in a function space, weak convergence of one-dimensional and finite-dimensional distributions, respectively. The following result is a combination of Theorems 3.1 and 3.2 in [14], see also Section 3 in [6] for an earlier weaker version. Put

$$V_j(t) := \mathbb{E}N_j(t), \quad j \in \mathbb{N}, \quad t \geq 0.$$

**Proposition 1.1.** *Assume that* $\mathtt{s}^2 = \mathrm{Var}\,\xi \in (0,\infty)$ *and* $\mathbb{E}\eta < \infty$. *Let* $j = j(t)$ *be any positive integer-valued function satisfying* $j(t) \to \infty$ *and* $j(t) = o(t^{1/2})$ *as* $t \to \infty$. *Then, as* $t \to \infty$,

$$\left(\frac{\lfloor j(t)\rfloor^{1/2}(\lfloor j(t)u\rfloor - 1)!}{(\mathtt{s}^2\mathtt{m}^{-2\lfloor j(t)u\rfloor-1}t^{2\lfloor j(t)u\rfloor-1})^{1/2}}\left(N_{\lfloor j(t)u\rfloor}(t) - V_{\lfloor j(t)u\rfloor}(t)\right)\right)_{u>0} \overset{\mathrm{f.d.d.}}{\longrightarrow} \left(\int_{[0,\infty)} e^{-uy}\mathrm{d}\mathscr{S}_2(y)\right)_{u>0}, \qquad (1)$$

*where* $\mathtt{m} := \mathbb{E}\xi < \infty$ *and* $\mathscr{S}_2 := (\mathscr{S}_2(v))_{v\geq 0}$ *is a standard Brownian motion.*

In this article we intend to prove a counterpart of Proposition 1.1 under the assumptions that $\mathtt{m} < \infty$, $\mathtt{s}^2 = \infty$ and that the distribution of $\xi$ belongs to the domain of attraction of a stable distribution. More precisely, we assume that one of the following conditions holds:

CONDITION RW I: $\mathtt{s}^2 = \infty$ and, for some $\ell$ slowly varying at infinity,

$$\mathbb{E}(\xi^2\,\mathbb{1}_{\{\xi\leq t\}}) \;\sim\; \ell(t), \quad t \to \infty, \qquad (2)$$

in which case the distribution of $\xi$ belongs to the (non-normal) domain of attraction of a normal distribution, or

CONDITION RW II: for some $\alpha \in (1,2)$ and some $\ell$ slowly varying at infinity,

$$\mathbb{P}\{\xi > t\} \;\sim\; t^{-\alpha}\ell(t), \quad t \to \infty, \qquad (3)$$

in which case the distribution of $\xi$ belongs to the domain of attraction of an $\alpha$-stable distribution.

Assume that (3) holds with $\alpha = 1$. There exist slowly varying $\ell$ for which $\mathtt{m} < \infty$. Thus, in principle, this situation could have also been considered. However, we do not treat the case $\alpha = 1$, for it is technically more complicated than the others and does not shed any new light on weak convergence that we are interested in.

We shall write $N$ for $N_1$, that is, $N(t) := \sum_{i\geq 1}\mathbb{1}_{\{T_i\leq t\}}$ for $t \geq 0$. Denote by $D$ the Skorokhod space of right-continuous functions defined on $[0,\infty)$ with finite limits from the left at positive points. For later needs, we recall the following functional limit theorems, obtained in Theorem 3.2 of [1], for the process $(N(ty))_{y\geq 0}$ as $t \to \infty$: under the additional assumption $\mathbb{E}\eta^a < \infty$ for some $a > 0$,

$$\Big(\frac{N(ty) - \mathtt{m}^{-1}\int_0^{ty}\mathbb{P}\{\eta \leq x\}\mathrm{d}x}{\mathtt{m}^{-1-1/\alpha}c_\alpha(t)}\Big)_{y\geq 0} \;\Longrightarrow\; (\mathscr{S}_\alpha(y))_{y\geq 0}, \quad t \to \infty. \qquad (4)$$

Here,

- under Condition RW I $\alpha = 2$, $\mathscr{S}_2$ is a standard Brownian motion, $c_2(t)$ is a positive function satisfying

  $$\lim_{t\to\infty} t\ell(c_2(t))/c_2^2(t) = 1,$$

  and the convergence takes place in the $J_1$-topology on $D$;

- under Condition RW II $\mathscr{S}_\alpha := (\mathscr{S}_\alpha(u))_{u\geq 0}$ is a spectrally negative $\alpha$-stable Lévy process such that $\mathscr{S}_\alpha(1)$ has the characteristic function
$$\mathbb{E}\exp(\mathrm{i}z\mathscr{S}_\alpha(1)) = \exp\{-|z|^\alpha\Gamma(1-\alpha)(\cos(\pi\alpha/2)+\mathrm{i}\,\mathrm{sgn}(z)\sin(\pi\alpha/2))\},\quad z\in\mathbb{R}, \tag{5}$$
where $\Gamma(\cdot)$ denotes Euler's gamma function, $c_\alpha(t)$ is a positive function satisfying
$$\lim_{t\to\infty} t\ell(c_\alpha(t))/c_\alpha^\alpha(t) = 1,$$
the convergence takes place in the $M_1$-topology on $D$.

Comprehensive information concerning the $J_1$- and $M_1$-convergence on $D$ can be found in the monographs [3, 16] and [18], respectively.

We recall that $\mathbb{E}\xi^\gamma<\infty$ for all $\gamma\in(0,\alpha)$ whenever either Condition RW I or RW II holds. Further, $\mathbb{E}\xi^2=\infty$ under Condition RW I. In contrast, $\mathbb{E}\xi^\alpha$ may be finite or infinite under Condition RW II. After this discussion we are ready to state assumptions on the distribution of $\eta$.

CONDITION PERT($\gamma$). If $\alpha\in(1,2)$ and $\mathbb{E}\xi^\alpha<\infty$, we set $\gamma:=\alpha$ and assume that
$$\mathbb{E}(\eta\wedge t) = O(t^{2-\gamma}),\quad t\to\infty. \tag{6}$$
If $\alpha\in(1,2]$ and $\mathbb{E}\xi^\alpha=\infty$ we assume that (6) holds for some $\gamma\in(2-1/\alpha,\alpha)$.

Here is our main result.

**Theorem 1.2.** *Assume that $\mathbb{E}\xi^2=\infty$, that the distribution of $\xi$ belongs to the domain of attraction of an $\alpha$-stable distribution, $\alpha\in(1,2]$ and that Condition* PERT($\gamma$) *holds. Let $j=j(t)$ be any positive integer-valued function satisfying $j(t)\to\infty$ and $j(t)=o(t^{(\gamma-1)/2})$ as $t\to\infty$. Then, as $t\to\infty$,*
$$\left(\frac{(\lfloor j(t)u\rfloor-1)!\mathtt{m}^{\lfloor j(t)u\rfloor+1/\alpha}}{t^{\lfloor j(t)u\rfloor-1}c_\alpha(t/j(t))}\Big(N_{\lfloor j(t)u\rfloor}(t)-V_{\lfloor j(t)u\rfloor}(t)\Big)\right)_{u>0} \overset{\text{f.d.d.}}{\longrightarrow} \left(\int_{[0,\infty)} e^{-uy}\mathrm{d}\mathscr{S}_\alpha(y)\right)_{u>0}, \tag{7}$$
*where $\mathtt{m}=\mathbb{E}\xi<\infty$, $\mathscr{S}_2$ is a standard Brownian motion and $\mathscr{S}_\alpha$ is a spectrally negative $\alpha$-stable Lévy process with characteristic function* (5)*.*

*Remark* 1.3. One can check that the inequality $\mathbb{E}\eta^{\gamma-1}<\infty$ ensures Condition PERT($\gamma$), and that Condition PERT($\gamma$) guarantees that $\mathbb{E}\eta^{\gamma-1-\delta}<\infty$ for any $\delta\in(0,\gamma-1)$. The latter means that under the assumptions of Theorem 1.2 relation (4) holds.

*Remark* 1.4. The limit process in Theorem 1.2, that we denote by $L_\alpha$, is actually defined as the result of integration by parts:
$$L_\alpha(u) = u\int_0^\infty e^{-uy}\mathscr{S}_\alpha(y)\mathrm{d}y,\quad u>0.$$
One can check that this definition produces the same process as an alternative definition appearing in Theorem 1.2 in which $L_\alpha$ is understood as the stochastic integral with the integrator being a semimartingale. Note that the process $L_\alpha$ is a.s. continuous and self-similar with *negative* index $-1/\alpha$, that is, for any $a>0$, any $r\in\mathbb{N}$ and any $0<u_1<\ldots<u_r<\infty$, the vector $(L_\alpha(au_1),\ldots,L_\alpha(au_r))$ has the same distribution as $a^{-1/\alpha}(L_\alpha(u_1),\ldots,L_\alpha(u_r))$.

