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STABLE MAPS AND SCHWARTZ MAPS

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1. **Introduction.** In the present paper M and N will denote two von Neumann algebras where $N \subset M$. If $\mathscr{A}$ is any von Neumann algebra, $\mathscr{A}'$ will denote the commutant of $\mathscr{A}$. N^c will denote the relative commutant of N in M, i.e. $N^c = N' \cap M$. $U(N)$ will denote all unitary operators of N. Let G be a group of unitaries of M. Let ϕ be a linear map of M into M. ϕ is called G-stable if $\phi(UXU^{-1}) = \phi(X)$ for all X in M and all U in G. $S(G, M)$ will denote all Schwartz maps which are G-stable. The purpose of this paper is to study the existence and properties of G-stable expectations. The main results contained here are:

THEOREM 1. *Let* Tr *be a faithful, semifinite trace on M. Let L be a von Neumann subalgebra of M such that* Tr *restricted to L is semifinite. Then there exists a normal, faithful, $U(L^c)$ stable expectation ϕ of M on L such that* $\mathrm{Tr}\,(A\phi(X)) = \mathrm{Tr}\,(AX)$ *for all X in M and all A in L for which* $\mathrm{Tr}\,|A| < \infty$.

THEOREM 2. *Suppose M has a faithful, normal, semifinite trace, call it* Tr. *Suppose $S(G, M)$ is sufficiently large, then there exists a faithful, normal, $U(N^{cc})$ stable expectation of M on N^c.*

As corollary to the above theorem, it follows that with the hypothesis of Theorem 2, N is finite, N^c can not be purely infinite. Moreover if M is of type I so is N^c. Another corollary to Theorem 2 is that if $S(G, L(h))$ has sufficiently many maps, then the von Neumann algebra N generated by G is atomic.

Next a notion of equivalence of two unitary groups will be defined. Two groups of unitary operators are equivalent if they generate the same von Neumann algebra.

THEOREM 3. *Assume $S(G, L(h))$ contains a normal map, then G is equivalent to a countable direct sum of finite groups.*

The next result is a sort of converse to Theorem 3.

THEOREM 4. *If G has the property (F), then G is a countable direct sum of finite groups and $S(G, M)$ has sufficiently many maps.*

A corollary to Theorem 3 is that if N is a finite atomic von Neumann algebra, then N is generated by a direct sum of finite unitary groups.

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Next uniqueness of expectations of certain type will be considered. The main result of this section is:

THEOREM 5. *Assume that*
(1) $N^c \subset N$,
(2) *N is finite*,
(3) *M is semifinite.*
Then there exists at most one normal expectation ϕ *of M on N.*

2. **Preliminaries.**

DEFINITION. Let ϕ be a map of M into N which preserves the identity. Assume that ϕ is a positive linear map and that $\phi(AX)=A\phi(X)$ for all A in N and X in M. ϕ will then be called an expectation of M in N.

It is trivial to see that ϕ is onto N and that ϕ is a bounded map. The motion of expectations in von Neumann algebras was studied in [2], [7], and [9].

DEFINITION. Let ϕ be an expectation of M on N, ϕ is called normal if $\phi(\text{Sup } A_\alpha) = \text{Sup } \phi(A_\alpha)$ for any increasing net of uniformly bounded selfadjoint operators.

ϕ is called faithful if, given a positive operator A such that $\phi(A)=0$, then $A=0$.

Let ϕ_α be a set of expectations of M onto N. The set ϕ_α is called complete if given a positive operator A such that $\phi_\alpha(A)=0$ then $A=0$.

DEFINITION. Let G be a subgroup of $U(M)$. By a Schwartz map relative to (G, M) one means a linear map of M into itself such that

(1) $P(X)=UP(X)U^{-1}$ for all U in G and all X in M,

(2) $P(X)$ is in $C_G[X]$ where $C_G[X]$ denotes the weak closure of the convex hull generated by elements of the type UXU^{-1} as U ranges over G.

For more information on Schwartz maps see [6].

$S(G, M)$ will denote all Schwartz maps relative to (G, M) which are G-stable, i.e. $P(X)=P(VXV^{-1})$ for all V in G. $S(G, M)$ will be called sufficient if for any positive operator X in M such that $P(X)=0$ for all P in $S(G, M)$ then $X=0$.

