Stable Inversion of MIMO Linear Discrete Time Non-Minimum Phase Systems

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Stable Inversion of MIMO Linear Discrete Time Non-Minimum Phase Systems*

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Abstract

A novel technique to achieve output tracking via stable inversion of non-minimum phase linear systems is presented wherein the desired signal is obtained from field measurements, and hence corrupted by noise. The earlier approach to stable inversion does not take into account the noise in the system. The unknown input decoupled observer approach is applicable only to minimum phase systems. Moreover, the unobservable states are inadequately constructed resulting in inferior output tracking in the presence of noise. In this paper we extend this procedure to non-minimum phase systems. We present the novel Stable Dynamic model Inversion (SDI) approach which is applicable to non-minimum phase systems, and takes into account the presence of noise in target time histories.

1   Introduction

Precision output tracking has been one of the fundamental problems for control engineers; increasingly stringent performance requirements are to be satisfied in a variety of applications notably in the robotics and aerospace industries. In the context of linear systems, it is well-known that perfect tracking is relatively easy to achieve in minimum phase systems. However, output tracking for non-minimum phase systems remains a challenging problem due to the fundamental limitations on transient tracking performance characterised by the number and location of the zeros which are non-minimum phase (Qiu and Davison, 1993).

In this paper we deal with the problem of output tracking wherein the desired signal is obtained through a data acquisition system, and hence corrupted by noise. One such application is that of time waveform replication which concerns with accurate reproduction of real or synthesised target time histories. Thus complex vibration environments (such as automobile crashes) may be recreated in a test laboratory by simulating field measurements thereby saving precious resources. Other applications include durability tests of, for instance, automobile components,

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and driving comfort assessment. Thus, given a corrupted version of the true desired signal y_k^d, and the following system,

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + Gw_k \\
y_k &= Cx_k + Du_k \\
z_k &= y_k + v_k
\end{aligned}
$$

our objective is to obtain the desired input u_k^d such that it satisfies the following system of equations

$$
\begin{aligned}
x_{k+1}^d &= Ax_k^d + Bu_k^d \\
y_k^d &= Cx_k^d + Du_k^d
\end{aligned}
$$

thereby overcoming the effect of the noise w_k and v_k.

For linear continuous systems, Francis and Wonham (1976) show that the asymptotic tracking problem is solvable if, and only if, a set of linear matrix equations is solvable. This was later generalised to nonlinear systems by replacing the linear matrix equations by a set of first order partial differential equations (Isidori and Byrnes, 1990). These approaches asymptotically track any member in a given family of signals generated by an exosystem. The stable inversion approach was introduced by Devasia *et al.* (1996) to avoid the use of exosystems, and, in the case of non-minimum phase systems, mitigate the poor transient performance by using pre-actuation.

The basic idea in the stable inversion approach (Devasia *et al.*, 1996) is to use a dichotomic split of a related system of equations in the case of a non-minimum phase plant to compute an inverse trajectory u_k^d for a desired output trajectory y_k^d. This inverse trajectory becomes a feed-forward signal used in conjunction with a more conventional feedback control law in order to make it attractive. (We note that such an approach is related to the classical Hirschorn inverse (Hirschorn, 1979) in the case of minimum phase systems.) The resulting desired input trajectory u_k^d is evidently non-causal in the case of non-minimum phase systems, and hence pre-actuation. From an engineering perspective, pre-actuation is applicable since the inversion is not expressed as a solution of differential equations but as a map. The stable inversion approach has been developed for continuous time systems. The starting point in this paper is to present a discrete-time version of this technique for linear systems. However, this procedure, just like the continuous time case, does not handle desired signals corrupted by noise, and, therefore, does not fulfil our objective.

The problem of trajectory tracking can also be dealt with from the point of view of estimating the state of a system subjected to unknown inputs. System uncertainties, quite often, arise from linearisation errors, unmodelled nonlinearities, or from unmeasurable disturbances which cannot satisfactorily be described as stochastic signals with known statistics. In order to facilitate a simple linear system description, such uncertainties are frequently modelled as 'unknown-inputs'. The problem of decoupling the unknown input affecting the underlying system from the state estimation error has attracted considerable attention from researchers leading to several applications typically in fault detection and isolation (see, (Hou and Patton, 1998), and the references cited therein). Using a unified unknown-input decoupling technique, Hou and Patton (1998) solve this filtering problem for time-varying linear systems in a rigorous and straightforward manner. The approach consists of first building an 'equivalent system' which is decoupled from the unknown inputs, and then designing a minimum covariance estimator for this equivalent system.

In this paper, we use the Hou-Patton technique to estimate the desired state of an input-decoupled system, and hence compute the desired input. However, this approach is valid only for

minimum phase systems. In this paper we also generalise this technique to non-minimum phase systems. This generalisation is obtained via an inner-outer factorisation of the given system, and hence avoids the computation of dichotomies. In principle, this generalisation consists of first modifying the target time history by the inverse of the inner factor, and then applying the Hou-Patton approach to the outer factor with the modified desired output trajectory.

Thus, we now have two techniques to solve the output tracking problem. It seems apparent that the major advantage of the Hou-Patton approach as compared to the stable inversion approach is that it takes into account the noise in the system. However, under fairly typical conditions, we show that the equivalent system built in the Hou-Patton approach is related to the system matrix of the inverse system. It is well-known that the internal dynamics, which is part of the inverse system and related to the zeros of the given plant, is not observable from its output. Therefore, the unobservable states are inadequately reconstructed resulting in inferior output tracking in the presence of noise.

