Compositio Mathematica **130:** 345–364, 2002.

Stable Maps and Branch Divisors

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(Received: 10 July 2000)

Abstract. We construct a natural branch divisor for equidimensional projective morphisms where the domain has lci singularities and the target is nonsingular. The method involves generalizing a divisor construction of Mumford from sheaves to complexes. The construction is valid in flat families. The generalized branch divisor of a stable map to a nonsingular curve X yields a canonical morphism from the space of stable maps to a symmetric product of X. This branch morphism (together with virtual localization) is used to compute the Hurwitz numbers of covers of the projective line for all genera and degrees in terms of Hodge integrals.

Mathematics Subject Classifications (2000). 14H10, 14C20.

Key words. Branch divisor, Hurwitz number.

0. Introduction

Let $f: X \to Y$ be a surjection of nonsingular projective varieties of the same dimension. The ramification divisor R of f on X is defined by requiring the sequence

$$0 \to f^*\omega_Y \to \omega_X \to \omega_X\Big|_R \to 0 \tag{1}$$

to be exact. The branch divisor $br(f)$ on Y is then defined by pushing forward: $br(f) = f_*(R)$. The support of $br(f)$ is the locus of points $y \in Y$ such that f is not étale in any neighborhood of $f^{-1}(y)$.

If $f: C \to D$ is a degree d map of nonsingular curves, then $br(f)$ is a divisor on D of degree

$$r = 2g(C) - 2 - d(2g(D) - 2)$$

by the Riemann–Hurwitz formula. Let $M_g(D, d)$ the moduli stack of degree d maps from nonsingular genus $g = g(C)$ curves to D. The branch divisor yields a morphism of Deligne–Mumford stacks

$$\gamma: M_g(D, d) \to \mathrm{Sym}^r(D). \tag{2}$$

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For moduli points $[f\colon C \to D] \in M_g(D, d)$, $\gamma([f] = br(f)$. A natural extension of γ to the compactification by stable maps

$$M_g(D, d) \subset \overline{M}_g(D, d)$$

is the main result of the paper.

THEOREM 1. *The branch divisor by* $br(F)$ *induces a morphism:*

$$\gamma\colon \overline{M}_g(D, d) \to \mathrm{Sym}^r(D).$$

Consider first the following situation. Let $f\colon X \to Y$ be a projective morphism of S-schemes where:

(i) X is a local complete intersection over S of relative dimension n.
(ii) Y is smooth over S of relative dimension n.
(iii) All geometric fibers of X over S are reduced.

Under these conditions, a functorial relative Cartier divisor $br(f)$ on Y over S is constructed in Section 2. The divisor $br(f)$ is supported on the locus of points $y \in Y$ such that f is not étale in any neighborhood of $f^{-1}(y)$. In this generality, $br(f)$ need not be an effective Cartier divisor. However, $br(f)$ is invariant under base change and coincides with the branch divisor defined by (1) when $X \to S$ is smooth and every component of X dominates one of Y.

The branch divisor $br(f)$ is constructed by studying the complex

$$Rf_*[f^*\omega_{Y/S} \to \omega_{X/S}], \tag{3}$$

well-defined up to isomorphism in $D^-_{coh}(Y)$. By generalizing to complexes a classical construction of Mumford for sheaves ([Mu], §5.3), we can associate to (3) a Cartier divisor on Y. Section 1 contains the required generalization of Mumford's results.

In Section 3, we apply our branch divisor construction to the universal family:

$$F\colon \mathcal{C} \to D \times \overline{M}_g(D, d)$$

over the moduli stack of stable maps $\overline{M}_g(D, d)$ for $d > 0$. Certainly this universal family (as a Deligne–Mumford stack) satisfies conditions (i)–(iii). It is shown $br(F)$ in this case is an *effective* relative Cartier divisor on $D \times \overline{M}_g(D, d)$ of relative degree r. The branch divisor $br(F)$ then yields a canonical morphism

$$\gamma\colon \overline{M}_g(D, d) \to \mathrm{Sym}^r(D) \tag{4}$$

extending (2).

The morphism γ has an appealing point theoretic description on the boundary of $\overline{M}_g(D, d)$. Let $[f\colon C \to D]$ be a moduli point where C is a singular curve. Let $N \subset C$ be the cycle of nodes of C. Let $\nu\colon \tilde{C} \to C$ be the normalization of C. Let $A_1, \dots, A_a$, be the components of $\tilde{C}$ which dominate D, and let $\{a_i\colon A_i \to D\}$ denote the natural maps. As a_i is a surjective map between nonsingular curves, the branch divisor $br(a_i)$ is defined by (1). Let $B_1, \ldots, B_b$ be the components of $\tilde{C}$ contracted over D and let $f(B_j) = p_j \in D$. We prove the formula

$$\gamma([f]) = br(f) = \sum_i br(a_i) + \sum_j (2g(B_j) - 2)[p_j] + 2f_*(N). \tag{5}$$

It is easy to see that formula (5) associates an effective divisor of degree r on D to every moduli point $[f]$. However, the construction of γ as a scheme-theoretic morphism requires the relative branch divisor results over arbitrary reducible, nonreduced bases S.

In Section 4, the morphism γ is used to study the classical simple Hurwitz numbers $H_{g,d}$ via Gromov–Witten theory. $H_{g,d}$ is the number of nonsingular, genus g curves expressible as d-sheeted covers of $\mathbf{P}^1$ with a *fixed* simple branch divisor. The Hurwitz numbers were first computed in [Hu] by combinatorical techniques. A simple analysis of the moduli space of stable maps to $\mathbf{P}^1$ shows:

$$H_{g,d} = \int_{[\overline{M}_g(\mathbf{P}^1,d)]^{vir}} \gamma^*(\xi^{2g-2+2d}), \tag{6}$$

where ξ is the hyperplane class on $\mathrm{Sym}^{2g-2+2d}(\mathbf{P}^1) = \mathbf{P}^{2g-2+2d}$. It is then possible to directly evaluate the integral (6) using the virtual localization formula [GrP] to obtain a Hodge integral expression for the Hurwitz numbers:

THEOREM 2.

$$H_{d,g} = \frac{(2g-2+2d)!}{d!} \int_{\overline{M}_{d,g}} \frac{1 - \lambda_1 + \lambda_2 - \lambda_3 + \cdots + (-1)^g \lambda_g}{\prod_{i=1}^{d} (1 - \psi_i)} \tag{7}$$

for $(g, d) \neq (0, 1), (0, 2)$.

The integral on the right is taken over the moduli space of pointed stable curves $\overline{M}_{g,d}$. The classes ψ_i and λ_j are the cotangent line classes and the Chern classes of the Hodge bundle respectively. The values $H_{0,1} = 1$ and $H_{0,2} = 1/2$ are degenerate cases from the point of view of the right side of (7).

