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Stable Phase Retrieval in Infinite Dimensions

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Abstract
The problem of phase retrieval is to determine a signal $f \in \mathcal{H}$, with $\mathcal{H}$ a Hilbert space, from intensity measurements $|F(\omega)|$, where $F(\omega) := \langle f, \varphi_\omega \rangle$ are measurements of f with respect to a measurement system $(\varphi_\omega)_{\omega\in\Omega} \subset \mathcal{H}$. Although phase retrieval is always stable in the finite-dimensional setting whenever it is possible (i.e. injectivity implies stability for the inverse problem), the situation is drastically different if $\mathcal{H}$ is infinite-dimensional: in that case phase retrieval is never uniformly stable (Alaifari and Grohs in SIAM J Math Anal 49(3):1895–1911, 2017; Cahill et al. in Trans Am Math Soc Ser B 3(3):63–76, 2016); moreover, the stability deteriorates severely in the dimension of the problem (Cahill et al. 2016). On the other hand, all empirically observed instabilities are of a certain type: they occur whenever the function $|F|$ of intensity measurements is concentrated on disjoint sets $D_j \subset \Omega$, i.e. when $F = \sum_{j=1}^k F_j$ where each F_j is concentrated on D_j (and $k \geq 2$). Motivated by these considerations, we propose a new paradigm for stable phase retrieval by considering the problem of reconstructing F up to a phase factor that is not global, but that can be different for each of the subsets D_j, i.e. recovering F up to the equivalence

$$F \sim \sum_{j=1}^{k} e^{\mathrm{i}\alpha_j} F_j.$$

We present concrete applications (for example in audio processing) where this new notion of stability is natural and meaningful and show that in this setting stable phase retrieval can actually be achieved, for instance, if the measurement system is a Gabor frame or a frame of Cauchy wavelets.

Keywords Phase retrieval · Fourier optics · Stability · Entire functions

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1 Introduction

1.1 Problem Formulation

Suppose we are given a complex-valued function $F : \Omega \to \mathbb{C}$ on some (discrete or continuous) domain Ω, and we can observe only its absolute values $|F|$. The problem of phase retrieval is to reconstruct F from these measurements, up to a global phase (meaning that the functions F and $e^{\mathrm{i}\alpha}F$, $\alpha \in \mathbb{R}$, are not distinguished).

Such problems are encountered in a wide variety of applications, ranging from X-ray crystallography and microscopy to audio processing and deep learning algorithms [15,26,36,39]; accordingly, a large body of literature treating the mathematical and algorithmic solution of phase retrieval problems exists, with new approaches emerging in recent years [6,9,11,27,40].

In these applications, the domain of definition Ω is a finite set, for example $\Omega = \{1, \ldots, N\}$, and the function F arises from a finite number of linear measurements

$$F(k) = \langle x, a_k \rangle := \sum_{l=1}^{d} x_l \, \overline{(a_k)_l}, \quad a_k \in \mathbb{C}^d, \; k = 1, \ldots, N$$

of some signal $x \in \mathbb{C}^d$ which one seeks to recover. Such problems arise as finite approximations to various real-world problems; in diffraction imaging, for instance, the set-up can be interpreted as measuring the diffraction pattern of x, modulated with a number of different filters.

Classically, the numerical solution of phase retrieval problems is treated via alternating projection algorithms that are simple to implement but lack a theoretical understanding [17,19]. More recent work [11] has introduced an algorithm named PhaseLift, based on a reformulation of the N-dimensional phase retrieval problem as a semidefinite optimization problem in an N^2-dimensional space. As shown in [11], PhaseLift succeeds with high probability in recovering the signal x, up to a global phase, in a randomized setting (meaning that the vectors $a_1, \ldots, a_N$ are drawn at random); moreover, PhaseLift is stable if the measurements $|\langle x, a_n \rangle|$ are corrupted by additive noise. More recently, it has been shown that gradient descent algorithms, together with a careful guess for their starting value, achieve the same theoretical guarantees while being vastly more efficient [12].

1.2 Infinite-Dimensional Phase Retrieval

The vector $x \in \mathbb{C}^d$ typically arises as a digital representation of a physical quantity. For instance, x could represent a finite-dimensional approximation of a continuous function describing an infinite-dimensional object. This naturally leads one to consider the more general infinite-dimensional phase retrieval problem, where one seeks to recover a signal $f \in \mathcal{H}$, with $\mathcal{H}$ a (possibly infinite-dimensional) Hilbert space, from the phaseless measurements $|F(\omega)|$, with

$$F(\omega) := \langle f, \varphi_\omega \rangle, \quad \omega \in \Omega, \tag{1.1}$$

and where $(\varphi_\omega)_{\omega\in\Omega} \subset \mathcal{H}$ is a (possibly infinite) parameterized family of measurement functions, typically normalized so that $\|\varphi_\omega\| = 1$ for all $\omega \in \Omega$.

We mention a few examples.

- Consider the classical n-dimensional phase retrieval problem of reconstructing a function f from intensity measurements of its Fourier transform $\widehat{f}$. For a compact subset $D \subset \mathbb{R}^n$, let $\mathcal{H} = L^2(D)$ and consider $f \in \mathcal{H}$. Let $F(\omega) = \widehat{f}(\omega)$, $\omega \in \Omega$, where Ω is either all of $\mathbb{R}^n$ or a suitable discrete subset of $\mathbb{R}^n$ (since f has compact support, there exist $\varphi_\omega \in \mathcal{H}$ such that $F(\omega) = \langle f, \varphi_\omega \rangle$). Applications of this setup include coherent diffraction imaging, X-ray crystallography and many more, in which one typically can measure only intensities, corresponding to $|\langle f, \varphi_\omega \rangle|^2$. The classical phase retrieval problem is in general not uniquely solvable [1]; recent work [34] has established the uniqueness of the solution, if the intensities of the Fourier transforms of certain structured modulations of f are measured instead.
- Related to the previous example, the work [38] studies the reconstruction of a bandlimited *real-valued* function f from unsigned samples $(|f(\omega)|)_{\omega\in\Omega}$ with Ω a suitable (discrete) sampling set; more general settings are considered in [2,14]. Note that the real-valued case (where only the sign ± 1 is missing from each measurement) is qualitatively simpler than the complex-valued case where each measurement lacks a phase factor $e^{\mathrm{i}\alpha}$, $\alpha \in \mathbb{R}$.
- In order to overcome the problem of nonuniqueness of the classical phase retrieval problem and to be able to apply techniques in diffraction imaging also to extended objects, one often records local illuminations of different overlapping parts of the object, which mathematically amounts to a windowed (or short-time) Fourier transform (STFT) $F = V_g f$, where for $f \in L^2(\mathbb{R})$

$$V_g f(x, y) := \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi \mathrm{i} t y}dt \tag{1.2}$$

  is defined by the window $g \in L^2(\mathbb{R})$ and the parameters (x, y) may vary over a discrete or continuous subset of $\mathbb{R}^2$. See [36] for an excellent survey on phase retrieval from STFT measurements.
- Another instance of phase retrieval from STFT measurements arises in audio processing applications involving phase vocoders. A phase vocoder [18] is a tool that allows to modify an audio signal f by transforming its STFT. Given f, a phase vocoder first calculates $V_g f(x, y)$ and then modifies it to some $H(x, y)$ before it transforms back to the time domain by taking the inverse (discrete) STFT of H. Typical modifications include time scaling and pitch shifting. In general, the modified H may not result in an STFT of *any* signal. This leads to the so-called *phase coherence* problem [30] in which one aims to make modifications such that the modified H is an approximate STFT. One possible approach is to modify the amplitude $|F(x, y)|$ only in a first step to obtain $|H(x, y)|$ and then to recover the phase of $H(x, y)$ in a coherent way.
- More recent work [39] seeks to reconstruct a signal $f \in L^2(\mathbb{R})$ from the magnitudes $|F(x, 2^j)|$ of semidiscrete wavelet measurements, where $F(x, 2^j) =$

$|w_\psi f(x, 2^j)|$, with $j \in \mathbb{N}$, $x \in \mathbb{R}$ and $w_\psi f(x, y) := \int_{\mathbb{R}} f(t)|y|^{1/2}\overline{\psi(y(t-x))}dt$;[1] the collection of these magnitudes is sometimes called the *scalogram*. The corresponding phase retrieval problem arises in, e.g. the reconstruction of f from the output of its so-called scattering transform as defined in [31].