Evidently, there is a need for a different approach that is applicable to non-minimum phase systems, and accounts for the noise on the desired trajectory. We present a simple approach to this problem called the Stable Dynamic model Inversion (SDI) technique. Here, the given system model is first augmented by a realistic model for the input time history, followed by designing a Kalman filter for the augmented system. This technique is more general than the earlier schemes in that it is not limited by the presence of zeros on the entire unit circle. Moreover, by suitably designing the Kalman filter, we can easily take into account the presence of noise in target time histories.

This paper is organised as follows. In Section 2 we present the discrete time version of the stable inversion technique. The Hou-Paton approach is illustrated in Section 3 and we compare these methods in Section 4. We present the generalisation of the Hou-Patton technique to non-minimum phase systems and the novel Stable Dynamic model Inversion procedure in Section 5. An illustrative benchmark example supporting the contributions of this paper is treated in Section 6.

2 Inversion of the Input-Output Map

The stable inversion approach aims at providing a bounded inverse of the system even for non-minimum phase systems. This bounded inverse is computed by solving a two point boundary value problem obtained via a dichotomic split of the internal dynamics of the system. This results in an acausal input u_k^d in the case of non-minimum phase systems; the anti-causal part of the input sets up the desired initial condition. We note that for minimum phase systems this inverse is related to the classical system inverse. Although the procedure has been developed for continuous time systems (Devasia *et al.*, 1996), it can be extended, *mutatis mutandis*, to linear discrete time systems.

Consider the following state space realisation of a linear time invariant system:

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$
$$y_k = Cx_k + Du_k \tag{2}$$

We assume that $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \triangleq \begin{pmatrix} y_{1,k} & y_{2,k} & \cdots & y_{p,k} \end{pmatrix}' \in \mathbb{R}^p$. Let $p = m$ and $D = 0$. (The extension to the non-zero D case is rather obvious.) If c_i denotes the ith row of C, then the system is said to have a well-defined relative degree $r \triangleq \begin{pmatrix} r_1 & r_2 & \cdots & r_p \end{pmatrix}'$ if

$$c_i A^l B = 0 \quad \forall l < r_i - 1, \quad 1 \le i \le p$$

and the matrix

$$\tilde{D} \triangleq \begin{pmatrix} c_1 A^{r_1-1} B \\ c_2 A^{r_2-1} B \\ \vdots \\ c_p A^{r_p-1} B \end{pmatrix}$$

is nonsingular. Thus,

$$y_{i,k+r_i} = c_i A^{r_i} x_k + c_i A^{r_i-1} B u_k, \qquad 1 \le i \le p \tag{3}$$

which implies that each output $y_{i,k}$, when shifted forward r_i samples, has a non-zero feed-through term. Define

$$y_{k+r} \triangleq \begin{pmatrix} y_{1,k+r_1} & y_{2,k+r_2} & \cdots & y_{p,k+r_p} \end{pmatrix}'$$

Therefore,

$$y_{k+r} = \tilde{C} x_k + \tilde{D} u_k \tag{4}$$

where

$$\tilde{C} \triangleq \begin{pmatrix} c_1 A^{r_1} \\ c_2 A^{r_2} \\ \vdots \\ c_p A^{r_p} \end{pmatrix}$$

For a given a desired output trajectory y_k^d, we can obtain from eqn. (4) a relationship for the desired input trajectory u_k^d as follows:

$$u_k^d = \tilde{D}^{-1} \left(y_{k+r}^d - \tilde{C} x_k \right) \tag{5}$$

Therefore,

$$x_{k+1} = \left(A - B\tilde{D}^{-1}\tilde{C} \right) x_k + B\tilde{D}^{-1} y_{k+r}^d \tag{6}$$

$$y_{k+r} = y_{k+r}^d \tag{7}$$

and exact tracking is maintained. In particular, if D is non-singular, then the system clearly has a relative degree $r = 0$, and we simply replace $\tilde{D}$ by D and $\tilde{C}$ by C in the above equations. Evidently, the boundedness of x_k is related to the eigenvalues of $\tilde{A} \triangleq A - B\tilde{D}^{-1}\tilde{C}$, which, in the case of a nonsingular D, are precisely the zeros of the original system. It is well-known (Isidori, 1995) that there exists a transformation T such that

$$\begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix} = T x_k$$

where

$$\xi_k \triangleq \begin{pmatrix} y_{1,k} & y_{1,k+1} & \cdots & y_{1,k+r_1-1} & y_{2,k} & \cdots & y_{2,k+r_2-1} & \cdots & y_{p,k} & y_{p,k+1} & \cdots & y_{p,k+r_p-1} \end{pmatrix}'$$

and

$$\begin{pmatrix} \xi_{k+1} \\ \eta_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} y_{k+r}^d$$

$$y = \begin{pmatrix} \tilde{C}_1 & 0 \end{pmatrix} \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}$$

The matrix $\tilde{A}$ has $\hat{r} \triangleq \sum_{i=1}^{i=p} r_i$ eigenvalues, corresponding to those of $\tilde{A}_{11}$, located at the origin, and the remaining $n - \hat{r}$ eigenvalues, corresponding to those of $\tilde{A}_{22}$, located at the zeros of the original system. Clearly, the states η_k represent the internal dynamics of the system, and are not observable from y_k. However, we require the knowledge of the complete state x_k in order to compute u_k^d from eqn. (5). We note that the desired trajectory of ξ_k, denoted ξ_k^d, is fixed by the desired output trajectory. Therefore, given a desired output trajectory, only the desired η_k remains to be computed.