Assume that $\alpha\in(1,2)$. The process $\mathscr{S}_\alpha$ which describes the limit fluctuations of $N_1=N$ (in the first generation) is a.s. discontinuous. The structure of the process $L_\alpha$ indicates that the limit fluctuations of $N_j$ (in the intermediate generations $j$) are driven by two factors: (i) the fluctuations of the input process $N_1$ which are governed by $\mathscr{S}_\alpha$; (ii) the renewal structure of the tree which is reflected in the function $u\mapsto e^{-uy}$. Furthermore, we see that the renewal structure of the tree makes the limit $L_\alpha$ continuous, thereby smoothing out the fluctuations of the input process.

The remainder of the paper is structured as follows. Some auxiliary results are stated and proved in Section 2. The proof of Theorem 1.2 is given in Section 3.

## 2. AUXILIARY RESULTS

The Lebesgue–Stieltjes convolution of functions $r,s:[0,\infty)\to[0,\infty)$ of locally bounded variation is given by

$$(r*s)(t)=\int_{[0,t]}r(t-y)\mathrm{d}s(y)=\int_{[0,t]}s(t-y)\mathrm{d}r(y),\quad t\geq 0.$$

We write $r^{*(j)}$ for the $j$-fold Lebesgue–Stieltjes convolution of $r$ with itself.

We proceed by recalling an extended version of Proposition 3.1 in [5]. Inequality (9) is not a part of the cited result, it is contained in its proof.

**Lemma 2.1.** *Let $f:\mathbb{R}\to[0,\infty)$ be a nondecreasing right-continuous function vanishing on the negative half-line and satisfying*

$$f(t)=at+O(t^{\beta}),\quad t\to\infty \tag{8}$$

*for some $a>0$ and $\beta\in[0,1)$. Then, for some constant $C\geq 1$,*

$$\Big|f^{*(j)}(t)-\frac{a^jt^j}{j!}\Big|\leq\sum_{i=0}^{j-1}\binom{j}{i}\frac{a^iC^{j-i}(t+1)^{\beta(j-i)+i}}{i!},\quad j\in\mathbb{N},\ t\geq 0. \tag{9}$$

*In particular, for any integer-valued function $j=j(t)$ satisfying $j(t)=o(t^{(1-\beta)/2})$ as $t\to\infty$,*

$$f^{*(j)}(t)\ \sim\ \frac{a^jt^j}{j!},\quad t\to\infty.$$

Of principal importance for what follows is the decomposition:

$$N_j(t)=\sum_{k\geq 1}N_{j-1}^{(k)}(t-T_k)\,\mathbb{1}_{\{T_k\leq t\}},\quad j\geq 2,\quad t\geq 0, \tag{10}$$

where $N_{j-1}^{(r)}(t)$ is the number of successors in the $j$th generation with birth times within $[T_r,t+T_r]$ of the first generation individual with birth time $T_r$. In what follows, we write $V$ for $V_1$. Note that (10) entails $\mathbb{E}N_j(t)=V_j(t)=V^{*(j)}(t)$ for $j\in\mathbb{N}$ and $t\geq 0$.

Corollary 2.2 is our important technical tool to be used in all subsequent proofs.

**Corollary 2.2.** *Assume that the assumptions of Theorem 1.2 hold. Then, for some constant $C\geq 1$,*

$$\Big|V_j(t)-\frac{t^j}{j!\mathtt{m}^j}\Big|\leq\sum_{i=0}^{j-1}\binom{j}{i}\frac{C^{j-i}(t+1)^{(2-\gamma)(j-i)+i}}{i!\mathtt{m}^i},\quad j\in\mathbb{N},\quad t\geq 0 \tag{11}$$

*with the same $\gamma$ as in Condition* PERT($\gamma$). *In particular, for any integer-valued function $j=j(t)$ satisfying $j(t)=o(t^{(\gamma-1)/2})$ as $t\to\infty$,*

$$V_j(t)\ \sim\ \frac{t^j}{j!\mathtt{m}^j},\quad t\to\infty. \tag{12}$$

*Let $j\in\mathbb{N}$ and $s\geq 0$ satisfy $(s+1)^{\gamma-1}\geq 2C\mathtt{m}j^2$. Then, for $1\leq k\leq j$,*

$$V_k(s)\leq\frac{2(s+1)^k}{k!\mathtt{m}^k}, \tag{13}$$

$$\sum_{i=0}^{k-1}\binom{k}{i}\frac{C^{k-i}(s+1)^{(2-\gamma)(k-i)+i}}{i!\mathtt{m}^i}\leq\frac{2Ck(s+1)^{k+1-\gamma}}{(k-1)!\mathtt{m}^{k-1}}. \tag{14}$$

*and*

$$\sum_{i=0}^{k-1}\binom{k}{i}\frac{C^{k-i}(s+1)^{(2-\gamma)(k-i)+i+1}}{(i+1)!\mathtt{m}^{i+1}}\leq\frac{2C(s+1)^{k+2-\gamma}}{(k-1)!\mathtt{m}^{k}}. \tag{15}$$

*Proof.* We shall show that the function $V$ satisfies the assumptions of Lemma 2.1 with $a=\mathtt{m}^{-1}$ and $\beta=2-\gamma$. Then (11) and (12) are an immediate consequence of Lemma 2.1.

Let $S_0^*$ be a random variable with distribution

$$\mathbb{P}\{S_0^*\in\mathrm{d}x\}=\mathtt{m}^{-1}\mathbb{P}\{\xi>x\}\,\mathbb{1}_{(0,\infty)}(x)\mathrm{d}x.$$

Then, according to formula (2) in [7],

$$U(t)-\mathtt{m}^{-1}t=\int_{[0,t]}\mathbb{P}\{S_0^*>t-y\}\mathrm{d}U(y),\quad t\geq 0,$$

where $U(t):=\sum_{i\geq 0}\mathbb{P}\{S_i\leq t\}$ for $t\geq 0$, that is, $U$ is the renewal function of $(S_i)_{i\in\mathbb{N}_0}$. Since the assumption $\mathbb{E}\xi^2=\infty$ is equivalent to $\mathbb{E}S^*=\infty$, we conclude that

$$U(t)-\mathtt{m}^{-1}t\ \sim\ \mathtt{m}^{-1}\int_0^t\mathbb{P}\{S_0^*>y\}\mathrm{d}y,\quad t\to\infty$$

by Theorem 4 in [17].

Assume that $\mathbb{E}\xi^{\alpha}<\infty$. Then $\gamma=\alpha\in(1,2)$, $\mathbb{E}(S_0^*)^{\alpha-1}<\infty$ and, by Markov's inequality,

$$\int_0^t\mathbb{P}\{S_0^*>y\}\mathrm{d}y\leq(2-\alpha)^{-1}\mathbb{E}(S_0^*)^{\alpha-1}t^{2-\alpha}.$$

This together with Condition $\mathrm{P{\small ERT}}(\gamma)$ which reads $\mathbb{E}(\eta\wedge t)=O(t^{2-\alpha})$ entails

$$V(t)-\mathtt{m}^{-1}t=\int_{[0,t]}(U(t-y)-\mathtt{m}^{-1}(t-y))\mathrm{d}\mathbb{P}\{\eta\leq y\}-\mathtt{m}^{-1}\mathbb{E}(\eta\wedge t)=O(t^{2-\alpha}),\quad t\to\infty.$$

Assume that $\mathbb{E}\xi^{\alpha}=\infty$. Since $\mathbb{E}\xi_1^{\gamma}<\infty$, hence $\mathbb{E}(S_0^*)^{\gamma_1-1}<\infty$ for all $\gamma_1\in(0,\alpha)$, the same reasoning as above leads to the conclusion

$$\int_{[0,t]}(U(t-y)-\mathtt{m}^{-1}(t-y))\mathrm{d}\mathbb{P}\{\eta\leq y\}=O(t^{2-\gamma_1}),\quad t\to\infty.$$

In conjunction with (6) this yields $V(t)-\mathtt{m}^{-1}t=O(t^{2-\gamma})$. In particular, there exists a constant $c>0$ such that

$$|V(t)-\mathtt{m}^{-1}t|\leq c(t+1)^{2-\gamma},\quad t\geq 0. \tag{16}$$

Next, we prove (13). According to (11), it is enough to check that

$$\sum_{i=0}^{k-1}\binom{k}{i}\frac{C^{k-i}(s+1)^{(2-\gamma)(k-i)+i}}{i!\mathtt{m}^{i}}\leq\frac{(s+1)^k}{k!\mathtt{m}^k},\quad 1\leq k\leq j,\quad (s+1)^{\gamma-1}\geq 2C\mathtt{m}j^2.$$

