# STABLE GROTHENDIECK POLYNOMIALS AND K-THEORETIC FACTOR SEQUENCES 

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#### Abstract

We give a nonrecursive combinatorial formula for the expansion of a stable Grothendieck polynomial in the basis of stable Grothendieck polynomials for partitions. The proof is based on a generalization of the EdelmanGreene insertion algorithm. This result is applied to prove a number of formulas and properties for K-theoretic quiver polynomials and Grothendieck polynomials. In particular we formulate and prove a K-theoretic analogue of Buch and Fulton's factor sequence formula for the cohomological quiver polynomials.


## 1. Introduction

1.1. Stable Grothendieck polynomials. For each permutation $w$ there is a symmetric power series $G_{w}=G_{w}\left(x_{1}, x_{2}, \ldots\right)$ called the stable Grothendieck polynomial for $w$. These power series were defined by Fomin and Kirillov [13, 12] as a limit of the ordinary Grothendieck polynomials of Lascoux and Schützenberger [18]. We recall this definition in Section 2. The term of lowest degree in $G_{w}$ is the Stanley function (or stable Schubert polynomial) $F_{w}$. The Stanley coefficients in the Schur expansion of a Stanley function are interesting combinatorial invariants which generalize the Littlewood-Richardson coefficients. It was proved by Edelman and Greene [10] and Lascoux and Schützenberger [19] that Stanley coefficients are nonnegative. Various combinatorial rules have been given for these coefficients [11, 15, 23].

Given a partition $\lambda=\left(\lambda_{1} \geq \cdots \geq \lambda_{k} \geq 0\right)$, the Grassmannian permutation $w_{\lambda}$ for $\lambda$ is uniquely defined by the requirement that $w_{\lambda}(i)=i+\lambda_{k+1-i}$ for $1 \leq i \leq k$ and $w_{\lambda}(i)<w_{\lambda}(i+1)$ for $i \neq k$. The power series $G_{\lambda}:=G_{w_{\lambda}}$ play a role in combinatorial $K$-theory similar to the role of Schur functions in cohomology. Buch has shown [3] that any stable Grothendieck polynomial $G_{w}$ can be written as a finite linear combination

$$
\begin{equation*}
G_{w}=\sum_{\lambda} c_{w, \lambda} G_{\lambda} \tag{1}
\end{equation*}
$$

of stable Grothendieck polynomials indexed by partitions, using integer coefficients $c_{w, \lambda}$ that generalize the Stanley coefficients [2]. Lascoux gave a recursive formula for stable Grothendieck polynomials which confirms a conjecture that these coefficients have signs that alternate with degree, i.e. $(-1)^{|\lambda|-\ell(w)} c_{w, \lambda} \geq 0$ [17]. The central result of this paper is a new formula for the coefficients $c_{w, \lambda}$ which generalizes Fomin and Greene's rule [11] for Stanley coefficients.

[^0]To state our formula, we need the 0-Hecke monoid, which is the quotient of the free monoid of all finite words in the alphabet $\{1,2, \ldots\}$ by the relations

$$
\begin{align*}
p p & \equiv p & & \text { for all } p  \tag{2}\\
p q p & \equiv q p q & & \text { for all } p, q  \tag{3}\\
p q & \equiv q p & & \text { for }|p-q| \geq 2 . \tag{4}
\end{align*}
$$

There is a bijection between the 0 -Hecke monoid and the infinite symmetric group $S_{\infty}=\bigcup_{n \geq 1} S_{n}$. Given any word $a$ there is a unique permutation $w \in S_{\infty}$ such that $a \equiv b$ for some (or equivalently every) reduced word $b$ of $w$. In this case we write $w(a)=w$ and say that $a$ is a Hecke word for $w$. Notice that the reduced words for $w$ are precisely the Hecke words for $w$ that are of minimum length. The Hecke product $u \cdot v$ of two permutations $u, v \in S_{\infty}$ is the element $w(a b) \in S_{\infty}$ where $a$ and $b$ are words such that $w(a)=u$ and $w(b)=v$.

We use the English notation for partitions and tableaux. A decreasing tableau ${ }^{1}$ is a Young tableau whose rows decrease strictly from left to right, and whose columns decrease strictly from top to bottom. The (row reading) word of a tableau is obtained by reading the rows of the tableau from left to right, starting with the bottom row, followed by the next-to-bottom row, etc. We shall identify a tableau with its word.

Theorem 1. For any permutation $w$ we have

$$
G_{w}=\sum_{\lambda} c_{w, \lambda} G_{\lambda}
$$

where $c_{w, \lambda}$ is equal to $(-1)^{|\lambda|-\ell(w)}$ times the number of decreasing tableaux $T$ of shape $\lambda$ such that $w(T)=w$.