Under the condition that $\tilde{A}_{22}$ does not have any eigenvalues on the unit circle, and hence a similar condition on the zeros of the original system, we can choose, without loss of generality, the state coordinates such that

$$\tilde{A}_{22} = \left(\begin{array}{cc} A_s & 0 \\ 0 & A_u \end{array} \right)$$

where A_s is completely stable, and A_u anti-stable. We use this dichotomic split of $\tilde{A}_{22}$ to obtain the solution of the following two point boundary value problem:

$$\eta_{k+1} = \tilde{A}_{22}\eta_k + \left(\tilde{B}_2 y_{k+r}^d + \tilde{A}_{21}\xi_k^d \right)$$

subject to the boundary conditions $\eta_\infty = \eta_{-\infty} = 0$. In essence, the stable part of $\tilde{A}_{22}$ is evolved forward in time, and its unstable part backward in time. It can be verified that

$$\eta_k^s = \sum_{i=-\infty}^{k-1} A_s^{k-1-i} \left(\tilde{B}_2 y_{i+r}^d + \tilde{A}_{21}\xi_i^d \right)_s$$

$$\eta_k^u = -\sum_{i=k}^{\infty} A_u^{k-1-i} \left(\tilde{B}_2 y_{i+r}^d + \tilde{A}_{21}\xi_i^d \right)_u$$

where η_k^s and η_k^u respectively are the states corresponding to A_s and A_u, and the input appropriately partitioned. Thus, the desired η_k can be computed by stable inversion of the system. The input to the system necessary for exact output tracking of a desired trajectory is then computed using eqn. (5).

In this section we have considered the discrete time version of the stable inversion technique. The necessary feed-forward input required can be computed by obtaining first the desired state trajectory. We emphasise that the stable inversion technique does not take into account the noise in the system, and is restricted to systems that do not have zeros on the unit circle. Evidently, the stable inversion technique is dependent on a model of a given plant, and hence not robust. We note that in the continuous time framework, an attempt to make the stable inversion technique robust is considered by Monsees *et al.* (1999). In the next section we compute the desired input trajectory by estimating the state trajectory.

3   The Unknown-Input Decoupled Observer Approach

The principal idea in this approach is to estimate the state of a system subjected to unknown inputs. Here, the state estimation error is decoupled from the unknown inputs affecting the underlying system by first computing an equivalent system (Hou and Patton, 1998). A minimum covariance estimator is then obtained for this equivalent system. In this section, we summarise this approach for time invariant systems.

Consider the following linear discrete time system:

$$\begin{align} x_{k+1} &= Ax_k + Bu_k + Gw_k \tag{8} \\ y_k &= Cx_k + Du_k + Hw_k \tag{9} \end{align}$$

where u_k is the unknown input and $\{w_k\}$ is a white noise process with zero mean. Without loss of generality, we assume that it has covariance $\mathcal{E}w_k w'_l = \delta_{k,l}I$. We have the following result:

Lemma 1 ((Hautus, 1983)) *There exists an unknown input decoupled observer if and only if*

1. $\text{rank} \begin{pmatrix} D & CB \\ 0 & D \end{pmatrix} = \text{rank} \begin{pmatrix} B \\ D \end{pmatrix} + \text{rank}\, D$

2. $\text{rank} \begin{pmatrix} -zI + A & B \\ C & D \end{pmatrix} = n + \text{rank} \begin{pmatrix} B \\ D \end{pmatrix}, \quad \forall\, |z| \geq 1$

$\blacksquare$

Most practical systems have more outputs than inputs ($p \geq m$), and the model is typically chosen with no redundant inputs (rank of B is full). Thus, for such systems, each of the above rank conditions are simplified by the following fact:

$$\text{rank} \begin{pmatrix} B \\ D \end{pmatrix} = \text{rank}\, B$$

Moreover, the first rank condition in Lemma 1 is satisfied whenever the matrix D has full rank. For systems with rank $D < m$, this condition is satisfied whenever the matrix product CB has full rank. If this rank condition fails, we can consider a modified system provided that the original plant model has a well-defined relative degree. Similar to eqn. (3), we can then shift each of the outputs sufficiently forward until the rank condition is satisfied; we build an equivalent system for this modified system. Further, the second rank condition in Lemma 1 is equivalent to the fact that the system (A, B, C, D) should not have any non-minimum phase zeros.

Under the conditions stated in Lemma 1, the unknown-input decoupled equivalent system exists (Hou and Patton, 1998) and is as follows:

$$\begin{aligned} x_{k+1} &= \bar{A}x_k + \bar{B}_1 y_k + \bar{B}_2 y_{k+1} + \bar{G}_1 w_k + \bar{G}_2 w_{k+1} & (10) \\ \bar{y}_k &= \bar{C}x_k + \bar{H}w_k & (11) \end{aligned}$$

where

$$\begin{aligned} \bar{A} &= (I - \bar{B}\bar{C})\left(A - BD^\dagger C\right) \\ \bar{B} &= B(I - D^\dagger D)\left(\bar{C}B(I - D^\dagger D)\right)^\dagger \\ \bar{C} &= (I - DD^\dagger)C \\ \bar{B}_1 &= (I - \bar{B}\bar{C})\,BD^\dagger \\ \bar{B}_2 &= \bar{B}(I - DD^\dagger) \\ \bar{G}_1 &= (I - \bar{B}\bar{C})\left(G - BD^\dagger H\right) \\ \bar{G}_2 &= -\bar{B}\bar{H} \\ \bar{H} &= (I - DD^\dagger)H \end{aligned}$$

Evidently, the equivalent system is independent of the unknown input u_k.

