Differential and Integral Equations Volume 15, Number 8, August 2002, Pages 923–944

STABLE PERIODIC MOTION OF A SYSTEM WITH STATE DEPENDENT DELAY

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Abstract. We consider an autonomous system of a differential and a functional equation for one-dimensional motion of an object which attempts to regulate its distance from a given point by means of reflected signals. In a suitable, compact state space the forward initial value problem is well-posed. For certain configurations of the parameters involved we prove that there exist periodic orbits which are exponentially stable with asymptotic phase.

1. Introduction

In the present paper it is shown that an autonomous delay differential equation, with the delay depending on present and past states, has a periodic solution whose orbit is exponentially stable with asymptotic phase. The proof extends a method first used in [17] for equations of the form

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)),\ \mu \geq 0,$$

with constant delay and continuous nonlinearity $f : \mathbb{R} \to \mathbb{R}$ varying not too much outside an interval $(-\beta, \beta)$. A set of initial data is found to which solutions return, after an excursion into phase space during which they first come closer together, then diverge from each other, and come closer together once again. Lipschitz continuity of the nonlinearity permits to find Lipschitz constants for the associated return maps. If the nonlinearity is sufficiently flat outside $(-\beta, \beta)$ the return map becomes a contraction, and the fixed point defines an attracting periodic orbit of the differential equation. Compared to [17] the situation studied in the present paper is more complicated. It requires another choice of a set of initial data, the proof that solutions return is different, and the Lipschitz estimates are more involved. The approach from [17] has also been extended into other directions. In [18], sharper estimates for nonlinearities with local monotonicity properties lead

Accepted for publication: August 2001.
AMS Subject Classifications: 34K15; 58F22.

to a result which can be applied to analytic nonlinearities. M.R.W. Martin [13] investigates another combination of instantaneous and delayed feedback. State-dependent delays arise in various circumstances, but it seems not obvious how to single out a tractable class of equations which contains a large set of examples which are well motivated. We model a simple, slightly idealized real world situation: An object moves along a line and regulates its position relative to an obstacle by means of signals which are reflected by the obstacle. Let $x(t) \in \mathbb{R}$ denote the position of an object at time $t \in \mathbb{R}$. The object should not collide with the obstacle located at $-w < 0$. A preferable position of the object is at distance w from the obstacle, i.e., at $0 \in \mathbb{R}$. Signals travel from the object to the obstacle at a speed $c > 0$ and are reflected. The object senses the reflected signals and measures the signal running time $s(t)$ between emission and detection at time t :

$$s(t) = \frac{1}{c}(|x(t - s(t)) + w| + |x(t) + w|).$$

Then it computes a distance d from s. It seems reasonable to consider the case $d = \frac{c}{2}s$ since this gives the true distance $d(t)$ at time t at least if at times $t - s(t)$ and t the object is in the preferred position at zero. Depending on the deviation of the computed distance from the preferred distance the object adjusts its speed in size and direction, after a reaction lag $r > 0$:

$$\dot{x}(t) = v(d(t - r) - w).$$

So we are led to study the system of the equations

$$s(t) = \frac{1}{c}(|x(t - s(t)) + w| + |x(t) + w|) \tag{1}$$

and

$$\dot{x}(t) = v\Big(\frac{c}{2}s(t - r) - w\Big) \tag{2}$$

for positive parameters c, w, r and a *response function* $v : \mathbb{R} \to \mathbb{R}$. Negative feedback with respect to the preferred position at $0 \in \mathbb{R}$ is expressed by the condition

$$\delta v(\delta) < 0 \quad \text{for all} \quad \delta \neq 0.$$

The natural assumption that the signals are faster than the object corresponds to a bound $\sup_{\delta \in \mathbb{R}} |v(\delta)| \le b$ for some $b \in (0, c)$. A similar model was mentioned earlier by R.D. Nussbaum [15]. Other related models, which take into account some fraction of the signal running time, are due to M. Büger and M.R.W. Martin [3, 4], and to J.A. Messer [14]. In the next section it is shown that for continuous and bounded response functions the system generates a continuous semiflow on an open subset of a compact metric space. This is particularly easy since we restrict attention to solutions with values in a bounded set, which allows to work in a Banach space of continuous functions on a compact interval. Section 3 deals with sets of continuous odd response functions which outside a bounded interval satisfy the negative feedback condition and do not vary much, as in [17]. Under a series of conditions on the parameters involved, the semiflow connects a set I_β of initial data to a set $-I_{\beta\epsilon}$ with compact closure in $-I_\beta$, and $-I_\beta$ to $I_{\beta\epsilon}$. Section 4 contains the main result. For Lipschitz continuous response function v the Lipschitz constant of the return map on I_β constructed in Section 3 is estimated in terms of the (global) Lipschitz constant of v and of the Lipschitz constant of its restrictions to rays $[\beta, \infty)$, $\beta > 0$. Theorem 1 describes a set of response functions v for which we obtain periodic orbits from contracting return maps. Corollary 3 adds that these periodic orbits are exponentially stable with asymptotic phase. Let us mention that several of the hypotheses made in the sequel serve only to keep the paper reasonably short. This refers in particular to the assumption that the response function is odd, and to others which make the situation as symmetric as possible. Existence of periodic solutions of autonomous differential equations with state-dependent delays has been proved earlier by several authors, see [1, 2, 7, 8, 10, 11] and the very general result in [12]. A result which, in addition to existence, involves stability properties is contained in work of Krisztin and Arino [6], where a periodic orbit in the (manifold) boundary of an attractor must be stable with respect to nearby data inside the attracting set, as in [16, 19] on equations with constant delay. Open questions concern the dynamics further away from the periodic orbit, solutions which are not bounded as in Section 2 below, and the range of applicability of Theorem 1 and Corollary 3. For example, is it possible to derive sharper estimates of the Lipschitz constants for monotone response functions, as in [18] for equations with constant delay? This might lead to results for analytic response functions like for example $\delta \mapsto -\tanh(\gamma\delta)$, $\delta \mapsto -\arctan(\gamma\delta)$ with $\gamma > 0$ sufficiently large.

Notation. For a map $f : A \to F, A \subset E, E$ and F Banach spaces, we set

$$Lip(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} \leq \infty.$$

2. Solutions

In this section we assume that c, w, r are positive constants, that $v : \mathbb{R} \to \mathbb{R}$ is continuous, and that v is bounded by some $b \in (0, c)$. Equation (1) implies $0 \leq s(t) < \frac{4w}{c}$ in case $x(t - s(t))$ and $x(t)$ belong to the interval $(-w, w)$. For the total delay $s(t) + r$ we get the upper bound $R = r + h$, where $h = \frac{4w}{c}$. This suggests to study the system (1, 2) for initial data in the Banach space $C = C([-R,0],\mathbb{R})$ with the norm given by $\|\phi\| = \max_{t\in[-R,0]} |\phi(t)|$, as long as only solutions with $|x(t)| < w$ are concerned. As such solutions will be sufficient for our purpose, we define a solution of the system (1, 2) to be either a pair (x,s) of continuous real functions on $\mathbb{R}$ with values in $(-w,w)$ and $[0,h]$, respectively, so that x is differentiable and both equations (1, 2) hold on $\mathbb{R}$, or a pair of continuous functions $x : [T_0 - R, T) \to (-w,w)$ and $s : [T_0 - r, T) \to [0,h]$, with $T_0 \in \mathbb{R}$ and $T_0 < T \le \infty$, so that (1) is satisfied for all $t \in [T_0 - r, T)$, x is differentiable on (T_0, T), and (2) holds on (T_0, T). In the first case we speak of a solution on $\mathbb{R}$, in the second case of a solution on $[T_0 - R, T)$. We shall often use that for these solutions equation (1) is equivalent to $c\, s(t) = x(t - s(t)) + x(t) + 2w$. The subset $M \subset C$ of all $\phi \in C$ with $Lip(\phi) \le b$ and $\|\phi\| \le w$ is compact, due to the Ascoli-Arzela theorem. Let I denote the open subset of M consisting of initial data $\phi \in M$ with $\|\phi\| < w$. Whenever $t \in \mathbb{R}$ and $[t-R,t]$ is in the domain of a map $x : D \to \mathbb{R}$ we define the segment $x_t : [-R,0] \to \mathbb{R}$ by $x_t(u) = x(t+u)$.