DEFINITION. A group G is said to be amenable as a discrete group if there exists a finitely additive probability measure μ on the field of all subsets of G such that $\mu(xE)=\mu(E)$. For more information on amenable groups see [4] and [5].

3. **Stable maps and Schwartz maps.**

LEMMA 1. *If there exists a complete set of U-stable expectations of M on N then U is in* N^c.

Proof. Let V be a unitary of N. Let ϕ_α be a complete set of U-stable expectations, then $\phi_\alpha(UVU^{-1}V^{-1})=\phi_\alpha(VU^{-1}V^{-1}U)=V\phi_\alpha(U^{-1}V^{-1}U)$. This by stability. Also $V\phi_\alpha(V^{-1})=VV^{-1}=I$. Let $W=UVU^{-1}V^{-1}$. Then $\phi_\alpha[(W-I)^*(W-I)]=0$. By completeness $W=I$ or $UV=VU$. So U is in N^c.

LEMMA 2. *A normal* $U(N)$ *stable expectation of M on* N^c *is faithful.*

Proof. Let ϕ be the expectation. Let $I=\{A/A\in M,\ \phi(A^*A)=0\}$ clearly as $\phi[(XA)^*(XA)]\leqq\|X\|^2\phi(A^*A)=0$ and $(A+B)^*(A+B)\leqq 2(A^*A+B^*B)$. I is a left ideal.

Now to show that I is ultra-weakly closed. The ultra-weak closure of I coincides with its ultra-strong closure. Let X_α be a net in I converging ultra-strongly to X, then $(X_\alpha-X)^*(X_\alpha-X)$ converges ultra-weakly to 0. Hence $\phi[(X_\alpha-X)^*(X_\alpha-X)]$ converges to 0 ultra-weakly (normality). As

$$\phi(X_\alpha-X)^*\phi(X_\alpha-X)\leqq\phi(X_\alpha-X)^*(X_\alpha-X)$$

it follows that $\phi(X_\alpha)$ converges to $\phi(X)$. X^*X_α and $X_\alpha X^*$ have the same limit so $\phi(X^*X)=0$. Hence I is a left ultra-weakly closed ideal. So there exists a unique projection E in M such that $I=\{T/TE=T\}$. $UTU^{-1}\in I$ for all $U\in U(N)$ by stability. So $UEU^{-1}=E$. So $E\in N^c$. So $E=\phi(E)=0$. So if $\phi(X^*X)=0$ then $X=XE=0$, so ϕ is faithful.

Now let G be a subgroup of $U(M)$. Let N be the von Neumann algebra generated by G.

LEMMA 3. *A Schwartz map relative to* (G, M) *is an expectation onto* N^c.

Proof. Let P be the Schwartz map. As $P(X)$ commutes with all unitaries of G, $P(X)$ is in N^c. Now if A is in N^c, $C_G[A]$ reduces to the element A. So $P(A)=A$. So $P^2=P$. N^c is hence the range of P and $P(I)=I$. Now to show that $\|P\|<1$. Let $T=\sum_{i=1}^n\alpha_iU_iAU_i^{-1}$ where $\alpha_i>0$ and $\sum\alpha_i=1$. Then $\|T\|<\|A\|$. Because $P(A)$ is in $C_G[A]$ this means that there exists a net T_α of the same form as T such that T_α converges strongly to $P(A)$. Let X be a vector of norm one. $\|T_\alpha X\|$ converges to $\|P(A)X\|$ but $\|T_\alpha X\|<\|A\|$. So $\|P(A)\|<\|A\|$. By a result of J. Tomiyama [7], this implies that P is an expectation.

LEMMA 4. *If G is amenable*, $S(G, M)$ *is nonvoid.*

Proof. Let λ be a mean. Let ξ and η be 2 vectors. Considering U as the variable, $\lambda(U^{-1}XU\xi,\eta)$ is a bounded hermitian form. By the Riez Lemma there exists an operator E_λ such that $\lambda(U^{-1}XU\xi,\eta)=(E_\lambda(X)\xi,\eta)$. It was shown in [1] that E_λ is in $S(G, M)$.