Lemma 2 ((Hou and Patton, 1998)) *The unknown input decoupled linear minimum covariance estimator for the system is given by:*

$$
\begin{aligned}
\hat{x}_{k+1|k+1} &= \bar{x}_{k+1} + K_{k+1}\left(\bar{y}_{k+1} - \bar{C}\bar{x}_{k+1}\right) \\
\bar{x}_{k+1} &= \bar{A}\hat{x}_{k|k} + \bar{B}_1 y_k + \bar{B}_2 y_{k+1} + \bar{G}_1\left(\bar{H}' + \bar{G}_2'\bar{C}'\right)R_k^\dagger\left(\bar{y}_k - \bar{C}\bar{x}_k\right) \\
K_{k+1} &= \left(\Sigma_{k+1|k}\bar{C}' + \bar{G}_2\bar{H}'\right)R_{k+1}^\dagger \\
R_{k+1} &= \bar{C}\Sigma_{k+1|k}\bar{C}' + \bar{H}\bar{H}' + \bar{C}\bar{G}_2\bar{H}' + \bar{H}\bar{G}_2'\bar{C}' \\
\Sigma_{k+1|k+1} &= \Sigma_{k+1|k} - \left(\Sigma_{k+1|k}\bar{C}' + \bar{G}_2\bar{H}'\right)R_{k+1}^\dagger\left(\Sigma_{k+1|k}\bar{C}' + \bar{G}_2\bar{H}'\right)' \\
\Sigma_{k+1|k} &= \bar{A}\Sigma_{k|k}\bar{A}' + \bar{G}_1\bar{G}_1' + \bar{G}_2\bar{G}_2' + S_k + S_k' - Q_k \\
Q_k &= \bar{G}_1\left(\bar{H} + \bar{G}_2\bar{C}\right)'R_k^\dagger\left(\bar{H} + \bar{G}_2\bar{C}\right) \\
S_k &= \bar{A}\left(\bar{G}_2 - K_k\bar{C}\bar{G}_2 - K_k\bar{H}\right)\bar{G}_1'
\end{aligned}
$$

∎

We remark that the term Q_k in the equation for $\Sigma_{k+1|k}$ has perhaps been inadvertently omitted in (Hou and Patton, 1998). The desired input can easily be computed from the estimated state. Define the following matrices:

$$
\begin{aligned}
X_{1,N-1} &= \left(\begin{array}{cccc} \hat{x}_{1|1} & \hat{x}_{2|2} & \cdots & \hat{x}_{N-1|N-1} \end{array}\right)' \\
X_{2,N} &= \left(\begin{array}{cccc} \hat{x}_{2|2} & \hat{x}_{3|3} & \cdots & \hat{x}_{N|N} \end{array}\right)' \\
U_{1,N-1} &= \left(\begin{array}{cccc} u_1 & u_2 & \cdots & u_{N-1} \end{array}\right)'
\end{aligned}
$$

where N is the number of samples of the desired output. From the original system we have

$$
U_{1,N-1} = B^\dagger\left(X_{2,N} - AX_{1,N-1}\right) \tag{12}
$$

We, therefore, have an alternative method to compute the desired input u_k^d for a particular given desired output y_k^d; this approach apparently takes into account the noise in the system. We compare the two methods in the next section.

4 Connections Between Stable Inversion and Hou-Patton Approaches

Albeit the Hou-Patton approach is not equivalent to the stable inversion technique, we can compare them, for minimum phase systems, under conditions that are fairly typical.

1. Suppose that D is square and nonsingular. Clearly, such a system has a well-defined relative degree $r = 0$, and the equivalent system in the Hou-Patton approach reduces to the following:

$$
x_{k+1} = \left(A - BD^{-1}C\right)x_k + BD^{-1}y_k + \left(G - BD^{-1}H\right)w_k
$$

(Note that $\bar{y}_k \equiv 0$.) However, the decoupled input observer, in this case is as follows:

$$
\hat{x}_{k+1|k+1} = \left(A - BD^{-1}C\right)\hat{x}_{k|k} + BD^{-1}y_k
$$

Evidently, for such systems, the observer is in an open loop as there is no correction for the noise. Observe that if the initial conditions are the same, the state of the inverse system (refer eqn. (6)) and the estimated state are the same. Therefore, the performance of both the stable inverse and Hou-Patton approaches are comparable. However, we emphasise that whilst the stable inverse can be computed for non-minimum phase systems, the Hou-Patton approach is applicable only to minimum phase systems.

2. Similarly, for a square system with $D = 0$, we have $\bar{A} = A - B(CB)^{-1}CA$. Moreover, $\text{rank}\,CB = \text{rank}\,B$ if, and only if, the system has a well-defined relative degree $r = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$. Therefore, under this condition, $\bar{A}$ is similar to $\tilde{A}$ (refer eqn. (6)), and hence has m zero eigenvalues, and the remaining $n - m$ eigenvalues correspond to the zeros of the system. Thus, for SISO systems with $D = 0$, the estimator has only one state observable from the output; for MIMO systems, only m states are observable. Evidently, for the states corresponding to the unobservable eigenvalues, there is no correction for the noise through the innovation process. Therefore, the observer fails to overcome the effect of noise in these $n - m$ states.