Proposition 1. (Maximal solutions) *For every $\phi \in I$ there is a solution (x,s) on $[-R,t_\infty)$, $0 < t_\infty \le \infty$, with $x_0 = \phi$ so that for every other solution $(\hat{x},\hat{s})$ defined on some interval $[-R,T)$, $0 < T \le \infty$, and satisfying $\hat{x}_0 = \phi$,*

$$T \le t_\infty, \ \hat{x}(t) = x(t) \quad on \quad [-R,T), \ \hat{s}(t) = s(t) \quad on \quad [-r,T).$$

We have $Lip(x) \le b$.

Proof. (Sketch) For $\phi \in I$ and $t \in [-r,0]$ the map $[0,h] \ni \bar{s} \mapsto \frac{1}{c}(\phi(t-\bar{s}) + \phi(t) + 2w) \in \mathbb{R}$ has all its values in $[0,h]$ and is a contraction since $Lip(\phi) \le b < c$. The fixed points of these maps define a map $s : [-r,0] \to [0,h]$ which satisfies

$$\begin{aligned} |s(t) - s(u)| &= \frac{1}{c}|\phi(t - s(t)) + \phi(t) - \phi(u - s(u)) - \phi(u)| \\ &\le \frac{b}{c}(2|t-u| + |s(t) - s(u)|) \end{aligned}$$

for all t,u in $[-r,0]$, hence $Lip(s) \le \frac{2b}{c-b}$. Use equation (2) to define

$$x(t) = \phi(0) + \int_0^t v\left(\frac{c}{2}s(u-r) - w\right) du$$

for $0 < t \le r$. Notice that b is a Lipschitz constant for this map on $[0,r]$. In case $|x(t)| < w$ for all $t \in (0,r]$, proceed by iteration. □

We denote the maximal solution obtained in the preceding proposition by (x^ϕ, s^ϕ), and the supremum of its domain of definition by t_∞^ϕ.

Corollary 1. (Positive invariance of I) *For all $\phi \in I$ and $t \in [0,t_\infty^\phi)$, $x_t^\phi \in I$. The maximal solutions constitute a semiflow $F : \Delta \to I$ on I in the usual* *way:*

$$\Delta = \bigcup_{\phi \in I} [0, t_\infty^\phi) \times \{\phi\}, \;\; F(t,\phi) = x_t^\phi$$

Proposition 2. (i) *If* $(x,s), (\bar{x},\bar{s})$ *are solutions on* $[-R,T), [-R,\bar{T})$, *respectively, and if* $-r \le t < T \le \bar{T}$, *then*

$$|s(t) - \bar{s}(t)| \le \frac{2}{c-b} \max_{u \in [t-h,t]} |x(u) - \bar{x}(u)|.$$

(ii) *The domain* Δ *is open in* $I \times [0,\infty)$, *and* F *is continuous.*
(iii) *If* $\phi \in I, \psi \in I, t_\infty^\phi \le t_\infty^\psi, n \in \mathbb{N}, (n-1)r < t_\infty^\phi$, *then*

$$\sup_{t \in [(n-1)r, nr] \cap [0, t_\infty^\phi)} |x^\phi(t) - x^\psi(t)| \le \big(1 + r\,Lip(v)\frac{c}{c-b}\big)^n \|\phi - \psi\|.$$

Proof. 1. To prove (i), observe

$$\begin{aligned} c|s(t) - \bar{s}(t)| &= |x(t-s(t)) + x(t) - \bar{x}(t-\bar{s}(t)) - \bar{x}(t)| \\ = |x(t-s(t)) - \bar{x}(t-s(t)) &+ \bar{x}(t-s(t)) - \bar{x}(t-\bar{s}(t)) + x(t) - \bar{x}(t)| \\ &\le 2 \max_{u \in [t-h,t]} |x(u) - \bar{x}(u)| + b|s(t) - \bar{s}(t)|. \end{aligned}$$

2. The openness of Δ and the continuity of F follow as in the theory of ordinary differential equations (compare Chapter VII in[5]) from the following fact.

2.1. For every $\phi \in I$ there exists $\eta = \eta(\phi) > 0$ with

$$\eta < t_\infty^\psi \quad \text{for all} \quad \psi \in I \quad \text{with} \quad \|\psi - \phi\| < \eta,$$

or equivalently, $[0,\eta(\phi)] \times \{\psi \in I : \|\psi - \phi\| < \eta\} \subset \Delta$, so that for every $\epsilon > 0$ there is $\delta \in (0,\eta)$ with

$$|x^\psi(t) - x^{\bar{\psi}}(t)| < \epsilon \quad \text{on} \quad [0,\eta]$$

for all $\psi, \bar{\psi}$ in I with $\|\psi - \phi\| < \eta$, $\|\bar{\psi} - \phi\| < \eta$, $\|\bar{\psi} - \psi\| < \delta$. In particular, each map $\{\psi \in I : \|\psi - \phi\| < \eta\} \ni \psi \mapsto F(t,\psi) \in I$, $t \in [0,\eta]$, is continuous.

Proof of 2.1. Let $\phi \in I$ be given. Choose $\eta \in (0,r] \cap (0, \frac{w}{2})$ with

$$-w + b\eta + \eta < \phi(0) < w - b\eta - \eta.$$

For $\psi \in I$ with $\|\psi - \phi\| < \eta$ we proceed as in the proof of Proposition 1 and find $s : [-r,0] \to [0,h]$ so that

$$c\,s(t) = \psi(t - s(t)) + \psi(t) + 2w \quad \text{for} \quad -r \le t \le 0.$$

Then

$$x : [0,\eta] \ni t \mapsto \psi(0) + \int_0^t v\left(\frac{c}{2}s(u-r) - w\right) du \in \mathbb{R}$$

satisfies

$$|x(t) - \phi(0)| \le |\psi(0) - \phi(0)| + b\eta < \eta + b\eta \quad \text{for} \quad 0 \le t \le \eta$$

which gives

$$-w < \phi(0) - b\eta - \eta < x(t) < \phi(0) + b\eta + \eta < w \quad \text{for such} \quad t.$$

It follows that x is a restriction of x^{ψ}, hence $\eta < t_{\infty}^{\psi}$. Let $\epsilon > 0$ be given. Choose $\bar{\epsilon} > 0$ so that $\bar{\epsilon}\frac{c-b}{c} + \eta\bar{\epsilon} < \epsilon$, and then $\bar{\delta} = \bar{\delta}(\bar{\epsilon}) \in (0,\eta) \cap (0,\bar{\epsilon})$ according to the uniform continuity of v on $\left[-w, \frac{hc}{2} - w\right]$. Set $\delta = \bar{\delta}\frac{c-b}{c}$. Consider $\psi, \bar{\psi}$ in I with $\|\psi - \phi\| < \eta, \|\bar{\psi} - \phi\| < \eta, \|\psi - \bar{\psi}\| < \delta$. Part (i) gives

$$|s^{\psi}(t) - s^{\bar{\psi}}(t)| \le \frac{2}{c-b}\,\|\psi - \bar{\psi}\| \quad \text{for} \quad -r \le t \le 0.$$

For $0 \le t \le \eta$ we obtain

$$\begin{aligned} &|x^{\psi}(t) - x^{\bar{\psi}}(t)| \\ &\le |\psi(0) - \bar{\psi}(0)| + \int_0^t \left| v\left(\frac{c}{2}s^{\psi}(u-r) - w\right) - v\left(\frac{c}{2}s^{\bar{\psi}}(u-r) - w\right)\right| du \\ &\le \bar{\delta}\frac{c-b}{c} + \eta\bar{\epsilon} < \bar{\epsilon}\frac{c-b}{c} + \eta\bar{\epsilon} < \epsilon. \end{aligned}$$

3. Proof of (iii). Set $x = x^{\phi}, s = s^{\phi}, \bar{x} = x^{\psi}, \bar{s} = s^{\psi}$. For $\nu \in \{1, \ldots, n\}$ and $(\nu - 1)r \le t \le \min\{\nu r, t_{\infty}^{\phi}\}$,

$$\begin{aligned} |x(t) - \bar{x}(t)| &\le |x((\nu-1)r) - \bar{x}((\nu-1)r)| + \int_{(\nu-1)r}^t |\dot{x}(u) - \dot{\bar{x}}(u)| du \\ &\le |x((\nu-1)r) - \bar{x}((\nu-1)r)| \\ &\quad + \int_{(\nu-1)r}^t |v\left(\frac{c}{2}s(u-r) - w\right) - v\left(\frac{c}{2}\bar{s}(u-r) - w\right)| du \\ &\le |x((\nu-1)r) - \bar{x}((\nu-1)r)| + r\, Lip(v)\frac{c}{2} \max_{-r \le u \le (\iota-1)r} |s(u) - \bar{s}(u)|. \end{aligned}$$