Example 2. Let $w=31524$. The decreasing tableaux that are Hecke words for $w$ are:


So $G_{w}=G_{22}+G_{31}-G_{32}$.
When the permutation $w$ is 321-avoiding, Theorem 1 also generalizes Buch's rule for the coefficients $c_{w, \lambda}$ in terms of set-valued tableaux [3], in the sense that there is an explicit bijection between the relevant decreasing and set-valued tableaux. As a consequence, we obtain a new proof of the set-valued Littlewood-Richardson rule for $K$-theoretic Schubert structure constants on Grassmannians, as well as an alternative rule based on decreasing tableaux.
1.2. Hecke insertion. Fomin and Kirillov proved that the monomial coefficients of (stable) Grothendieck polynomials are counted by combinatorial objects called resolved wiring diagrams (also known as FK-graphs, pipe dreams, or nonreduced RC-graphs) [13, 12]. This formula was used in [3] to express the monomial coefficients of stable Grothendieck polynomials for partitions in terms of set-valued tableaux. We prove Theorem 1 by exhibiting an explicit bijection between the set of FK-graphs for a permutation $w$ and the set of pairs $(T, U)$ where $T$ is a decreasing tableau with $w(T)=w$ and $U$ is a set-valued tableau of the same shape as $T$. This

[^1]bijection, called Hecke insertion, is the technical core of our paper. It is a subtle extension of the Edelman-Greene insertion algorithm from the set of reduced words to the set of all (Hecke) words.

Hecke insertion allows us to define a product of decreasing tableaux $\left(T_{1}, T_{2}\right) \mapsto$ $T_{1} \cdot T_{2}$ (see section 3.5). This product is used in the definition of $K$-theoretic factor sequences in the next section.
1.3. Quiver coefficients. Our main application of Theorem 1 concerns quiver coefficients. A sequence of vector bundle morphisms $E_{0} \rightarrow E_{1} \rightarrow \cdots \rightarrow E_{n}$ over a variety $X$ together with a set of rank conditions $r=\left\{r_{i j}\right\}$ for $0 \leq i \leq j \leq n$ define a quiver variety $\Omega_{r} \subset X$ of points where each composition of bundle maps $E_{i} \rightarrow E_{j}$ has rank at most $r_{i j}$. We demand that the rank conditions can occur, which is equivalent to the requirement that $r_{i i}=e_{i}:=\operatorname{rank}\left(E_{i}\right)$ for all $i, 0 \leq$ $r_{i j} \leq \min \left(r_{i, j-1}, r_{i+1, j}\right)$ for $0 \leq i<j \leq n$, and $r_{i j}+r_{i-1, j+1} \geq r_{i-1, j}+r_{i, j+1}$ for $0<i \leq j<n$. We also demand that the bundle maps are generic, so that the quiver variety $\Omega_{r}$ obtains its expected codimension $d(r)=\sum_{i<j}\left(r_{i, j-1}-r_{i j}\right)\left(r_{i+1, j}-r_{i j}\right)$. Buch and Fulton proved a formula for the cohomology class of $\Omega_{r}$ [5] which was later generalized to $K$-theory by Buch [2]. The $K$-theory version states that the Grothendieck class of $\Omega_{r}$ is given by

$$
\begin{equation*}
\left[\mathcal{O}_{\Omega_{r}}\right]=\sum_{\mu} c_{\mu}(r) G_{\mu_{1}}\left(E_{1}-E_{0}\right) G_{\mu_{2}}\left(E_{2}-E_{1}\right) \cdots G_{\mu_{n}}\left(E_{n}-E_{n-1}\right) \tag{5}
\end{equation*}
$$

where the sum is over sequences $\mu=\left(\mu_{1}, \ldots, \mu_{n}\right)$ of partitions $\mu_{i}$ such that $\sum\left|\mu_{i}\right| \geq$ $d(r)$ and each partition $\mu_{i}$ can be contained in the rectangle $e_{i} \times e_{i-1}$ with $e_{i}$ rows and $e_{i-1}$ columns. The coefficients $c_{\mu}(r)$ in this formula are integers called quiver coefficients. When $\sum\left|\mu_{i}\right|=d(r)$, the coefficient $c_{\mu}(r)$ also appears in the cohomology formula from [5] and is called a cohomological quiver coefficient. It was conjectured that cohomological quiver coefficients are nonnegative, while $K$-theoretic quiver coefficients have signs that alternate with degree, i.e. $(-1)^{\sum\left|\mu_{i}\right|-d(r)} c_{\mu}(r) \geq 0$. The conjecture for cohomological quiver coefficients was proved by Knutson, Miller, and Shimozono [16], after which the general case was proved by Buch [4] and Miller [21]. Both of the latter proofs are based on the ratio formula from [16], which expresses the Grothendieck class of a quiver variety as a ratio of two double Grothendieck polynomials.

A more precise conjecture for cohomological quiver coefficients was posed in [5], which asserts that any such coefficient $c_{\mu}(r)$ counts the number of factor sequences of tableaux with shapes given by the sequence of partitions $\mu$. A factor sequence is a sequence of semistandard Young tableaux that can be obtained by performing a series of plactic factorizations and multiplications of chosen tableaux arranged in a tableau diagram. For a specific choice of tableau diagram, this more precise conjecture was also proved by Knutson, Miller and Shimozono [16]. It appears, however, that the original definition of factor sequences from [5] has no natural generalization to $K$-theory.

In this paper, we prove that $K$-theoretic quiver coefficients are counted by a new type of factor sequence. These sequences are constructed from a tableau diagram of decreasing tableaux using the same algorithm that defines the original factor sequences, except that the plactic product is replaced with a product $(U, T) \mapsto U \cdot T$ of decreasing tableaux which respects Hecke words (see section 3.5).