Using

$$\binom{k}{i}\leq\frac{k!}{i!}\leq k^{k-i} \tag{17}$$

and

$$(s+1)^k=(s+1)^{(2-\gamma)k}(s+1)^{(\gamma-1)k}, \tag{18}$$

this follows from

$$\begin{aligned}\frac{k!\mathtt{m}^k}{(s+1)^k}\sum_{i=0}^{k-1}\binom{k}{i}\frac{C^{k-i}(s+1)^{(2-\gamma)(k-i)+i}}{i!\mathtt{m}^i} &= \sum_{i=0}^{k-1}\binom{k}{i}\frac{k!}{i!}\Big(\frac{C\mathtt{m}}{(s+1)^{\gamma-1}}\Big)^{k-i}\leq\sum_{i=0}^{k-1}\Big(\frac{C\mathtt{m}k^2}{(s+1)^{\gamma-1}}\Big)^{k-i}\\ &\leq \sum_{i=0}^{k-1}\Big(\frac{C\mathtt{m}k^2}{2C\mathtt{m}j^2}\Big)^{k-i}=\sum_{i=1}^{k}\Big(\frac{k^2}{2j^2}\Big)^{i}\leq\sum_{i=1}^{\infty}2^{-i}=1 \qquad (19)\end{aligned}$$

because $k\leq j$.

Now we are passing to the proof of (14). Invoking once again (17) and (18) we arrive at

$$\begin{aligned}\frac{(k-1)!\mathtt{m}^{k-1}}{k(s+1)^{k+1-\gamma}}\sum_{i=0}^{k-2}\binom{k}{i}\frac{C^{k-i}(s+1)^{(2-\gamma)(k-i)+i}}{i!\mathtt{m}^i} &= \frac{(s+1)^{\gamma-1}}{\mathtt{m}k^2}\sum_{i=0}^{k-2}\binom{k}{i}\frac{k!}{i!}\Big(\frac{C\mathtt{m}}{(s+1)^{\gamma-1}}\Big)^{k-i}\\ &\leq \frac{(s+1)^{\gamma-1}}{\mathtt{m}k^2}\sum_{i=0}^{k-2}\Big(\frac{C\mathtt{m}k^2}{(s+1)^{\gamma-1}}\Big)^{k-i}\\ &\leq \frac{(s+1)^{\gamma-1}}{\mathtt{m}k^2}\sum_{i\geq 2}\Big(\frac{C\mathtt{m}k^2}{(s+1)^{\gamma-1}}\Big)^{i}\\ &= \frac{\mathtt{m}(Ck)^2}{(s+1)^{\gamma-1}}\Big(1-\frac{C\mathtt{m}k^2}{(s+1)^{\gamma-1}}\Big)^{-1}\leq C,\end{aligned}$$

and (14) follows. The proof of (15) is analogous, hence omitted. The proof of the corollary is complete. □

Lemma 2.3 will be used in the proof of relation (25) below.

**Lemma 2.3.** *Let $u>0$ be fixed. Under the assumptions of Theorem 1.2,*

$$\lim_{t\to\infty}\frac{(\lfloor j(t)u\rfloor-1)!\mathtt{m}^{\lfloor j(t)u\rfloor-1}}{t^{\lfloor j(t)u\rfloor-1}}V_{\lfloor j(t)u\rfloor-1}(t(1-y/j))=\ e^{-uy} \qquad (20)$$

*for each fixed $y\geq 0$, and*

$$\lim_{T\to\infty}\limsup_{t\to\infty}\frac{(\lfloor j(t)u\rfloor-1)!\mathtt{m}^{\lfloor j(t)u\rfloor}}{t^{\lfloor j(t)u\rfloor-1}c_\alpha(t/j)}\int_{(Tt/j,t]}c_\alpha(y)\mathrm{d}_y(-V_{\lfloor j(t)u\rfloor-1}(t-y))=0. \qquad (21)$$

*Proof.* For notational simplicity, we only treat the case $u=1$. We first prove (20). According to (11),

$$\Big|V_j(t)-\frac{t^j}{\mathtt{m}^j j!}\Big|\leq g_j(t),\quad j\in\mathbb{N},\ t\geq 0,$$

where

$$g_j(t):=\sum_{i=0}^{j-1}\binom{j}{i}\frac{C^{j-i}(t+1)^{(2-\gamma)(j-i)+i}}{\mathtt{m}^i i!},\quad j\in\mathbb{N},\ t\geq 0.$$

It suffices to prove that, for each fixed $y>0$,

$$\lim_{t\to\infty}\frac{(j-1)!\mathtt{m}^{j-1}}{t^{j-1}}\frac{(t(1-y/j))^{j-1}}{\mathtt{m}^{j-1}(j-1)!}=e^{-y}$$

and

$$\lim_{t\to\infty}\frac{(j-1)!\mathtt{m}^{j-1}}{t^{j-1}}g_{j-1}(t(1-y/j))=0. \qquad (22)$$

The first of these is immediate. To prove the second, we first recall that $j(t)=o(t^{(\gamma-1)/2})$ as $t\to\infty$. Hence, for $t$ large enough, $\frac{C\mathtt{m}(j-1)^2}{(t+1)^{\gamma-1}}\leq 1/2$, say. Write, for such $t$, with the help of (19)

$$\frac{(j-1)!\mathtt{m}^{j-1}}{(t+1)^{j-1}}g_{j-1}(t)\leq\sum_{i=0}^{j-2}\Big(\frac{C\mathtt{m}(j-1)^2}{(t+1)^{\gamma-1}}\Big)^{j-1-i}\leq\frac{C\mathtt{m}(j-1)^2}{(t+1)^{\gamma-1}}\Big(1-\frac{C\mathtt{m}(j-1)^2}{(t+1)^{\gamma-1}}\Big)^{-1}.$$

Since

$$\frac{(j-1)^2}{(t(1-y/j)+1)^{\gamma-1}}\ \sim\ \frac{j^2}{t^{\gamma-1}}\ \to\ 0,\quad t\to\infty,$$

the last inequality entails (22).

Next, we intend to prove (21). The function $c_\alpha$ is regularly varying at infinity of index $1/\alpha$, see, for instance, Lemma 6.1.3 in [11]. By Theorem 1.8.3 in [4] and its proof, there exists an infinitely differentiable function $g_\alpha$ with nonincreasing derivative $g'_\alpha$ which varies regularly at infinity of index $1/\alpha-1$. Without loss of generality, we can and do assume that $c_\alpha$ itself enjoys all these properties. As a consequence,

$$\lim_{t\to\infty}\frac{tc'_\alpha(t)}{c_\alpha(t)}=\frac{1}{\alpha}. \tag{23}$$

Integrating by parts we infer

$$\int_{(Tt/j,t]}c_\alpha(y)\mathrm{d}_y(-V_{j-1}(t-y))=V_{j-1}(t(1-T/j))c_\alpha(Tt/j)+\int_{Tt/j}^t V_{j-1}(t-y)c'_\alpha(y)\mathrm{d}y.$$

In view of (20),

$$\lim_{t\to\infty}\frac{(j-1)!\mathtt{m}^j}{t^{j-1}c_\alpha(t/j)}V_{j-1}(t(1-T/j))c_\alpha(Tt/j)=\mathtt{m}T^{1/\alpha}e^{-T}.$$

The right-hand side converges to 0 as $T\to\infty$. Using (11) we obtain

$$\begin{aligned}\frac{(j-1)!\mathtt{m}^j}{t^{j-1}c_\alpha(t/j)}\int_{Tt/j}^t V_{j-1}(t-y)c'_\alpha(y)\mathrm{d}y&\leq\frac{\mathtt{m}}{t^{j-1}c_\alpha(t/j)}\int_{Tt/j}^t(t-y)^{j-1}\mathrm{d}c_\alpha(y)\\&+\frac{(j-1)!\mathtt{m}^j}{t^{j-1}c_\alpha(t/j)}\sum_{i=0}^{j-2}\binom{j-1}{i}\frac{C^{j-1-i}}{\mathtt{m}^i i!}\int_{Tt/j}^t(t+1-y)^{(2-\gamma)(j-1-i)+i}c'_\alpha(y)\mathrm{d}y=:a_j(t)+b_j(t).\end{aligned}$$