Let $H_{g,(\alpha_1,\ldots,\alpha_l)}$ denote the Hurwitz numbers of degree $d = \sum_i \alpha_i$ covers with branching profile $(\alpha_1, \ldots, \alpha_l)$ over $\infty \in \mathbf{P}^1$ (and simple branching elsewhere). A generalization of formula (7) relating $H_{g,(\alpha_1,\ldots,\alpha_l)}$ to Hodge integrals was announced independently by Ekedahl, Lando, Shapiro, and Vainshtein [ELSV]:

$$\begin{aligned} H_{g,(\alpha_1,\ldots,\alpha_l)} = {} & \frac{(2g-2+d+l)!}{|\mathrm{Aut}(\alpha_1,\ldots,\alpha_l)|} \prod_{i=1}^{l} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \\ & \times \int_{\overline{M}_{g,d}} \frac{1 - \lambda_1 + \lambda_2 - \lambda_3 + \cdots + (-1)^g \lambda_g}{\prod_{i=1}^{l} (1 - \alpha_i \psi_i)}. \end{aligned} \tag{8}$$

The Hodge integral expression on the right side of (8) is directly identified as a vertex term in the virtual localization formula. A proof of (8) using the branch morphism

https://doi.org/10.1023/A:1014347115536 Published online by Cambridge University Press

γ and localization was later found in [GV]. The argument of [GV] requires a significantly more subtle localization analysis and generalizes the proof of Theorem 2 given here.

Finally, we note the branch morphism γ plays an essential role in the Hodge integral computations in [FaP2]. The class $\gamma^*(\xi)$ is a new torus equivariant class on $\overline{M}_g(\mathbf{P}^1, d)$. As explained in [FaP1], such equivariant classes yield relations among Hodge integrals. In [FaP2], the 1-point Hodge integral series in the tautological ring $R^*(M_g)$ is explicitly calculated settling several previous conjectures of Faber.

1. Perfect Torsion Complexes

1.1. CARTIER DIVISORS

The base field $\mathbb{C}$ of complex numbers will be fixed for the entire paper. However, all the results of Sections 1.1–2.2 are valid over any algebraically closed base field. The characteristic 0 condition is required for generic smoothness in the construction of the branch divisor.

Let A be an algebra of finite type over C. Let $S \subset A$ be the multiplicative system of elements which are not zero divisors. Recall, the set of zero divisors of A equals the union of all associated primes of A ([Ma], p. 50). A prime ideal $\mathfrak{p} \subset A$ is *depth* 0 if all non-units of $A_\mathfrak{p}$ are zero divisors. The associated primes of A are exactly the depth 0 primes ([Ma], p. 102). Let $K(A) = S^{-1}(A)$ be the total quotient ring of A. It is easy to check for $f \in A$, $K(A_f) = K(A)_f$.

Let X be a scheme (always taken here to be quasi-projective over $\mathbb{C}$). We distinguish the points of X (integral subschemes) from the geometric points of X ($Spec(\mathbb{C})$ subschemes). Let $\mathcal{K}$ be the sheaf of rings on X defined by associating $K(A_i)$ to the basis of all affine open sets $Spec(A_i)$ of the Zariski topology of X. The equality

$$\Gamma(Spec(A_i), \mathcal{K}) = K(A_i)$$

follows from the property $K(A_f) = K(A)_f$. Let $\mathcal{K}^*$ denote the sheaf of invertible elements of $\mathcal{K}$. A Cartier divisor is an element of $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$. This discussion follows Hartshorne ([Ha1], §II.6).

A Cartier divisor is defined by the data $\{(f_i, W_i)\}$ where the open sets $W_i = Spec(R_i)$ cover X and

$$f_i \in K(R_i)^*, \qquad f_i/f_j \in \Gamma(W_i \cap W_j, \mathcal{O}^*).$$

A Cartier divisor D is *effective* if there exist defining data as above satisfying $f_i \in R_i \subset K(R_i)$. An effective Cartier divisor naturally defines a locally free ideal sheaf of $\mathcal{O}_X$.

LEMMA 1. *Let $U \subset X$ be an open set containing all depth 0 points of X. Let $f \in \Gamma(U, \mathcal{O}_U^*)$. Then, f defines a canonical element of $\Gamma(X, \mathcal{K}^*)$.*

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Proof. Let $Z = U^c \subset X$. Let $\{W_i = Spec(R_i)\}$ be an open affine cover of X. Let $U_i = U \cap W_i$, $Z_i = Z \cap W_i$, and $f_i = f|_{U_i}$. Let $I \subset R_i$ be the radical ideal determined by closed set Z_i. Since Z_i contains no depth 0 points, I must contain a element x of R_i which is not a zero divisor.

Since $Spec((R_i)_x) \subset U_i$, we see f_i is naturally an element of $(R_i)_x^*$. As $K(R_i)$ is obtained from $(R_i)_x$, by further localization, f_i yields a canonical element of $K(R_i)^*$. These local sections over W_i patch to yield a canonical element of $\Gamma(X, \mathcal{K}^*)$. □

1.2. THE DIVISOR CONSTRUCTION (LOCAL)

We recall here a construction of Mumford ([Mu], §5.3). For our general branch divisor construction, we must extend these results from sheaves to complexes.

Let $D^-_{coh}(X)$ be the derived category of bounded (from above) complexes of quasi-coherent $\mathcal{O}_X$-modules with coherent cohomology on a scheme X. We will identify a sheaf with a complex in degree zero; we will identify a morphism with a complex in degrees $[-1, 0]$. By convention, *free* and *locally free* sheaves will have finite rank. An object $E^\bullet$ of $D^-_{coh}(X)$ is *perfect* if it is locally isomorphic to a finite complex of locally free sheaves. $E^\bullet$ is *torsion* if for all $i \in \mathbb{Z}$ the support of $H^i(E^\bullet)$ does not contain any point of depth zero of X.

Let $E^\bullet = [E^a \to E^{a+1} \to \cdots \to E^b]$ be a finite complex of free sheaves on X, and let $rank(E^i) = r_i$. Let

$$\Lambda(E^\bullet) = \bigotimes_{i=a}^{b} (\Lambda^{r_i} E^i)^{(-1)^i}.$$

Following [Mu], a *choice* of an explicit isomorphism $E^i = \mathcal{O}_X^{r_i}$ for each i yields an isomorphism $\psi\colon \Lambda(E^\bullet) \to \mathcal{O}_X$. Because of the choice of the trivializations of E^i, ψ is determined only up to multiplication by a section of $\mathcal{O}_X^*$. However, if $E^\bullet$ is exact, there is a canonical isomorphism $\kappa\colon \Lambda(E^\bullet) \to \mathcal{O}_X$. These isomorphisms ψ and κ will together determine a Cartier divisor in the torsion case.

Let $E^\bullet = [E^a \to E^{a+1} \to \cdots \to E^b]$ be a finite torsion complex of free sheaves on X. We define the associated Cartier divisor $div(E^\bullet)$ following [Mu]. Let U be the complement of the union of the supports of $H^i(E^\bullet)$. On U there is a canonical isomorphism $\kappa_U\colon \Lambda(E^\bullet)|_U \to \mathcal{O}_U$.

Let $\psi\colon \Lambda(E^\bullet) \to \mathcal{O}_X$ be an isomorphism defined by trivializations as above. Then, $\psi_U \circ (\kappa|_U)^{-1}$ is a section of $\mathcal{O}_U^*$. As U contains all points of depth zero of X, we obtain a unique section f of $\mathcal{K}_X^*$ by Lemma 1. Different trivializations of E^i over X change the isomorphism ψ by multiplication by an element of $\mathcal{O}_X^*$. Hence, f yields a well-defined section of $\mathcal{K}_X^*/\mathcal{O}_X^*$. Let $div(E^\bullet)$ denote this canonically associated Cartier divisor on X. Note if $E^\bullet$ is an exact finite complex of free sheaves, then $div(E^\bullet)$ is zero.