For each $0 \leq i<j \leq n$ let $R_{i j}$ be a rectangle with $r_{i+1, j}-r_{i j}$ rows and $r_{i, j-1}-r_{i j}$ columns. Let $U_{i j}$ be the unique decreasing tableau of shape $R_{i j}$ such that the lower left box contains the number $r_{i, j-1}$, and each box is one larger than the box below it and one smaller than the box to the left of it. For example, if $r_{i, j-1}=6, r_{i+1, j}=5$, and $r_{i j}=2$ then

$$
U_{i j}=\begin{array}{|l|l|l|l|}
\hline 8 & 7 & 6 & 5 \\
\hline 7 & 6 & 5 & 4 \\
\hline 6 & 5 & 4 & 3 \\
\hline
\end{array} .
$$

These tableaux $U_{i j}$ can be arranged in a triangular tableau diagram as in [5, §4]. We define a $K$-theoretic factor sequence for the rank conditions $r$ by induction on $n$. If $n=1$ then the only factor sequence is the sequence ( $U_{01}$ ) consisting of the only tableau in the tableau diagram. If $n \geq 2$ then the numbers $\bar{r}=\left\{\bar{r}_{i j}: 0 \leq i \leq\right.$ $j \leq n-1\}$ defined by $\bar{r}_{i j}=r_{i, j+1}$ form a valid set of rank conditions corresponding to a sequence of $n-1$ bundle maps. In this case, a factor sequence for $r$ is any sequence of the form $\left(U_{01} \cdot A_{1}, \ldots, B_{i-1} \cdot U_{i-1, i} \cdot A_{i}, \cdots, B_{n-1} \cdot U_{n-1, n}\right)$, for a choice of decreasing tableaux $A_{i}$ and $B_{i}$ such that $\left(A_{1} \cdot B_{1}, \ldots, A_{n-1} \cdot B_{n-1}\right)$ is a factor sequence for $\bar{r}$.
Theorem 3. The $K$-theoretic quiver coefficient $c_{\mu}(r)$ is equal to $(-1)^{\sum\left|\mu_{i}\right|-d(r)}$ times the number of $K$-theoretic factor sequences $\left(T_{1}, \ldots, T_{n}\right)$ for the rank conditions $r$, such that $T_{i}$ has shape $\mu_{i}$ for each $i$.

Using results about Demazure characters it was proved in [16] that cohomological quiver coefficients are special cases of the Stanley coefficients associated to the Zelevinsky permutation $z(r)[24,16]$. We recall the definition of this permutation in Section 4. In this paper we prove more generally that the $K$-theoretic quiver coefficients are special cases of the coefficients $c_{z(r), \lambda}$ in the expansion (1) of the stable Grothendieck polynomial for $z(r)$. This result also sharpens the fact from $[4,8]$ that quiver coefficients are special cases of the decomposition coefficients of Grothendieck polynomials studied in [6] (see section 1.4.1). Given a sequence of partitions $\mu=\left(\mu_{1}, \ldots, \mu_{n}\right)$ such that $\mu_{i}$ is contained in the rectangle $e_{i} \times e_{i-1}$, let $\lambda(\mu)$ be the partition obtained by concatenating the partitions $\left(e_{0}+\cdots+e_{n-2-i}\right)^{e_{i}}+\mu_{n-i}$ for $i=0, \ldots, n-1$. Our proof of the following identity is based on a bijection between the $K$-theoretic factor sequences for $r$ and the decreasing tableaux representing $z(r)$.

Theorem 4. For any set of rank conditions $r$ and sequence of partitions $\mu$ we have $c_{\mu}(r)=c_{z(r), \lambda(\mu)}$.

Central to the proof of the nonnegativity of cohomological quiver coefficients given in [16] is the stable component formula, which writes the cohomology class of a quiver variety as a sum of products of Stanley functions. This sum is over all lace diagrams representing the rank conditions $r$, which have the smallest possible number of crossings. The $K$-theoretic version of the component formula from [4, 21] states that

$$
\begin{equation*}
\left[\mathcal{O}_{\Omega_{r}}\right]=\sum_{\left(w_{1}, \ldots, w_{n}\right)}(-1)^{\sum \ell\left(w_{i}\right)-d(r)} G_{w_{1}}\left(E_{1}-E_{0}\right) \cdots G_{w_{n}}\left(E_{n}-E_{n-1}\right) \tag{6}
\end{equation*}
$$

where the sum is over a generalization of minimal lace diagrams, which was named $K M S$-factorizations in [4]. We recall this definition in Section 4. The $K$-theoretic factor sequences also have the following characterization.

Theorem 5. A sequence of decreasing tableaux $\left(T_{1}, \ldots, T_{n}\right)$ is a $K$-theoretic factor sequence for the rank conditions $r$ if and only if $\left(w\left(T_{1}\right), \ldots, w\left(T_{n}\right)\right)$ is a $K M S$ factorization for $r$.