Since $\lim_{t\to\infty}(c_\alpha(ty/j)/c_\alpha(t/j))=y^{1/\alpha}$ for each $y>0$, we infer

$$\begin{aligned}a_j(t)=\frac{\mathtt{m}}{c_\alpha(t/j)}\int_T^j(1-y/j)^{j-1}\mathrm{d}_y c_\alpha(ty/j)&\leq\frac{\mathtt{m}}{c_\alpha(t/j)}\int_T^j e^{-(j-1)y/j}\mathrm{d}_y c_\alpha(ty/j)\\&\leq\frac{\mathtt{m}}{c_\alpha(t/j)}\int_T^j e^{-y/2}\mathrm{d}_y c_\alpha(ty/j)\to\frac{\mathtt{m}}{\alpha}\int_T^\infty e^{-y/2}y^{1/\alpha-1}\mathrm{d}y,\quad t\to\infty.\end{aligned}$$

Here, the limit relation is justified by the continuity theorem for Laplace-Stieltjes trasnforms. The the right-hand side of the last centered formula converges to 0 as $T\to\infty$. We claim that $\lim_{t\to\infty}b_j(t)=0$. To prove this, we first observe that

$$\begin{aligned}\frac{1}{c'_\alpha(Tt/j)}\int_{Tt/j}^t(t+1-y)^{(2-\gamma)(j-1-i)+i}c'_\alpha(y)\mathrm{d}y&\leq\int_{Tt/j}^t(t+1-y)^{(2-\gamma)(j-1-i)+i}\mathrm{d}y\\&=\frac{(t(1-T/j)+1)^{(2-\gamma)(j-1-i)+i+1}-1}{(2-\gamma)(j-1-i)+i+1}\leq\frac{t^{(2-\gamma)(j-1-i)+i+1}}{i+1},\end{aligned}$$

where the first inequality follows from the fact that $c'_\alpha$ is nonincreasing, and the last inequality holds for $t$ so large that $Tt/j \geq 1$ and, as a consequence, $t(1-T/j)+1 \leq t$. Further, in view of (23),

$$\lim_{t\to\infty} \frac{(t/j)c'_\alpha(Tt/j)}{c_\alpha(t/j)} = \alpha^{-1}T^{1/\alpha-1}.$$

Hence, for large $t$ and some constant $A(T)>0$,

$$\frac{(t/j)c'_\alpha(Tt/j)}{c_\alpha(t/j)} \leq A(T).$$

With these at hand, we infer, for large $t$,

$$\begin{aligned} b_j(t) &\leq \frac{(t/j)c'_\alpha(t/j)}{c_\alpha(t/j)} \frac{j!\mathtt{m}^j}{t^j} \sum_{i=0}^{j-2} \binom{j-1}{i} \frac{C^{j-1-i}t^{(2-\gamma)(j-1-i)+i+1}}{(i+1)!\mathtt{m}^i} \leq A(T)\mathtt{m} \sum_{i=0}^{j-2} \Big(\frac{C\mathtt{m}j^2}{t^{\gamma-1}}\Big)^{j-1-i} \\ &\leq A(T)\frac{C\mathtt{m}^2 j^2}{t^{\gamma-1}}\Big(1-\frac{C\mathtt{m}j^2}{t^{\gamma-1}}\Big)^{-1} \to 0, \quad t\to\infty. \end{aligned}$$

We have used (17) for the second inequality. □

## 3. Proof of Theorem 1.2

3.1. **Preparation.** We shall use a decomposition of $N_j - V_j$ into a 'martingale' part and a 'shot-noise' part obtained with the help of (10):

$$\begin{aligned} N_j(t)-V_j(t) &= \Big(\sum_{k\geq 1}(N_{j-1}^{(k)}(t-T_k)-V_{j-1}(t-T_k))\,\mathbb{1}_{\{T_k\leq t\}}\Big) \\ &\quad + \Big(\sum_{k\geq 1} V_{j-1}(t-T_k)\,\mathbb{1}_{\{T_k\leq t\}} - V_j(t)\Big), \quad j\geq 2, \quad t\geq 0. \end{aligned}$$

We shall prove that, as $t\to\infty$,

$$\frac{(\lfloor j(t)u\rfloor-1)!\mathtt{m}^{\lfloor j(t)u\rfloor}}{t^{\lfloor j(t)u\rfloor-1}c_\alpha(t/j(t))} \sum_{k\geq 1}\big(N_{\lfloor j(t)u\rfloor-1}^{(k)}(t)-V_{\lfloor j(t)u\rfloor-1}(t-T_k)\big)\,\mathbb{1}_{\{T_k\leq t\}} \overset{\text{f.d.d.}}{\longrightarrow} (\Theta(u))_{u>0}, \tag{24}$$

where $\Theta(u):=0$ for $u>0$, and

$$\begin{aligned} &\Bigg(\frac{(\lfloor j(t)u\rfloor-1)!\mathtt{m}^{\lfloor j(t)u\rfloor+1/\alpha}}{t^{\lfloor j(t)u\rfloor-1}c_\alpha(t/j(t))}\Big(\sum_{k\geq 1} V_{\lfloor j(t)u\rfloor-1}(t-T_k)\,\mathbb{1}_{\{T_k\leq t\}} - V_{\lfloor j(t)u\rfloor}(t)\Big)\Bigg)_{u>0} \\ &\qquad \overset{\text{f.d.d.}}{\longrightarrow} \Big(\int_{[0,\infty)} e^{-uy}\,\mathrm{d}\mathscr{S}_\alpha(y)\Big)_{u>0}, \end{aligned} \tag{25}$$

thereby showing that the asymptotics in focus is driven by the 'shot-noise' part, whereas the contribution of the 'martingale' part is negligible.

We start with several preparatory results which are needed for the proof of (25). Lemma 3.1 is a version of limit relation (4) with a different centering.

**Lemma 3.1.** *Under the assumptions and notation of Theorem 1.2, as $t\to\infty$,*

$$\Big(\frac{N(ut)-V(ut)}{\mathtt{m}^{-(\alpha+1)/\alpha}c_\alpha(t)}\Big)_{u\geq 0} \implies (\mathscr{S}_\alpha(u))_{u\geq 0} \tag{26}$$

*in the $J_1$-topology on $D$ if $\alpha=2$ and in the $M_1$-topology on $D$ if $\alpha\in(1,2)$.*

*Proof.* Put $\nu(t) := \#\{k \in \mathbb{N}_0 : S_k \le t\}$ for $t \ge 0$, so that $U(t) = \mathbb{E}\nu(t)$. According to Wald's identity, $U(t) = \mathtt{m}^{-1}\mathbb{E}S_{\nu(t)} \ge \mathtt{m}^{-1}t$ for $t \ge 0$. It is shown in the proof of Corollary 2.2 (see a few lines preceding (16)) that

$$U(t) - \mathtt{m}^{-1}t = O(t^{2-\gamma}), \quad t \to \infty. \tag{27}$$

As a consequence,

$$0 \le V(t) - \mathtt{m}^{-1}\int_0^t \mathbb{P}\{\eta \le y\}\mathrm{d}y = \int_{[0,t]}(U(t-y) - \mathtt{m}^{-1}(t-y))\mathrm{d}\mathbb{P}\{\eta \le y\} = O(t^{2-\gamma}), \quad t \to \infty.$$

Hence, relation (26) follows from (4) if we can show that

$$\lim_{t\to\infty} \frac{t^{2-\gamma}}{c_\alpha(t)} = 0. \tag{28}$$

To prove (28), recall that the function $c_\alpha$ is regularly varying at infinity of index $1/\alpha$ and that the $\gamma$ appearing in Condition $\textsc{Pert}(\gamma)$ satisfies $\gamma \in (2-1/\alpha, \alpha]$. Thus, $2-\gamma < 1/\alpha$. This justifies (28) and completes the proof of Lemma 3.1. □

**Lemma 3.2.** *Under the assumptions and notation of Theorem 1.2,*

$$\lim_{t\to\infty} \mathbb{E}|N(t) - V(t)|/c_\alpha(t) = \mathtt{m}^{-(\alpha+1)/\alpha}\mathbb{E}|\mathscr{S}_\alpha(1)|.$$

*Proof.* Putting $u = 1$ in (26) yields

$$\frac{N(t) - V(t)}{\mathtt{m}^{-(\alpha+1)/\alpha}c_\alpha(t)} \xrightarrow{\mathrm{d}} \mathscr{S}_\alpha(1), \quad t \to \infty. \tag{29}$$

Fix any $r \in (1, \alpha)$. Assume that we can show that

$$\mathbb{E}|N(t) - V(t)|^r = O((c_\alpha(t))^r), \quad t \to \infty. \tag{30}$$

Then the family $((N(t) - V(t))/c_\alpha(t))_{t\ge1}$ is uniformly integrable. This together with (29) is sufficient for completing the proof.