In all these examples, it is extremely challenging to establish whether f is uniquely determined, up to a global phase; the problem is still not well understood except in special cases.

1.3 (In-)stability of (In-)finite-Dimensional Phase Retrieval

Even if the uniqueness problem was completely solved, this would, however, not yet be sufficient for applications. Since physical measurements are always corrupted by noise and/or uncertainties and numerical algorithms always introduce rounding errors, solving a real-world phase retrieval problem mandates a reconstruction that is stable, meaning that there should exist a (moderate) constant $C > 0$ such that

$$\inf_{\alpha \in \mathbb{R}} \left\| F - e^{\mathrm{i}\alpha} G \right\|_{\mathcal{B}} \leq C \,\| |F| - |G| \|_{\mathcal{B}'} , \tag{1.3}$$

for $\mathcal{B}$, $\mathcal{B}'$ suitable Hilbert (or Banach) spaces.

For phase retrieval problems in spaces of finite (and fixed) dimensions, stability and uniqueness typically go hand in hand [8,11]. The situation changes drastically when we consider infinite-dimensional spaces. A central finding of [4,10] is that *all infinite-dimensional phase retrieval problems are unstable* and that *the stability of finite-dimensional phase retrieval problems deteriorates severely as the dimension grows*.

Example 1.1 (*Stability deterioration as the dimension grows*) We borrow the following example from recent work [10, Example 2.11] to which we refer for more detail. Consider the real-valued Paley–Wiener space

$$PW = \{f \in L^2(\mathbb{R}, \mathbb{R}) : \operatorname{supp} \widehat{f} \subseteq [-\pi, \pi]\},$$

and the measurement vectors $\{\varphi_n\}_{n\in\mathbb{Z}}$ of elements $\varphi_n := \operatorname{sinc}(\cdot - \frac{n}{4})$. As shown in [38], each $f \in PW$ is uniquely determined by $\{|\langle f, \varphi_n\rangle|\}_{n\in\mathbb{Z}}$, up to a global sign ± 1 (note that this set-up is real-valued). More precisely, suppose that $f, g \in PW$ with $|\langle f, \varphi_n\rangle| = |\langle g, \varphi_n\rangle|$ for all $n \in \mathbb{Z}$. Then, there exists $\sigma \in \{-1, 1\}$ with $f = \sigma \cdot g$.

Now, we consider an approximate problem restricted to the finite-dimensional subspaces $V_n \subset PW$, defined as

$$V_n := \operatorname{span}\{\varphi_{4\ell} : \ell \in [-n, n]\}.$$

The space V_n consists of $f \in PW$ for which $\widehat{f}$ is the restriction to $[-\pi, \pi]$ of a trigonometric polynomial of degree n. Then, [10, Example 2.11] gives the explicit construction of f_m, $g_m \in V_{2m}$ such that, for some $m-$independent constant $c > 0$,

[1] Note that our $w_\psi f(x, y)$ corresponds to $W_\psi f(x, 1/y)$ in the notation of [39].

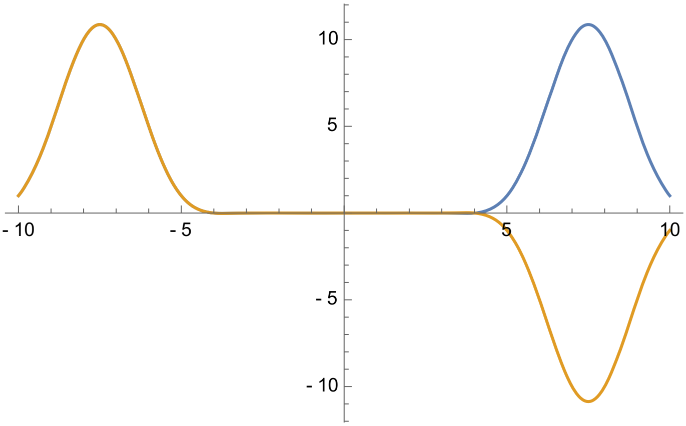

Fig. 1 Functions f_5 and g_5 satisfying (1.4) and with $\operatorname{supp} \widehat{f_5} = \operatorname{supp} \widehat{g_5} = [-\pi, \pi]$. While $\sup_{n\in\mathbb{Z}} \big| |f_5(\frac{n}{4})| - |g_5(\frac{n}{4})| \big|$ is small, $\|f_5 - g_5\|_{L^2(\mathbb{R})}$ and $\|f_5 + g_5\|_{L^2(\mathbb{R})}$ are not

$$\min_{\tau\in\{\pm 1\}} \|f_m - \tau g_m\|_{L^2(\mathbb{R})} > c(m+1)^{-1} 2^{3m} \, \|(|\langle f_m, \varphi_n\rangle| - |\langle g_m, \varphi_n\rangle|)_{n\in\mathbb{Z}}\|_{\ell^2(\mathbb{Z})}, \quad \forall m \in \mathbb{N}. \qquad (1.4)$$

Comparing this with (1.3), we find that the corresponding Lipschitz constant C thus decays at least exponentially fast as the dimension of the problem grows.

Figure 1 shows the plot of the functions f_n and g_n for $n = 5$, illustrating that the two functions have almost identical absolute value despite being significantly different from each other. (Note that the two functions in this example are large on two distant domains and small in between. For the real-valued setting it was shown in [4] that this is the generic form of instabilities; in [23] it is shown that multi-component signals are likewise the generic example for instability in the Gabor transform case.) Consequently, *stable phase retrieval is not possible for infinite-dimensional problems, or even for their fine-grained (and thus finite- but high-dimensional) approximations*.

1.4 Three Observations and a New Paradigm

The instability for infinite-dimensional phase retrieval problems and for their high-resolution approximations makes one wonder whether phase retrieval is even advisable in these situations. It is instructive, however, to take a closer look at how this instability manifests itself in concrete phase retrieval attempts. We offer the following three observations.

1. One way to construct phase retrieval problems leading to instabilities is to consider functions $F = \sum_{j=1}^{k} F_j$ with F_j concentrated on disjoint sets D_j that are far apart from each other. In the sequel, we will occasionally refer to functions of this form as multi-component functions. Clearly, any function of the form

$$G := \sum_{j=1}^{k} e^{\mathrm{i}\alpha_j} F_j \tag{1.5}$$

for any $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, will result in an instability: the absolute values of F, G will be very close, due to the fact that the F_j's are concentrated on well-separated disjoint sets, but $F - e^{\mathrm{i}\gamma} G$ need not be small at all, even for the optimal choice of γ.

The functions constructed in Example 1.1 are of this form with $k = 2$. In fact, in the general real-valued case it can be shown that *all* instabilities arise in this way [4]. In the complex case, it is not known whether this is the case as well.

2. One can investigate how existing concrete phase retrieval algorithms deal with finite-dimensional approximations to the multi-component F introduced above, under item 1. Figure 2 gives a typical albeit simplistic example. Consider an analytic[2] signal f, e.g. as in Fig. 2a, whose Gabor transform $F = V_\varphi f$ (as in Definition 1.2 with $\varphi = e^{-\pi t^2}$) has two disconnected components F_1, F_2, s.t. $F = F_1 + F_2$, see Fig. 2b. Given the Gabor transform measurements $|F| = |V_\varphi f|$, a reconstruction f^{rec} is obtained using the phase retrieval algorithm in [39], and the corresponding code from http://www.di.ens.fr/~waldspurger/wavelets_phase_retrieval.html.[3] The relative error $\|f - f^{rec}\|/\|f\|$ in time domain is 8.61×10^{-1}, whereas the relative error $\||F| - |F^{rec}|\|/\|F\|$ in the Gabor transform measurements is 1.27×10^{-5}. The large difference in the time domain (the ratio of the relative errors exceeds 5×10^4; see also Fig. 2c) is due to a non-uniform but piecewise constant phase shift in the time–frequency domain. Let F_1^{rec}, F_2^{rec} be the two components of F^{rec} corresponding to F_1, F_2. As shown in Fig. 2d, F_1 and F_1^{rec} differ by only a phase factor $e^{\mathrm{i}\alpha_1}$; similarly, F_2 and F_2^{rec} differ by $e^{\mathrm{i}\alpha_2}$; however, $\alpha_1 \neq \alpha_2$. So although it is hopeless to expect that any numerical algorithm could stably distinguish such a multi-component function from $\sum_{j=1}^k e^{\mathrm{i}\alpha_j} F_j$, algorithmic reconstruction *up to the equivalence* $\sum_{j=1}^k F_j \sim \sum_{j=1}^k e^{\mathrm{i}\alpha_j} F_j$ seems to work quite well.
3. Being able to reconstruct (if this is indeed feasible) multi-component functions of the type $\sum_{j=1}^k F_j$ up to the equivalence $\sum_{j=1}^k F_j \sim \sum_{j=1}^k e^{\mathrm{i}\alpha_j} F_j$ is of interest only if this equivalence is itself meaningful.