We will use the statement of Theorem 5 as our definition of $K$-theoretic factor sequences. When this definition is used, Theorem 3 is an immediate consequence of Theorem 1 combined with the $K$-theoretic stable component formula (6). The above inductive construction of factor sequences is then derived from a similar construction of KMS-factorizations proved in [4].
1.4. Other applications. We list other applications for the methods presented in this extended abstract that are not developed here but which will appear in the full version of this paper.
1.4.1. Decomposition coefficients of Grothendieck polynomials. Fulton's universal Schubert polynomials from [14] describe certain quiver varieties associated to a sequence of vector bundles $E_{1} \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_{n} \rightarrow F_{n} \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_{1}$ over $X$, such that $\operatorname{rank}\left(E_{i}\right)=\operatorname{rank}\left(F_{i}\right)=i$ for each $i$. Given a permutation $w \in S_{n+1}$, we let $\Omega_{w} \subset X$ be the degeneracy locus of points where the rank of each composed map $E_{q} \rightarrow F_{p}$ is at most equal to the number of integers $1 \leq i \leq p$ such that $w(i) \leq q$. The quiver formula (5) can be applied to give a formula

$$
\begin{equation*}
\left[\mathcal{O}_{\Omega_{w}}\right]=\sum_{\mu} c_{w, \mu}^{(n)} G_{\mu_{1}}\left(E_{2}-E_{1}\right) \cdots G_{\mu_{n}}\left(F_{n}-E_{n}\right) \cdots G_{\mu_{2 n-1}}\left(F_{1}-F_{2}\right) \tag{7}
\end{equation*}
$$

for the Grothendieck class of $\Omega_{w}$, where the coefficients $c_{w, \mu}^{(n)}$ are special cases of quiver coefficients. It was shown in [2] that the coefficients $c_{w, \lambda}$ of the expansion (1) of the stable Grothendieck polynomial for $w$ can be obtained as the specializations $c_{w,\left(\emptyset^{n-1}, \lambda, \emptyset^{n-1}\right)}^{(n)}$, where $\emptyset^{n-1}$ denotes a sequence of $n-1$ empty partitions. More generally, it was proved in [6, Thm. 4] that the coefficients $c_{w, \lambda}^{(n)}$ can be used to expand a double Grothendieck polynomial as a linear combination of products of stable Grothendieck polynomials applied to disjoint intervals of variables. In [6], the formula (7) was also used to prove that

$$
\left[\mathcal{O}_{\Omega_{w}}\right]=\sum(-1)^{\ell\left(u_{1} \cdots u_{2 n-1} w\right)} G_{u_{1}}\left(E_{2}-E_{1}\right) \cdots G_{u_{n}}\left(F_{n}-E_{n}\right) \cdots G_{u_{2 n-1}}\left(F_{1}-F_{2}\right)
$$

where this sum is over all sequences of permutations $\left(u_{1}, \ldots, u_{2 n-1}\right)$ such that $u_{i} \in S_{\min (i, 2 n-i)+1}$ and $w$ is equal to the Hecke product $u_{1} \cdot u_{2} \cdots u_{2 n-1}$. Combining this with Theorem 1, we obtain the following generalization of [7, Thm. 1].
Theorem 6. The coefficient $c_{w, \mu}^{(n)}$ of (7) is equal to $(-1)^{\sum\left|\mu_{i}\right|-\ell(w)}$ times the number of sequences $\left(T_{1}, \ldots, T_{2 n-1}\right)$ of decreasing tableaux of shapes $\left(\mu_{1}, \ldots, \mu_{2 n-1}\right)$, such that the entries of $T_{i}$ are at most $\min (i, 2 n-i)$ and $w\left(T_{1} T_{2} \cdots T_{n}\right)=w$.
1.4.2. Expansion of Grothendieck polynomials. Theorem 1 may be refined to give an expansion of Grothendieck polynomials. The cohomological analogue is the combinatorial rule $[20,22,23]$ for the expansion of a Schubert polynomial as a positive sum of Demazure characters [9]. Taking a suitable limit, the Schubert polynomial becomes a Stanley function and each Demazure character becomes a Schur function. Using divided difference operators, one may introduce a new basis of $\mathbb{Z}\left[x_{1}, x_{2}, \ldots\right]$ called Grothendieck-Demazure polynomials. We have a conjectural expansion of a Grothendieck polynomial as an alternating sum of GrothendieckDemazure polynomials. In the limit this expansion becomes that in Theorem 1.

## 2. Grothendieck polynomials

2.1. Definition. Lascoux and Schützenberger's original definition of Grothendieck polynomials was based on divided difference operators [19]. In this paper we will use Fomin and Kirillov's construction of these polynomials [12], in notation that generalizes Billey, Jockusch, and Stanley's formula for Schubert polynomials [1].

Define a compatible pair to be a pair $(a, i)$ of words $a=a_{1} a_{2} \cdots a_{p}$ and $i=$ $i_{1} i_{2} \cdots i_{p}$, such that $i_{1} \leq i_{2} \leq \cdots \leq i_{p}$, and so that $i_{j}<i_{j+1}$ whenever $a_{j} \leq a_{j+1}$. For $w \in S_{\infty}$, the stable Grothendieck polynomial for $w$ is given by [12]

$$
\begin{equation*}
G_{w}=\sum_{(a, i)}(-1)^{\ell(i)-\ell(w)} x^{i} \tag{8}
\end{equation*}
$$

where the sum is over all compatible pairs $(a, i)$ such that $w(a)=w$. Here $\ell(i)$ is the common length of $a$ and $i$, and $x^{i}=x_{i_{1}} x_{i_{2}} \cdots x_{i_{\ell(i)}}$. The ordinary Grothendieck polynomial $\mathfrak{G}_{w}$ is given by the same sum (8), but only including the compatible pairs $(a, i)$ for which $a_{j} \geq i_{j}$ for each $j$. The Schubert polynomial for $w$ is equal to the lowest term of $\mathfrak{G}_{w}$, while the Stanley function $F_{w}$ is the lowest term of $G_{w}$.