Use part (i) and induction to complete the proof. □

3. Recurrence

Let constants $c > 0, w > 0, r > 0, b > 0$ be given and in addition $a \in (0, b)$. For $0 < \epsilon < a$ and $0 < \beta < w$ consider the set $V = V(\beta, \epsilon)$ of odd continuous functions $v : \mathbb{R} \to \mathbb{R}$ with

$$\sup_{\xi \in \mathbb{R}} |v(\xi)| \le b \quad \text{and} \quad |v(\xi) + a| \le \epsilon \quad \text{for all} \quad \xi \ge \beta.$$

Incidentally, notice that V contains functions for which the negative feedback condition from Section 1 is violated in $(-\beta,\beta)$. Recall $h=\frac{4w}{c}$. We need additional hypotheses, beginning with

$$h\big(1+\frac{b}{a}\big)<r<\frac{w}{b}-h. \tag{3}$$

The left part of (3) implies $ha<ra-hb$. Let $d>0$ be given with

$$ha<d<ra-hb. \tag{4}$$

Using (3) we get $d<ra<rb<w$. Inequality (4) permits to find $\epsilon_0\in(0,a)$ and $\beta_0\in(0,w)$ so that for $0<\epsilon<\epsilon_0$ and $0<\beta<\beta_0$ the inequalities

$$h(a+\epsilon)-d\le-\beta \tag{5}$$

and

$$d<-\beta+\Big(r+\frac{d-\beta}{a+\epsilon}-\frac{d-\beta}{a-\epsilon}\Big)(a-\epsilon)-b\Big(h+\frac{d+\beta}{a-\epsilon}-\frac{d-\beta}{a+\epsilon}\Big) \tag{6}$$

hold. By (3), $b(r+h)<w$. Therefore we may finally assume

$$b\Big(\frac{d+\beta}{a-\epsilon}-\frac{d-\beta}{a+\epsilon}+r+h\Big)<w \tag{7}$$

for $0<\beta<\beta_0, 0<\epsilon<\epsilon_0$. Inequality (7) will be used to guarantee that components $x(t)$ of certain solutions eventually are contained in a compact subinterval of $(-w,w)$. Observe that due to (5),

$$-d<-\beta. \tag{8}$$

Let $\epsilon\in(0,\epsilon_0), \beta\in(0,\beta_0)$, and $v\in V(\beta,\epsilon)$ be given. In the sequel we follow a solution $(x,s)=(x^\phi,s^\phi)$ which starts in the set $I_\beta\subset I$ of initial data $\phi\in I$ with the properties $\phi(t)\le-\beta$ on $[-R,0]$ and $\phi(0)=-d$. The first aim is to show that the segments x_t^ϕ reach the set $-I_\beta$ at some $t=q(\phi)>0$. Set $t_\infty=t_\infty^\phi$.

Proposition 3. *We have* $r<t_\infty$, $0<a-\epsilon<\dot{x}(t)<a+\epsilon$ *on* $(0,r]$, *and* $\beta<x(r)$. *The constants* $s_+=\frac{d-\beta}{a+\epsilon}$, $s_-=\frac{d-\beta}{a-\epsilon}$, $u_-=\frac{d+\beta}{a-\epsilon}$ *and the arguments* $t_<=t_<(\phi)$, $z=z(\phi)$, $t_>=t_>(\phi)$ *in* $(0,r)$ *given by* $x(t_<)=-\beta$, $x(z)=0$, $x(t_>)=\beta$ *satisfy* $h\le s_+<t_<<s_-<u_-$ *and* $t_<<z<t_><u_-<r$. *In particular,* $x(t)\le-\beta$ *on* $[0,h]$ *and* $\beta\le x(t)$ *on* $[u_-,r]$.

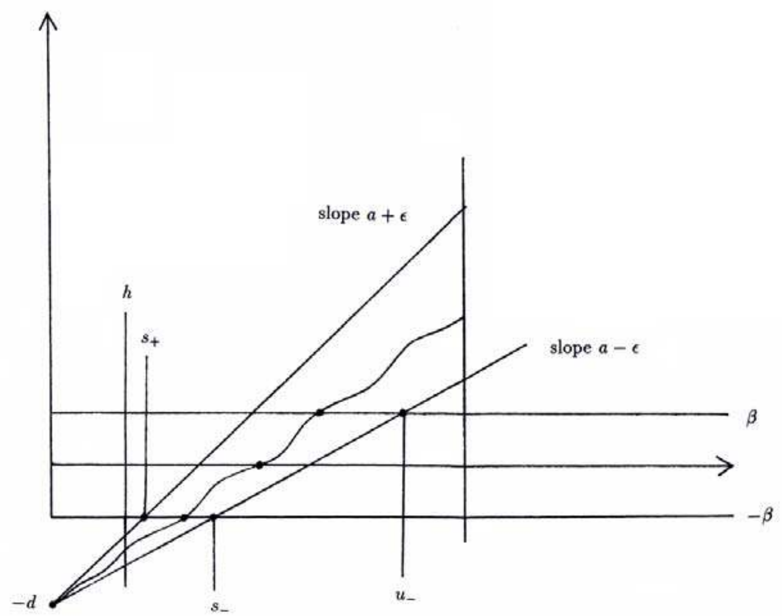

Figure 1

Proof. For $-r \leq t \leq 0$, equation (1) yields

$$\frac{c}{2}s(t) - w = \frac{1}{2}\left(\phi(t - s(t)) + \phi(t)\right) \leq -\beta.$$

Consequently,

$$v\left(\frac{c}{2}s(t-r) - w\right) \in [a-\epsilon, a+\epsilon] \quad \text{for} \quad 0 \leq t \leq r.$$

Using $rb < w$ (see the inequality following (4)) we infer from equation (2) and from $x(0) = -d \leq -\beta < 0$ that $r < u$ and $a - \epsilon < \dot{x}(t) < a + \epsilon$ on $(0, r]$. So the restriction of x to $(0, r]$ is between the straight lines with slopes $a - \epsilon, a + \epsilon$ passing through the point $(0, -d) \in \mathbb{R}^2$. The inequality (5) shows that the upper one of these lines reaches the level $-\beta$ not before $t = h$. The lower one passes through the point $(r, -d + r(a + \epsilon)) \in \mathbb{R}^2$; from (6) we infer $\beta < -d + r(a + \epsilon)$. Therefore, the arguments s_+, s_-, u_- where the upper line reaches the level $-\beta$ and where the lower line reaches $-\beta$ and then β, respectively, satisfy $h \leq s_+ < s_- < u_- < r$. The remaining inequalities now become obvious. □

Proposition 4. *We have* $u_- + R < t_\infty$ *and*

$$a - \epsilon < \dot{x}(t) < a + \epsilon \quad \textit{on} \quad (0, s_+ + r],$$

$$-\beta+\left(\frac{d-\beta}{a+\epsilon}+r-\frac{d-\beta}{a-\epsilon}\right)(a-\epsilon)\le x(s_++r),$$

$$d<x(t)\quad on\quad [s_++r,u_-+R],$$

$$x(t)\le b\left(\frac{d+\beta}{a-\epsilon}+R-\frac{d-\beta}{a+\epsilon}\right)\quad on\quad [z,u_-+R].$$

Proof. On $[s_+-R,s_+]$, $x(t)\le-\beta$. Arguing as in the preceding proof we find $s_++r<t_\infty$ and $a-\epsilon<\dot x(t)<a+\epsilon$ on $(s_+,s_++r]$. Recall

$$s_+<s_-<u_-<r<s_++r<u_-+R.$$

Using $-\beta\le x(s_-)$ and $a-\epsilon<\dot x(t)$ on $[s_-,s_++r]$ we infer

$$x(s_++r)\ge-\beta+\int_{s_-}^{s_++r}\dot x(t)dt\ge-\beta+\left(\frac{d-\beta}{a+\epsilon}+r-\frac{d-\beta}{a-\epsilon}\right)(a-\epsilon).$$