LEMMA 5. *Let M be finite and countably decomposable, let G be any subgroup of* $U(N)$, *then* $S(G, M)$ *is nonvoid.* (*In particular if N is any von Neumann subalgebra of M, then* $S(U(N), M)$ *is nonvoid.*)

Proof. Let Tr be a faithful, normal, finite trace on M [3]. By finiteness there exists a faithful, normal expectation ϕ of M on N^c such that $\mathrm{Tr}\,(XB)=\mathrm{Tr}\,(\phi(X)B)$ for all X in M and all B in N^c. Hence $\phi(VXV^{-1})=\phi(X)$ for all X in M and all V in $U(N^{cc})\supset U(N)$. Now to show $\phi(X)$ is in $C_G[X]$. $C_G[X]$ intersects N^c [3]. Let T be in $C_G[X]\cap N^c$, then by normality $T=\phi(T)=\phi(X)$. Hence ϕ is in $S(G, M)$.

Let G be a subgroup of $U(M)$. Let N be the von Neumann algebra generated by G.

LEMMA 6. *If $S(G, M)$ contains a normal map ϕ, then $S(G, M)$ reduces to ϕ, and so does $S(U(N), M)$. Moreover $C_G[X]$ intersects N^c in just one point.*

Proof. Let T be in $C_G[X]$, by normality $\phi(T)=\phi(X)$. Now let T be in $C_G[X] \cap N^c$. Then $T=\phi(T)$ by Lemma 3. So T is the unique point in $C_G[X] \cap N^c$. By normality ϕ is $U(N)$ stable, so $S(U(N), M)=\{\phi\}$.

LEMMA 7. *Let* Tr *be a faithful, normal, semifinite trace on M. Let G be a subgroup of $U(M)$ and N the von Neumann algebra generated by G. Suppose $S(G, M)$ is sufficient, then the restriction of* Tr *to N^c is semifinite.*

Proof. In this proof the notation of [3] will be used. Let $\mathcal{M}$ be the ideal whose positive part consists of positive operators A such that $\mathrm{Tr}\, A<\infty$. Consider $\mathcal{M}^{1/2}$. If A is in $\mathcal{M}^{1/2}$, $C_G[A]\subset\mathcal{M}^{1/2}$ and $C_G[A] \cap N^c$ is nonvoid [3]. Let S be a positive operator in N^c, $S\neq 0$. To show that there exists $S_1\neq 0$, $S_1\leqq S$ where S_1 is a positive operator of $N^c \cap M$. Let A be in $\mathcal{M}$ such that $0\leqq A\leqq I$. Let P_α be in $S(G, M)$. Then $S\geqq \sqrt{S}\, P_\alpha(A)\sqrt{S}=P_\alpha(\sqrt{S}\, A\sqrt{S})$. A can be picked such that $\sqrt{S}\, A\sqrt{S}\neq 0$ or else $A\sqrt{S}=0$ for all A positive in $\mathcal{M}$. By semifiniteness there would exist a net A_α converging weakly to I so $I\sqrt{S}=0$. So $S=0$, a contradiction. Pick A then so that $\sqrt{S}\, A\sqrt{S}\neq 0$. Let $H=\sqrt{S}\, A\sqrt{S}$ then H is in $\mathcal{M}^{1/2}$. $P_\alpha(\sqrt{H})$ is in $\mathcal{M}^{1/2} \cap N^c$. So $[P_\alpha(\sqrt{H})]^2$ is in $\mathcal{M} \cap N^c$. So $(P_\alpha(\sqrt{H}))^2\leqq P_\alpha(H)\leqq S$. By sufficiency, there exists an α_0 such that $P_{\alpha_0}(\sqrt{H})\neq 0$. Choose $S_1=(P_{\alpha_0}(\sqrt{H}))^2$.

THEOREM 1. *Let* Tr *be a faithful, semifinite trace of M. Let N be a von Neumann subalgebra of M and assume that the restriction of* Tr *to N is semifinite, then there exists a normal, faithful $U(N^c)$-stable expectation ϕ of M on N such that* $\mathrm{Tr}\,(A\phi(X)) =\mathrm{Tr}\,(AX)$ *for all X in M and all A in N such that* $\mathrm{Tr}\,|A|<\infty$.