We can therefore conclude that, for a large class of systems, the two techniques are similar in their noise handling capabilities. We illustrate this drawback with an example in the next section. The second major drawback for the Hou-Patton approach is that it is applicable only for minimum phase systems. In the next section, we propose a method that would overcome these drawbacks. At this juncture, we note that although Lemma 1 preclude the presence of zeros on the unit circle, in practice, however, the technique seems, by experience, applicable to systems with such zeros.

5   Stable Inversion for General Systems

In this section, we first present a method that would make the Hou-Patton technique applicable for non-minimum phase systems. We avoid the use of dichotomies, and hence the computation of the eigenvalues, by performing an inner-outer factorisation. We then present a different approach to the problem of computing the desired input which not only is applicable for non-minimum phase systems, but as well takes into account the noise.

5.1   Hou-Patton Approach for Non-minimum Phase Systems

Consider the following, possibly non-minimum phase, linear time invariant discrete time system:

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + Gw_k \\
y_k &= Cx_k + Du_k + Hw_k
\end{aligned}
$$

Obviously, in the z-domain, we have the transfer function

$$
P(z) = C\,(zI - A)^{-1} \begin{pmatrix} B & G \end{pmatrix} + \begin{pmatrix} D & H \end{pmatrix}
$$

Suppose $P(z) = P_i(z)P_o(z)$ constitute an inner-outer factorisation of the transfer function $P(z)$ where P_i and P_o respectively are the inner and outer factors. Thus,

$$
\begin{aligned}
y &= P_i P_o \begin{pmatrix} u \\ w \end{pmatrix} \\
\Longrightarrow P_i^{\sim} y &= P_o \begin{pmatrix} u \\ w \end{pmatrix}
\end{aligned}
$$

We first compute

$$\tilde{y} \triangleq P_i^{\sim} y$$

and then apply the Hou-Patton procedure to the following system:

$$\tilde{y} = P_o \left(\begin{array}{c} u \\ w \end{array} \right)$$

Evidently, the pre-actuation necessitated by the non-minimum phase zeros are incorporated in $\tilde{y}$. The length of this pre-actuation is related to the non-minimum phase zeros by the following result:

Lemma 3 *Suppose that the unstable zeros lie outside a circle of radius ρ, and that the support of y^d is contained in $[k_0, \infty)$ for some k_0. Then $\exists M > 0$ such that*

$$\left\| u_k^d \right\|_\infty \leq M \rho^{k-k_0} \quad \forall\, k < k_0$$

∎

The proof of the lemma is similar to the corresponding result for continuous time systems (Devasia, 1997).

We now remark on the choice of the method to compute the inner-outer factorisation. Presently, there are two different ways to obtain this factorisation. In one method, a right invertible outer factor is first factored followed by the corresponding inner factor. This method requires the solution of a Riccati equation, and simple and explicit formulae can be provided in terms of the solution of this Riccati equation. However, as remarked in an earlier section, the square and invertible feed-forward term corresponding to the outer factor leads to a Hou-Patton observer without the innovation process. Moreover, such a factorisation technique usually has problems when there are zeros on the unit circle. A second method to compute an inner-outer factorisation is to first extract an elementary square inner factor. In a recent paper, Varga (1998) provides such a procedure which relies on the dislocation of unstable zeros in an efficient recursive manner, and then reflect them into the stable region in the complex plane. Any zeros on the unit circle are included in the outer factor. This latter method is more amenable for the generalised the Hou-Patton approach.

Example: We will now illustrate this technique with the following simplified model of a lunar roving vehicle (Dorf and Bishop, 1995):

$$G_c(s) = \frac{2e^{-0.1s}}{0.2s + 1} \tag{13}$$

The objective is to make the steering angle track a desired trajectory. After approximating the delay by a second order Pade approximation, and discretising the system, we obtain the following model:

$$G(z) = \frac{0.0359z^2 - 0.0833z + 0.0487}{z^3 - 2.6903z^2 + 2.4134z - 0.7225}$$

The two zeros $1.1595 \pm 0.1005i$ are located outside the unit circle. We perform an inner-outer factorisation, and apply the general approach to this example. The results are shown in Fig. 1. The desired steering angle is shown in Fig. 1(a), and the signal corrupted by white noise is shown in Fig. 1(b). As expected, the procedure faithfully reproduces the signal, as shown in Fig. 1(c); the figure shows the superposition of both the desired and actual trajectories. However, as mentioned earlier, the technique cannot be used for time waveform replication of signals that are measurements of field tests, as the noise content is also faithfully reproduced.

5.2 The Stable Dynamic Model Inversion (SDI) Procedure

We now describe a new approach that overcomes the main drawback of the generalised Hou-Patton approach in terms of its noise handling capability. The Stable Dynamic model Inversion technique is based on augmenting the state space model of the given plant by a reasonable model for the input sequence, and then designing a Kalman filter to provide an estimate from the measurements.