For any continuous real function $\bar d$ on $[s_++r,u_-+R]$ the map

$$\bar x:[s_++r,u_-+R]\ni t\mapsto x(s_++r)+\int_{s_++r}^{t}v(\bar d(u))du\in\mathbb{R}$$

is bounded from below by

$$-\beta+\left(\frac{d-\beta}{a+\epsilon}+r-\frac{d-\beta}{a-\epsilon}\right)(a-\epsilon)-b(u_-+R-s_+-r)$$

$$=-\beta+\left(\frac{d-\beta}{a+\epsilon}+r-\frac{d-\beta}{a-\epsilon}\right)(a-\epsilon)-b\left(\frac{d+\beta}{a-\epsilon}-\frac{d-\beta}{a+\epsilon}+h\right)>d$$

(see (6)). For all continuous functions

$$\hat d:[z,u_-+R]\to\mathbb{R}\quad\text{and}\quad \hat x:[z,u_-+R]\ni t\mapsto\int_z^t v(\hat d(u))du\in\mathbb{R}$$

an upper bound is

$$b(u_-+R-z)\le b(u_-+R-s_+)=b\left(\frac{d+\beta}{a-\epsilon}+R-\frac{d-\beta}{a+\epsilon}\right)<w\quad\text{(see (7))}.$$

Recall the construction of maximal solutions in the proof of Proposition 1 and use the a-priori estimates for the functions $\bar x,\hat x$ to show $u_-+R<t_\infty$ and the last 2 inequalities of the assertion. □

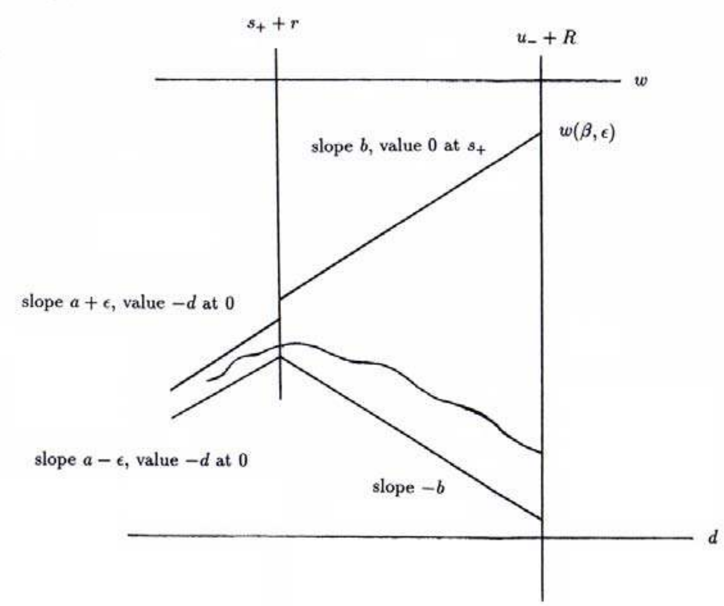

Figure 2

Set

$$w(\beta, \epsilon) = b\big(R + \frac{d+\beta}{a-\epsilon} - \frac{d-\beta}{a+\epsilon}\big) < w.$$

Proposition 5. *There exists $q = q(\phi) \in (u_- + R, t_\infty)$ so that $x(q) = d$,*

$$\dot{x}(t) < 0 \quad on \quad [u_- + R, q], \ \beta \le x(t) \le w(\beta, \epsilon) \quad on \quad [q - R, q],$$

and

$$q \le u_- + R + \frac{w-d}{a-\epsilon}.$$

Proof. From Proposition 3, $\beta \le x(t)$ on $[u_-, r]$. The second inequality from Proposition 4 and $\beta \le x(u_-)$ combined yield $\beta \le x(t)$ on $[u_-, s_+ + r]$. The fourth inequality from Proposition 4 gives $\beta < d < x(t)$ on $[s_+ + r, u_- + R]$. Altogether, $\beta \le x(t)$ on $[u_-, u_- + R]$. Arguing as in the proof of Proposition 3 we obtain from the last two inequalities that for every $t_0 \in (u_- + R, t_\infty)$ with $\beta \le x(t)$ on $[u_- + R, t_0]$, we have $t_0 + r < t_\infty$ and

$$-a - \epsilon < \dot{x}(t) < -a + \epsilon < 0 \quad \text{on} \quad [u_- + R, t_0 + r].$$

This property combined with the preceding propositions yields all assertions except of the last inequality. To prove the latter observe $\beta < d \le x(t)$ on

$[u_- + R, q]$. For $t_0 = q$ the property stated above guarantees $\dot{x}(t) < -a + \epsilon$ on $[u_- + R, q]$. It follows that

$$(q - (u_- + R))(-a + \epsilon) \geq \int_{u_- + R}^{q} \dot{x}(t)dt = x(q) - x(u_- + R) > d - w. \quad \square$$

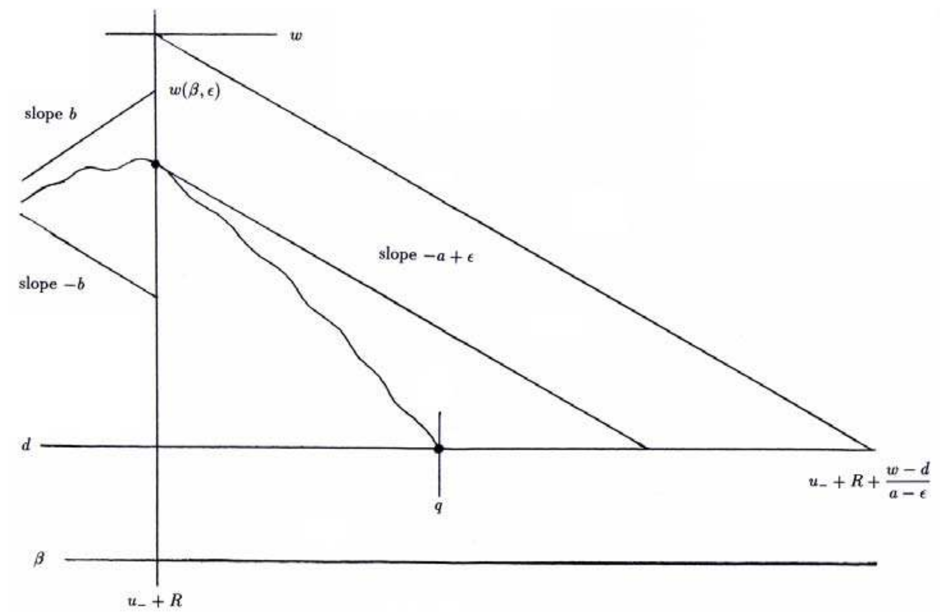

Figure 3

The subset $I_{\beta\epsilon} = \{\phi \in I_\beta : -w(\beta, \epsilon) \leq \phi(t) \text{ on } [-R, 0]\}$ of I_β is closed. The map $I_\beta \ni \phi \mapsto q(\phi) \in (0, \infty)$ given by Proposition 5 has the property that $F(q(\phi), \phi) \in -I_{\beta\epsilon}$ for all $\phi \in I_\beta$. Proceeding as before one shows that for initial data $\phi \in -I_\beta$ and for $0 < \beta < \beta_0, 0 < \epsilon < \epsilon_0, v \in V(\beta, \epsilon)$ there exists $q(\phi) \in (u_- + R, t_\infty^\phi)$ so that x^ϕ satisfies

$$x^\phi(q(\phi)) = -d, \ \dot{x}^\phi(t) > 0,$$

and

$$-w(\beta, \epsilon) \leq x^\phi(t) \leq -\beta \quad \text{on} \quad [q(\phi) - R, q(\phi)].$$

Furthermore,

$$q(\phi) \leq u_- + R + \frac{w - d}{a - \epsilon} \quad \text{and} \quad \beta \leq x^\phi(t) \quad \text{on} \quad [0, h].$$

Therefore, we may consider the map $q : I_\beta \cup (-I_\beta) \ni \phi \mapsto q(\phi) \in (0, \infty)$. The associated map $Q : I_\beta \cup (-I_\beta) \ni \phi \mapsto F(q(\phi), \phi) \in I$ satisfies $Q(I_\beta) \subset -I_{\beta\epsilon}$,

$Q(-I_\beta) \subset I_{\beta\epsilon}$. Notice that $t_\infty^\phi = \infty$ for all $\phi \in I_\beta$. The iterate $P = Q \circ Q$ maps I_β into the closed set $I_{\beta\epsilon} \subset I_\beta$. Observe that fixed points ϕ of P define periodic solutions of the system (1), (2), with minimal period $q(\phi)+q(Q(\phi))$.