1.3. THE DIVISOR CONSTRUCTION (GLOBAL)

Let ϕ: $E^\bullet \to F^\bullet$ be a chain map of complexes. The mapping cone of ϕ is the complex $M(\phi)^\bullet$ with sheaves $M(\phi)^i = E^{i+1} \oplus F^i$ and differentials $M(\phi)^{i-1} \to M(\phi)^i$ determined by $(e, f) \mapsto (\mathrm{d}e, \mathrm{d}f + (-1)^i \phi(e))$ (where d denotes differentials on $E^\bullet$ and $F^\bullet$). Note there are natural morphisms of complexes

$$E^\bullet \to F^\bullet \to M(\phi)^\bullet \to E^\bullet[1] \tag{9}$$

where $F^i M(\phi)^i$ is given by $f \mapsto (0, f)$ and $M(\phi)^i \to E^{i+1}$ is defined by $(e, f) \mapsto e$. Any sequence of morphisms $E \to F \to G \to E[1]$ in $D^-_{coh}(X)$ which is isomorphic to (9) in $D^-_{coh}(X)$ is called a *distinguished triangle*.

The morphisms (9) induce a long exact sequence of cohomology

$$\cdots \to H^i(E) \to H^i(F) \to H^i(M(\phi))H^{i+1}(E) \to \cdots$$

In particular, if ϕ is a quasi-isomorphism, then $M(\phi)^\bullet$ is exact.

LEMMA 2. *Let $E^\bullet$ and $F^\bullet$ be finite torsion complexes of free sheaves, and let ϕ: $E^\bullet \to F^\bullet$ be a chain map. Then, the mapping cone $G^\bullet$ of ϕ is also a finite torsion complex of free sheaves, and*

$$div(F^\bullet) = div(E^\bullet) + div(G^\bullet).$$

Proof. $G^\bullet$ is certainly a finite complex of free sheaves. Let $Z \subset X$ be the union of the supports of the cohomology sheaves of $E^\bullet$ and $F^\bullet$. As the latter supports do not contains points of depth zero, neither does Z. Let $U = Z^c$. Both $E^\bullet$ and $F^\bullet$ are exact on U, so $\phi|_U$ is a quasi-isomorphism and $G^\bullet|_U$ is also exact. Hence, $G^\bullet$ is torsion.

There is a canonical isomorphism of $\Lambda(F^\bullet)$ with $\Lambda(E^\bullet) \otimes \Lambda(G^\bullet)$, which proves the lemma. $\square$

COROLLARY 1. *Let $E_1^\bullet$ and $E_2^\bullet$ be finite torsion complexes of free sheaves. If they are isomorphic in $D^-_{coh}(X)$, then the induced Cartier divisors $div(E_1^\bullet)$ and $div(E_2^\bullet)$ are equal.*

Proof. If $E_1^\bullet$ and $E_2^\bullet$ are isomorphic in $D^-_{coh}(X)$, then there exists an object $L^\bullet \in D^-_{coh}(X)$ and chain maps $L^\bullet \to E_i^\bullet$ which are quasi-isomorphisms. We may prove the Corollary locally on X. Locally, we can find a free complex $F^\bullet$ with a chain map $F^\bullet \to L^\bullet$ which is a quasi-isomorphism ([Ha1], Lemma 12.3). As $E_i^\bullet$ are finite and free, $F^\bullet$ may be cut-off from below to yield a finite and free complex with quasi-isomorphisms: $F^\bullet_{cut} \to E_i^\bullet$.

It is therefore enough to prove the Corollary in case there exists a quasi-isomorphism ϕ: $E_1^\bullet \to E_2^\bullet$, but then it follows from Lemma 2. $\square$

Let $E^\bullet$ be a perfect torsion complex on X. As $E^\bullet$ is locally a finite torsion complex of free sheaves, Cartier divisors may be associated locally to $E^\bullet$ via local

https://doi.org/10.1023/A:1014347115536 Published online by Cambridge University Press

trivializations and the construction of Section 1.2. By Corollary 1, these locally associated divisors agree and define a canonical Cartier divisor $div(E^\bullet)$ on X.

PROPOSITION 1. *Let $E^\bullet$ be a perfect torsion complex on X. Then $div(E^\bullet)$ satisfies the following properties:*

(i) *$div(E^\bullet)$ depends only on the isomorphism class of $E^\bullet$ in $D^-_{coh}(X)$,*
(ii) *If F is a coherent torsion sheaf on X admitting locally a finite free resolution, then $div(F)$ is the divisor constructed in [Mu]. Moreover, $div(F)$ is an effective Cartier divisor.*
(iii) *If D is an effective Cartier divisor in X, $div(\mathcal{O}_D) = D$.*
(iv) *The divisor is additive for distinguished triangles.*
(v) *If $f\colon X' \to X$ is a base change, such that $f^*E^\bullet$ is torsion, then $f^*(div(E^\bullet))$ is a Cartier divisor. Moreover, in this case*

$$div(f^*(E^\bullet)) = f^*(div(E^\bullet)).$$

(vi) *$div(E^\bullet[-1]) = -div(E^\bullet)$.*
(vii) *If L is a line bundle on X, $div(E^\bullet) = div(E^\bullet \otimes L)$.*

Proof. For the most part, these properties are simple consequences of the construction. Property (i) follows immediately from local considerations and Corollary 1. The equivalence with Mumford's construction (ii) is true by definition. The effectivity of $div(F)$ is a subtle issue proven in [Mu]. An easy computation using the isomorphism between $[\mathcal{O}_D]$ and $[\mathcal{O}_X(-D) \to \mathcal{O}_X]$ in $D^-_{coh}(X)$ proves (iii). Lemma (2) and local analysis together imply (iv). Property (v) may be checked locally on X and Y where the divisor construction is seen to be compatible with the definition of the pull-back of Cartier divisors. Properties (vi) and (vii) are trivial consequences of the definitions. Property (vi) shows $div(E^\bullet)$ is not an effective Cartier divisor for all perfect torsion complexes. □

The following example will be required. Let X be a projective scheme, and let Y be a nonsingular curve. Let $f\colon X \to Y$ be a constant morphism with image $y \in Y$.

LEMMA 3. *For any coherent sheaf F on X, $Rf_*(F)$ is a perfect torsion complex in $D^-_{coh}(Y)$, and $div(Rf_*(F)) = \chi(F)[y]$.*

Proof. $Rf_*(F)$ defines a complex in $D^-_{coh}(Y)$ with coherent cohomology, nonzero in finitely many degrees. By the nonsingularity of Y, $Rf_*(F)$ is perfect. That $Rf_*(F)$ is torsion is clear. As $Rf_*(F)$ is exact outside of y, $div(Rf_*(F))$ is a multiple of the point $[y]$. The Lemma then follows from a local calculation. □

1.4. TORSION CRITERION

Let $q\colon Y \to S$ be a smooth morphism with irreducible fibers. Let $Ass(Y)$ and $Ass(S)$ be the sets of depth 0 points of the schemes Y and S respectively. A point $\mathfrak{p}$ of

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S corresponds to an integral subscheme $V_{\mathfrak{p}} \subset S$. Since q is smooth with irreducible fibers, $q^{-1}(V_{\mathfrak{p}})$ is an integral subscheme of Y determining a point $\mathfrak{q}$ of Y. Let $\imath(\mathfrak{p}) = \mathfrak{q}$.

LEMMA 4. $\imath(Ass(S)) = Ass(Y)$.