We also require Buch's formula [3] for the stable Grothendieck polynomial $G_{\lambda}$. A set-valued tableau of shape $\lambda$ is a filling of the boxes of $\lambda$ with finite nonempty sets of positive integers, such that these sets are weakly increasing along rows and strictly increasing down columns. In other words, all integers in a box must be smaller than or equal to the integers in the box to the right of it, and strictly smaller than the integers in the box below it. For a set-valued tableau $S$, let $x^{S}$ denote the monomial where the exponent of $x_{i}$ is equal to the number of boxes containing the integer $i$, and let $|S|$ be the degree of this monomial. We have [3]

$$
\begin{equation*}
G_{\lambda}=\sum_{S}(-1)^{|S|-|\lambda|} x^{S} \tag{9}
\end{equation*}
$$

where $S$ runs over all set-valued tableaux of shape $\lambda$.
2.2. The required bijection. For any permutation $w \in S_{n}$, it follows from (8) and the symmetry of stable Grothendieck polynomials that $G_{w}=G_{w_{0} w^{-1} w_{0}}$, where $w_{0}=w_{0}^{(n)}$ is the longest permutation in $S_{n}$. Theorem 1 is therefore equivalent to the following statement. Define an increasing tableau to be a Young tableau with strictly increasing rows and columns.

Theorem 7. The coefficient $c_{w, \lambda}$ is equal to $(-1)^{|\lambda|-\ell(w)}$ times the number of increasing tableaux $T$ of shape $\lambda$ such that $w(T)=w^{-1}$.

In light of (8) and (9), to prove this theorem, it suffices to establish a bijection $(a, i) \mapsto(T, U)$ between all compatible pairs $(a, i)$ such that $w(a)=w$, and all pairs of tableaux $(T, U)$ of the same shape, such that $T$ is increasing with $w(T)=w^{-1}$ and $U$ is set-valued. In addition, this bijection must satisfy that $x^{U}=x^{i}$.

## 3. Hecke Insertion

Let $M_{1}$ be the set of pairs $(Y, x)$ where $Y$ is an increasing tableau and $x$ is a letter. Let $M_{2}$ be the set of triples $(Z, r, \alpha)$ where $Z$ is an increasing tableau, $Z$ has a corner cell $(r, c)$ in the $r$-th row, and $\alpha \in\{0,1\}$. Hecke insertion defines a
bijection

$$
\begin{align*}
\Phi: M_{1} & \rightarrow M_{2} \\
(Y, x) & \mapsto(Z, r, \alpha) \tag{11}
\end{align*}
$$

such that either
(1) $\alpha=0$ and $\operatorname{shape}(Z)=\operatorname{shape}(Y)$.
(2) $\alpha=1$ and $\operatorname{shape}(Z)=\operatorname{shape}(Y) \bigsqcup\{(r, c)\}$.

This bijection defines the Hecke insertion of $x$ into the increasing tableau $Y$, resulting in the increasing tableau $Z$, ending at the corner cell $(r, c)$ of $Z$. Unlike ordinary Schensted insertion, it is possible for a Hecke insertion not to add a cell to the tableau: a new cell is created if and only if $\alpha=1$.
3.1. Hecke (Row) Insertion. Let $\left(Y, x^{\prime}\right) \in M_{1}$. Inductively for $i \geq 1$, suppose that the first $i-1$ rows of $Y$ have been suitably modified, and that the number $x$ is being inserted into the $i$-th row. Let $y$ be the smallest entry in the $i$-th row such that $y>x$. If no such element exists, set $y=\infty$ and let $z$ be the last entry in the $i$-th row. If the $i$-th row is empty then set $z=0$.
(H1) If $y=\infty$ and $z=x$ : The insertion terminates. Let $(r, c)$ be the corner cell in the same column as $z$ and $\alpha=0$.
(H2) If $y=\infty$ and $z<x$ : The insertion terminates. (a) If adjoining $x$ to the end of the $i$-th row, results in an increasing tableau, then do so, with $r=i$ and $\alpha=1$. (b) If not (and this can only happen if $i>1$ ), let $(r, c)$ be the corner cell in the same column as $z$ and $\alpha=0$.
(H3) If $y<\infty$, replace $y$ by $x$ if the result is an increasing tableau and otherwise leave the row unchanged. Continue by inserting $y$ into the $(i+1)$-th row.
Let $Z$ be the resulting increasing tableau. This algorithm produces a triple $\Phi\left(Y, x^{\prime}\right)=(Z, r, \alpha) \in M_{2}$. Write $Z=\left(Y \stackrel{H}{\longleftarrow} x^{\prime}\right)$.

Remark 8. A increasing tableau is produced in cases (H2) and (H3) unless $x$ would be placed just to the right of, or just below, another $x$.

## Example 9.

$$
\begin{array}{|l|l|l|l}
\hline 1 & 2 & 3 & 5 \\
\hline 2 & 5 & \\
\hline 3 & 6 & \\
\hline 4 & & H \\
\hline
\end{array}
$$

3 is inserted into the first row, which contains 3 . So 5 is inserted into the second row, whose largest value is $z=5$. This is case (H1). Then $\alpha=0$ and $r=3$, since $(3,2)$ is the cell at the bottom of the column of $z$.

## Example 10.

\[

\]

2 is inserted into the first row, which contains 2.4 is inserted into the second row, displacing the 5 . The 5 is inserted into the third row, where it comes to rest. This is case (H2a). Then $\alpha=1$ and $r=3$.