Comment. As $v \in V(\beta,\epsilon)$ is odd the symmetry property

$$x^\phi = -x^{-\phi}$$

comes to mind, as well as the possibility to reduce solutions starting in $-I_\beta$ to solutions starting in I_β. The following example, however, shows that in general the symmetry just mentioned is violated. Consider an injective function $v \in V(\beta,\epsilon)$ and $\phi \in I_\beta$ so that for some $\alpha \in (0,c)$ and $t \in (0,r)$, $\phi(\tau) = \alpha(\tau + r - t) - \beta$ on $[-R, -r+t]$.

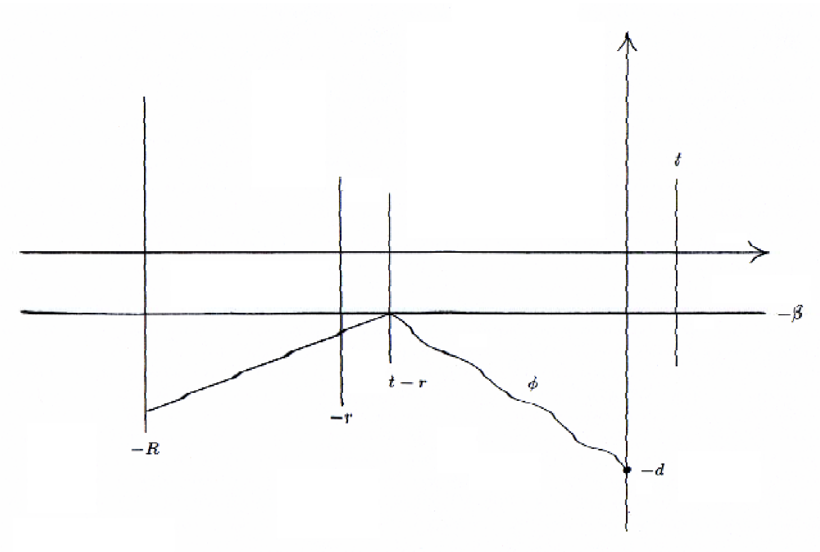

Figure 4

Then $cs^\phi(t-r) = \phi(t-r-s^\phi(t-r)) + \phi(t-r) + 2w$, which gives

$$s^\phi(t-r) = 2\frac{w-\beta}{c+\alpha},$$

and thereby

$$\frac{c}{2}s^\phi(t-r) - w = -\frac{\beta c + \alpha w}{c+\alpha}, \quad \dot{x}^\phi(t) = v\big(-\frac{\beta c + \alpha w}{c+\alpha}\big).$$

Also,

$$c\, s^{-\phi}(t-r) = \alpha\, s^{-\phi}(t-r)) + 2(\beta + w), \quad s^{-\phi}(t-r) = 2\frac{w+\beta}{c-\alpha},$$

and thereby

$$\frac{c}{2}s^{-\phi}(t-r)-w=\frac{\beta c+\alpha w}{c-\alpha},\quad \dot{x}^{-\phi}(t)=v\Big(\frac{\beta c+\alpha w}{c-\alpha}\Big).$$

Consequently,

$$-\dot{x}^{\phi}(t)=-v\Big(-\frac{\beta c+\alpha w}{c+\alpha}\Big)=v\Big(\frac{\beta c+\alpha w}{c+\alpha}\Big)\neq v\Big(\frac{\beta c+\alpha w}{c-\alpha}\Big)=\dot{x}^{-\phi}(t).$$

The next proposition prepares estimates of $Lip(q), Lip(Q), Lip(P)$ in terms of $Lip(v)$ and

$$Lip(v,\beta)=Lip(v|[\beta,\infty)).$$

Proposition 6. *Let* $0<\epsilon<\epsilon_0, 0<\beta<\beta_0, v\in V(\beta,\epsilon)$ *and* $\phi\in I_\beta, \psi\in I_\beta$ *be given. Then* $x=x^\phi, s=s^\phi$ *and* $\bar{x}=x^\psi, \bar{s}=s^\psi$ *have the following properties.*

(*i*) *For all* $t>0$,

$$|\dot{x}(t)-\dot{\bar{x}}(t)|\le Lip(v)\,\frac{c}{c-b}\max_{u\in[t-R,t-r]}|x(u)-\bar{x}(u)|.$$

(*ii*) *If* $t>0$ *and if* $x(t-r-s(t-r))$, $x(t-r)$, $\bar{x}(t-r-\bar{s}(t-r))$, $\bar{x}(t-r)$ *all belong to* $[\beta,\infty)$ *or all belong to* $(-\infty,-\beta]$, *then*

$$|\dot{x}(t)-\dot{\bar{x}}(t)|\le Lip(v,\beta)\,\frac{c}{c-b}\max_{u\in[t-R,t-r]}|x(u)-\bar{x}(u)|.$$

(*iii*) *If* $t_0>0$ *and* $t_1>t_0+R$, *if*

$$\beta\le x(t)\quad\text{and}\quad\beta\le\bar{x}(t)\quad\text{on}\quad[t_0,t_1]$$

or

$$-\beta\ge x(t)\quad\text{and}\quad-\beta\ge\bar{x}(t)\quad\text{on}\quad[t_0,t_1],$$

and if $n\in\mathbb{N}$ *satisfies* $t_0+R+(n-1)r\le t_1\le t_0+R+nr$, *then*

$$\max_{t\in[t_0,t_1]}|x(t)-\bar{x}(t)|\le\Big(1+r\,Lip(v,\beta)\,\frac{c}{c-b}\Big)^n\max_{t\in[t_0,t_0+R]}|x(t)-\bar{x}(t)|.$$

Proof. 1. Proof of (i). Use Proposition 2 (i) and equation (2).

2. Proof of (ii). If $x(t-r-s(t-r))$, $x(t-r)$, $\bar{x}(t-r-\bar{s}(t-r))$, $\bar{x}(t-r)$ all belong to $[\beta,\infty)$, then

$$\frac{c}{2}\,s(t-r)-w=\frac{1}{2}(x(t-r-s(t-r))+x(t-r))$$

and $\frac{c}{2}\,\bar{s}(t-r)-w$ both belong to $[\beta,\infty)$. Use Proposition 2 (i), equation (2), and the definition of $Lip(v,\beta)$. The other case is analogous.

3. Proof of (iii). Suppose x and $\bar{x}$ are above β on $[t_0, t_1]$. Consider first $t \le t_1$ in $[t_0 + R, t_0 + R + r]$. For such t,

$$|x(t) - \bar{x}(t)| \le |x(t_0 + R) - \bar{x}(t_0 + R)| + |\int_{t_0+R}^{t} |\dot{x}(u) - \dot{\bar{x}}(u)|du.$$

Using (ii) we conclude that the last integral is majorized by

$$r Lip(v, \beta)\,\frac{c}{c-b}\max_{u\in[t_0,t-r]}|x(u)-\bar{x}(u)| \le r Lip(v, \beta)\frac{c}{c-b}\max_{u\in[t_0,t_0+R]}|x(u)-\bar{x}(u)|.$$

It follows that

$$|x(t) - \bar{x}(t)| \le \big(1 + r\,Lip(v, \beta)\,\frac{c}{c-b}\big)\ \max_{u\in[t_0,t_0+R]} |x(u) - \bar{x}(u)|.$$

Induction yields

$$\max_{t\in[t_0+R,t_1]} |x(t) - \bar{x}(t)| \le \big(1 + r\,Lip(v, \beta)\,\frac{c}{c-b}\big)^n \max_{u\in[t_0,t_0+R]} |x(u) - \bar{x}(u)|$$

for the integer n with $t_0 + R + (n-1)r \le t_1 \le t_0 + R + nr$. Obviously we also have

$$\max_{t\in[t_0,t_1]} |x(t) - \bar{x}(t)| \le \big(1 + r\,Lip(v, \beta)\,\frac{c}{c-b}\big)^n \max_{u\in[t_0,t_0+R]} |x(u) - \bar{x}(u)|.$$