PROOF OF (30). We shall use a decomposition

$$N(t) - V(t) = \sum_{k\ge0}(\mathbb{1}_{\{S_k+\eta_{k+1}\le t\}} - G(t-S_k)) + \int_{[0,t]} G(t-x)\mathrm{d}(\nu(x) - U(x)),$$

where $G(x) := \mathbb{P}\{\eta \le x\}$ for $x \ge 0$. In view of

$$|x+y|^r \le 2^{r-1}(|x|^r + |y|^r), \quad x, y \in \mathbb{R},$$

it suffices to check that

$$\mathbb{E}\Big|\sum_{k\ge0}(\mathbb{1}_{\{S_k+\eta_{k+1}\le t\}} - G(t-S_k))\Big|^r = O((c_\alpha(t))^r), \quad t \to \infty \tag{31}$$

and

$$D(t) := \mathbb{E}\Big|\int_{[0,t]} G(t-x)\mathrm{d}(\nu(x) - U(x))\Big|^r = O((c_\alpha(t))^r), \quad t \to \infty. \tag{32}$$

We first prove (31). By Jensen's inequality, $(\mathbb{E}|X|^r)^{1/r} \le (\mathbb{E}X^2)^{1/2}$ for any real-valued random variable $X$. Thus, (31) follows if we can check that

$$\mathbb{E}\Big(\sum_{k\ge0}(\mathbb{1}_{\{S_k+\eta_{k+1}\le t\}} - G(t-S_k))\Big)^2 = O((c_\alpha(t))^2), \quad t \to \infty.$$

Actually, we shall prove even more, namely, that the right-hand side is $O(c_\alpha(t))$. The last expectation is equal to

$$\int_{[0,t]} G(t-x)(1-G(t-x))\mathrm{d}U(x) \le \int_{[0,t]} (1-G(t-x))\mathrm{d}U(x) \ \sim\ \mathtt{m}^{-1}\mathbb{E}(\eta \wedge t), \quad t \to \infty,$$

where the asymptotic relation is secured by Theorem 4 in [17]. Recall that the function $c_\alpha$ is regularly varying at infinity of index $1/\alpha$. According to Condition $\mathrm{P{\small ERT}}(\gamma)$ and (28),

$$\lim_{t\to\infty} \frac{\mathbb{E}(\eta\wedge t)}{c_\alpha(t)} = 0,$$

which proves (31).

Next, we intend to prove (32). As has already been mentioned in the proof of Lemma 2.3, we can assume that $c_\alpha$ is a nondecreasing function. Integration by parts in (32) followed by an application of Jensen's inequality yields

$$D(t) = \mathbb{E}\Big|\int_{[0,t]} (\nu(t-x) - U(t-x))\mathrm{d}G(x))\Big|^r \le \int_{[0,t]} \mathbb{E}|\nu(t-x) - U(t-x)|^r \mathrm{d}G(x).$$

By Theorems 1.1 and 1.4 in [13],

$$\lim_{t\to\infty} \frac{\mathbb{E}|\nu(t) - \mathtt{m}^{-1}t|^r}{(c_\alpha(t))^r} = \mathbb{E}|\mathscr{S}_\alpha(1)|^r < \infty. \tag{33}$$

Recalling (27) and (28), we conclude that

$$\lim_{t\to\infty} \frac{U(t) - \mathtt{m}^{-1}t}{c_\alpha(t)} = 0.$$

This together with (33) shows that

$$\lim_{t\to\infty} \frac{\mathbb{E}|\nu(t) - U(t)|^r}{(c_\alpha(t))^r} = \mathbb{E}|\mathscr{S}_\alpha(1)|^r < \infty.$$

Modifying $c_\alpha$ if needed in the right vicinity of 0 we infer $\mathbb{E}|\nu(t) - \mathtt{m}^{-1}t|^r \le A(c_\alpha(t))^r$ for some constant $A > 0$ and all $t \ge 0$. With this at hand,

$$D(t) \le \int_{[0,t]} \mathbb{E}|\nu(t-x) - U(t-x)|^r \mathrm{d}G(x) \le A\int_{[0,t]} (c_\alpha(t-x))^r \mathrm{d}G(x) = O((c_\alpha(t))^r), \quad t\to\infty.$$

We have used monotonicity of $c_\alpha$ for the last equality. □

Lemma 3.3 is a slight reformulation of Lemma A.5 in [10].

**Lemma 3.3.** *Let* $0 \le a < b < \infty$ *and, for each* $n \in \mathbb{N}$, $y_n : [0,\infty) \to [0,\infty)$ *be a right-continuous bounded and nondecreasing function. Assume that* $\lim_{n\to\infty} x_n = x$ *in the* $J_1$*- or* $M_1$*-topology on* $D$ *and that, for each* $t \ge 0$, $\lim_{n\to\infty} y_n(t) = y(t)$, *where* $y : [0,\infty) \to [0,\infty)$ *is a bounded continuous function. Then*

$$\lim_{n\to\infty} \int_{[a,b]} x_n(t)\mathrm{d}y_n(t) = \int_{[a,b]} x(t)\mathrm{d}y(t).$$

We are ready to prove (24) and (25).

3.2. **Proof of** (24)**.** This proof proceeds along the lines of the proof of Theorem 3.1 in [14].

For $j \in \mathbb{N}$ and $t \geq 0$, put $D_j(t) := \mathrm{Var} N_j(t)$ and

$$I_j(t) := \mathbb{E}\Big(\sum_{r\geq 1} V_{j-1}(t-T_r)\,\mathbb{1}_{\{T_r\leq t\}} - V_j(t)\Big)^2$$

with the convention that $V_0(t)=1$ for $t\geq 0$. Our starting point is the recursive formula which is a consequence of (10): for $j\geq 2$ and $t\geq 0$,

$$\begin{aligned} D_j(t) &= \mathbb{E}\Big(\sum_{r\geq 1}\big(N_{j-1}^{(r)}(t-T_r) - V_{j-1}(t-T_r)\big)\,\mathbb{1}_{\{T_r\leq t\}}\Big)^2 \\ &+ \mathbb{E}\Big(\sum_{r\geq 1} V_{j-1}(t-T_r)\,\mathbb{1}_{\{T_r\leq t\}} - V_j(t)\Big)^2 = \int_{[0,t]} D_{j-1}(t-y)\mathrm{d}V(y) + I_j(t). \end{aligned} \tag{34}$$

Starting with $D_1(t) = I_1(t)$ and iterating (34) we obtain

$$\int_{[0,t]} D_{j-1}(t-y)\mathrm{d}V(y) = \sum_{k=1}^{j-1}\int_{[0,t]} I_k(t-y)\mathrm{d}V_{j-k}(y),\quad j\geq 2,\quad t\geq 0. \tag{35}$$

Our purpose is to show that whenever $j=j(t)\to\infty$ and $j(t)=o(t^{(\gamma-1)/2})$ as $t\to\infty$,

$$\int_{[0,t]} D_{j-1}(t-y)\mathrm{d}V(y) = O\Big(\frac{t^{2j-\gamma}}{(j-2)!(j-1)!\mathtt{m}^{2j-2}}\Big),\quad t\to\infty. \tag{36}$$

We proceed via two steps. First, we show that $I_j$ is upper bounded by a nonnegative and nondecreasing function $h_j$, say, and that the corresponding inequality is valid for all nonnegative arguments. This leads by virtue of (35) to a useful inequality for $D_j$ which holds for all nonnegative arguments. Second, we derive an upper bound for both $h_j$ and $D_j$ which is valid for large arguments.

STEP 1. Throughout this step it is tacitly assumed that both $j\in\mathbb{N}$ and $t\geq 0$ are arbitrary.