Proof. The Lemma may be checked locally on Y and S, so we may take $Y = Spec(B)$ and $S = Spec(A)$. Since q is smooth, q is flat. If M is a Noetherian R-module, let $Ass_R(M)$ denote the set of primes of R associated to M. An algebraic result from Bourbaki is now required (also [Ma], Theorem 12):

$$Ass_B(B) = \bigcup_{\mathfrak{p} \in Ass_A(A)} Ass_B(B/\mathfrak{p}B). \tag{10}$$

As discussed above, $\mathfrak{p}B \subset B$ is a prime ideal. Hence $Ass_B(B/\mathfrak{p}B) = \{\mathfrak{p}B\}$. Moreover, $\imath(\mathfrak{p}) = \mathfrak{p}B$ by definition. □

Let $E^\bullet$ be a perfect object of $D^-_{coh}(Y)$. We will require the following criterion for torsion.

LEMMA 5. *Let q: $Y \to S$ be a smooth morphism with irreducible fibers. If for every geometric point $s \in S$, the complex $i_s^*(E^\bullet)$ is torsion on Y_s (where i_s: $Y_s \to Y$ is the inclusion), then $E^\bullet$ is torsion on Y.*

Proof. We again may take $Y = Spec(B)$ and $S = Spec(A)$. Let $\mathfrak{q} = \imath(\mathfrak{p})$ be a depth 0 point of Y. By Lemma 4, all depth 0 points of Y may be so expressed. Let $y \in V_{\mathfrak{q}}$, be a geometric point of Y with $s = q(y)$ satisfying: $i_s^*(E^\bullet)$ has cohomology supported away from y in Y_s. Such a y can be found since $V_{\mathfrak{q}}$, contains fibers of q. As $E^\bullet$ is perfect on Y, we can take a finite locally free representative

$$E^\bullet = [E^a \to E^{a+1} \to \cdots \to E^b]$$

locally at $y \in Y$. Since the fiber sequence

$$0 \to E^a_y \to E^{a+1}_y \to \cdots \to E^b_y \to 0$$

is exact by the torsion condition on $i_s^*(E^\bullet)$, $E^\bullet$ is exact in a Zariski neighborhood of y in Y. In particular, the point y does not lie in the support of the cohomology of $E^\bullet$ on Y. Since y is in the closure of the point $\mathfrak{q}$, we see $\mathfrak{q}$ does not lie in the cohomology support. □

We first note $i_s^*(E^\bullet)$ is the pull-back in the derived category. For a complex of free objects (or, more generally a complex of S-flat objects), this pull-back is determined by the simple pull-back of sheaves. Second, we note the irreducibility hypothesis on the fibers of q: $Y \to S$ can be easily removed in Lemma 5 by generalizing Lemma 4 slightly. We leave the details to the reader.

2. Branch Divisors

2.1. NOTATION

Let X, Y, and S be schemes. Let $p\colon X \to S$, $q\colon Y \to S$ be morphisms satisfying:

(i) X is a local complete intersection over S of relative dimension n.
(ii) Y is smooth over S of relative dimension n.
(iii) All geometric fibers of X over S are reduced.

Let $f\colon X \to Y$ be a projective morphism over S. This data will be fixed for the entire section. We will construct a relative Cartier divisor $br(f)$ on Y generalizing the standard branch divisor.

2.2. DIRECT IMAGES

We review here the natural map

$$Rf_*\colon D^-_{coh}(X) \to D^-_{coh}(Y)$$

obtained from direct images. Let $\mathcal{U}$ be an f-relative Cech cover of X (over every affine open in $Y, \mathcal{U}$ restricts to a usual Cech covering). For any quasi-coherent sheaf E on X, let $C^\bullet(\mathcal{U}, E)$ be the associated Cech complex of quasi-coherent sheaves on Y. Let $E^\bullet$ be an object of $D^-_{coh}(X)$. Then, $Rf_*(E^\bullet)$ is defined to be the simple complex on Y obtained from the double complex $C^p(\mathcal{U}, E^q)$. The complex $Rf_*(E^\bullet)$ is certainly bounded from above. Moreover, the cohomology of $Rf_*(E^\bullet)$ may be computed by a spectral sequence with E_2 term $R^pf_*(H^q(E^\bullet))$. Since, $R^pf_*(H^q(E^\bullet))$ is a grid of coherent sheaves on Y with only finitely many objects on each line of slope -1, the cohomology of $Rf_*(E^\bullet)$ is coherent. Hence, $Rf_*(E^\bullet)$ defines an element of $D^-_{coh}(Y)$. To show this construction is well-defined in the derived category, see [Ha2].

LEMMA 6. $Rf_*\colon D^-_{coh}(X) \to D^-_{coh}(Y)$ *carries perfect complexes to perfect complexes.*

Proof. The statement is local, so we assume Y is affine. Since f is projective and $E^\bullet$ is perfect, we can assume $E^\bullet$ is a finite complex of locally free sheaves globally on X. By Lemma 5.8 of [Mu], each of the Cech sheaves $C^p(\mathcal{U}, E^q)$ has finite *Tor*-dimension and hence admits a finite flat resolution by quasi-coherent sheaves on Y. Therefore $Rf_*(E^\bullet)$ is isomorphic in the derived category to a finite complex of quasi-coherent flat sheaves and hence is *Tor*-finite. As $Rf_*(E^\bullet)$ is bounded from above and has coherent cohomology, we can construct an isomorphic complex of locally free sheaves, indexed in $(-\infty, a]$ for some a. Then the *Tor*-finiteness implies the cut-off the complex below at a sufficient negative value will be locally free: the added sheaf will be flat and finitely generated, hence locally free. □

We now study the required base change properties. Let $\psi\colon \tilde{Z} \to Z$ be a projective morphism of schemes. We assume Z has enough locally frees (certainly Z

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quasi-projective over $\mathbb{C}$ suffices). The functor ψ^* induces a natural derived functor

$$L\psi^*\colon D^-_{coh}(Z) \to D^-_{coh}(\tilde{Z})$$

which sends perfect complexes to perfect complexes.

Let $\phi\colon \tilde{S} \to S$ be a base change of schemes and consider the Cartesian diagram:

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow[\tilde{f}]{} & \tilde{Y} & \longrightarrow & \tilde{S} \\ {\scriptstyle\phi_X}\downarrow & & {\scriptstyle\phi_Y}\downarrow & & {\scriptstyle\phi}\downarrow \\ X & \xrightarrow[f]{} & Y & \longrightarrow & S. \end{array}$$

In this case, $L\phi^*_X$ and $L\phi^*_Y$ may be defined on complexes of S-flat sheaves by ϕ^*_X and ϕ^*_Y respectively.

LEMMA 7. *For each complex $E^\bullet \in D^-_{coh}(X)$, there is a natural iso-morphism*

$$L\phi^*_Y(Rf_*(E^\bullet)) \to R\tilde{f}_*(L\phi^*_X(E^\bullet)). \tag{11}$$

of complexes in $D^-_{coh}(\tilde{Y})$.