Our third observation is that this is indeed the case for some applications. We list two examples here.

Our first example is concerned with coherent diffraction imaging. Measurements of X-ray diffraction intensities by complicated objects allow reconstruction of the object under certain constraints on the object; see, e.g. [29] for a mathematical uniqueness result, or [32] for an algorithm effective for fine-grained reconstruction on physical data sets that are supported in a finite volume, without the exact location of this support being known. In its most stripped-down form, the problem

[2] i.e. $\widehat{f}(\omega) = 0, \forall \omega < 0$,

[3] The original algorithm works on magnitude measurements of wavelet transforms such as Morlet wavelets and Cauchy wavelets. Here we apply it to dyadic Gabor wavelet, where the phenomenon of phase difference between the initial and reconstructed signal persists.

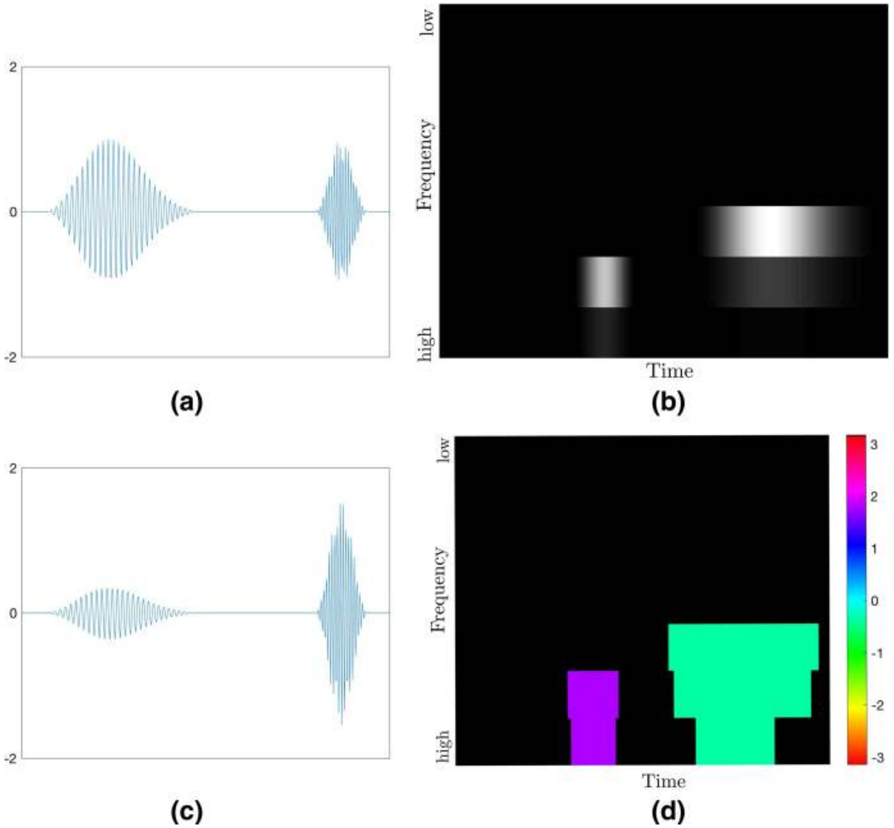

Fig. 2 Phase retrieval on the Gabor measurements $|V_\varphi f|$ of an analytic function f; note that in **d**, $\arg(V_\varphi f / V_\varphi f^{rec}) = \alpha_j$, on the domain where F_j is large, $j = 1, 2$. The Gabor measurements $|V_\varphi f|$ consist of two components that are localized and well separated in time, as illustrated by **a** and **b**. On the measurements of $|V_\varphi f|$ shown in (2b), we applied the algorithm in [39] to reconstruct a candidate f^{rec}, which is markedly different from f, as shown by **c**. However, a careful analysis of each of the components separately shows that the only difference lies in a different phase factor (see **d**): $f^{rec} = e^{i\gamma_1} f_1 + e^{i\gamma_2} f_2$ for some $\gamma_1 \neq \gamma_2$, whereas $f = f_1 + f_2$. **a** real(f) in time domain. **b** $|V_\varphi f|$ in time–frequency (TF) domain of Gabor transform. **c** real$(f - f^{rec})$ in time domain. **d** $\arg(V_\varphi f / V_\varphi f^{rec})$ in TF domain

consists in reconstruction of a function f supported on a compact domain Ω from measurements of the magnitude of its Fourier transform, $|\widehat{f}(\xi)|$. For the plain-vanilla scattering implementation, the physical object to be reconstructed is illuminated by a plane wave. If the object is more extended, illumination by more narrowly concentrated beams might be easier to achieve; one then acquires scattering intensity data for each of several different beam illuminations, which corresponds to replacing the Fourier transform by an STFT. The methodology which we just described is widely used, for example, in Fourier ptychography [25,35,42].

If the scene to be reconstructed consisted of several disjoint objects, separated by "empty" space (the example in Fig. 1 in [32] illustrates such an example), then reconstruction of the individual objects might be numerically and mathematically much easier if it were allowed to reconstruct each object up to a uniform phase (for complex f) or up to a uniform sign (for real f). The simulation illustrated in Fig. 2, for a one-dimensional Gabor transform, suggests as much.

Our second example is concerned with audio processing. It is well known that human audio perception is insensitive to a "global phase change". One way to show this is to start with a (real-valued) audio signal $f(t)$, with Fourier transform $\widehat{f}(\xi)$, and carry out the following operations: first, take its analytic representation f_a by disregarding its negative frequency components: $\widehat{f_a}(\xi) := \widehat{f}(\xi)\chi_{\xi>0}$; next multiply it by an arbitrary (but fixed) phase $e^{\mathrm{i}\alpha}$, $\widehat{f_a^\alpha}(\xi) := e^{\mathrm{i}\alpha}\widehat{f_a}(\xi)$. Finally, we turn it back into the Fourier transform of a real-valued function f^α by "symmetrizing", i.e. by setting $\widehat{f^\alpha}(\xi) = e^{\mathrm{i}\alpha}\widehat{f}(\xi)\chi_{\xi>0} + e^{-\mathrm{i}\alpha}\overline{\widehat{f}(-\xi)}\chi_{\xi<0}$ (note that $\overline{\widehat{f}(-\xi)} = \widehat{f}(\xi)$ because f is real-valued). Equivalently, f^α can be expressed in terms of the original signal f as $f^\alpha(t) = \cos\alpha \cdot f(t) + \sin\alpha \cdot (Hf)(-t)$, where Hf is the Hilbert transform of f. Then, even though the plot of f is typically very different from that of f^α (if α differs significantly from a multiple of 2π), the two sound the same to the human ear, making them equivalent for most practical applications. Consider now an audio signal f consisting of two "bursts" of sound, separated by a short stretch of silence, i.e. $f(t) = f_1(t) + f_2(t)$, with supp $f_1 = [t_1, T_1]$ and supp $f_2 = [t_2, T_2]$ where $t_2 - T_1 > \tau$ for some pre-assigned positive τ (typically of the order of a few tenths of seconds). Figure 3a plots such an example, for the utterance "cup, luck", retrieved from the database at http://www.antimoon.com/how/pronunc-soundsipa.htm, with "cup" corresponding to f_1, "luck" to f_2. Because both f_1 and f_2 are highly oscillatory (as is customary for audio signals), Hf_1 and Hf_2 both have fast decay and are negligibly small outside supp $f_1 = [t_1, T_1]$ and supp $f_2 = [t_2, T_2]$, respectively. For such signals f, one can pick two *different* phases α_1 and α_2 and construct $f^{\alpha_1,\alpha_2} = f_1^{\alpha_1} + f_2^{\alpha_2}$; the resulting audio signals again sound exactly the same as the original f. On https://services.math.duke.edu/~rachel/research/PhaseRetrieval/acoustic_result/acoustic_result.html, one can download and/or listen to f and f^{α_1,α_2}.