Consider the following state space realization of a linear time invariant system:

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + Gw_k \\
y_k &= Cx_k + Du_k \\
z_k &= y_k + v_k
\end{aligned}
\tag{14}
$$

We assume that $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$, w_k and v_k are white noise processes with covariances Q_w and R_v respectively, uncorrelated with each other, and with the initial condition x_0. From intuitive and physical reasoning, it seems realistic to model the input signal u_k as follows

$$
u_{k+1} = u_k + \eta_k
\tag{15}
$$

for some η_k. For simplicity, we assume that η_k is white noise with covariance Q_η, and uncorrelated with w_k and v_k. The resulting augmented system is as follows:

$$
\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + \begin{pmatrix} G & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} w_k \\ v_k \\ \eta_k \end{pmatrix}
\tag{16}
$$

$$
z_k = \begin{pmatrix} C & D \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + \begin{pmatrix} 0 & I & 0 \end{pmatrix} \begin{pmatrix} w_k \\ v_k \\ \eta_k \end{pmatrix}
\tag{17}
$$

In compact form, we have,

$$
\begin{aligned}
x_{a,k+1} &= A_a x_{a,k} + G_a w_{a,k} \\
z_k &= C_a x_{a,k} + H_a w_{a,k}
\end{aligned}
$$

where the definition of each of the individual matrices and vectors is rather obvious. We note that the system matrix A_a is different from that of $\tilde{A}$ of the stable inverse approach, and $\bar{A}$ of the Hou-Patton approach. Therefore, it is expected that the properties are different from that of the earlier techniques. Indeed, the following result shows that if the original system is observable, then the augmented system is observable for almost all points in the complex plane; the proof of the result readily follows from the definitions of observability and the zeros of a system.

Lemma 4 *Suppose the pair (C, A) is observable. The pair (C_a, A_a) is observable if, and only if, $z = 1$ is not a zero of the system (A, B, C, D).*

∎

Therefore, we can set up a Kalman filter for the augmented system whenever the model has no zeros at $z = 1$. Evidently, this method is easily and directly applicable for a larger class of systems as compared to the earlier approaches. Besides, under the conditions of Lemma 4, the augmented system is completely observable. Therefore, the Kalman filter is able to overcome the effect of noise in each and every state.

For systems that do have a zero at unity, we can easily dislocate these zeros and factor them into an invertible transfer function by using the numerically efficient technique developed by Van Dooren (1990). (We note that for numerical efficiency, it is recommended that any zeros located within a circle of radius ϵ centred at unity be dislocated.) This generalisation is similar to the one considered earlier for the Hou-Patton approach via an inner-outer factorisation. The details of this approach will appear elsewhere (George *et al.*, 1999). Further, it is well-known that the Kalman filter can be viewed from an $\mathcal{H}_2$ perspective. This is summarised by Westwick *et al.* (1999), and is not included here for brevity.

Example: We now apply the SDI approach to the example lunar-vehicle considered earlier in this section. Evidently, since the discretised model does not have any zeros at unity, we can apply the method directly to this example, and the results are as shown in Fig. 2. Fig. 2(c) shows the superposition of both desired and actual trajectories. Minor deviations from the desired trajectory are observed in Fig. 2(d). It is clear from the response plots that this technique performs rather reasonably even in the presence of noise in the desired output signal: the standard deviation of the tracking error is an order of magnitude smaller than that of the noise level in z_k.

6 Application to Helicopter Hover Control

In this section we apply to the hover control of a Bell 205 helicopter the generalised Hou-Patton and the SDI approaches for output tracking. The example considered in this paper is a case wherein the dynamics was trimmed at a nominal 5 degrees pitch attitude with a midrange weight and a mid-position centre of gravity, and operating at near sea level (see (Devasia, 1997), and the references cited therein).

We discretise the linearised model given by Devasia (1997) to obtain

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned}$$

This is an eighth order system with four inputs. The states of the system are the forward, vertical and lateral velocities, pitch and roll attitudes, and the roll, pitch and yaw rates. The inputs to the system are collective, longitudinal and lateral cyclic, and the tail rotor collective. The system has zeros at $0.9997 \pm 0.0114i$, and $1.0000 \pm 0.0215i$, which are clearly very close to $z = 1$.

The objective is to control the forward, vertical and lateral velocities, and the yaw rate of the helicopter. The forward velocity and the yaw rate are to be maintained at zero, and the desired profiles of vertical and lateral velocities are respectively shown in Fig. 3(a) and Fig. 3(b). These profiles are similar to the ones given in (Devasia, 1997). We apply both the generalised Hou-Patton and SDI approaches to this model in order to track the desired profiles. The results are as shown in Fig. 3. In each of the response plots, we superimpose the desired and actual trajectories. Evidently, both techniques perform reasonably well in the absence of noise, and are comparable to the results in (Devasia, 1997).

7 Conclusions

The stable inversion technique is applicable to a large class of systems. However, it consists of numerical computation of a dichotomy in the case of non-minimum phase systems, and it does not take into account noise in the system. A numerically more efficient procedure is the

combination of the inner-outer factorisation with the Hou-Patton approach. However, it is evident that this generalisation is still unable to reasonably handle noise. The Stable Dynamic model Inversion technique overcomes these main drawbacks; it is both numerically efficient and tries to overcome the system noise, thereby making it rather suitable for time waveform replication of target time histories. The extension of SDI for the task of output tracking of nonlinear time-varying linear systems is currently being investigated. Further, methods to make SDI robust towards modelling uncertainties and parameter variations are also being considered.