Proof. As f is projective, $E^\bullet$ may be taken to be a complex of locally free sheaves (bounded from above). Let $\mathcal{U}$ be an f-relative Cech covering of X. Then the pull-back covering $\tilde{\mathcal{U}}$ is a $\tilde{f}$-relative Cech covering of $\tilde{X}$ (as may be checked locally on Y). As $E^\bullet$ is a locally free complex, $L\phi^*_X(E^\bullet)$ is just $\phi^*_X(E^\bullet)$ in $D^-_{coh}(\tilde{X})$. Hence $R\tilde{f}_*(L\phi^*_X(E^\bullet))$ is represented by the simple complex on Y associated to

$$C^p(\tilde{\mathcal{U}}, \phi^*_X E^q) \tag{12}$$

On the other hand, $Rf_*(E^\bullet)$ is the simple complex on Y associated to the double complex

$$C^p(\mathcal{U}, E^q). \tag{13}$$

The double complex (12) is easily seen to be the ϕ_Y pull-back of the complex (13). As a consequence, there is a natural map

$$L\phi^*_Y(C^p(\mathcal{U}, E^q)) \to C^p(\tilde{\mathcal{U}}, \phi^*_X E^q). \tag{14}$$

As X is flat over S, the complex (13) is also S-flat. Hence, the map (14) is a quasi-isomorphism. $\square$

2.3. THE BRANCH DIVISOR CONSTRUCTION

Let $\omega_{X/S}$ and $\omega_{Y/S}$ denote the relative dualizing sheaves of the structure maps p and q respectively. After constructing a natural perfect torsion complex

$$E^\bullet = [f^*\omega_{Y/S} \to \omega_{X/S}],$$

the branch divisor is defined by $br(f) = div(Rf_*(E^\bullet))$ on Y.

LEMMA 8. *There is a natural morphism* $f^*\omega_{Y/S} \to \omega_{X/S}$.

Proof. The canonical morphism $f^*\Omega_{Y/S} \to \Omega_{X/S}$ induces a morphism

$$f^*\omega_{Y/S} = \Lambda^n f^*\Omega_{Y/S} \to \Lambda^n\Omega_{X/S}. \tag{15}$$

Locally on X, we have an S-embedding $X \to M$, where M is smooth of relative dimension $n+r$ over S and X is a local complete intersection. Let $I = I_{X/M}$. There is an exact sequence

$$0 \to I/I^2 \to \Omega_{M/S} \otimes \mathcal{O}_X \to \Omega_{X/S} \to 0,$$

where I/I^2 and $\Omega_{M/S} \otimes \mathcal{O}_X$ are locally free sheaves on X of ranks r and $n+r$. This sequence yields a morphism

$$\Lambda^n\Omega_{X/S} \otimes \Lambda^r(I/I^2) \to \Lambda^{n+r}\Omega_{M/S} \otimes \mathcal{O}_X. \tag{16}$$

On the other hand, there is a canonical isomorphism

$$\omega_{X/S} \xrightarrow{\sim} Hom(\Lambda^r(I/I^2), \Lambda^{n+r}\Omega_{M/S} \otimes \mathcal{O}_X). \tag{17}$$

The morphisms (16) and (17) above induce a morphism

$$\Lambda^n\Omega_{X/S} \to \omega_{X/S}. \tag{18}$$

It is easily checked the locally defined morphism (18) is canonical and hence yields a global morphism on X. The Lemma is established by composing (15) with (18). □

LEMMA 9. *Let* $E^\bullet = [f^*\omega_{Y/S} \to \omega_{X/S}]$. *Then* $Rf_*(E^\bullet)$ *is a perfect torsion complex in* $D^-_{coh}(Y)$.

Proof. Since $E^\bullet$ is perfect, $Rf_*(E^\bullet)$ is perfect by Lemma 6. To prove $Rf_*(E^\bullet)$ is torsion on Y, we may use Lemmas 5 and 7 to reduce to the case in which S is a geometric point. Then, by property (iii), X is reduced. Let $\nu\colon \tilde{X} \to X$ be a resolution of singularities. Let Z_1 be the image in Y of the singular locus of X, and let $Z_2 \subset Y$ be the locus where $f \circ \nu$ is not étale. Let $Z = Z_1 \cup Z_2$. As $Rf_*(E^\bullet)$ is exact on $Y \setminus Z$, the cohomology of $Rf_*(E^\bullet)$ is supported on Z. Since Z is a closed subset Y of dimension at most $n-1$, Z does not contain any point of depth zero (a generic point of a component of Y). □

DEFINITION. *Let* $br(f) = div(Rf_*(E^\bullet))$. *We call* $br(f)$ *the generalized branch divisor of* f.

BASE CHANGE. *Let* $\phi\colon \tilde{S} \to S$ *be any morphism. Properties* (i)–(iii) *hold for* $\tilde{X} \to \tilde{Y} \to \tilde{S}$ *and* $\phi^*_Y(br(f)) = br(\tilde{f})$.

Proof. By Lemma 7, $L\phi^*_Y(Rf_*(E^\bullet_f)) = R\tilde{f}_*(L\phi^*_X(E^\bullet)) = R\tilde{f}^*(E^\bullet_{\tilde{f}})$. The result then follows from Proposition 1(v). □

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By the base change property, the generalized branch divisor $br(f)$ is a relative Cartier divisor on Y. By *relative* we mean here the restriction to any geometric fiber of Y over S is a well-defined Cartier divisor.

If $p\colon X \to S$ is smooth and every component of X dominates a component of Y, then $br(f)$ is the standard branch divisor of f.

3. Stable Maps

3.1. MODULI POINTS

Let $\overline{M}_g(D, d)$ be the moduli space of genus g, degree $d > 0$ stable maps to a nonsingular curve D. Let

$$F\colon \mathcal{C} \to D \times \overline{M}_g(D, d) \tag{19}$$

be the universal family of maps over $\overline{M}_g(D, d)$. These objects and morphisms naturally lie in the category of Deligne–Mumford stacks. We could instead utilize equivariant constructions in the category of schemes to study these universal objects (See [FuP], [GrP]). In any case, conditions (i)–(iii) of Section 2.1 are valid for (19). Hence, there exists a relative Cartier divisor $br(F)$ on $D \times \overline{M}_g(D, d)$ over D.

Let $[f\colon C \to D] \in \overline{M}_g(D, d)$ be a moduli point. We first calculate $br(f)$ on D. Let $\nu\colon \tilde{C} \to C$ be the normalization map, and let $\tilde{f} = f \circ \nu$. Let N be the singular locus of C (N is the union of the nodal points).

There are canonical exact sequences

$$0 \to f^*\omega_D \to \nu_*\tilde{f}^*\omega_D \to \mathcal{O}_N \otimes f^*\omega_D \to 0, \tag{20}$$

$$0 \to \nu_*(\omega_{\tilde{C}}) \to \omega_C \to \mathcal{O}_N \to 0. \tag{21}$$

We will use these sequences to express the branch divisor $br(f)$ as a sum over component contributions.

LEMMA 10. $br(f) = br(\tilde{f}) + 2f_*(N)$.

Proof. Since ν is a finite map,

$$R\tilde{f}_*([\tilde{f}^*\omega_D \to \omega_{\tilde{C}}]) \xrightarrow{\sim} Rf_*([\nu_*\tilde{f}^*\omega_D \to \nu_*\omega_{\tilde{C}}]). \tag{22}$$

Using 20) and (21), there are a natural distinguished triangles in $D^-_{coh}(C)$:

$$[f^*\omega_D \to \nu_*\omega_{\tilde{C}}] \to [\nu_*\tilde{f}^*\omega_D \to \nu_*\omega_{\tilde{C}}] \to [\mathcal{O}_N \otimes f^*\omega_D \to 0],$$
$$[f^*\omega_D \to \nu_*\omega_{\tilde{C}}] \to [\tilde{f}^*\omega_D \to \omega_{\tilde{C}}] \to [0 \to \mathcal{O}_N].$$

Push-forward by Rf_* preserves distinguished triangles. By (22) and the first triangle,

$$br(\tilde{f}) = div\, Rf_*([f^*\omega_D \to \nu_*\omega_{\tilde{C}}]) - f_*(N)$$

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(using also properties (iv) and (vi) of Proposition 1). The second triangle yields

$$br(f) = div\, Rf_*([f^*\omega_D \to \nu_*\omega_{\tilde{C}}]) + f_*(N).$$

The Lemma now follows. □

Let $A_1, \ldots, Aa$ be the components of $\tilde{C}$ which dominate D, and let $B_1, \ldots, B_b$ be the components of $\tilde{C}$ contracted over D. Let

$$\{a_i : A_i \to D\},\ \{b_j : B_j \to D\}$$

denote the natural restrictions of f. As a_i is a surjective map between nonsingular curves, the branch divisor $br(a_i)$ is defined by (1). Let $b_j(B_j) = p_j \in D$.