Proof. Using the notations of the above lemma, let A and B be in $\mathcal{M}^{1/2} \cap N$ (that intersection is nonvoid), define $(A, B)=\mathrm{Tr}\,(AB^*)$. Choose X positive in M and define $[A, B]=\mathrm{Tr}\,(AB^*X)$. $[\ ,\ ]$ is a bounded hermitian form respectively to $(\ ,\ )$. Let k be the completion of $\mathcal{M}^{1/2}$ under $(\ ,\ )$. By the Riez Lemma there exists an operator $\phi(X)$ in $L(k)$ such that $[A, B]=(\phi(X)(A), B)$. Now: Let R_c denote the right multiplication by C, where C is in $\mathcal{M}^{1/2}$.

$$(R_c\phi(X)(A), B) = (\phi(X)(A), BC^*) = [A, BC^*] = \mathrm{Tr}\,(ACB^*X),$$
$$(\phi(X)R_c(A), B) = [R_c(A), B] = [AC, B] = \mathrm{Tr}\,(ACB^*X)$$

so $R_c\phi(X)=\phi(X)R_c$.

By the commutation theorem [3] this implies that $\phi(X)(A)$ is a left multiplication by an element of N. Call that element $\phi(X)$. Then $\mathrm{Tr}\,(AB^*X)=\mathrm{Tr}\,(\phi(X)AB^*) =\mathrm{Tr}\,(AB^*\phi(X))$ for all A and B in $\mathcal{M}^{1/2} \cap N$ and all X positive in M. ϕ can then be extended in the obvious fashion to all of M. As Tr is faithful, normal, it is easy to see that ϕ is faithful, normal, and $U(N^c)$ stable. For example to check that ϕ is $U(N^c)$ stable; let V be in $U(N^c)$, let A be in N, then:

$$\operatorname{Tr}(A\phi(VXV^{-1})) = \operatorname{Tr}(AVXV^{-1}) = \operatorname{Tr}(VAXV^{-1}) = \operatorname{Tr}(AX) = \operatorname{Tr}(A\phi(X)).$$

So $\operatorname{Tr}[A(\phi VXV^{-1})\cdot\phi(X))]=0$ for all A in $N\cap\mathscr{M}$. Since Tr is semifinite on N, let P_α be a family of orthogonal projections of N such that $\operatorname{Tr} P_\alpha<\infty$ and $\sum P_\alpha=I$. Make $A=(\phi(VXV^{-1})-\phi(X))^*P_\alpha$. One has $P_\alpha(\phi(VXV^{-1})-\phi(X))=0$ for all α, i.e. $\phi(VXV^{-1})=\phi(X)$.

THEOREM 2. *Suppose M has a faithful, normal, semifinite trace* Tr. *Suppose $S(G, M)$ is sufficient, then there exists a faithful, normal $U(N^{cc})$ stable expectation of M on N^c. (N is the algebra generated by G.)*

Proof. By Lemma 7 the restriction of Tr to N^c is semifinite. By Theorem 1 there exists a normal, faithful, $U(N^{cc})$ stable expectation ϕ of M on N^c such that $\operatorname{Tr}(AX)=\operatorname{Tr}(A\phi(X))$ for all A in N^c such that $\operatorname{Tr}|A|<\infty$. Now ϕ is in $S(G, M)$. Indeed ϕ is G-stable and if P is in $S(G, M)$ then $\phi(P(X))=P(X)$ (as ϕ is the identity on N^c). By normality $\phi(P(X))=\phi(X)$. So $P=\phi$. Hence ϕ is a normal, faithful, $U(N^{cc})$ stable expectation of M on N^c by Lemma 3.

The above theorem says that if there is a sufficient number of G-stable expectations of M on N^c, there is a faithful, normal one which in fact is more than G-stable it is $U(N^{cc})$ stable.

COROLLARY 1. *With the above hypothesis N^{cc} is finite.*

Proof. By the above theorem $S(U(N^{cc}), M)$ is nonvoid. Let P be in $S(U(N^{cc}), M)$. Let A be in N^{cc}, let $C(A)$ be the norm closure of the convex hull K_A of points of the form UAU^{-1} as U ranges over $U(N^{cc})$. Consider $C(A)\cap Z$ where Z is the center of N^{cc}. $C(A)\cap Z$ is nonvoid [3]. By [3] it is sufficient to show that $C(A)\cap Z$ reduces to one point. P is constant on K_A hence on $C(A)$. Let T_1 and T_2 be in $C(A)\cap Z$, then $T_1=P(T_1)=P(T_2)=T_2$, so N^{cc} is finite. In particular N is finite.