The case that x and $\bar{x}$ are below $-\beta$ on $[t_0, t_1]$ is analogous. □

4. Lipschitz return maps and stable periodic orbits

Let $0 < \epsilon < \epsilon_0$, $0 < \beta < \beta_0$. Assume that $v \in V(\beta, \epsilon)$ is Lipschitz continuous, i.e., $Lip(v) < \infty$. In order to derive an estimate of $Lip(Q|I_\beta)$ in terms of $Lip(v), Lip(v, \beta), \beta, \epsilon$ we first write the map $Q|I_\beta$ as a composition. Set $F_R = F(R, \cdot)|I_\beta$, $F_- = F(u_-, \cdot)|F_R(I_\beta)$, and consider $F_j : F(u_- + R, I_\beta) \ni \psi \mapsto F(j(\psi), \psi) \in -I_\beta$, where $j : F(u_- + R, I_\beta) \to (0, \infty)$ is given by $j(\psi) = q(\phi) - u_- - R$ for all $\phi \in I_\beta$ with $\psi = F(u_- + R, \phi)$. Notice that for all $\phi, \bar{\phi}$ in I_β with $F(u_- + R, \phi) = F(u_- + R, \bar{\phi})$, $q(\phi) = q(\bar{\phi})$. We have $Q|I_\beta = F_j \circ F_- \circ F_R$. We begin with estimates for $Lip(F_R)$ and $Lip(F_-)$.

Proposition 7. *We have*

$$(c - b - Lip(v, \beta)\,c\,R)Lip(F_R) \le Lip(v, \beta)\,c\,R$$

and

$$Lip(F_-) \le 1 + Lip(v)\,\frac{cr}{c-b}.$$

Proof. 1. Proof of the first assertion. Let $\phi, \bar{\phi}$ in I_β be given. Set $x = x^\phi, s = s^\phi, \bar{x} = x^{\bar{\phi}}, \bar{s} = s^{\bar{\phi}}$. For $-R \le t \le 0$,

$$|(F_R(\phi) - F_R(\bar{\phi}))(t)| \le \int_0^{R+t} |\dot{x}(u) - \dot{\bar{x}}(u)|du \le \int_0^R |\dot{x}(u) - \dot{\bar{x}}(u)|du.$$

Recall $x(u) \le -\beta$ on $[-R, h]$. For $0 < u < R$ we infer

$$x(u - r - s(u-r)) \le -\beta \quad \text{and} \quad x(u-r) \le -\beta.$$

Analogously,

$$\bar{x}(u - r - \bar{s}(u-r)) \le -\beta \quad \text{and} \quad \bar{x}(u-r) \le -\beta$$

for such u. Proposition 6 (ii) gives

$$|\dot{x}(u) - \dot{\bar{x}}(u)| \le Lip(v,\beta)\,\frac{c}{c-b} \max_{\tau \in [u-R, u-r]} |x(\tau) - \bar{x}(\tau)|$$

on $(0, R)$. It follows that

$$\|F_R(\phi) - F_R(\bar{\phi})\| \le Lip(v,\beta)\,\frac{c}{c-b} \max_{t \in [-R,h]} |x(t) - \bar{x}(t)|.$$

Finally, use that the last maximum is bounded by

$$\|\phi - \bar{\phi}\| + \max_{t \in [0,h]} |x(t) - \bar{x}(t)| \le \|\phi - \bar{\phi}| + \|F_R(\phi) - F_R(\bar{\phi})\|.$$

2. Proof of the second assertion. Consider $\psi = F(R,\phi)$ and $\bar{\psi} = F(R, \bar{\phi})$ with $\phi, \bar{\phi}$ in I_β, and $x = x^\phi, \bar{x} = x^{\bar{\phi}}$. For $-R \le t \le 0$,

$$|(F_-(\psi) - F_-(\bar{\psi}))(t)| = |x(R + u_- + t) - \bar{x}(R + u_- + t)|.$$

In case $u_- + t \le 0$ the last term is majorized by $\|\psi - \bar{\psi}\|$. In case $0 < u_- + t$ we have

$$|x(R+u_-+t) - \bar{x}(R+u_-+t)| \le |x(R) - \bar{x}(R)| + \int_R^{R+u_-+t} |\dot{x}(u) - \dot{\bar{x}}(u)|du$$

$$\le \|\psi - \bar{\psi}\| + \int_R^{R+u_-} |\dot{x}(u) - \dot{\bar{x}}(u)|du$$

Proposition 6 (i) permits to majorize the last integral by

$$u_-\, Lip(v)\,\frac{c}{c-b} \max_{u \in [0, R+u_- - r]} |x(u) - \bar{x}(u)|$$

$$\le r\, Lip(v)\,\frac{c}{c-b} \max_{u \in [0,R]} |x(u) - \bar{x}(u)| = Lip(v)\,\frac{cr}{c-b}\,\|\psi - \bar{\psi}\|. \qquad \square$$

Comment. As

$$Lip(v) \ge Lip(v|[-\beta, \beta]) \ge \frac{a - \epsilon}{\beta}$$

becomes large for β small the second inequality of Proposition 7 permits that segments x_t of solutions starting in I_β strongly diverge as long as $R \le t \le R+u_-$. This is also seen from equation (2) with $s(t-r)$ replaced by the right hand side of equation (1): The passage of solutions through the interval $[-\beta,\beta]$ where the response has a large slope leads to large derivatives later, after the delay.

In order to obtain an estimate of $Lip(F_j)$ we first need an estimate of $Lip(j)$.

Proposition 8. *We have*

$$(a-\epsilon)\,Lip(j) \le 1 + Lip(v,\beta)\,\frac{c(w-d)}{(c-b)(a-\epsilon)}\,\big(1+Lip(v,\beta)\,\frac{cr}{c-b}\big)^{1+\frac{w-d}{r(a-\epsilon)}}$$

and

$$Lip(F_j) \le b\,Lip(j) + \big(1 + r\,Lip(v,\beta)\,\frac{c}{c-b}\big)^{1+\frac{w-d}{r(a-\epsilon)}}.$$

Proof. 1. Consider $\psi = F(u_- + R,\phi)$ and $\bar\psi = F(u_- + R,\bar\phi)$ with $\phi,\bar\phi$ in I_β. Set $x = x^\phi, s = s^\phi, \bar x = x^{\bar\phi}, \bar s = s^{\bar\phi}$. Set $q_* = q(\phi), j_* = j(\psi), \bar q = q(\bar\phi), \bar j = j(\bar\psi)$. Assume $\bar j \le j_*$. The equations

$$x(q_*) = d \quad \text{and} \quad j_* = q_* - R - u_-$$

and their analogues for $\bar x, \bar q, \bar j$ yield the estimate

$$\|\psi - \bar\psi\| \ge |x(R+u_-) - \bar x(R+u_-)| = \Big|\int_{R+u_-}^{R+u_-+j_*}\dot x(t)dt - \int_{R+u_-}^{R+u_-+\bar j}\dot{\bar x}(t)dt\Big|$$

$$\ge |\int_{R+u_-+\bar j}^{R+u_-+j_*}\dot x(t)dt| - \int_{R+u_-}^{R+u_-+\bar j}|\dot{\bar x}(t) - \dot x(t)|dt.$$

Propositions 3, 4, 5 combined show that for $R+u_-+\bar j < t < R+u_-+j_*$,

$$\beta \le x(t-r-s(t-r)) \quad \text{and} \quad \beta \le x(t-r).$$

Hence

$$\begin{aligned}\frac{c}{2}\,s(t-r) - w &= \frac{1}{2}\,(x(t-r-s(t-r)) + w + x(t-r) + w) - w\\ &= \frac{1}{2}\,(x(t-r-s(t-r)) + x(t-r)) \ge \beta,\end{aligned}$$

and consequently

$$\dot x(t) = v\left(\frac{c}{2}\,s(t-r) - w\right) \le -a+\epsilon < 0 \quad \text{for such} \quad t.$$