We further note that signals remain undistinguishable to the human ear under a more general class of transformations: even for signals $f = \sum_{j=1}^J f_j$ with $J > 2$ components, in which the f_j correspond to components F_j that are separated in the time–frequency domain (but not necessarily in time, or in frequency) replacing each F_j by $e^{\mathrm{i}\alpha_j}F_j$ results in a signal that sounds exactly like the original signal f (see Fig. 4 for an example of such a signal and its Gabor transform; on https://services.math.duke.edu/~rachel/research/PhaseRetrieval/acoustic_result/acoustic_result.html one can listen to this example and component-wise phase-shifted versions).

If one seeks to reconstruct f only within the equivalence class of audio signals that are indistinguishable from f by human perception, then it is thus natural to treat all the functions of type (1.5) as equivalent, for all choices of α_j.

These observations suggest a new paradigm for stable phase retrieval: rather than aiming for bounds of the form (1.3) (which we know do not exist), we investigate a weaker form of stability that would be sufficient for this type of application: we study

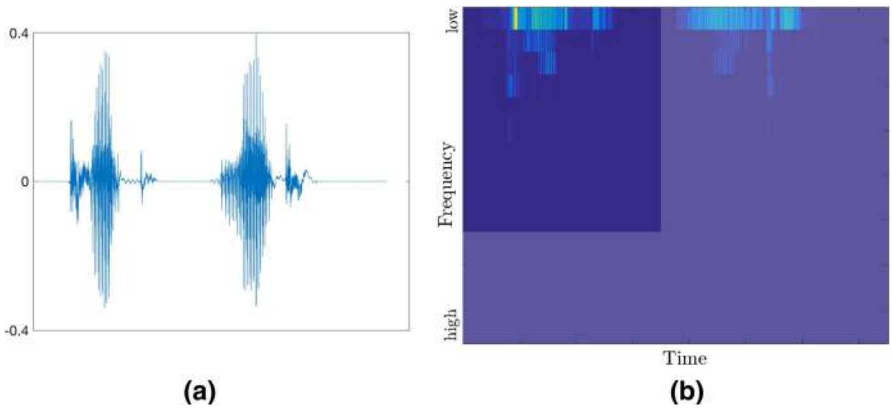

Fig. 3 Audio signal "cup luck" and its Gabor measurements; both in the time domain and in the time–frequency plane, the two components are well separated. **a** Audio signal f in time domain. **b** Time–frequency plot of the magnitude $|F|$ of the Gabor representation of f, with one separated component highlighted

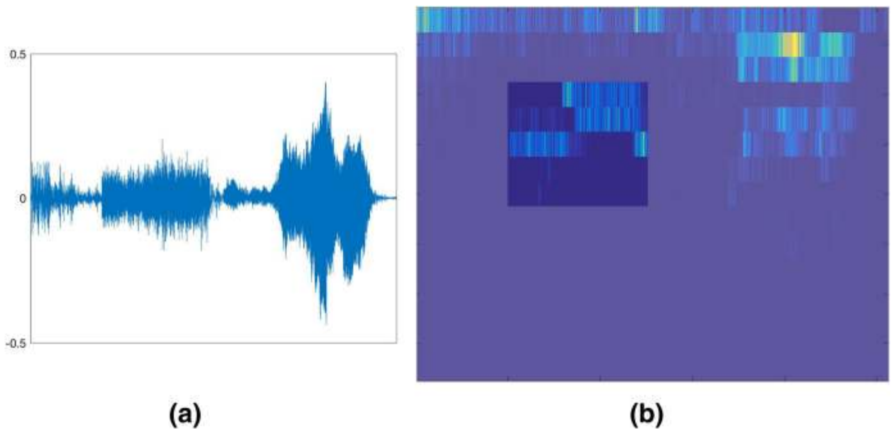

Fig. 4 Audio signal of a sound mixture of thunder, a bird call and a baby crying, together with its Gabor measurements; although in this example there is no clear separation in either time or frequency, one can carve out separated components in the time–frequency plane (one of them is highlighted here). **a** Audio signal f in time domain. **b** Time–frequency plot of the magnitude $|F|$ of the Gabor representation of f, with one separated component highlighted

the stability of phase retrieval *subject to the equivalence* $\sum_{j=1}^{k} F_j \sim \sum_{j=1}^{k} e^{\mathrm{i}\alpha_j} F_j$, that is, bounds of the form

$$\inf_{\alpha_1,\ldots,\alpha_k\in\mathbb{R}} \sum_{j=1}^{k} \left\| F_j - e^{\mathrm{i}\alpha_j} G_j \right\|_{\mathcal{B}} \le C \, \| |F| - |G| \|_{\mathcal{B}'} \,, \tag{1.6}$$

where $\mathcal{B}$, $\mathcal{B}'$ are suitable Hilbert (or Banach) spaces and F_j, G_j any pairs of functions which have their essential support contained in sets D_j.

The question of whether bounds of the form (1.6) can actually be established for examples of practical interest will be the main subject of this article.

1.5 Stability for Atoll Functions

To study this question mathematically, we first need to make it more precise. Figure 4 suggests that a realistic model for Gabor transform measurements on acoustic signals are functions $\sum_{j=1}^{k} F_j$ where each F_j is "large" on a domain D_j, which we shall interpret as a strictly positive lower bound on $|F_j|$. In practice, we expect that F_j may still have zeroes within D_j, which means that there could be "holes" in D_j (reasonably small neighbourhoods of these zeroes) on which $|F_j|$ could *not* be bounded below away from zero. This motivates the following definition:

Definition 1.2 (Atoll domains) Let $D \subset \mathbb{C}$ be a domain. A domain $D_0 \subset D$ is called a *hole* of D if D_0 is simply connected and $\overline{D_0} \subset D$. By definition, D is called a domain with disjoint holes $(D_0^i)_{i=1}^l$ if D_0^i is a hole of D for all $i = 1, \dots, l$ and the sets $\overline{D_0^i}$, $i = 1, \dots, l$ are pairwise disjoint. For a set D with disjoint holes $(D_0^i)_{i=1}^l$, we call $D_+ := D \setminus (\bigcup_{i=1}^l \overline{D_0^i})$ an *atoll domain*. The holes $(\overline{D_0^i})_{i=1}^l$ are called *lagoons* of the atoll domain.

A prototypical domain with one hole is an annulus. More precisely, if for $z \in \mathbb{C}$ and $s > r > 0$ we denote by $B_{r,s}(z)$ the annulus

$$B_{r,s}(z) := \{w \in \mathbb{C} : \ r < |w - z| < s\},$$

then $B_{r,s}(z)$ is an atoll domain with one hole. (We shall use the notation $B_r(z)$ for the open disc with radius r and centre z.) Associated with a domain with holes, we define the following class of functions which will act as our model for the functions F_j mentioned in Sect. 1.4.

Definition 1.3 Suppose that D is a bounded atoll domain with disjoint lagoons $(D_0^i)_{i=1}^l$ and let $\Delta \geq \delta > 0$. Then, we define the function class $\mathcal{H}(D, (D_0^i)_{i=1}^l, \delta, \Delta)$ of *atoll functions* associated with D and $(D_0^i)_{i=1}^l$ as follows:

$$\mathcal{H}(D, (D_0^i)_{i=1}^l, \delta, \Delta) := \left\{ F \in C^1(D) : \ \max\{|F(z)|, |\nabla|F|(z)|\} \leq \Delta \text{ for all } z \in D, \ |F(z)| \geq \delta \text{ for all } z \in D_+ \right\}. \tag{1.7}$$

The interpretation of Definition 1.3 is straightforward. It consists of functions on D which are large on an atoll D_+ and possibly small on a number of lagoons D_0^i which are encircled by an atoll D_+. See Fig. 5 for an illustration.

The functions we want to consider for phase retrieval (and for which we will show that phase retrieval is uniformly stable) will correspond to a linear combination of atoll functions, each supported on different atolls. Furthermore, as proposed in Sect. 1.4, the reconstruction will be allowed to assign different phases to components supported on different atolls.