We start with

$$\begin{aligned} &\mathbb{E}\sum_{r\geq 2}\sum_{1\leq i<r} V_{j-1}(t-T_i)\,\mathbb{1}_{\{T_i\leq t\}} V_{j-1}(t-T_r)\,\mathbb{1}_{\{T_r\leq t\}} \\ &\quad\leq \mathbb{E}\sum_{i\geq 1}\mathbb{E}\big(V_{j-1}(t-T_i)\,\mathbb{1}_{\{T_i\leq t\}}\big(V_{j-1}(t-\eta_{i+1}-S_i)\,\mathbb{1}_{\{\eta_{i+1}\leq t-S_i\}} \\ &\quad+ V_{j-1}(t-\eta_{i+2}-\xi_{i+1}-S_i)\,\mathbb{1}_{\{\eta_{i+2}+\xi_{i+1}\leq t-S_i\}}+\ldots\big)|(\xi_k,\eta_k)_{1\leq i\leq k}\big)\,\mathbb{1}_{\{S_i\leq t\}} \\ &\quad= \mathbb{E}\sum_{i\geq 1} V_{j-1}(t-T_i)\,\mathbb{1}_{\{T_i\leq t\}} V_j(t-S_i)\,\mathbb{1}_{\{S_i\leq t\}} \leq \mathbb{E}\sum_{i\geq 0} V_{j-1}(t-S_i)V_j(t-S_i)\,\mathbb{1}_{\{S_i\leq t\}}. \end{aligned}$$

Hence,

$$\begin{aligned} I_j(t) &= \mathbb{E}\sum_{r\geq 1} V_{j-1}^2(t-T_r)\,\mathbb{1}_{\{T_r\leq t\}} + 2\mathbb{E}\sum_{r\geq 2}\sum_{1\leq i<r} V_{j-1}(t-T_i)\,\mathbb{1}_{\{T_i\leq t\}} V_{j-1}(t-T_r)\,\mathbb{1}_{\{T_r\leq t\}} - V_j^2(t) \\ &\leq V_{j-1}(t)\mathbb{E}\sum_{r\geq 1} V_{j-1}(t-T_r)\,\mathbb{1}_{\{T_r\leq t\}} + 2\int_{[0,t]} V_{j-1}(t-y)V_j(t-y)\mathrm{d}U(y) - V_j^2(t) \\ &= V_{j-1}(t)V_j(t) + 2\int_{[0,t]} V_{j-1}(t-y)V_j(t-y)\mathrm{d}U(y) - V_j^2(t). \end{aligned} \tag{37}$$

Put $\tilde U(t) := \sum_{i\geq 1}\mathbb{P}\{S_i\leq t\}$ for $t\geq 0$. Using $\tilde U(t) = U(t)-1$ for $t\geq 0$ and (27) we conclude that there exists a constant $\tilde c>0$ such that, for all $t\geq 0$,

$$|\tilde U(t) - \mathtt{m}^{-1}t| \leq \tilde c(t+1)^{2-\gamma}.$$

With this at hand integration by parts yields

$$\int_{[0,t]} V_{j-1}(t-y)V_j(t-y)\mathrm{d}U(y) = V_{j-1}(t)V_j(t) + \int_{[0,t]} V_{j-1}(t-y)V_j(t-y)\mathrm{d}\tilde{U}(y)$$

$$= V_{j-1}(t)V_j(t) + \int_{[0,t]} \tilde{U}(t-y)\mathrm{d}(V_{j-1}(y)V_j(y)) \le (\tilde{c}(t+1)^{2-\gamma}+1)V_{j-1}(t)V_j(t) + \mathtt{m}^{-1}\int_0^t V_{j-1}(y)V_j(y)\mathrm{d}y,$$

whence, by (37),

$$I_j(t) \le (2\tilde{c}(t+1)^{2-\gamma}+3)V_{j-1}(t)V_j(t) + 2\mathtt{m}^{-1}\int_0^t V_{j-1}(y)V_j(y)\mathrm{d}y - V_j^2(t)$$

$$\le (2\tilde{c}+3)(t+1)^{2-\gamma}V_{j-1}(t)V_j(t) + 2\mathtt{m}^{-1}\int_0^t V_{j-1}(y)V_j(y)\mathrm{d}y - V_j^2(t).$$

Invoking (11) yields

$$2\mathtt{m}^{-1}\int_0^t V_{j-1}(y)V_j(y)\mathrm{d}y \le 2\mathtt{m}^{-1}\int_0^t \Big(\frac{y^{j-1}}{(j-1)!\mathtt{m}^{j-1}} + \sum_{i=0}^{j-2}\binom{j-1}{i}\frac{C^{j-1-i}(y+1)^{(2-\gamma)(j-1-i)+i}}{i!\mathtt{m}^i}\Big) \tag{38}$$

$$\times\Big(\frac{y^j}{j!\mathtt{m}^j} + \sum_{i=0}^{j-1}\binom{j}{i}\frac{C^{j-i}(y+1)^{(2-\gamma)(j-i)+i}}{i!\mathtt{m}^i}\Big)\mathrm{d}y$$

$$\le \frac{t^{2j}}{(j!)^2\mathtt{m}^{2j}} + 2\frac{(t+1)^{j+1}}{j!\mathtt{m}^{j+1}}\sum_{i=0}^{j-2}\binom{j-1}{i}\frac{C^{j-1-i}(t+1)^{(2-\gamma)(j-1-i)+i}}{((2-\gamma)(j-1-i)+j+1+i)i!\mathtt{m}^i}$$

$$+2\frac{(t+1)^j}{(j-1)!\mathtt{m}^j}\sum_{i=0}^{j-1}\binom{j}{i}\frac{C^{j-i}(t+1)^{(2-\gamma)(j-i)+i}}{((2-\gamma)(j-i)+j+i)i!\mathtt{m}^i}$$

$$+2\Big(\sum_{i=0}^{j-2}\binom{j-1}{i}\frac{C^{j-1-i}(t+1)^{(2-\gamma)(j-1-i)}+i}{i!\mathtt{m}^i}\Big)\int_0^t\sum_{i=0}^{j-1}\binom{j}{i}\frac{C^{j-i}(y+1)^{(2-\gamma)(j-i)+i}}{i!\mathtt{m}^{i+1}}\mathrm{d}y$$

$$\le \frac{t^{2j}}{(j!)^2\mathtt{m}^{2j}} + 2\frac{(t+1)^{j+1}}{(j+1)!\mathtt{m}^{j+1}}\sum_{i=0}^{j-2}\binom{j-1}{i}\frac{C^{j-1-i}(t+1)^{(2-\gamma)(j-1-i)+i}}{i!\mathtt{m}^i}$$

$$+2\frac{(t+1)^j}{j!\mathtt{m}^j}\sum_{i=0}^{j-1}\binom{j}{i}\frac{C^{j-i}(t+1)^{(2-\gamma)(j-i)+i}}{i!\mathtt{m}^i}$$

$$+2\Big(\sum_{i=0}^{j-2}\binom{j-1}{i}\frac{C^{j-1-i}(t+1)^{(2-\gamma)(j-1-i)}+i}{i!\mathtt{m}^i}\Big)\Big(\sum_{i=0}^{j-1}\binom{j}{i}\frac{C^{j-i}(t+1)^{(2-\gamma)(j-i)+i+1}}{(i+1)!\mathtt{m}^{i+1}}\Big)$$

$$=: \frac{t^{2j}}{(j!)^2\mathtt{m}^{2j}} + \tilde{f}_j(t). \tag{39}$$

Appealing to (11) once again we obtain

$$V_j^2(t) - \frac{t^{2j}}{(j!)^2\mathtt{m}^{2j}} = \Big(V_j(t)+\frac{t^j}{j!\mathtt{m}^j}\Big)\Big(V_j(t)-\frac{t^j}{j!\mathtt{m}^j}\Big) \ge -\Big(V_j(t)+\frac{t^j}{j!\mathtt{m}^j}\Big)\sum_{i=0}^{j-1}\binom{j}{i}\frac{C^{j-i}(t+1)^{(2-\gamma)(j-i)+i}}{i!\mathtt{m}^i}$$

$$= -\Big(V_j(t)+\frac{t^j}{j!\mathtt{m}^j}\Big)g_j(t) := -\tilde{g}_j(t).$$

Note that both $\tilde{f}_j$ and $\tilde{g}_j$ are nonnegative nondecreasing functions. Summarizing

$$I_j(t) \le (2\tilde{c}+3)(t+1)^{2-\gamma}V_{j-1}(t)V_j(t) + \tilde{f}_j(t) + \tilde{g}_j(t) =: \tilde{h}_j(t). \tag{40}$$

Since $\tilde{h}_j$ is a nondecreasing function, we further infer

$$D_{j-1}(t)=\sum_{k=1}^{j-1}\int_{[0,t]} I_k(t-y)\mathrm{d}V_{j-k-1}(y)\leq \tilde{h}_{j-1}(t)+\sum_{k=1}^{j-2}\tilde{h}_k(t)V_{j-k-1}(t),\quad j\geq 2,\quad t\geq 0.$$

STEP 2. Fix now $j\in\mathbb{N}$ and $s\geq 0$ satisfying $(s+1)^{\gamma-1}\geq 2C\mathtt{m}j^2$ and let $1\leq k\leq j$. Here, $C$ is the same as in (11). Throughout this step we tacitly assume that all formulae hold true for this range of parameters.