LEMMA 11. *Let $b\colon B \to p \in D$ be a contracted component. Then, $br(b) = (2g(B) - 2)[p]$.*

Proof. The complex $[b^*\omega_D \to \omega_B]$ is isomorphic to $[\mathcal{O}_B \to \omega_B]$ with the zero map. By Lemma 3, $div\, Rf_*(\mathcal{O}_B) = \chi(\mathcal{O}_B)[p] = (1 - g(B))[p]$, and $div\, Rf_*(\omega_B) = \chi(\omega_B)[p] = (g(B) - 1)[p]$. As there is a distinguished triangle in $D^-_{coh}(B)$:

$$[\mathcal{O}_B] \to [\omega_B] \to [\mathcal{O}_B \to \omega_B],$$

we find $br(b) = (2g(B) - 2)[p]$. □

Lemmas 10 and 11 prove

$$br(f) = \sum_i br(a_i) + \sum_j (2g(B_j) - 2)[p_j] + 2f_*(N). \tag{23}$$

The only negative contributions in (23) occur for contracted genus 0 components of $\tilde{C}$. However, by stability such components must contain at least 3 nodes of C. The Cartier divisor $br(f)$ is therefore effective for every moduli point $[f\colon C \to D]$.

3.2. UNIVERSAL EFFECTIVITY

The effectivity of $br(F)$ over each closed point of $\overline{M}_g(D, d)$ does not guarantee $br(F)$ is an effective Cartier divisor on $D \times \overline{M}_g(D, d)$. The latter effectivity will now be established.

Let $\pi\colon \mathcal{C} \to \overline{M}_g(D, d)$ be the structure map of the universal curve. There is a canonical exact sequence on $\mathcal{C}$:

$$0 \to K \to F^*\omega_D \to \omega_\pi \to Q \to 0. \tag{24}$$

The following vanishing statement will be proven in Section 3.3.

LEMMA 12. $R^0F_*(K) = 0$ *and* $R^1F_*(Q) = 0$.

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Let $E^\bullet = [F^*\omega_D \to \omega_\pi]$ in $D^-_{coh}(C)$. By definition,

$$br(F) = div(RF_*(E^\bullet)).$$

The cohomology of $RF_*(E^\bullet)$ may be computed via a spectral sequence with E_2 term:

$$\begin{matrix} R^1F_*(K) & R^1F_*(Q) \\ R^0F_*(K) & R^0F_*(Q) \end{matrix}$$

where the grading is -1 for the bottom left corner, 0 for the diagonal, and $+1$ for the top right corner. By Lemma 12, we see $RF_*(E^\bullet)$ has cohomology only at grade 0. Hence, locally on $D \times \overline{M}_g(D, d)$, the complex $RF_*(E^\bullet)$ is isomorphic in the derived category to a finite resolution of the coherent torsion sheaf $H^0(RF_*(E^\bullet)$. By Mumford's effectivity result (Proposition l(ii)), $div(RF_*(E^\bullet)$ is an effective Cartier divisor on $D \times \overline{M}_g(D, d)$.

As $br(F)$ is effective *and* π-relatively effective, $br(F)$ determines a π-flat subscheme of $D \times \overline{M}_g(D, d)$. The relative degree of $br(f)$ is $r = 2g - 2 - d(2g(D) - 2)$. We have proven:

THEOREM 1. *The branch divisor $br(F)$ induces a morphism:*

$$\gamma\colon \overline{M}_g(D, d) \to Hilb^r(D) = Sym^r(D).$$

3.3. PROOF OF LEMMA 12

We follow here the notation of Section 3.1. The first step in the proof is:

LEMMA 13. *The vanishings $R^0F_*(K) = 0$, $R^1F_*(K) = 0$ are equivalent to the vanishings $R^0\pi_*(K) = 0$, $R^1\pi_*(Q) = 0$ respectively.*

Proof. Let $p\colon D \times \overline{M}g(D, d) \to \overline{M}g(D, d)$ be the projection. Consider first K. Since $\pi = p \circ F$, there is a spectral sequences with E_2 term:

$$\begin{matrix} R^1p_*(R^0F_*(K)) & R^1p_*(R^1F_*(K)) \\ R^0p_*(R^0F_*(K)) & R^0p_*(R^1F_*(K)) \end{matrix}$$

which calculates the sheaves $R^i\pi(K)$. As both $R^0F_*(K)$ and $R^1F_*(K)$ have support *finite* over $\overline{M}_g(D, d)$, the first row of the above spectral sequence vanishes. Hence,

$$R^0\pi_*(K) = R^0p_*(R^0F_*(K)).$$

Moreover, as the support of $R^0F_*(K)$ is p-finite, $R^0F_*(K)$ vanishes if and only if $R^0p_*(R^0F_*(K))$ does. The proof for Q is identical as the supports of the sheaves $R^iF_*(Q)$ are also p-finite. □

LEMMA 14. $R^0\pi_*(K) = 0$.

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Proof. Let $[f\colon C \to D]$ be a moduli point. Let $[f] \in U \subset \overline{M}_g(D, d)$ where U is an open set. Let $z \in \Gamma(\pi^{-1}(U), K)$. The element z is naturally a section of $F^*\omega_D$ over $\pi^{-1}(U)$ which lies in the kernel of $F^*\omega_D \to \omega_\pi$.

Let $Spec(A) \subset \overline{M}_g(D, d)$ be any Artinian local subscheme supported at $[f]$. We will show the restriction of z to the closed scheme $\mathcal{C}_A = \pi^{-l}(Spec(A))$ vanishes for all such Artinian local subschemes. This vanishing suffices to prove $z = 0$ over a Zariski open neighborhood of $[f]$ by the Theorem on formal functions (see [Ha1]).

For notational simplicity, let $L = F^*\omega_D$ on $\mathcal{C}$. Let $B \subset C$ be the union of subcurves contracted by f. Since $L|_B$ is trivial, we find the vanishing condition: *a section of* $\Gamma(C, L_{[f]})$ *which has support on* B *must vanish identically*.

Let $Spec(A) \subset \overline{M}_g(D, d)$ be an Artinian local subscheme as above. Let z_A be the restriction of z to $\mathcal{C}_A$. Let $m \subset A$ be the maximal ideal. We note z_A must have support on B as the sheaf map $L_A \to \omega_{\pi_A}$ is an isomorphism on the open set $B^c \subset \mathcal{C}_A$. By the flatness of π, there is an exact sequence

$$0 \to mL_A \to L_A \to L_{[f]} \to 0$$

on $\mathcal{C}_A$. By the vanishing condition we see $z_A \in \Gamma(\mathcal{C}_A, mL_A)$. We then use the exact sequences

$$0 \to m^{k+1}L_A \to m^kL_A \to m^k/m^{k+1} \otimes L_{[f]} \to 0$$

to inductively prove $z_A \in \Gamma(\mathcal{C}_A, m^kL_A)$ for all k. Thus $z_A = 0$ by the Artinian condition. □

LEMMA 15. $R^1\pi_*(Q) = 0$.