It follows that

$$|\int_{R+u_-+\bar j}^{R+u_-+j_*}\dot x(t)dt| \ge (j_* - \bar j)(a-\epsilon).$$

We infer

$$|j_* - \bar{j}| \le \frac{1}{a-\epsilon}(\|\psi - \bar{\psi}\| + \int_{R+u_-}^{R+u_-+\bar{j}} |\dot{\bar{x}}(t) - \dot{x}(t)|dt).$$

2. Estimate of the last integral. Propositions 3, 4, 5 and the assumption $\bar{j} \le j_*$ permit to apply Proposition 6 (ii). This yields the upper bound

$$\bar{j}\, Lip(v,\beta)\, \frac{c}{c-b} \max_{t\in[u_-,R+u_-+\bar{j}-r]} |x(t) - \bar{x}(t)|$$
$$\le \bar{j}\, Lip(v,\beta)\, \frac{c}{c-b} \max_{t\in[u_-,R+u_-+\bar{j}]} |x(t) - \bar{x}(t)|.$$

Due to Propositions 3, 4, 5, $\beta \le \bar{x}(t)$ on $[u_-, R+u_-+\bar{j}]$ and $\beta \le x(t)$ on $[u_-, R+u_-+j_*] \supset [u_-, R+u_-+\bar{j}]$. Therefore, Proposition 6 (iii) is applicable to $x, \bar{x}, t_0 = u_-, t_1 = u_- + R + \bar{j}$. The integer n with $(n-1)r \le \bar{j} < nr$ satisfies

$$t_0 + R + (n-1)r \le u_- + R + \bar{j} = t_1 < t_0 + R + nr.$$

Proposition 6 (iii) gives

$$\max_{t\in[u_-,R+u_-+\bar{j}]} |x(t) - \bar{x}(t)| \le \big(1 + r\, Lip(v,\beta)\frac{c}{c-b}\big)^n \max_{t\in[u_-,R+u_-]} |x(t) - \bar{x}(t)|$$
$$= \big(1 + rLip(v,\beta)\frac{c}{c-b}\big)^n \|\psi - \bar{\psi}\| \le \big(1 + rLip(v,\beta)\frac{c}{c-b}\big)^{1+\frac{w-d}{r(a-\epsilon)}} \|\psi - \bar{\psi}\|$$

since

$$(n-1)r \le \bar{j} \le \frac{w-d}{a-\epsilon} \quad \text{(Proposition 5)}.$$

It follows that

$$\int_{R+u_-}^{R+u_-+\bar{j}} |\dot{x}(t) - \dot{\bar{x}}(t)|dt$$
$$\le \frac{w-d}{a-\epsilon}\, Lip(v,\beta)\, \frac{c}{c-b}\Big(1 + r\, Lip(v,\beta)\, \frac{c}{c-b}\Big)^{1+\frac{w-d}{r(a-\epsilon)}} \|\psi - \bar{\psi}\|,$$

and the asserted estimate of $Lip(j)$ becomes obvious.

3. Proof of the estimate of $Lip(F_j)$. For $-R \le t \le 0$,

$$|(F_j(\psi) - F_j(\bar{\psi}))(t)| = |x(R+u_-+j_*+t) - \bar{x}(R+u_-+\bar{j}+t)|$$
$$\le |x(R+u_-+j_*+t) - x(R+u_-+\bar{j}+t)| + |x(R+u_-+\bar{j}+t) - \bar{x}(R+u_-+\bar{j}+t)|$$
$$\le b|j_* - \bar{j}| + \max_{u\in[u_-,R+u_-+\bar{j}]} |x(u) - \bar{x}(u)|$$

since $Lip(x) \le b$. Use the estimate of the last term obtained in part 2 to complete the proof. □

Corollary 2. *We have*

$$(c-b-Lip(v,\beta)\,c\,R)Lip(Q|I_\beta) \le Lip(v,\beta)\,c\,R\left(1+Lip(v)\frac{cr}{c-b}\right)$$
$$\times\left(\frac{b}{a-\epsilon}\left(1+Lip(v,\beta)\,\frac{c(w-d)}{(c-b)(a-\epsilon)}\left(1+r\,Lip(v,\beta)\,\frac{c}{c-b}\right)^{1+\frac{w-d}{r(a-\epsilon)}}\right)\right.$$
$$\left.+\left(1+r\,Lip(v,\beta)\,\frac{c}{c-b}\right)^{1+\frac{w-d}{r(a-\epsilon)}}\right).$$

Denote the upper bound of Corollary 2 by $k(v,\beta,\epsilon)$. Proceeding as before one can derive the estimate

$$(c-b-Lip(v,\beta)\,c\,R)Lip(Q|(-I_\beta)) \le k(v,\beta,\epsilon).$$

The relations $P=Q\circ Q$ and $Q(I_\beta)\subset -I_\beta$ then yield

$$(c-b-Lip(v,\beta)\,c\,R)^2 Lip(P|I_\beta) \le k(v,\beta,\epsilon)^2$$

This implies the following result.

Theorem 1. *Let $0<\epsilon<\epsilon_0, 0<\beta<\beta_0$. For every $L\ge\frac{a-\epsilon}{\beta}$ there exists a positive constant $L_\beta\le\frac{c-b}{cR}$ so that for all $v\in V(\beta,\epsilon)$ with*

$$Lip(v)\le L \quad \textit{and} \quad Lip(v,\beta)\le L_\beta$$

we have

$$Lip(P|I_\beta)<1,$$

and the unique fixed point ϕ of P in the closed set $I_{\beta\epsilon}$ coincides with the segment x_0 of a periodic solution (x,s) of the system (1), (2).

Notice that each set $V(\beta,\epsilon)$ contains functions v with $L(v,\beta)>0$ arbitrarily small and functions with $L(v,\beta)=0$. Theorem 1 yields a variety of examples of systems (1, 2) with periodic solutions, including cases where the negative feedback condition

$$\delta v(\delta)<0 \quad \text{for all} \quad \delta\ne 0$$

holds and cases where it is violated. The minimal period of the periodic solution obtained in Theorem 1 is $p=q(\phi)+q(Q(\phi))$. The orbit $\mathcal{O}=\{x_t\in I: t\in[0,p)\}$ of the periodic solution obtained in Theorem 1 from the fixed point of the contraction $P|I_{\beta\epsilon}$ is exponentially stable with asymptotic phase in the following sense.

Corollary 3. *For every $\delta>0$ there is a neighbourhood N of $\mathcal{O}$ in the open subset I of the compact metric space $M\subset C$ so that for all $t\ge 0$ and for all $\psi\in N$,*

$$(t,\psi)\in\Delta \quad \textit{and} \quad dist(F(t,\psi),\mathcal{O})<\delta.$$

There are constants $k_N > 0$ and $\gamma > 0$ so that for every $\psi \in N$ there exists $\alpha \in [0, p)$ with

$$\|F(t, \psi) - x_{t+\alpha}\| \le k_N e^{-\gamma t} \quad \textit{for all} \quad t \ge 0.$$

Corollary 3 is proved by arguments familiar from ordinary differential equations. An earlier version, for delay differential equations with constant delay, is Proposition 3.3 in [9].

Proof of Corollary 3. 1. Set $L = Lip(P|I_\beta) \in [0, 1)$. Let L_0 denote the Lipschitz constant of the map $p_\beta : I_\beta \ni \psi \mapsto q(\psi) + q(Q(\psi)) \in \mathbb{R}$. Notice $p_\beta(\phi) = p$. Let n_0 denote the smallest integer such that $\frac{L_0}{1-L} < n_0 p$. Then $n_0 > 1$. Proposition 2 (iii) yields $c_0 \ge 0$ so that for all $t \in [0, 1 + (2n_0 + 1)p]$ and for all $\psi, \bar{\psi}$ in I with $1 + (2n_0 + 1)p < t_\infty^\psi \le t_\infty^{\bar{\psi}}$,

$$\|F(t, \psi) - F(t, \bar{\psi})\| \le c_0 \|\psi - \bar{\psi}\|.$$

In particular,

$$|x^\psi(t) - x^{\bar{\psi}}(t)| \le c_0 \|\psi - \bar{\psi}\| \quad \text{on} \quad [-R, 1 + (2n_0 + 1)p].$$