## Example 11.

2 is inserted into the first row, which contains a 2.4 is inserted into the second row, which has largest entry $z=3$. 4 can't come to rest at the cell $(2,3)$ since that is just below the 4 in cell (1,3). Case (H2b) holds. Then $\alpha=0$ and $r=3$ because $(3,2)$ is the cell at the bottom of the column of $z$.

## Example 12.

$$
\begin{array}{|l|l|}
\hline 1 & 3 \\
\hline 2 & 4 \\
\hline 3 & 5 \\
\hline
\end{array} \left\lvert\, \stackrel{H}{4} 1=\right.
$$

1 is inserted into the first row, which already contains a 1 . So 3 is inserted into the second row. It would have replaced 4 , but this replacement would place a 3 directly below another 3, violating the increasing tableau condition. So the second row is unchanged and 4 is inserted into the third row. Similarly 4 cannot replace 5 . So 5 is inserted into the fourth row, where it comes to rest in the cell $(4,1)$ with $\alpha=1$.
3.2. Reverse Hecke insertion. The inverse map $\Psi: M_{2} \rightarrow M_{1}$ is defined as follows. Let $(Z, r, \alpha) \in M_{2},(r, c)$ the corner cell in the $r$-th row of $Z$, and $y=Z_{r, c}$. If $\alpha=1$ then remove $y$. In any case, reverse insert $y$ up into the previous row.

Whenever a value $y$ is reverse inserted into a row, let $x$ be the largest entry in the row such that $x<y$. If replacing $x$ by $y$ yields an increasing tableau then do so; otherwise leave the row unchanged. In any case, reverse insert $x$ into the previous row.

Eventually a value $x^{\prime}$ reverse inserted out of the first row, leaving behind an increasing tableau $Y$. Call $x^{\prime}$ the output value. Define $\Psi(Z, r, \alpha)=\left(Y, x^{\prime}\right)$.

Remark 13. Note that the only obstructions for replacing $x$ by $y$, are when the entry below or to the right of $x$ already contains $y$.

Example 14. Let us apply reverse Hecke insertion to the tableau computed in Example 12 at the cell $(4,1)$ with $\alpha=1$. Since $\alpha=1$ the entry 5 in cell $(4,1)$ is removed. Then 5 is reverse inserted into the third row. Since 5 is already in the third row, the third row remains unchanged and 3 is reverse inserted into the second row. 3 cannot replace 2 because this would place a 3 directly atop a 3 , creating a vertical violation of the increasing tableau condition. The second row is unchanged and 2 is reverse inserted into the first row. 2 cannot replace 1 for the same reason. The first row is unchanged and 1 is the output value. This recovers the initial tableau of Example 12.

Proposition 15. The maps $\Phi$ and $\Psi$ are mutually inverse bijections.
3.3. Properties of Hecke insertion. Hecke insertion respects Hecke words.

Lemma 16. Suppose reverse Hecke insertion of the tableau $T$ at some corner cell results in the tableau $T^{\prime}$ and the output value $x$. Then $w(T)=w\left(T^{\prime} x\right)$.

Hecke insertion also satisfies the following Pieri property.

Lemma 17. Suppose we first reverse Hecke insert starting from one corner $C_{1}$ of $T$, and then reverse Hecke insert from a corner $C_{2}$ of the modification of $T$. Then the first output value is strictly smaller than the second output value if and only if $C_{1}$ is strictly lower than $C_{2}$.
3.4. Proof of Theorem $\mathbf{7}$ via column Hecke Robinson-Schensted. In this section we give the bijection that was sought in section 2.2. We may define Hecke column insertion by switching the roles of rows and columns in Hecke row insertion. Write $\Phi^{\prime}: M_{1} \rightarrow M_{2}$ for this bijection.

Let $(a, i)$ be as in section 2.2 with $a=a_{1} a_{2} \cdots a_{p}$ and $i=i_{1} i_{2} \cdots i_{p}$. We start with the empty tableau pair $\left(T_{0}, U_{0}\right)=(\varnothing, \varnothing)$. If $\left(T_{j-1}, U_{j-1}\right)$ has been defined for some $j \geq 1$, let $\left(T_{j}, s_{j}, \alpha_{j}\right)=\Phi^{\prime}\left(T_{j-1}, a_{j}\right)$. Let $U_{j}$ be obtained from $U_{j-1}$ by adjoining a new cell to the end of the $s_{j}$-th row containing the singleton set $\left\{i_{j}\right\}$ if $\alpha_{j}=1$. Otherwise $U_{j}$ is obtained from $U_{j-1}$ by putting $i_{j}$ into the existing set in the corner cell in row $s_{j}$. Define $(T, U)=\left(T_{p}, U_{p}\right)$. The map $(a, i) \mapsto(T, U)$ has the desired properties. $U$ is a set-valued tableau by Lemma 17 and $x^{i}=x^{U}$ by definition. The fact that $w(T)=w^{-1}$ follows from Lemma 16 combined with the fact that the reversal of a word gives a bijection between the Hecke words for $w$ and those for $w^{-1}$. This proves Theorems 7 and 1.
3.5. Product of decreasing tableaux. For use with factor sequences, we define the product of the decreasing tableaux $T_{1}$ and $T_{2}$. Consider the variant of Hecke insertion in which larger numbers bump smaller numbers. In other words, we reverse the order of the positive integers in the algorithm of Section 3.1. Let $T_{1} \cdot T_{2}$ be the decreasing tableau obtained by inserting the word of $T_{2}$ into $T_{1}$ using this variant of Hecke insertion. More precisely, if $a_{1} a_{2} \cdots a_{p}$ is the word of $T_{2}$ then $T_{1} \cdot T_{2}=\left(\left(\left(T_{1} \stackrel{H}{\longleftarrow} a_{1}\right) \stackrel{H}{\longleftarrow} a_{2}\right) \cdots\right) \stackrel{H}{\longleftarrow} a_{p}$. This product has the following properties.