Proof. From sequence (24), we obtain

$$R^1\pi_*(F^*\omega_D) \xrightarrow{i} R^1\pi_*(\omega_\pi) \to R^1\pi_*(Q) \to 0$$

on $\overline{M}_g(D, d)$. It suffices to prove i is a surjection of sheaves. As before, let $[f\colon C \to D]$ be a moduli point. Consider the standard diagram

$$\begin{array}{ccc} R^1\pi_*(F^*\omega_D)_{[f]} & \xrightarrow{i_{[f]}} & R^1\pi_*(\omega_\pi)_{[f]} \\ {\scriptstyle s}\downarrow & & {\scriptstyle t}\downarrow \\ H^1(C, F^*\omega_D) & \xrightarrow{j} & H^1(C, \omega_C). \end{array}$$

Here, the top line denotes the fiber of the sheaves at the point $[f]$. As $R^1\pi_*(\omega_\pi)$ is locally free and Serre dual to $R^0\pi_*(\mathcal{O}_\mathcal{C})$ on $\overline{M}_g(D, d)$, the map t is an isomorphism. As $F^*\omega_D$ is π-flat, we may apply the cohomology and base change Theorem (see [Ha1]) to deduce s is surjective. (As $R^2\pi_*(F^*\omega_D)_{[f]} \to H^2(C, F^*\omega_D)$ is trivially surjective and $R^2\pi_*(F^*\omega_D)$ is locally free, the surjectivity of s follows.) Surjectivity of i locally at $[f]$ is equivalent to the surjectivity of $i_{[f]}$ by Nakayama's Lemma. Therefore, the Lemma may be proven by showing j is surjective.

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It suffices finally to prove $H^1(C, Q) = 0$. Again, let $B \subset C$ be the union of subcurves contracted by f. Let $I_B \subset \mathcal{O}_C$ be the ideal sheaf of B. As the map $F^*\omega_D \to \omega_C$ is 0 on B, we see $Image(F^*\omega_D) \subset I_B \otimes \omega_C$. Hence, there is a sequence

$$0 \to T \to Q \to \omega_C\Big|_B \to 0$$

where T is easily seen to be a torsion sheaf. Then,

$$h^1(C, Q) = h^1\left(C, \omega_C\Big|_B\right) = h^0\left(C, Hom\left(\omega_C\Big|_B, \omega_C\right)\right).$$

The last equality is by Serre duality. As B is a proper subcurve, the last cohomology group certainly vanishes. $\square$

Lemmas 13–15 combine to prove Lemma 12.

4. Hurwitz Numbers

4.1. INTEGRALS

Let $g \geqslant 0$ and $d > 0$ be integers. Let b be a fixed general divisor of degree $2g - 2 + 2d$ on $\mathbf{P}^1$. Let $H_{g,d}$ be the number of nonsingular genus g curves expressible as d sheeted covers of $\mathbf{P}^1$ with branch divisor b. There is a long history of the study of $H_{g,d}$ in geometry and combinatorics. The first approach to these numbers via the combinatorics of the symmetric group was pursued by Hurwitz in [Hu].

PROPOSITION 2. *The Hurwitz numbers are integrals in Gromov–Witten theory:*

$$H_{d,g} = \int_{[\overline{M}_g(\mathbf{P}^1, d)]^{vir}} \gamma^*(\xi^{2g-2+2d}),$$

where ξ is the hyperplane class on $Sym^{2g-2+2d}(\mathbf{P}^1) = \mathbf{P}^{2g-2+2d}$.

Proof. We first prove the locus $M_g(\mathbf{P}^1, d) \subset \overline{M}_g(\mathbf{P}^1, d)$ is nonsingular (of the expected dimension). It suffices to prove the obstruction space $\mathrm{Obs}(f)$ vanishes. Let $[f: C \to \mathbf{P}^1]$ be a moduli point with C nonsingular. There is a canonical right exact sequence:

$$H^1(C, T_C) \xrightarrow{i} H^1(C, f^*T_{\mathbf{P}^1}) \to \mathrm{Obs}(f) \to 0.$$

Since $d > 0$, the sheaf map $\mathrm{T}_C \to f^*T_{\mathbf{P}^1}$ has a torsion quotient. Hence, i is surjective and $\mathrm{Obs}(f) = 0$. The virtual class of $\overline{M}_g(\mathbf{P}^1, d)$ must then restrict to the ordinary fundamental class of the open set $M_g(\mathbf{P}^1, d)$.

Let $r = 2g - 2 + 2d$. Let $p_1, \ldots, p_r \in \mathbf{P}^1$ be distinct points. By the computation of γ on singular curves (23), we find $\gamma^{-1}(\sum p_i) \subset M_g(\mathbf{P}^1, d)$. By Bertini's Theorem applied to γ: $M_g(\mathbf{P}^1, d) \to \mathbf{P}^r$, a general divisor $\sum p_i$ intersects the stack $M_g(\mathbf{P}^1, d)$

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transversely via γ in a finite number of points. These points are simply the finitely many Hurwitz covers ramified over $\sum p_i$. $\square$

The first approach to the Hurwitz numbers is via divisor linear equivalences in $\overline{M}_g(\mathbf{P}^1, d)$. In genus 0 and 1, the divisor of map $D_p \subset \overline{M}_g(\mathbf{P}^1, d)$ ramified over a fixed point $p \in \mathbf{P}^1$ may be expressed in terms of the boundary divisors of $\overline{M}_g(\mathbf{P}^1, d)$: $D_p = \sum_i \alpha_i \Delta_i$. The equation

$$H_{d,g} = D_p \cap \gamma(\xi^{r-1}) = \sum_i \alpha_i \Delta_i \cap \gamma^*(\xi^{r-1})$$

then immediately yields recursive relations for $H_{g,d}$:

$$H_{0,d} = \frac{2d-3}{d} \sum_{i=1}^{d-1} \binom{2d-4}{2i-2} i^2 (d-i)^2 H_{0,i} H_{0,d-i},$$

$$H_{1,d} = \frac{d}{6}\binom{d}{2}(2d-1)H_{0,d} + \sum_{i=1}^{d-1} \binom{2d-2}{2i-2}(4d-2)i^2(d-i)H_{0,i}H_{1,d-i}.$$

The above recursions were derived by the second author and T. Graber. R. Vakil has extended these formulas in genus 0 and 1 by refining the method to include non-simple branching. We omit the proofs here since a uniform treatment may be found in [Va]. Following the shape of these equations, the recursion

$$\begin{aligned} H_{2,d} = {} & d^2\left(\frac{97}{136}d - \frac{20}{17}\right)H_{1,d} + \sum_{i=1}^{d-1}\binom{2d}{2i-2}\left(8d - \frac{115}{17}i\right)i(d-i)H_{0,i}H_{2,d-i} \\ & + \sum_{i=1}^{d-1}\binom{2d}{2i}\left(\frac{11697}{34}i(d-i) - \frac{3899}{68}d^2\right)i(d-i)H_{1,i}H_{1,d-i} \end{aligned}$$

was conjectured by the second author and T. Graber in 1997. Using a completely different combinatorial approach, Goulden and Jackson have proven the genus 2 conjecture in [GoJ].