By (13),

$$V_{k-1}(s)V_k(s)\leq \frac{4(s+1)^{2k-1}}{(k-1)!k!\mathtt{m}^{2k-1}}\leq \frac{4(s+1)^{2k-1}}{((k-1)!)^2\mathtt{m}^{2k-1}}.$$

Next, we show that

$$\tilde{f}_k(s)\leq \frac{12C(s+1)^{2k+1-\gamma}}{((k-1)!)^2\mathtt{m}^{2k-1}}.$$

Indeed, according to (14),

$$2\frac{(s+1)^{k+1}}{(k+1)!\mathtt{m}^{k+1}}\sum_{i=0}^{k-2}\binom{k-1}{i}\frac{C^{k-1-i}(s+1)^{(2-\gamma)(k-1-i)+i}}{i!\mathtt{m}^i}\leq 2\frac{(s+1)^{k+1}}{(k+1)!\mathtt{m}^{k+1}}\frac{2C(k-1)(s+1)^{k-\gamma}}{(k-2)!\mathtt{m}^{k-2}}$$
$$\leq \frac{4C(s+1)^{2k+1-\gamma}}{((k-1)!)^2\mathtt{m}^{2k-1}}.$$

Analogously,

$$2\frac{(s+1)^k}{k!\mathtt{m}^k}\sum_{i=0}^{k-1}\binom{k}{i}\frac{C^{k-i}(s+1)^{(2-\gamma)(k-i)+i}}{i!\mathtt{m}^i}\leq \frac{4C(s+1)^{2k+1-\gamma}}{((k-1)!)^2\mathtt{m}^{2k-1}}.$$

Finally, the third summand in the definition of $\tilde{f}_k$ can be treated as follows:

$$2\Big(\sum_{i=0}^{k-2}\binom{k-1}{i}\frac{C^{k-1-i}(s+1)^{(2-\gamma)(k-1-i)+i}}{i!\mathtt{m}^i}\Big)\Big(\sum_{i=0}^{k-1}\binom{k}{i}\frac{C^{k-i}(s+1)^{(2-\gamma)(k-i)+i+1}}{(i+1)!\mathtt{m}^{i+1}}\Big)$$
$$\leq 2\frac{2C(k-1)(s+1)^{k-\gamma}}{(k-2)!\mathtt{m}^{k-2}}\frac{2C(s+1)^{k+2-\gamma}}{(k-1)!\mathtt{m}^k}=\frac{8C^2(s+1)^{2k+2-2\gamma}}{((k-2)!)^2\mathtt{m}^{2k-2}}\leq \frac{4C(s+1)^{2k+1-\gamma}}{((k-1)!)^2\mathtt{m}^{2k-1}}.$$

Here, we have used (14) and (15) to bound the first and second factor, respectively, and the inequality $(s+1)^{\gamma-1}\geq 2C\mathtt{m}(k-1)^2$ for the last passage. Finally,

$$\tilde{g}_k(s)\leq \frac{6C(s+1)^{2k+1-\gamma}}{((k-1)!)^2\mathtt{m}^{2k-1}}$$

by (13) and (14). Summarizing, we have shown that

$$\tilde{h}_k(s)\leq \frac{A(s+1)^{2k+1-\gamma}}{((k-1)!)^2\mathtt{m}^{2k-1}}, \tag{41}$$

where $A:=12+8\tilde{c}+18C$.

Further, we obtain, for $s$ satisfying $(s+1)^{\gamma-1}\geq 2\max(c,1)\mathtt{m}j^2=:a_j$, where $c$ is as given in (16),

$$\begin{aligned}D_{j-1}(s)&\leq \tilde{h}_{j-1}(s)+\sum_{k=1}^{j-2}\tilde{h}_k(s)V_{j-k-1}(s)\\&\leq \frac{A(s+1)^{2j-1-\gamma}}{((j-2)!)^2\mathtt{m}^{2j-3}}+2A\sum_{k=1}^{j-2}\frac{(s+1)^{j+k-\gamma}}{(j-k-1)!((k-1)!)^2\mathtt{m}^{j+k-2}}\\&=\frac{A(s+1)^{2j-1-\gamma}}{((j-2)!)^2\mathtt{m}^{2j-3}}\Big(1+2\sum_{k=1}^{j-2}\binom{j-2}{k-1}\frac{(j-2)!}{(k-1)!}\Big(\frac{\mathtt{m}}{s+1}\Big)^{j-k-1}\Big)\\&\leq \frac{A(s+1)^{2j-1-\gamma}}{((j-2)!)^2\mathtt{m}^{2j-3}}\Big(1+2\frac{\mathtt{m}j^2}{s+1}\Big(1-\frac{\mathtt{m}j^2}{s+1}\Big)^{-1}\Big)\leq\frac{3A(s+1)^{2j-1-\gamma}}{((j-2)!)^2\mathtt{m}^{2j-3}}.\end{aligned}\tag{42}$$

Here, the first inequality is just formula (40), the second inequality is implied by (13) and (41), and the third inequality is justified by (17).

Assume now that $j=j(t)\to\infty$ and $j(t)=o(t^{(\gamma-1)/2})$ as $t\to\infty$, so that the inequality $t\geq a_j$ holds true for large enough $t$. We intend to prove (36). To this end, we write

$$\begin{aligned}\int_{[0,t]}D_{j-1}(t-y)\mathrm{d}V(y)&=\int_{[0,t-a_j]}D_{j-1}(t-y)\mathrm{d}V(y)+\int_{(t-a_j,t]}D_{j-1}(t-y)\mathrm{d}V(y)\\&\leq\frac{3A}{((j-2)!)^2\mathtt{m}^{2j-3}}\int_{[0,t+1]}(t+1-y)^{2j-1-\gamma}\mathrm{d}V(y)+\Big(\max_{s\in[0,a_j]}D_{j-1}(s)\Big)U(a_j)\\&\leq\frac{3A(t+1)^{2j-\gamma}}{((j-2)!)^2(2j-\gamma)\mathtt{m}^{2j-2}}+\frac{3Ac(t+1)^{2j+1-2\gamma}}{((j-2)!)^2\mathtt{m}^{2j-3}}+\Big(\max_{s\in[0,a_j]}D_{j-1}(s)\Big)U(a_j)\end{aligned}$$

having utilized (42) and $V(x+y)-V(x)\leq U(y)$ for $x,y\in\mathbb{R}$ (for the proof, see formula (40) in [5]) for the first inequality and integration by parts together with (16) for the second. The asymptotic relation

$$\frac{3Ac(t+1)^{2j+1-2\gamma}}{((j-2)!)^2\mathtt{m}^{2j-3}}=o\Big(\frac{t^{2j-\gamma}}{(j-2)!(j-1)!\mathtt{m}^{2j-2}}\Big),\quad t\to\infty$$

is a consequence of $j(t)=o(t^{\gamma-1})$ as $t\to\infty$. Using (13) and (42) for the second inequality below we further obtain

$$\Big(\max_{s\in[0,a_j]}D_{j-1}(s)\Big)U(a_j)\leq(D_{j-1}(a_j)+V_{j-1}^2(a_j))U(a_j)\leq\Big(\frac{3A(a_j+1)^{2j-1-\gamma}}{((j-2)!)^2\mathtt{m}^{2j-3}}+\frac{4(a_j+1)^{2j-2}}{((j-1)!)^2\mathtt{m}^{2j-2}}\Big)U(a_j).$$

By the elementary renewal theorem, with $a=2\max(c,1)\mathtt{m}$,

$$\frac{(j-2)!(j-1)!\mathtt{m}^{2j-2}}{t^{2j-\gamma}}\frac{(a_j+1)^{2j-2}}{((j-1)!)^2\mathtt{m}^{2j-2}}U(a_j)\ \sim\ \Big(\frac{aj^2}{t}\Big)^{2j-\gamma}\frac{aj}{\mathtt{m}}\frac{1}{(aj^2)^{2-\gamma}}\ \to\ 0,\quad t\to\infty$$

because $\lim_{t\to\infty}j^b(aj^2/t)^{2j-2}=0$ for any $b>0$. The last two limit relation hold true whenever $j(t)=o(t^{1/2})$ and particularly under the assumption $j(t)=o(t^{(\gamma-1)/2})$. Analogously,

$$\frac{(j-2)!(j-1)!\mathtt{m}^{2j-2}}{t^{2j-\gamma}}\frac{(a_j+1)^{2j-1-\gamma}}{((j-2)!)^2\mathtt{m}^{2j-3}}U(a_j)\ \sim\ \Big(\frac{aj^2}{t}\Big)^{2j-\gamma}j\ \to\ 0,\quad t\to\infty.$$

Thus,

$$\Big(\max_{s\in[0,a_j]}D_{j-1}(s)\Big)U(a_j)=o\Big(\frac{t^{2j-\gamma}}{(j-2)!(j-1)!\mathtt{m}^{2j-2}}\Big),$$

and (36) follows.