The present paper establishes such results; as an appetizer, we mention the following stability result which applies to the reconstruction of a function $f \in L^2(\mathbb{R})$ from

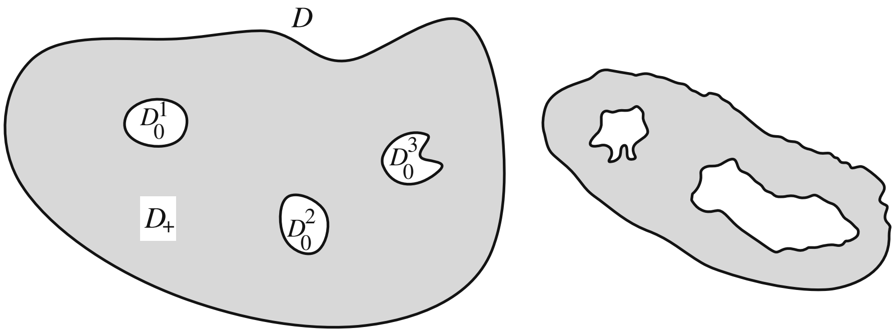

Fig. 5 Left: An atoll domain with three lagoons: D is the open domain enclosed by the outer curve, D_0^1, D_0^2 and D_0^3 are the three "holes" or *lagoons*, and D_+, the shaded area, is the *atoll domain*. Right: although most atoll islands (in their standard geographic meaning) are sickle-shaped, with lagoons in the shape of large bays, narrowly connected with the sea or ocean surrounding the island, some are indeed similar to the figure on the left; given here is the shape of Teeraina island, a coral atoll that is part of Kiribati, at about 4.71° North latitude and 160.76° West longitude

measuring absolute values of its Gabor transform $V_\varphi f$ as defined in (1.2), with window function $\varphi(t) := e^{-\pi t^2}$.

Let us suppose that we know a priori that the function f to be recovered can be written as a sum $f = \sum_{j=1}^{k} f_j$ with functions f_j each having time–frequency (TF) concentration in an annulus or a disc, i.e.

$$V_\varphi f_j \in \mathcal{H}(D_j, D_{0,j}, \delta_j, \Delta_j), \quad \text{for } j = 1, \ldots, k, \tag{1.8}$$

where the D_j are (possibly disjoint) discs $D_j := B_{s_j}(z_j)$, each with one hole, $D_{0,j} := B_{r_j}(z_j)$ for $0 \le r_j < s_j$ and $z_j \in \mathbb{C}$ for $j = 1, \ldots, k$. In audio processing, each of the f_j's may be interpreted as different tones and in different periods of time, each having its TF concentration on the set D_j in the following sense.

Definition 1.4 For $B \subset \mathbb{R}^2$ and $\varepsilon > 0$, we say that $f \in L^2(\mathbb{R})$ is ε-concentrated in B, if

$$\int_{\mathbb{R}^2 \setminus B} |V_\varphi f(x, y)|^2 dx dy \le \varepsilon^2 \|f\|^2_{L^2(\mathbb{R})}.$$

We use the notation $W^{1,p}(D)$ for the Sobolev space with norm

$$\|F\|_{W^{1,p}(D)} = \|F\|_{L^p(D)} + \|\nabla F\|_{L^p(D)}.$$

With these definitions and notation, we can now formulate the following theorem that states one of our stability results:

Theorem 1.5 *Suppose that* $f = \sum_{j=1}^{k} f_j \in L^2(\mathbb{R})$ *such that (1.8) holds true with each* f_j ε_j*-concentrated in* D_j.

Suppose that $g \in L^2(\mathbb{R})$ *can likewise be written as* $g = \sum_{j=1}^k g_j$ *with each* g_j ε_j*-concentrated in* D_j*. Then, there exists a continuous function* $\rho : [0, 1) \to \mathbb{R}_+$ *and a uniform constant* $c > 0$ *so that the following estimate holds:*

$$\inf_{\alpha_1,\ldots,\alpha_k \in \mathbb{R}} \sum_{j=1}^k \left\| f_j - e^{\mathrm{i}\alpha_j} g_j \right\|_{L^2(\mathbb{R})}$$
$$\leq c \cdot \left(\sum_{j=1}^k \frac{\Delta_j^2}{\delta_j^2} (1 + \rho(r_j/s_j) \cdot s_j) \cdot \left(1 + (r_j/s_j)^{1/2} \cdot \rho(r_j/s_j) \cdot (s_j + 1) \cdot e^{r_j^2 \pi/2}\right) \right.$$
$$\left. \cdot \left\| |V_\varphi f| - |V_\varphi g| \right\|_{W^{1,2}((D_j)_+)} + \sum_{j=1}^k \varepsilon_j \left(\|f_j\|_{L^2(\mathbb{R})} + \|g_j\|_{L^2(\mathbb{R})} \right) \right).$$

The theorem states that a function that is the sum of components, each of which has a Gabor transform of type (1.8), can be stably reconstructed from the absolute values of its Gabor transform, whenever its Gabor transform is concentrated on a number of atolls with lagoons that are not too large.

Note that as the lagoons get large, more precisely, if we let r_j grow while keeping the ratios r_j/s_j fixed, the stability of reconstruction degenerates *at most* exponentially in their area. This is completely in line with the results of [10], and in particular with the example mentioned in Sect. 1.3 for which the stability of the reconstruction degenerates *at least* exponentially in the size of its corresponding lagoon. Therefore, we believe that such a decay is not a proof artefact but a fundamental barrier to stable phase retrieval, related to the TF-localization properties of the window φ, see also Remark 3.10 in Sect. 3.4.

One can construct an example of phase retrieval from Gabor measurements in the spirit of Example 1.1 of real-valued measurements in 1D: in [3], some of the authors construct two functions f_a^+, f_a^-, for which the (Gabor transform) measurements are close to each other in absolute value but such that $\|f_a^+ - e^{\mathrm{i}\alpha} f_a^-\|_{L^2(\mathbb{R})}$ is not small for any phase factor $e^{\mathrm{i}\alpha}$, $\alpha \in \mathbb{R}$. The functions f_a^+, f_a^- are constructed such that their Gabor transforms are concentrated on two separated discs $B_{r_0}((-a, 0))$ and $B_{r_0}((a, 0))$, so that they can be viewed as atoll functions. Applying Theorem 1.5 to this example gives stability of the phase retrieval problem with a stability constant that is independent of a. In contrast, in the classical sense (i.e. when $V_\varphi f_a^+$, $V_\varphi f_a^-$ are not treated as atoll functions), phase retrieval is unstable in this example with the stability constant deteriorating exponentially in a^2. We note, however, that the stability constant from Theorem 1.5 is not independent of the size of atolls, i.e. of the radius r_0. In fact, it grows exponentially in r_0^2. Recent work [23] by one of the authors has developed improved results that overcome this growth of the stability constant in the size of the atolls by replacing the exponential dependence on r_0^2 by a low-order polynomial dependence. While the aforementioned work provides a rather complete picture of the local continuity properties of the inverse map $f \mapsto |V_\varphi f|$, it is not immediate what can be shown in the noisy case where f is to be reconstructed from noisy measurements $|V_\varphi f|$ + noise, unless rather stringent assumption on the noise hold true, see [23, Corollary 2.10] and also [7] for some general results in the finite-dimensional case.

Theorem 1.5 is a special case of our much more general Theorem 3.1, proved in Sect. 3.3. Theorem 3.1, however, applies to a much wider class of measurement scenarios. Another application, discussed in Sect. 3.4, concerns the phase retrieval problem from measuring absolute values of the Cauchy wavelet transform of a signal.

1.6 Proof Strategy

We briefly describe the underlying mechanism in the proof of the above-mentioned stability theorem.

- At the backbone of Theorem 1.5 lies the well-known fact that the Gabor transform $F(x, y) := V_\varphi f(x, -y)$ is a holomorphic function, up to normalization. More precisely, there exists a function η such that the product $\eta \cdot F$ is holomorphic, see Theorem 3.6. In fact, in Theorem 3.1, we establish a general stability result for atoll functions which are, up to normalization, holomorphic.
- A key insight leading to this result is the observation that, for a holomorphic function F, the rate of change of F is dominated by the rate of change of $|F|$. This fact, which is Lemma 4.1, follows directly from the Cauchy–Riemann equations.
- Lemma 4.1 then allows us to prove a stability result for atoll functions, restricted to the atoll D_+ on which a lower bound on their absolute value holds true.
- In order to also establish a stability bound on the lagoons $(D_0^i)_{i=1}^l$, we use a version of the maximum principle and a trace theorem for Sobolev functions to prove that the reconstruction error on the lagoons $(D_0^i)_{i=1}^l$ can be dominated by the approximation error on the atoll D_+ which has been controlled in the previous step. These two steps are carried out in Sect. 4. The proof turns out to be involved and dependent on a number of preparatory results which are summarized in Sect. 2.