COROLLARY 2. *With the above hypothesis N^c can not be purely infinite.*

Proof. In [7] J. Tomiyama proved that if π is an expectation from a semifinite algebra M onto a purely infinite subalgebra $\mathscr{A}$, then π is always singular, i.e. π is not normal. Since there exists a normal expectation from M on N^c, N^c is not purely infinite.

COROLLARY 3. *With the above hypothesis if M is of type* I, *so is N^c.*

Proof. In [7] it has been shown that if there exists an expectation from M of type I to a subalgebra of type II, that expectation is not normal. By the above corollary N^c has no part of type III and hence no part II or III are present, so N^c is of type I.

Let G be a subgroup of $U(M)$. Let N be generated by G.

COROLLARY 4. *Let M be a countably decomposable von Neumann algebra and consider the following conditions*:

(1) *N is finite and there exists a faithful, normal expectation ϕ of M and N.*

(2) *There exists a faithful, normal state ρ of M such that $\rho(UXU^{-1})=\rho(X)$ for all U in G.*

(3) *There exists a faithful, normal expectation ψ of M on N^c such that $\psi(VXV^{-1})=\psi(X)$ for all V in $U(N)$.*

(4) *$S(G, M)$ is sufficient and M has a faithful, semifinite normal trace* Tr.

Then (1) *and* (2) *are equivalent. If $S(G, M)$ is nonvoid,* (2) *and* (3) *are equivalent. Finally* (4) *always implies* (3).

Proof. Assume (1), then there exists a faithful, normal finite trace λ on N. Let $r(X)=\lambda[\phi(X)]$. Clearly r is faithful, normal and bounded. Let U be in G, then $r(UXU^{-1})=\lambda\phi(UXU^{-1})=\lambda\phi(X)=r(X)$. Normalizing r, (2) is established.

Assume (2). By a classical Hilbert algebra argument one can show that there exists an expectation ϕ such that $\rho(AX)=\rho(A\phi(X))$ for all A in N and all X in M. ϕ will satisfy (1).

Assume now (2) together with the fact that $S(G, M)$ is nonvoid. Let P be in $S(G, M)$. ρ is constant on $C_G[A]$. Hence $\rho(A)=\rho(P(A))$. This shows that P is faithful, normal and satisfies $P(VAV^{-1})=P(A)$, for all V in $U(N)$. For example to check that $P(A)=P(VAV^{-1})$:

Let B be any element of N^c.

$$\rho(BVAV^{-1}) = \rho(P(BVAV^{-1})) = \rho(BP(VAV^{-1})),$$
$$\rho(BVAV^{-1}) = \rho(VBAV^{-1}) = \rho(BA) = \rho(BP(A)).$$

Choose $B=(P(VAV^{-1})-P(A))^*$, by faithfulness of ρ, $P(VAV^{-1})=P(A)$.

Assume now (3). By countable decomposability there exists a faithful, normal state σ of M (get a maximal set of orthogonal projections P_n of M where each P_n is the projection on $[M'x_n]$, and let $\sigma=\sum W_{x_n}$ (notation of [3]). Let $\rho(X)=\sigma\psi(X)$. ρ in the state of (2).

Finally to show that (4) implies (3). By Theorem 2 there exists a faithful, normal expectation of M on N^c, call it Ψ such that $\mathrm{Tr}\,(XA)=\mathrm{Tr}\,(\Psi(X)A)$ for all A in $\mathscr{M}$ N^c. As above one shows that $\Psi(VXV^{-1})=\Psi(X)$.

Corollary 5. *If $S(G, L(h))$ is sufficient, N, the algebra generated by G, is atomic.*

Proof. By Theorem 2 there exists a faithful, normal, expectation of $L(h)$ on N' which is $U(N)$ stable. By Corollary 3, N' is of Type I, hence so is N [3]. Also N is finite by Corollary 1. Let Z be the center of N. Any projection of N dominates an abelian projection in N, call it $P\neq 0$. If Q is a projection of N such that $Q\leqq P$, then $Q=PC$ where C is a projection of Z. Since Z is atomic [3], Q and hence P dominates a minimal projection. So N is atomic.