Lemma 18. (1) For decreasing tableaux $T_{1}, T_{2}$ we have $w\left(T_{1} \cdot T_{2}\right)=w\left(T_{1}\right) \cdot w\left(T_{2}\right)$.
(2) Suppose a decreasing tableau $T$ is cut along a vertical line into $T_{\text {left }}$ and $T_{\text {right }}$. Then $T=T_{\text {left }} \cdot T_{\text {right }}$.
(3) Suppose $T$ is cut along a horizontal line into tableaux $T_{\text {bottom }}$ and $T_{\text {top }}$. Then $T=T_{\text {bottom }} \cdot T_{t o p}$.

Our applications to factor sequences require that the product of decreasing tableaux satisfies the properties of this lemma. When the concatenation of the words of $T_{1}$ and $T_{2}$ is a reduced word of a permutation, then these conditions imply that $T_{1} \cdot T_{2}$ agrees with the Coxeter-Knuth product, but no such uniqueness statement holds in general. The product $T_{1} \cdot T_{2}$ also fails to be associative.

## 4. Quiver varieties

Let $r=\left\{r_{i j}\right\}$ be a set of rank conditions for $0 \leq i, j \leq n$, and set $N=e_{0}+\cdots+e_{n}$ where $e_{i}=r_{i i}$. A result of Zelevinsky shows that when the base variety $X$ is a product of matrix spaces, the quiver variety $\Omega_{r} \subset X$ is identical to a dense open subset of a Schubert variety [24]. The Zelevinsky permutation corresponding to this Schubert variety was used in [16] to prove the ratio formula for quiver varieties.

With the notation from [4], the Zelevinsky permutation can be constructed as a product of permutations as follows (see [16, Prop. 1.6] for a different construction). Extend the rank conditions $r=\left\{r_{i j}\right\}$ by setting $r_{i j}=e_{j}+\cdots+e_{i}$ for $0 \leq j<i \leq n$.

Then define decreasing tableaux $U_{i j}$ as in the introduction, but for all $0 \leq i<n$ and $0<j \leq n$. The corresponding permutation $W_{i j}=w\left(U_{i j}\right)$ is given by

$$
W_{i j}(p)= \begin{cases}p+r_{i, j-1}-r_{i j} & \text { if } r_{i j}<p \leq r_{i+1, j} \\ p-r_{i+1, j}+r_{i j} & \text { if } r_{i+1, j}<p \leq r_{i+1, j}+r_{i, j-1}-r_{i j} \\ p & \text { otherwise }\end{cases}
$$

The Zelevinsky permutation can now be defined by $z(r)=\prod_{j=1}^{n} \prod_{i=0}^{n-1} W_{i j}$.
For each $1 \leq j \leq n-1$ we set $\delta_{j}=W_{j j} W_{j+1, j} \cdots W_{n-1, j} \in S_{N}$. A $K M S$ factorization for the rank conditions $r$ is any sequence $\left(w_{1}, \ldots, w_{n}\right)$ of permutations with $w_{i} \in S_{e_{i-1}+e_{i}}$, such that the Zelevinsky permutation $z(r)$ is equal to the Hecke product

$$
w_{1} \cdot \delta_{1} \cdot w_{2} \cdot \delta_{2} \cdots \delta_{n-1} \cdot w_{n}
$$

These sequences of permutations generalize the notion of a minimal lace diagram from [16] and give the index set in the $K$-theoretic stable component formula (6) from [4, 21].

We define a $K$-theoretic factor sequence for the rank conditions $r$ to be any sequence $\left(T_{1}, \ldots, T_{n}\right)$ of decreasing tableaux, such that $\left(w\left(T_{1}\right), \ldots, w\left(T_{n}\right)\right)$ is a KMSfactorization for $r$. As noted in the introduction, this definition means that Theorem 3 is a consequence of Theorem 1 combined with the stable component formula (6). To obtain the inductive definition of factor sequences we need the following result proved in [4, Thm. 7], which shows that KMS-factorizations can themselves be defined as 'factor sequences'.

Theorem 19. (a) If $\left(w_{1}, \ldots, w_{n}\right)$ is a KMS-factorization for $r$, then each permutation $w_{i}$ has a reduced factorization $w_{i}=v_{i-1} \cdot W_{i-1, i} \cdot u_{i}$ with $v_{i-1} \in S_{e_{i-1}}$ and $u_{i} \in S_{e_{i}}$, such that $v_{0}=u_{n}=1$.
(b) Let $u_{1}, v_{1}, \ldots, u_{n-1}, v_{n-1}$ be permutations with $u_{i}, v_{i} \in S_{e_{i}}$. Then the sequence $\left(W_{01} \cdot u_{1}, v_{1} \cdot W_{12} \cdot u_{2}, \ldots, v_{n-1} \cdot W_{n-1, n}\right)$ is a KMS-factorization for $r$ if and only if $\left(u_{1} \cdot v_{1}, u_{2} \cdot v_{2}, \ldots, u_{n-1} \cdot v_{n-1}\right)$ is a KMS-factorization for $\bar{r}$.