Our main result is Theorem 3.1 which establishes a stability result for arbitrary atoll functions that arise from holomorphic measurements (up to normalization). Theorem 1.5 then comes as a corollary, but the machinery of Theorem 3.1 allows to deduce stability of phase retrieval for any type of measurements which depend holomorphically on its parameters. As a further example, we mention Cauchy wavelets which have been treated previously in [39].

1.7 Outline

The article is structured as follows. Section 2 provides a package of all the preparatory tools that will be needed later. In particular, we describe analytic Poincaré inequalities and the relation of the analytic Poincaré constant to the classical Poincaré constant in Sect. 2.1. Sections 2.2 and 2.3 outline the results that are needed to control the reconstruction error on the lagoons $(D_0^i)_{i=1}^l$. Stable point evaluations and the simultaneous control of two different constants that will appear in the main result of this paper are treated in Sect. 2.4.

Section 3 features our main result (Theorem 3.1) and gives its illustration for two concrete examples: the case of the domain $D = D_+$ being a disc (Sect. 3.1) and the case of D_+ being an annulus (Sect. 3.2). In the remainder of this section, the cases of magnitude measurements of the Gabor transform (Sect. 3.3) and of the Cauchy wavelet transform (Sect. 3.4) are studied and the stability constants are quantified. We give the proof of the main theorem (Theorem 3.1) in Sect. 4.

2 Preparatory Results

In the course of our work, we will use several auxiliary results that are summarized in this section. For an overview of the main results of this paper, the reader may want to visit Sect. 3 directly. We consider a path-connected domain $D \subset \mathbb{C}$ which is sufficiently nice (e.g. Lipschitz domain) and let $\mathcal{O}(D)$ denote the space of holomorphic functions from D to $\mathbb{C}$.

We will always write $z = x + \mathrm{i}y \in \mathbb{C}$ and $F(z) = u(x, y) + \mathrm{i}v(x, y)$. We denote $F'(z) = u_x(x, y) + \mathrm{i}v_x(x, y)$ and $\nabla F(z) = (\nabla u(x, y), \nabla v(x, y)) \in \mathbb{R}^{2\times 2}$.

Any $F \in \mathcal{O}(D)$ satisfies the Cauchy–Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \tag{2.1}$$

A key object of our study is the absolute value $|F| : D \to \mathbb{R}$ and its gradient $\nabla|F| = (|F|_x, |F|_y)^T$. For a subset $B \subset \mathbb{C}$, we denote by $|B|$ its area and by χ_B its indicator function. We write

$$\mathbb{R}_+ = \{x \in \mathbb{R} : \ x > 0\}$$

and

$$\mathbb{C}_+ = \{z = x + \mathrm{i}y : \ x \in \mathbb{R}, \ y \in \mathbb{R}_+\}.$$

2.1 Analytic Poincaré Inequalities

We shall rely several times on the validity of an analytic Poincaré inequality. A domain D is said to be an analytic p-Poincaré domain if for $z_0 \in D$, there exists a constant $C^a_{\mathrm{poinc}}(p, D, z_0) > 0$ such that

$$\|F - F(z_0)\|_{L^p(D)} \le C^a_{\mathrm{poinc}}(p, D, z_0) \left\|F'\right\|_{L^p(D)} \tag{2.2}$$

for all $F \in \mathcal{O}(D)$, and $1 \le p \le \infty$.

Such inequalities are studied in [37]. Although (2.2) features the point $z_0 \in D$, it turns out that whether or not the domain D is an analytic Poincaré domain is independent of z_0. However, [37], the best possible constant $C^a_{\mathrm{poinc}}(p, D, z_0)$ depends on the choice of z_0. Denote by $C_{\mathrm{poinc}}(p, D)$ the usual Poincaré constant of the domain D, i.e. the optimal constant C such that

$$\|F - F_D\|_{L^p(D)} \leq C\,\|\nabla F\|_{L^p(D)}\,,$$

where $F_D := \frac{1}{|D|}\int_D F(z)dz$. Then, we have the following estimate for $C^a_{\text{poinc}}(p, D, z_0)$:

Lemma 2.1

$$C^a_{\text{poinc}}(p, D, z_0) \leq C_{\text{poinc}}(p, D) \cdot \left(1 + \left(\frac{|D|}{\pi\,\text{dist}\,(z_0, \partial D)^2}\right)^{1/p}\right).$$

Proof In [37, p. 365], the case $p = 1$ is shown; the general case can be done analogously. Let $r := \text{dist}\,(z_0, \partial D)$ and consider the ball $B = B_r(z_0)$. By the mean value property, it holds that $F(z_0) = F_B$. Therefore, we have

$$\|F - F(z_0)\|_{L^p(D)} = \|F - F_B\|_{L^p(D)}\,.$$

With this, the triangle inequality yields

$$\|F - F(z_0)\|_{L^p(D)} \leq \|F - F_D\|_{L^p(D)} + \|F_D - F_B\|_{L^p(D)}\,.$$

Now, we observe that

$$|F_B - F_D| \leq \frac{1}{|B|}\int_B |F(z) - F_D|dz \leq \frac{1}{|B|}|B|^{1-1/p}\,\|F - F_D\|_{L^p(B)}\,,$$

where the last inequality follows from Hölder's inequality. Now, it remains to observe that $\|F - F_D\|_{L^p(B)} \leq \|F - F_D\|_{L^p(D)}$ and $|B| = \pi\,\text{dist}\,(z_0, \partial D)^2$ to arrive at the desired result. □

Essentially, Lemma 2.1 states that whenever z_0 lies in a central location of D (i.e. not too close to ∂D), the constant $C^a_{\text{poinc}}(p, D, z_0)$ can be controlled by the classical Poincaré constant $C_{\text{poinc}}(p, D)$ which is well studied. For instance, the following result is known [33].

Theorem 2.2 *Suppose that $D \subset \mathbb{C}$ is a bounded, convex domain with Lipschitz boundary. Then,*

$$C_{\text{poinc}}(2, D) \leq \frac{diam(D)}{\pi}.$$

For non-convex domains, the determination of the optimal Poincaré constant is more difficult. For the annulus $B_{r,s}(z)$, the following result is known.

Theorem 2.3 *Suppose that $D = B_{r,s}(z)$. Then, there exists a uniform constant $c > 0$ such that*

$$C_{\text{poinc}}(2, D) \leq c \cdot s.$$

Proof By a scaling argument, it is easily seen that

$$C_{\text{poinc}}(2, B_{r,s}(z)) = s \cdot C_{\text{poinc}}(2, B_{r/s,1}(0)).$$

The function $h : \tau \mapsto C_{\text{poinc}}(2, B_{\tau,1}(0))$ is continuous on $(0, 1)$ because the Poincaré constant depends continuously on the domain [21]. In [20], it is shown that the function h extends continuously to the endpoint $\tau = 1$, and in [24] it is shown that the function h extends continuously to the endpoint $\tau = 0$. Therefore, h is continuous on the closed interval $[0, 1]$, and hence bounded, which proves the statement. □

For more general domains which arise as a diffeomorphic image of a convex domain or an annulus, one can obtain estimates on the Poincaré constant by studying the Jacobian of the diffeomorphism, but in the present paper we are content with knowing the Poincaré constant on convex domains and on annuli.

2.2 Sobolev Trace Inequalities

In what follows, we will consider inequalities involving the L^p-norm of functions on the piecewise smooth boundary of a bounded domain $D \subset \mathbb{C}$. We define it as

$$\|F\|_{L^p(\partial D)} := \left(\int_a^b |F(\gamma(t))|^p \, |\gamma'(t)| dt\right)^{1/p},$$

where $\gamma : [a, b) \to \partial D$ can be any bijective parameterization of ∂D.

The Sobolev trace inequality [16] provides an upper bound for this norm, which will be important for our purposes:

Theorem 2.4 *Suppose that $D \subset \mathbb{C}$ is a bounded domain with Lipschitz boundary ∂D. Then, there exists a constant $C_{\text{trace}}(p, D)$ with*

$$\|F\|_{L^p(\partial D)} \leq C_{\text{trace}}(p, D) \, \|F\|_{W^{1,p}(D)} \, .$$

The next result provides concrete estimates of the trace constant for discs and annuli. It says that the trace constant behaves nicely for annuli that are not too thin.