Remarks. The following statements are trivial to see:

(1) If $S(G, L(h))$ is sufficient then there exists a normal expectation ϕ from $L(h)$ to N' such that $\phi(UXU^{-1})=\phi(X)$ for all U in G, this is part of Corollary 4.

(2) Assuming $S(G, L(h))$ contains a normal map, then $S(G, L(h))$ is sufficient.

Let π be a normal map, then π is faithful. Indeed: by normality $\pi(UXU^{-1})=\pi(X)$ for all U in $U(N)$. Assume that P is a projection such that $\pi(P)=0$. Let $Q=$ Sup UPU^{-1} is $U\in U(N)$. Then $Q=VQV^{-1}$ for all V in $U(N)$, so Q is in N'. Hence $Q=\pi(Q)=0$. So $P=0$.

DEFINITION. Two groups of unitaries are equivalent if they generate the same von Neumann algebra.

THEOREM 3. *Assume $S(G, L(h))$ contains a normal map π, then G is equivalent to a countable direct sum of finite groups.*

Proof. By Lemma 6, $S(G, L(h))=\{\pi\}$. By the above remark π is faithful and by normality π is in $S(U(N), L(h))$. By Corollary 1, N is finite. Let Z be the center of N. By Corollary 5, Z is atomic. Pick a maximal set of orthogonal minimal projections, C_n of Z such that $N=\bigoplus N_{c_n}$. N_{c_n} is a factor of Type I_n. N_{c_n} is isomorphic to $n\times n$ matrices, so N_{c_n} is generated by a finite group K_n of unitaries. Let $K=\bigoplus K_n$ (all components are the identity except a finite number). The algebra generated by K contains all N_{c_n}, so it contains N. Each K_n is a subgroup of $U(N)$. So the algebra generated by K is N.

Let M be a von Neuman algebra and let G be a subgroup of $U(M)$.

DEFINITION. G will satisfy condition (F) if

(1) There exists orthogonal projections C_α of N' (N is the algebra generated by G) such that $I=\sum C_\alpha$ and $|GC_\alpha|<\infty$.

(2) For every U in G, $UC_\alpha=C_\alpha$ for all but a finite number of α.

THEOREM 4. *If G has property (F), then G is a countable direct sum of finite groups, and $S(G, M)$ is sufficient.*

Proof. Define a map π_α on G by $\pi_\alpha(U)=UC_\alpha\cdot\pi_\alpha$ is clearly a homomorphism of G and $\pi_\alpha(G)$ is finite. Also the intersection of all kernels of π_α is I. Let $F_\alpha=\pi_\alpha(G)$, then by definition of condition (F), $G=\bigoplus F_\alpha$. As each F_α is finite G is amenable since it is locally finite. So $S(G, M)$ is nonvoid by Lemma 4. Now let A be a positive operator in M, let P be in $S(G, M)$ and suppose $P(A)=0$ for all P in $S(G, M)$. If $A\neq 0$, $C_\alpha AC_\alpha\neq 0$ for some C_α, call α_0 such an α. $C_{\alpha_0}P(A)C_{\alpha_0}=P(C_{\alpha_0}AC_{\alpha_0})\in C_G[C_{\alpha_0}AC_{\alpha_0}]$. Let H be all elements of G where the α_0 component is the identity. Then $G=HF_{\alpha_0}$. Let U be in G, then U is uniquely written as $U=VW$ where V is in H and W in F_{α_0}. $UC_{\alpha_0}AC_{\alpha_0}U^{-1}=WC_{\alpha_0}AC_{\alpha_0}W^{-1}$ but there is only a finite number of $UC_{\alpha_0}AC_{\alpha_0}U^{-1}$. Hence $C_G[C_{\alpha_0}AC_{\alpha_0}]$ is the convex hull of $WC_{\alpha_0}AC_{\alpha_0}W^{-1}$ as W ranges in F_{α_0}. So $0=P(C_{\alpha_0}AC_{\alpha_0})=\sum_{i=1}^n \alpha_i W_i C_{\alpha_0}AC_{\alpha_0}W_i^{-1}$. So $C_{\alpha_0}AC_{\alpha_0}=0$, a contradiction. So $A=0$ and $S(G, M)$ is sufficient.