We also need the following statement.
Lemma 20. Let $T$ be any decreasing tableau such that $w(T) \in S_{m}$, and for some integers $a, b<m$ we have $w(T)(p) \leq b$ for all $a<p \leq m$. Then $T$ contains the rectangle $R=(m-a) \times(m-b)$ in its upper left corner. The upper-left box of $R$ equals $m-1$, and the boxes of $R$ decrease by one for each step down or to the right.

Let $(U, T) \mapsto U \cdot T$ be the product of decreasing tableaux defined in section 3.5.
Corollary 21. A sequence of decreasing tableaux $\left(T_{1}, \ldots, T_{n}\right)$ is a $K$-theoretic factor sequence for the rank conditions $r$ if and only if there exist decreasing tableaux $A_{i}, B_{i}$ for $1 \leq i \leq n-1$, such that $T_{i}=B_{i-1} \cdot U_{i-1, i} \cdot A_{i}$ for each $i$ (with $B_{0}=$ $\left.A_{n}=\emptyset\right)$ and $\left(A_{1} \cdot B_{1}, \ldots, A_{n-1} \cdot B_{n-1}\right)$ is a $K$-theoretic factor sequence for $\bar{r}$.

Given a sequence $\left(T_{1}, \ldots, T_{n}\right)$ of decreasing tableaux, such that each tableau $T_{i}$ can be contained in the rectangle $e_{i} \times e_{i-1}$ and all entries of $T_{i}$ are smaller than $e_{i-1}+e_{i}$, we let $\Phi\left(T_{1}, \ldots, T_{n}\right)$ denote the decreasing tableau constructed from this
sequence as well as the tableaux $U_{i j}$ for $i \geq j$ as follows.


Notice that the upper-left box of $U_{n-1,1}$ is equal to $N-1$, and the boxes in the union of tableaux $U_{i j}$ decrease by one for each step down or to the right. Theorem 4 follows from the following proposition combined with Theorems 1 and 3.

Proposition 22. The $\operatorname{map}\left(T_{1}, \ldots, T_{n}\right) \mapsto \Phi\left(T_{1}, \ldots, T_{n}\right)$ gives a bijection of the set of all K-theoretic factor sequences for $r$ with the set of all decreasing tableaux representing $z(r)$.

Proof. Since the permutation of a decreasing tableau can be defined as the southwest to north-east Hecke product of the simple reflections given by the boxes of the tableau, it follows from the definition of KMS-factorizations that $\left(T_{1}, \ldots, T_{n}\right)$ is a factor sequence if and only if $\Phi\left(T_{1}, \ldots, T_{n}\right)$ represents the Zelevinsky permutation $z(r)$. It remains to show that any decreasing tableau $T$ representing $z(r)$ contains the arrangement of rectangular tableaux $U_{i j}$ in its upper-left corner, and has no boxes strictly south-east of the tableaux $U_{i i}$ for $1 \leq i \leq n-1$. The inclusion of the tableaux $U_{i j}$ in $T$ follows from Lemma 20 because $z(r) \in S_{N}$ and for each $0<i \leq n$ and $p>r_{n i}$ we have $z(r)(p) \leq r_{i 0}$, see [16, Prop 1.6] or [4, Lemma 3.1].

To see that $T$ contains no boxes strictly south-east of $U_{i i}$, we use that the Grothendieck polynomial $\mathfrak{G}_{\widehat{z}(r)}\left(x_{1}, \ldots, x_{N}\right)$ is separately symmetric in each group of variables $\left\{x_{p} \mid r_{n, i}<p \leq r_{n, i-1}\right\}$, where $\widehat{z}(r)=w_{0}^{(N)} z(r)^{-1} w_{0}^{(N)}$ and $w_{0}^{(N)}$ is the longest permutation in $S_{N}$. This is true because the descent positions of $\widehat{z}(r)$ are contained in the set $\left\{r_{n j} \mid 0<j \leq n\right\}$. It follows that the exponent of $x_{r_{n i}+1}$ in any monomial of $\mathfrak{G}_{\widehat{z}(r)}\left(x_{1}, \ldots, x_{N}\right)$ is less than or equal to $N-r_{n, i-1}=r_{i-2,0}$. Now $T$ can be used to construct a unique compatible pair $(a, k)$ for $\widehat{z}(r)$, such that $T$ contains the integer $p$ in some box of row $q$ if and only if $\left(a_{l}, k_{l}\right)=(N-p, q)$ for some $l$. Since this pair contributes the monomial $x^{k}$ to $\mathfrak{G}_{\overparen{z}(r)}\left(x_{1}, \ldots, x_{N}\right)$, it follows that row $r_{n i}+1$ of $T$ has at most $r_{i-2,0}$ boxes. This means exactly that $T$ contains no boxes south-east of $U_{i-1, i-1}$, as required.

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[^0]:    Date: November 28, 2004.

[^1]:    ${ }^{1}$ The use of decreasing tableaux rather than increasing, is merely for convenience in the definition of a $K$-theoretic factor sequence.