Theorem 2.5 *There exists a continuous function $\rho : [0, 1) \to \mathbb{R}$ with $\lim_{\tau \to 1_-} \rho(\tau) = \infty$ such that*

$$C_{\text{trace}}(2, B_{r,s}(z)) \leq \rho(r/s) \cdot (s^{1/2} + s^{-1/2}).$$

Proof By a scaling argument, one can verify that

$$C_{\text{trace}}(2, B_{r,s}(z)) \leq s^{1/2} \cdot C_{\text{trace}}(2, B_{r/s,1}(z)), \quad \text{for } s \geq 1,$$
$$C_{\text{trace}}(2, B_{r,s}(z)) \leq s^{-1/2} \cdot C_{\text{trace}}(2, B_{r/s,1}(z)), \quad \text{for } s < 1.$$

The statement then follows by noting that $C_{\text{trace}}(2, B_{\tau,1}(z)) < \infty$ for $\tau \in [0, 1)$. □

2.3 Boundary Values of Holomorphic Functions

Another key fact we shall use is that the L^p-norm of a holomorphic function on a simply connected domain is dominated by its L^p-norm on the boundary.

Theorem 2.6 *Suppose that $F \in \mathcal{O}(D)$, where $D \subset \mathbb{C}$ is a bounded and simply connected domain with smooth boundary. Then, there exists a constant $C_{\text{bound}}(p, D) > 0$ such that*

$$\|F\|_{L^p(D)} \leq C_{\text{bound}}(p, D)\,\|F\|_{L^p(\partial D)}$$

for all bounded functions $F \in \mathcal{O}(D)$.

Proof We assume without loss of generality that $D = B_1(0)$. The general case can be handled using the Riemann mapping theorem. We shall make use of the Hardy space H^p, consisting of all functions $F \in \mathcal{O}(B_1(0))$ with finite H^p-norm, defined by

$$\|F\|_{H^p}^p := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |F(r \cdot e^{\mathrm{i}\varphi})|^p d\varphi.$$

It is well known (see for instance [28]) that any $F \in H^p$ can be extended to the boundary $\partial B_1(0)$ and that

$$2\pi\,\|F\|_{H^p}^p = \|F\|_{L^p(\partial D)}^p. \tag{2.3}$$

We further note that

$$\frac{1}{2\pi}\,\|F\|_{L^p(B_1(0))}^p = \frac{1}{2\pi}\int_0^1 \int_0^{2\pi} |F(r \cdot e^{\mathrm{i}\varphi})|^p r dr d\varphi \leq \|F\|_{H^p}^p\,.$$

Combining this result with (2.3) yields the desired result. □

For discs $B_r(z)$, a simple scaling argument leads to the following result.

Theorem 2.7 *For all $r > 0$, $z \in \mathbb{C}$ and $D = B_r(z)$, we have $C_{\text{bound}}(p, D) \leq r^{1/p}$.*

For more general simply connected domains, the constant $C_{\text{bound}}(p, D)$ depends on upper and lower bounds on the Jacobian of the Riemann mapping from D to $B_1(0)$.

2.4 Stable Point Evaluations

Given a function $G \in L^p(D)$, the proof of our main result will require us to pick a point $z \in D$ with a small sampling constant which is defined as follows.

Definition 2.8 Let D be a domain and $G \in L^p(D)$. Then, we define, for $z_0 \in D$ and $1 \leq p \leq \infty$, the *sampling constant*

$$C_{\text{samp}}(p, D, z_0, G) := \inf\{C > 0 :\ \|G(z_0)\|_{L^p(D)} \leq C\,\|G\|_{L^p(D)}\}.$$

To control the constant $C(z_0, p, D_+, (D_0^i)_{i=1}^l)$ in our main result Theorem 3.1, it is necessary to control $C_{\text{samp}}(p, D_+, z_0, |F_2| - |F_1|)$ and $C^a_{\text{poinc}}(p, D_+, z_0)$ simultaneously.

The purpose of this subsection is to show that this can indeed be achieved for general domains D and functions $G \in L^p(D)$.

We start with the following lemma which shows that there exist "many" points with a given sampling constant.

Lemma 2.9 *Suppose that $D \subset \mathbb{C}$ is a domain and let $G \in L^p(D)$ for $1 \leq p \leq \infty$. For $C > 1$ we denote*

$$D_C(G) := \left\{ z_0 \in D : \ |G(z_0)||D|^{1/p} \leq C \, \|G\|_{L^p(D)} \right\}.$$

Then,

$$|D_C(G)| \geq |D| \cdot \left(1 - \frac{1}{C^p}\right).$$

Proof We compute

$$\int_{D \setminus D_C(G)} |G(x)|^p dx + \int_{D_C(G)} |G(x)|^p dx = \|G\|^p_{L^p(D)}.$$

By the definition of $D_C(G)$ we have that

$$|G(x)|^p > \frac{C^p}{|D|} \, \|G\|^p_{L^p(D)} \ \text{ for all } x \in D \setminus D_C(G),$$

and this implies that

$$|D \setminus D_C(G)| \frac{C^p}{|D|} \, \|G\|^p_{L^p(D)} + \int_{D_C(G)} |G(x)|^p dx \leq \|G\|^p_{L^p(D)}.$$

Consequently,

$$(|D| - |D_C(G)|) \frac{C^p}{|D|} \leq 1$$

and this yields the statement. □

Lemma 2.9 implies that if we define $C(t) := \frac{1}{(1-t)^{1/p}}$, then for any $0 < t < 1$ and $G \in L^p(D)$ we have

$$|D_{C(t)}(G)| \geq t|D|.$$

Next, we define (cf. Fig. 6)

$$s_t(D) := \inf_{S \subset D, \ |S| = t|D|} \sup_{z \in S} \operatorname{dist}(z, \partial D). \tag{2.4}$$

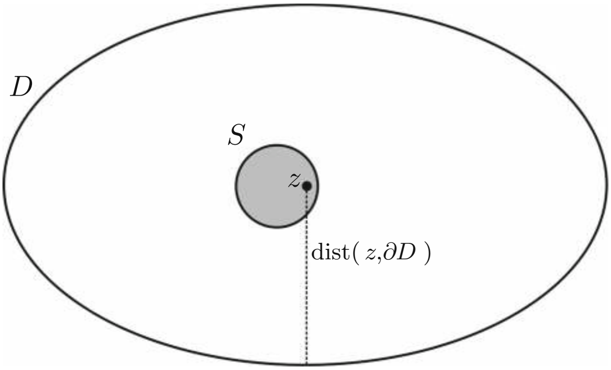

Fig. 6 A domain D, a subset S and an element $z \in S$ that maximizes $\operatorname{dist}(z, \partial D)$. The value of $s_t(D)$ is then obtained by taking the infimum of these quantities over all $S \subset D$ with the same area $|S| = t|D|$, i.e. $s_t(D) = \inf_{S \subset D,\, |S|=t|D|} \sup_{z \in S} \operatorname{dist}(z, \partial D)$

For "nice" domains, the quantity $s_t(D)$ can be controlled easily. We mention the following result; the proof is an elementary calculus computation.

Lemma 2.10 *For all $s > r > 0$ and $z \in \mathbb{C}$, we have the estimate*

$$s_{1/2}(B_r(z)) \geq c \cdot r \quad \text{and} \quad s_{1/2}(B_{r,s}(z)) \geq c \cdot (s - r),$$

with $c = 1 - \frac{1}{\sqrt{2}}$.

Control of s_t lets us gain control over both the sampling constant and the analytic Poincaré constant. As an immediate consequence of Lemma 2.9, we have the following result.