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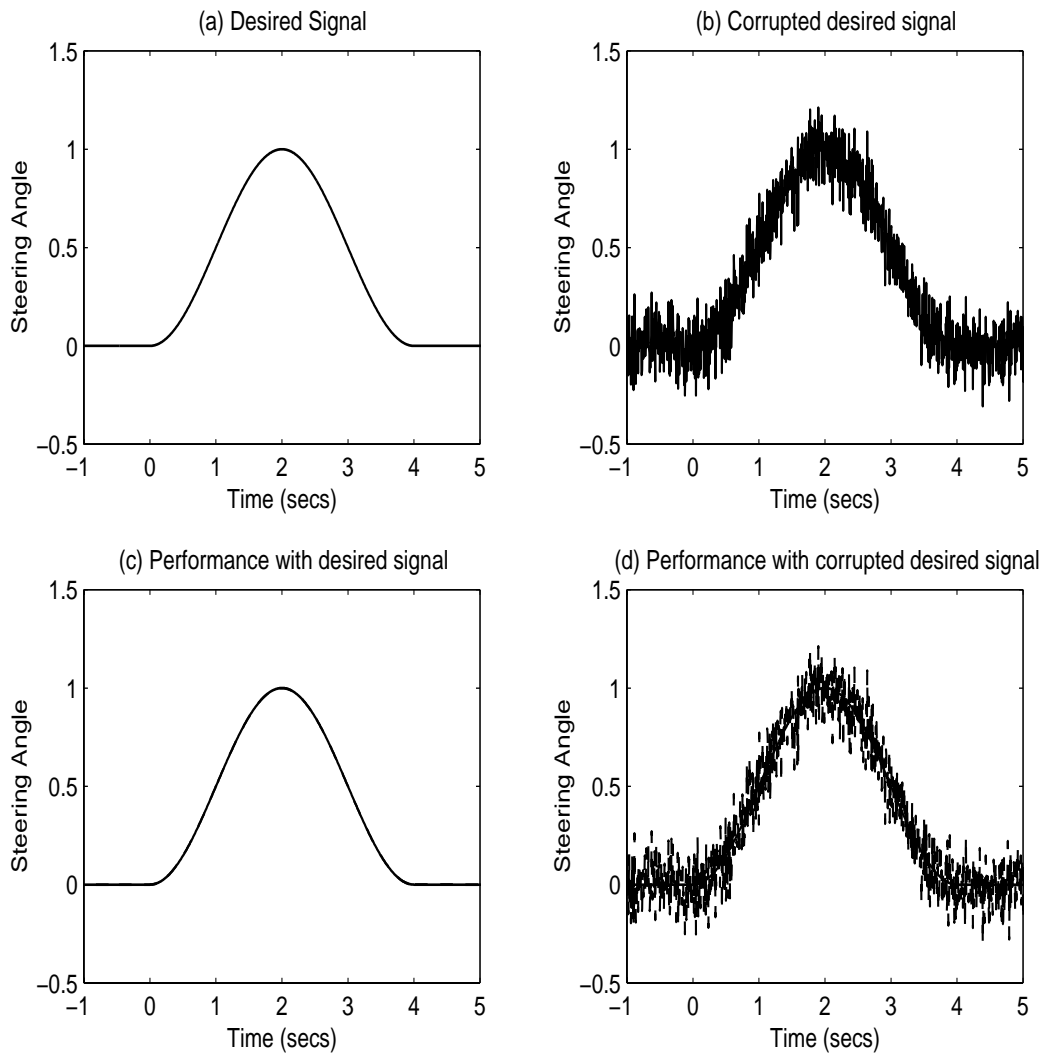
Figure 1: Time Waveform Replication: Application of the generalised Hou-Patton approach to a lunar roving vehicle; – – output of system; — desired signal.
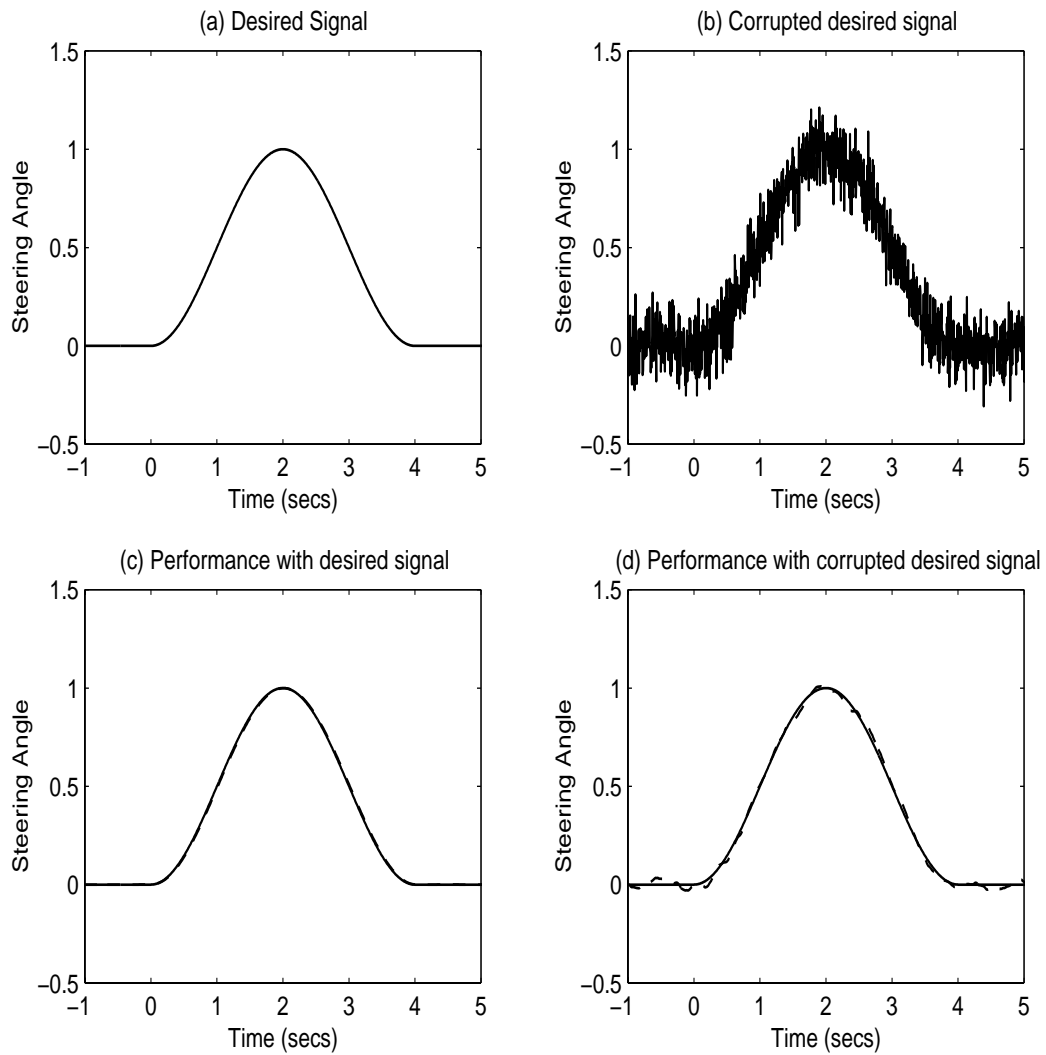