The existence of the genus 2-relation does not yet have any geometric explanation. In this sense, it analogous to the simple Virasoro prediction for the elliptic Gromov–Witten invariants of $\mathbf{P}^2$ (see [EHX], [P], [DZ]). Recently a combinatorial method was developed from (8) which proves the existence of many higher genus equations generalizing the above relations in genus 0, 1, and 2 [GoJV]. Viewed geometrically, the existence of these higher genus relations is a surprise.

4.2. LOCALIZATION

Let the torus $\mathbb{C}^*$ act on $V = \mathbb{C} \oplus \mathbb{C}$ diagonally with weights [0, 1] on a basis set $[v_1, v_2]$. This action induces natural γ-equivariant actions on $\overline{M}_g(\mathbf{P}(V), d)$ and

$$\mathbf{P}^r = \mathrm{Sym}(\mathbf{P}(V)) = \mathbf{P}(\mathrm{Sym}^r V^*).$$

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Moreover, the $\mathbb{C}^*$ action lifts equivariantly to the line bundle

$$L = \mathcal{O}_{\mathbf{P}^r}(1).$$

The choice of equivariant lift to L will be exploited below. The integral

$$H_{d,g} = \int_{[\overline{M}_g(\mathbf{P}(V),d)]^{vir}} \gamma^*(c_1(L)^r), \tag{25}$$

may then be evaluated via the virtual localization formula [GrP].

The connected components of the $\mathbb{C}^*$-fixed locus of $\overline{M}_g(\mathbf{P}(V), d)$ are indexed by a set of labelled connected graphs Γ first studied by Kontsevich [Ko]. The vertices of these graphs lie over the fixed points $p_1, p_2 \in \mathbf{P}^1$ and are labelled with genera (which sum over the graph to $g - h^1(\Gamma)$). The edges of the graphs lie over $\mathbf{P}^1$ and are labelled with degrees (which sum over the graph to d). The virtual localization formula of [GrP] yields the equation:

$$H_{d,g} = \int_{\overline{M}_g(\mathbf{P}(V),d)} \gamma^*(c_i(L)^r) = \sum_{\Gamma} \frac{1}{\mathrm{Aut}(\Gamma)} \int_{\overline{M}_\Gamma} \frac{\gamma^*(c_1(L)^r)}{e(N_\Gamma^{vir})} \tag{26}$$

where $\overline{M}_\Gamma$ is a product moduli spaces of stable pointed curves and $\overline{M}_\Gamma/\mathrm{Aut}(\Gamma)$ is the fixed locus associated to Γ (see [GrP]). Moreover, the equivariant Euler class of the virtual normal bundle, $e(N_\Gamma^{vir})$, is explicitly calculated in terms of the tautological ψ and λ classes on $\overline{M}_\Gamma$. Recall the Hodge integrals are the top intersection products of the, ψ and λ classes on the moduli spaces of curves (see [GeP], [FaP]). For each choice of equivariant lifting of $\mathbb{C}^*$ to L, formula (26) yields an explicit Hodge integral expression for $H_{g,d}$.

There are exactly $r+1$ distinct $\mathbb{C}^*$-fixed points of $\mathbf{P}^r = \mathbf{P}(\mathrm{Sym}^r V^*)$. For $0 \leqslant a \leqslant r$. Let p_a denote the fixed point $v_1^{*a} v_2^{*(r-a)}$. The canonical $\mathbb{C}^*$-linearization on $L = \mathcal{O}(1)$ has weight $w_a = r - a$ at p_a. Let L_i denote the unique $\mathbb{C}^*$-linearization of L satisfying $w_i = 0$. We note the weight at p_0 of L_i is i. We may rewrite (26) as:

$$H_{g,d} = \sum_{\Gamma} \frac{1}{\mathrm{Aug}(\Gamma)} \int_{\overline{M}_\Gamma} \frac{\prod_{i=1}^r \gamma^*(c_1(L_i))}{e(N_\Gamma^{vir})}. \tag{27}$$

This choice of linearization for the integrand will lead to the simplest localization formula.

The morphism γ associates a unique fixed point p_Γ to each graph Γ: $\gamma(\overline{M}_\Gamma/\mathrm{Aut}(\Gamma)) = p_\Gamma$. Let $p_\Gamma = p_i \neq p_0$. Then, $\gamma(L_i)$ is a trivial bundle with trivial linearization when restricted to the fixed locus $\overline{M}_\Gamma/\mathrm{Aut}(F)$. The Γ-contribution to the sum (27) therefore vanishes. We must only consider those graphs Γ satisfying $p_\Gamma = p_0$ in the sum (27).

The point $p_0 = [v_2^{*r}]$ corresponds to the divisor $r[v_1]$ on $\mathbf{P}(V)$. It is a very strong condition for a stable map $[f\colon C \to \mathbf{P}(V)]$ to have $br(f)$ supported at the single point $[v_1]$ – all nodes, collapsed components, and ramifications must lie over $[v_1]$. Hence, if $p_\Gamma = p_0$, the graph Γ may not have any vertices of positive genus or valence greater

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than 1 lying over $[v_2]$. Moreover, the degrees of the edges of Γ must all be 1. Exactly one graph Γ_0 satisfies these conditions: Γ_0 has a single genus g vertex lying over $[v_1]$ which is incident to exactly d degree 1 edges (the vertices over $[v_2]$ are degenerate of genus 0).

The sum (27) contains only one term:

$$H_{g,d} = \frac{1}{\mathrm{Aut}(\Gamma_0)} \int_{\overline{M}_{\Gamma_0}} \frac{\prod_{i=1}^{r} \gamma^*(c_1(L_i))}{e(N_{\Gamma_0}^{vir})}. \tag{28}$$

By definition (see [GrP]), $\overline{M}_{\Gamma_0} = \overline{M}_{g,d}$. Since the automorphism group of Γ_0 is the full permutation group of the edges, $\mathrm{Aut}(\Gamma_0) = d!$. The virtual normal bundle contribution is calculated in [GrP]:

$$\frac{1}{e(N_{\Gamma_0}^{vir})} = \frac{1 - \lambda_1 + \lambda_2 - \lambda_3 + \cdots + (-1)^g \lambda_g}{\prod_{i=1}^{d}(1 - \psi_i)}$$

Finally, the integrand $\prod_{i=1}^{r} \gamma^*(c_1(L_i))$ restricts to a term of pure weight $r!$.We have proven:

THEOREM 2.

$$H_{g,d} = \frac{(2g-2+2d)!}{d!} \int_{\overline{M}_{g,d}} \frac{1 - \lambda_1 + \lambda_2 - \lambda_3 + \cdots + (-1)^g \lambda_g}{\prod_{i=1}^{d}(1 - \psi_i)}$$

for $(g, d) \neq (0, 1), (0, 2)$.

In genus 0, there is a well-known formula for the ψ integrals:

$$\int_{\overline{M}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \binom{n-3}{a_1, \ldots, a_n}$$

(see [W]). The genus 0 formula

$$H_{0,d} = \frac{(2d-2)!}{d!} d^{d-3} \tag{29}$$

then follows immediately from Theorem 2. Equation (29) was first found by Hurwitz.

Acknowledgements

We thank T. Graber and R. Vakil for many discussions about Hurwitz numbers. The authors thank the Mittag–Leffler Institute where this research was partially carried out. The second author was partially supported by National Science Foundation grant DMS-9801574.

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