According to the Cramér-Wold device and Markov's inequality, relation (24) follows if we can show that, for any fixed $u>0$,

$$\lim_{t\to\infty}\frac{((\lfloor j(t)u\rfloor-1)!)^2\mathtt{m}^{2\lfloor j(t)u\rfloor}}{t^{2\lfloor j(t)u\rfloor-2}c_\alpha^2(t/j(t))}\mathbb{E}\Big(N_{\lfloor j(t)u\rfloor}(t)-\sum_{r\geq 1}V_{\lfloor j(t)u\rfloor-1}(t-T_r)\,\mathbb{1}_{\{T_r\leq t\}}\Big)^2=0. \tag{43}$$

In view of (34) and (36), the left-hand side is the big $O$ of

$$\frac{jt^{2-\gamma}}{c_\alpha^2(t/j)}=\frac{j^2}{t^{\gamma-1}}\frac{t/j}{c_\alpha^2(t/j)}.$$

The first factor on the right-hand side is $o(1)$ by assumption. We claim that

$$\lim_{x\to\infty}x^{-1}c_\alpha^2(x)=\infty, \tag{44}$$

so that the second factor on the right-hand side is $o(1)$, too, which proves (43).

To check (44), recall that the function $c_\alpha$ is regularly varying at infinity of index $1/\alpha$ which particularly entails

$$\lim_{x\to\infty}c_\alpha(x)=\infty. \tag{45}$$

In the case $\alpha\in(1,2)$, the regular variation implies (44). Assume now that $\alpha=1/2$. Then $c_2$ satisfies $\ell(c_2(x))\sim x^{-1}c_2^2(x)$ as $x\to\infty$, where $\ell$ is a slowly varying diverging to infinity function, see (2). Recalling (45) we infer (44) with $\alpha=2$.

3.3. **Proof of** (25)**.** In what follows we write $j$ for $j(t)$. According to the Cramér-Wold device, it is enough to show that for any $r\in\mathbb{N}$, any real $\alpha_1,\dots,\alpha_r$ and any $0<u_1<\ldots<u_r<\infty$, as $t\to\infty$,

$$\sum_{i=1}^r\alpha_i\frac{(\lfloor ju_i\rfloor-1)!\mathtt{m}^{\lfloor ju_i\rfloor+1/\alpha}Z(ju_i,t)}{t^{\lfloor ju_i\rfloor-1}c_\alpha(t/j)}\overset{\mathrm{d}}{\longrightarrow}\sum_{i=1}^r\alpha_iu_i\int_0^\infty\mathscr{S}_\alpha(y)e^{-u_iy}\mathrm{d}y, \tag{46}$$

where

$$Z(ju,t):=\sum_{k\geq 1}V_{\lfloor ju\rfloor-1}(t-T_k)\,\mathbb{1}_{\{T_k\leq t\}}-V_{\lfloor ju\rfloor}(t),\quad u>0.$$

For any $u,T>0$ and sufficiently large $t$,

$$\begin{aligned}\frac{(\lfloor ju\rfloor-1)!\mathtt{m}^{\lfloor ju\rfloor+1/\alpha}Z(ju,t)}{t^{\lfloor ju\rfloor-1}c_\alpha(t/j)}&=\frac{(\lfloor ju\rfloor-1)!\mathtt{m}^{\lfloor ju\rfloor+1/\alpha}}{t^{\lfloor ju\rfloor-1}c_\alpha(t/j)}\int_{[0,t]}V_{\lfloor ju\rfloor-1}(t-y)\mathrm{d}(N(y)-V(y))\\&=\frac{(\lfloor ju\rfloor-1)!\mathtt{m}^{\lfloor ju\rfloor-1}}{t^{\lfloor ju\rfloor-1}}\int_{[0,T]}\frac{N(yt/j)-V(yt/j)}{\mathtt{m}^{-(\alpha+1)/\alpha}c_\alpha(t/j)}\mathrm{d}_y(-V_{\lfloor ju\rfloor-1}(t(1-y/j)))\\&+\frac{(\lfloor ju\rfloor-1)!\mathtt{m}^{\lfloor j(t)u\rfloor+1/\alpha}}{t^{\lfloor ju\rfloor-1}c_\alpha(t/j)}\int_{(Tt/j,t]}(N(y)-V(y))\mathrm{d}_y(-V_{\lfloor ju\rfloor-1}(t-y)).\end{aligned}$$

By Lemma 3.1,

$$\Big(\frac{N(ut/j)-V(ut/j)}{\mathtt{m}^{-(\alpha+1)/\alpha}c_\alpha(t/j)}\Big)_{u\geq 0}\implies(\mathscr{S}_\alpha(u))_{u\geq 0}$$

in the $J_1$-topology on $D$ if $\alpha=2$ and in the $M_1$-topology on $D$ if $\alpha\in(1,2)$. Here, we have used the assumption $t/j(t)\to\infty$. By Skorokhod's representation theorem there exist versions $\widehat{N}_t$ and $\widehat{\mathscr{S}_\alpha}$ of $((N(ut/j)-V(ut/j))/(\mathtt{m}^{-(\alpha+1)/\alpha}c_\alpha(t/j))_{u\geq 0}$ and $\mathscr{S}_\alpha$, respectively such that

$$\lim_{t\to\infty}\widehat{N}_t(y)=\widehat{\mathscr{S}_\alpha}(y)\quad\text{a.s.} \tag{47}$$

in the $J_1$-topology on $D$ if $\alpha=2$ and in the $M_1$-topology on $D$ if $\alpha\in(1,2)$. In view of (20),

$$\lim_{t\to\infty}\frac{(\lfloor ju\rfloor-1)!\mathtt{m}^{\lfloor ju\rfloor-1}}{t^{\lfloor ju\rfloor-1}}V_{\lfloor ju\rfloor-1}(t(1-y/j))=e^{-uy},\quad t\to\infty$$

for each fixed $y\geq 0$. By Lemma 3.3, this in combination with (47) yields

$$\lim_{t\to\infty}\sum_{i=1}^r\alpha_i\frac{(\lfloor ju\rfloor-1)!\mathtt{m}^{\lfloor ju\rfloor-1}}{t^{\lfloor ju\rfloor-1}}\int_0^T\widehat{N}_t(y)\mathrm{d}_y(-V_{\lfloor ju\rfloor-1}(t(1-y/j)))=\sum_{i=1}^r\alpha_iu_i\int_0^T\widehat{\mathscr{S}_\alpha}(y)e^{-u_iy}\mathrm{d}y\quad\text{a.s.}$$

and thereupon

$$\sum_{i=1}^r\alpha_i\frac{(\lfloor ju\rfloor-1)!\mathtt{m}^{\lfloor ju\rfloor-1}}{t^{\lfloor ju\rfloor-1}}\int_{[0,T]}\frac{N(yt/j)-V(yt/j)}{\mathtt{m}^{-(\alpha+1)/\alpha}c_\alpha(t/j)}\mathrm{d}_y(-V_{\lfloor ju\rfloor-1}(t(1-y/j)))\overset{\mathrm{d}}{\longrightarrow}\sum_{i=1}^r\alpha_iu_i\int_0^T\mathscr{S}_\alpha(y)e^{-u_iy}\mathrm{d}y,$$

as $t\to\infty$. Since $\lim_{T\to\infty}\sum_{i=1}^r\alpha_iu_i\int_0^T\mathscr{S}_\alpha(y)e^{-u_iy}\mathrm{d}y=\sum_{i=1}^r\alpha_iu_i\int_0^\infty\mathscr{S}_\alpha(y)e^{-u_iy}\mathrm{d}y$ a.s. we are left with proving that

$$\lim_{T\to\infty}\limsup\nolimits_{t\to\infty}\mathbb{P}\Big\{\Big|\sum_{i=1}^r\alpha_i\frac{(\lfloor ju_i\rfloor-1)!\mathtt{m}^{\lfloor ju\rfloor+1/\alpha}}{t^{\lfloor ju_i\rfloor-1}c_\alpha(t/j)}\int_{(Tt/j,t]}(N(y)-V(y))\mathrm{d}(-V_{\lfloor ju\rfloor-1}(t-y))\Big|>\varepsilon\Big\}=0$$

for all $\varepsilon>0$. By Lemma 3.2, $\mathbb{E}|N(y)-V(y)|\sim\mathtt{m}^{-(\alpha+1)/\alpha}\mathbb{E}|\mathscr{S}_\alpha(1)|c_\alpha(y)$ as $y\to\infty$. With this at hand, the last limit relation follows from Markov's inequality and (21) The proof of (25) is complete.

**Acknowledgement**. The present work was supported by the National Research Foundation of Ukraine (project 2020.02/0014 'Asymptotic regimes of perturbed random walks: on the edge of modern and classical probability').

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