Figure 2: Time Waveform Replication: Application of the Stable Dynamic model Inversion approach to a lunar roving vehicle; – – output of system; — desired signal.
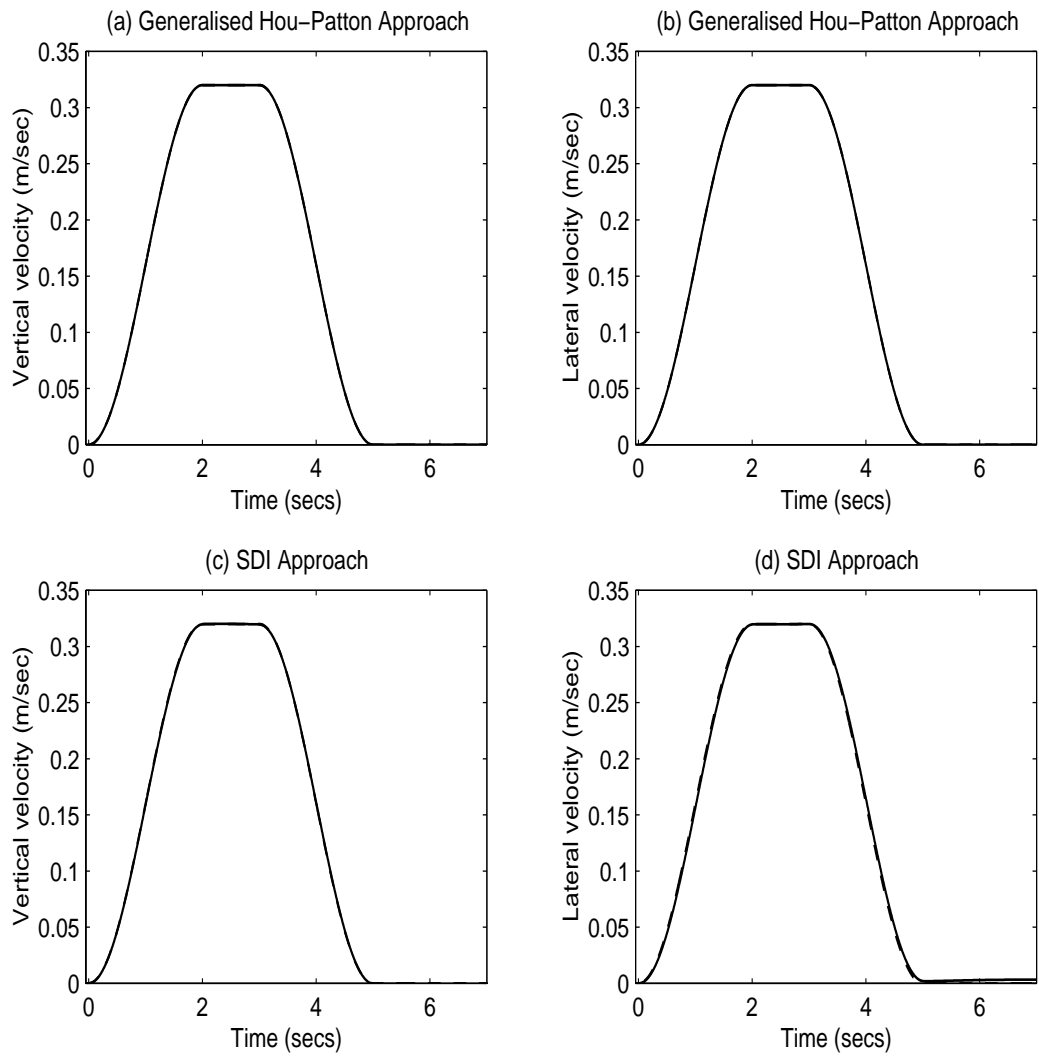
Figure 3: Application to helicopter example; – – output of system; — desired signal.