REMARK. While proving Theorem 3 it has been shown that if N is a finite atomic von Neumann algebra, then N is generated by a direct sum of finite groups K_n.

4. Uniqueness properties.

LEMMA 8. *If there exists only one faithful, normal expectation Φ of M on N then $N^c\subset N$.*

Proof. Let $\varepsilon>0$. Let H be a positive operator in N^c such that $H>\varepsilon I>0$, $\Phi(H)$ is in N, $\Phi(H)>\varepsilon I$. Let X be in N, then $X\Phi(H)=\Phi(XH)=\Phi(H)X$. So $\Phi(H)$ is in N', so in $N\cap N'=Z_N$. Define $\pi(X)=\Phi(H)^{-1}\Phi(H^{1/2}XH^{1/2})$.

Clearly π is another expectation of M on N, by uniqueness $\pi=\Phi$. So $\Phi(H^{1/2}XH^{1/2})=\Phi(H)\Phi(X)$, in particular if $X=H$ then $\Phi(H)^2=\Phi(H)^2$; this holds for any selfadjoint operator in N^c which is positive. Let H be any selfadjoint operator in N^c. Pick $C>0$ such that $CI+H>\varepsilon I$ then $[\Phi(CI+H)]^2=\Phi[(CI+H)^2]$ so $\Phi(H)^2=\Phi(H)^2$. Let P be a projection in N^c, then $(P-\Phi(P))^2>0$ so $\Phi(P-\Phi(P))^2=(\Phi(P)-\Phi(P))^2=0$. By faithfulness $P=\Phi(P)$ i.e. P is in N so $N^c\subset N$.

Lemma 9. *Let N be normal in M (i.e. $N^{cc}=N$) if a faithful expectation exists from M to N then $N^c\subset N$. If $N^c\subset N$ and if a normal expectation Φ exists from M to N then Φ is the only such normal expectation.*

Proof. The first part was shown in Lemma 1. Now to show the second part: As $N^c\subset N$, N^c is the center of N in particular N^c is abelian. Hence, $U(N^c)$ is amenable, so $S(U(N^c), M)$ is nonvoid by Lemma 4. Let P be in $S(U(N^c), M)$, then P is an expectation on $N^{cc}\cap M$ by Lemma 3. So P is an expectation on N. Let $\Phi(P(X))=\Phi(X)$. Also $\Phi(P(X))=P(X)$, so $\Phi=P$. This shows that if Φ exists, it is unique.

Theorem 5. *Assume the following conditions*:

(1) $N^c\subset N$,

(2) *N is finite*,

(3) *M is semifinite.*

Then there exists at most one normal expectation ϕ.

Proof. N^c is the center of N, by finiteness the map (notation of [3]) is defined from N to N^c. If X is in M, define $\Psi(X)=(\phi(X))^\natural$. Ψ is a normal map and $S(U(N^c), M)$ is nonvoid. Let P be in $S(U(N^c), M)$. If X is in $\mathscr{M}^{1/2}$, $C_{U(N)}[X]\cap\mathscr{M}^{1/2}$ and $C_{U(N)}[X]$ intersect N^c. Let T be in $C_{U(N)}[X]\cap N^c$. Ψ is invariant under $U(N)$, so $T=\Psi(T)=\Psi(X)$. So $N^c\cap C_{U(N)}[X]=\{\Psi(X)\}$. If ϕ_1 is another normal expectation of M on N, then define $\Psi_1(X)=[\phi_1(X)]^\natural$. Also $N^c\cap C_{U(N)}[X]=\{\Psi_1(X)\}$, so $\Psi=\Psi_1$ on $\mathscr{M}^{1/2}$, hence on M.

Let λ be any normal, finite trace on N. Then: $\lambda\phi(X)=\lambda\Psi(X)=\lambda\Psi_1(X)=\lambda\phi_1(X)$. Since the λ form a complete set $\phi=\phi_1$.

In conclusion consider the following problem. Let N be a von Neumann algebra. Suppose there exists sufficiently many expectations of M on N. Is N relatively semifinite? An answer to that problem was given when the expectations are of a certain type (Lemma 7).

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