2. Let $\delta_0 \in (0, 1]$ be given. Recall $-w(\beta, \epsilon) < x(t) < -\beta$ on $[p - R, p]$, $x(p) = -d$, $\dot{x}(p) > 0$. Let $\delta_1 \in (0, \delta_0)$ be given. Proposition 2 (ii) combined with a compactness argument permits to find an open neighbourhood V of ϕ in I and $\delta_2 \in (0, \delta_1)$ so that for all $\psi \in V$,

$$p + \delta_2 < t_\infty^\psi,$$

$$|x^\psi(t) - x(t)| < \delta_1 \quad \text{and} \quad -w(\beta, \epsilon) < x^\psi(t) < -\beta \quad \text{on} \quad [p - R - \delta_2, p + \delta - 2],$$

$$x^\psi(p - \delta_2) < d < x^\psi(p + \delta_2).$$

It follows that for such ψ there exists $p_\psi \in (p - \delta_2, p + \delta_2)$ with $x^\psi(p_\psi) = -d$ and $F(p_\psi, \psi) \in I_{\beta\epsilon}$. We have

$$\|F(p_\psi, \psi) - \phi\| \le \|F(p_\psi, \psi) - F(p_\psi, \phi)\| + \|F(p_\psi, \phi) - F(p, \phi)\|$$

$$\le \delta_1 + b|p_\psi - p| \le \delta_1 + b\delta_2 \quad \text{(see Proposition 1)}.$$

Therefore, we can choose V so small that in addition

$$\|F(p_\psi, \psi) - \phi\| < \delta_0 \quad \text{on} \quad V.$$

Using Proposition 2 (ii) once again we infer that for every $t \in [0, p)$ there exists an open neighbourhood V_t of x_t in I so that for all $\psi \in V_t$,

$$\|\psi - x_t\| < \delta_0 \quad \text{and} \quad F(p - t, \psi) \in V.$$

Set

$$N = \bigcup_{0 \le t < p} V_t.$$

It follows that for every $\psi \in N$ there exist $u^\psi \in [0, 2p+1]$ and $u_\psi \in [0, p)$ with

$$\|\psi - F(u_\psi, \phi)\| < \delta_0 \quad \text{and} \quad \|F(u^\psi, \psi) - \phi\| < \delta_0,\ F(u^\psi, \psi) \in I_{\beta\epsilon}.$$

In particular, $t_\infty^\psi = \infty$ for all $\psi \in N$. Using part 1 we infer that for $\psi \in N$ and $0 \le t \le 1 + (2n_0+1)p$,

$$\text{dist}(F(t,\psi), \mathcal{O}) \le c_0\|\psi - F(u_\psi, \phi)\| \le c_0\delta_0.$$

3. For $\chi \in I_{\beta\epsilon}$ with $\|\chi - \phi\| < \delta_0$, consider the sequences given by

$$\chi_0 = \chi, \quad \chi_{n+1} = P(\chi_n), \quad p_n = p_\beta(\chi_n).$$

Then $\chi_{n+1} = F(p_n, \chi_n)$, $\|\chi_n - \phi\| \le L^n\|\chi - \phi\| \le L^n\delta_0$,

$$|p_n - p| \le L_0\|\chi_n - \phi\| \le L_0L^n\|\chi - \phi\| \le L_0L^n\delta_0$$

and

$$\Big|\sum_{\nu=0}^{n}(p_\nu - p)\Big| \le \sum_{\nu=0}^{n}|p_\nu - p| \le \frac{L_0\delta_0}{1-L} \le \frac{L_0}{1-L} < n_0p.$$

Set $\alpha_0 = \sum_{\nu=0}^\infty(p_\nu - p)$. Observe that $\alpha_0 + n_0p > 0$. For every integer $n \ge 0$,

$$\begin{aligned}
&\|F((n+n_0)p, \chi) - F(\alpha_0 + n_0p, \phi)\| \\
&= \Big\|F\Big(n_0p + \sum_{\nu=0}^{n-1}(p - p_\nu) + \sum_{\nu=0}^{n-1}p_\nu, \chi\Big) - F(\alpha_0 + n_0p, \phi)\Big\| \\
&= \Big\|F\Big(n_0p + \sum_{\nu=0}^{n-1}(p - p_\nu), \chi_n\Big) - F(\alpha_0 + n_0p, \phi)\Big\| \\
&\le \Big\|F\Big(n_0p + \sum_{\nu=0}^{n-1}(p - p_\nu), \chi_n\Big) - F\Big(n_0p + \sum_{\nu=0}^{n-1}(p - p_\nu), \phi\Big)\Big\| \\
&+ \Big\|F\Big(n_0p + \sum_{\nu=0}^{n-1}(p - p_\nu), \phi\Big) - F(\alpha_0 + n_0p, \phi)\Big\| \\
&\le c_0\|\chi_n - \phi\| + b\Big|\sum_{\nu=0}^{n-1}(p - p_\nu) - \alpha_0\Big| \quad \text{(see part 1 and Proposition 1)} \\
&\le c_0L^n\delta_0 + b\sum_{\nu=0}^{\infty}L_0\delta_0L^\nu \le L^n\Big(c_0 + \frac{bL_0}{1-L}\Big)\delta_0.
\end{aligned}$$

4. For $\psi \in N$, the segment $\chi = F(u^\psi, \psi)$ belongs to $I_{\beta\epsilon}$, and $\|\chi - \phi\| < \delta_0$, $0 \le u^\psi \le 2p+1$. Consider $t \ge u^\psi + n_0p$. Then

$$\text{dist}(F(t,\psi), \mathcal{O}) \le \|F(t - u^\psi, \chi) - F(t - u^\psi + \alpha_0 + n_0p, \phi)\|.$$

For nonnegative integers n with $(n_0+n)p \le t-u^\psi < (n_0+(n+1))p$, we get

$$\begin{aligned}
&\operatorname{dist}(F(t,\psi),\mathcal{O}) \le \|F(t-u^\psi-n_0p-np, F((n_0+n)p,\chi))\\
&- F(t-u^\psi-n_0p-np, F(\alpha_0+n_0p,\phi))\|\\
&\le c_0\|F((n_0+n)p,\chi)-F(\alpha_0+n_0p,\phi)\| \quad \text{(see part 1)}\\
&\le c_0\big(c_0+\frac{bL_0}{1-L}\big)L^n\delta_0 \quad \text{(see part 3).}
\end{aligned}$$

The intervals $[0, 1+(2n_0+1)p]$ and $[u^\psi+n_0p,\infty)$ overlap. Now it becomes obvious how to complete the proof of the first statement in Corollary 3. To prove the last statement of Corollary 3, choose $n_1 \in \mathbb{N}$ with

$$\alpha = n_1p - u^\psi + \alpha_0 + n_0p > 0.$$

For $t \ge u^\psi + n_0p$ and $n \in \mathbb{N}_0$ as above we get

$$\begin{aligned}
&\|F(t,\psi)-x_{t+\alpha}\| = \|F(t-u^\psi-n_0p-np, F((n_0+n)p,\chi))\\
&\quad - F(t-u^\psi+n_1p+\alpha_0+n_0p,\phi)\|\\
&= \|F(t-u^\psi-n_0p-np, F((n_0+n)p,\chi))\\
&\quad - F(t-u^\psi-n_0p-np, F(\alpha_0+n_0p,\phi))\|\\
&\le c_0\big(c_0+\frac{bL_0}{1-L}\big)L^n\delta_0 \quad \text{(see the estimate above).}
\end{aligned}$$

We have $L^n = e^{n\log L}$ and $\frac{t-u^\psi}{p} - n_0 - 1 < n$ which gives

$$L^n < e^{\frac{t\log L}{p}} e^{-\left(\frac{u^\psi}{p}+n_0+1\right)\log L} \le e^{\frac{t\log L}{p}} e^{-\left(\frac{2p+1}{p}+n_0+1\right)\log L}.$$

Set $\gamma = -\frac{\log L}{p}$. Then

$$\|F(t,\psi)-x_{t+\alpha}\| \le e^{-\gamma t}c_0\big(c_0+\frac{bL_0}{1-L}\big)e^{-(\frac{2p+1}{p}+n_0+1)\log L}\ \delta_0$$

for $t \ge u^\psi + n_0p$. Using the compactness of $\mathcal{O}$ and $\operatorname{dist}(F(t,\psi),\mathcal{O}) \le c_0\delta_0$ on $[0, 1+(2n_0+1)p] \supset [0, u^\psi+n_0p]$ (see part 2), one easily completes the proof.

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