Lemma 2.11 *Let $0 < t < 1$, and let $D \subset \mathbb{C}$ be a domain and $G \in L^p(D)$. There exists $z_0 \in D$ with*

$$C_{\text{samp}}(p, D, z_0, G) \leq \frac{1}{(1-t)^{1/p}},$$

and

$$C^a_{\text{poinc}}(p, D, z_0) \leq C_{\text{poinc}}(p, D) \left(1 + \left(\frac{|D|}{\pi s_t(D)^2} \right)^{1/p} \right).$$

Proof Picking $C(t) = \frac{1}{(1-t)^{1/p}}$, we get that $|D_{C(t)}(G)| \geq t|D|$ by Lemma 2.9. Therefore,

$$\sup_{z \in D_{C(t)}(G)} \operatorname{dist}(z, \partial D) \geq s_t(D)$$

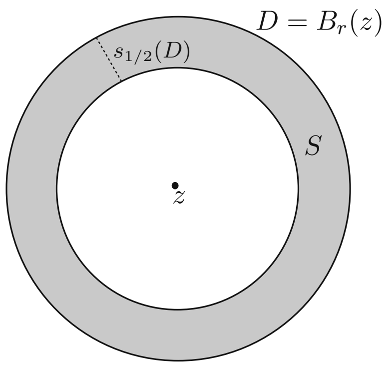

Fig. 7 For $D = B_r(z)$, $s_{1/2}B_r(z)$ is attained by the subset $S = B_{r_0,r}(z)$ with $r_0 = \frac{1}{\sqrt{2}}r$

and thus there exists $z_0 \in D_{C(t)}(G)$ with

$$\text{dist}\,(z_0, \partial D) \geq s_t(D).$$

Lemma 2.1 now immediately implies the claimed bound for $C^a_{\text{poinc}}(p, D, z_0)$ (Fig. 7).

On the other hand, by the definition of $C(t)$ and the fact that $z_0 \in D_{C(t)}(G)$, we get the desired bound on the sampling constant which proves the statement. □

In order to make use of Lemma 2.11 to estimate the constants $C_{\text{samp}}(p, D, z_0, G)$ and $C^a_{\text{poinc}}(p, D, z_0)$, we need to control only the quantity $s_t(D)$. For "nice" domains D, we expect that $s_t(D)$ behaves like the diameter $\text{diam}(D)$ and also that $\text{diam}(D)^2$ behaves like $|D|$; hence, the quotient $\frac{|D|}{\pi s_t(D)^2}$ would be uniformly bounded which implies that, for a suitable choice of $z_0 \in D$, the constant $C^a_{\text{poinc}}(p, D, z_0)$ is comparable to the classical Poincaré constant $C_{\text{poinc}}(p, D)$, while $C_{\text{samp}}(p, D, z_0, G)$ is bounded by a fixed constant. These considerations will give us full control of all underlying constants for sufficiently nice domains, needed in the estimates in the next section.

3 Stability of Phase Reconstruction from Holomorphic Measurements

The purpose of this section is to formulate the following fundamental result and discuss some of its implications.

Theorem 3.1 *Suppose that F_1 belongs to a class of atoll functions as in Definition 1.3, i.e. $F_1 \in \mathcal{H}(D, (D_0^i)_{i=1}^l, \delta, \Delta)$. Assume further that $F_2 \in C^1(D)$ such that there exists a continuous function $\eta : D \to \mathbb{C}$ for which both functions $\eta \cdot F_1,\ \eta \cdot F_2 \in \mathcal{O}(D)$. Suppose that $1 \leq p \leq \infty$.*

Pick $z_0 \in D_+$. We denote $C_{\text{samp}} := C_{\text{samp}}(p, D_+, z_0, |F_1| - |F_2|)$, meaning that

$$\||F_1(z_0)| - |F_2(z_0)|\|_{L^p(D_+)} \leq C_{\text{samp}} \cdot \||F_1| - |F_2|\|_{L^p(D_+)}. \tag{3.1}$$

Then, the following estimate holds:

$$\inf_{\alpha\in\mathbb{R}} \left\| F_1 - e^{\mathrm{i}\alpha} F_2 \right\|_{L^p(D)} \leq C(z_0, p, D_+, (D_0^i)_{i=1}^l) \frac{\Delta^2}{\delta^2} \, \| |F_1| - |F_2| \|_{W^{1,p}(D_+)} , \tag{3.2}$$

where for the constant $C(z_0, p, D_+, (D_0^i)_{i=1}^l)$ *we may choose (with a suitably large but uniform constant* $c > 0$*):*

$$\begin{aligned} C(z_0, p, D_+, (D_0^i)_{i=1}^l) = c \cdot (C^a_{\mathrm{poinc}}(D_+) + C_{\mathrm{samp}} \\ + \sum_{i=1}^{l} C_{\mathrm{bound}}(D_0^i) \cdot \mathrm{var}(\eta, D_0^i) \cdot \\ C_{\mathrm{trace}}(D_+)(C^a_{\mathrm{poinc}}(D_+) + C_{\mathrm{samp}})), \end{aligned} \tag{3.3}$$

where we have omitted the dependence of the various constants on p, z_0 *and denote*

$$\mathrm{var}(\eta, D_0^i) := \frac{\max_{z\in\partial D_0^i} |\eta(z)|}{\min_{z\in D_0^i} |\eta(z)|}, \quad i = 1, \ldots, l.$$

Remark 3.2 By Lemma 2.11, the two constants C_{samp} and $C^a_{\mathrm{poinc}}(D_+)$ depending on z_0 can be controlled simultaneously. To achieve the best possible $C(z_0, p, D_+(D_0^i)_{i=1}^l)$, z_0 should be picked s.t. dist $(z_0, \partial D_+)$ is large and $\| |F_1(z_0)| - |F_2(z_0)| \|$ is small.

Remark 3.3 In Theorem 3.1, we assume that there exists a normalization function η, s.t. $\eta \cdot F_1, \eta \cdot F_2 \in \mathcal{O}(D)$. In Sects. 3.3 and 3.4, we show for F in the image domain of the Gabor or Cauchy wavelet transform, respectively, the existence of explicit functions η such that $\eta \cdot F$ is holomorphic on the entire parameter domain and the results of [5] show that these are essentially the only functions which generate, up to normalization, holomorphic wavelet or STFT measurements. For more general measurements, such global η may not exist and for $F \in \mathcal{H}(D, D_0, \delta, \Delta)$, there might be accumulated zeros in D_0. In this case, if the accumulated zero set $D_O := \overline{\{z;\ F_1(z)F_2(z) = 0\}}^\circ \subset D_0$ is simply connected with smooth boundary, then the bound (3.2) in Theorem 3.1 still holds with the domain of the L^p-norm on the right-hand side changing from D_+ to D.[4]

Before we provide the lengthy proof of Theorem 3.1 in Sect. 4, we pause and provide some special examples which might be illuminating. To give two simple examples, in Sect. 3.1 we shall see how to gain explicit estimates for the quantity $C(z_0, p, D_+, D_0)$ for $D = D_+$ a disc (i.e. $D_0 = \emptyset$) and in Sect. 3.2 for D_+ an annulus.

These examples should make clear that similar results also hold for more general domains.

[4] This extension requires a generalized version of Theorem 2.6 for the annulus, which can be shown following the same idea of proof of the disc case but considering the Hardy space defined on an annulus instead, see Theorem 3 in [41].

3.1 Example I: A Disc

In this subsection, we shall treat the case $D = D_+ = B_r(z)$ and $D_0 = \emptyset$. The class $\mathcal{H}(D_+, D_0, \delta, \Delta)$ now consists of functions which are bounded from below by δ and which (together with their gradient) are bounded from above by Δ on all of $B_r(z)$. We have the following result.

Theorem 3.4 *Suppose that $F_1 \in \mathcal{H}(B_r(z), \emptyset, \delta, \Delta)$ for some $r > 0$ and $z \in \mathbb{C}$. We further assume that $F_2 \in C^1(B_r(z))$ such that there exists a continuous function $\eta : B_r(z) \to \mathbb{C}$ for which both functions $\eta \cdot F_1,\ \eta \cdot F_2 \in \mathcal{O}(B_r(z))$.*

Then, there exists a uniform constant $c > 0$ such that the following estimate holds.

$$\inf_{\alpha \in \mathbb{R}} \left\| F_1 - e^{\mathrm{i}\alpha} F_2 \right\|_{L^2(B_r(z))} \leq c \cdot (1 + r) \cdot \frac{\Delta^2}{\delta^2} \cdot \| |F_1| - |F_2| \|_{W^{1,2}(B_r(z))} . \tag{3.4}$$

Proof We let uniform constants c vary from line to line. First, we note that, by Lemma 2.10, there exists a uniform constant $c > 0$ such that $s_{1/2}(B_r(z)) \geq c \cdot r$ with $s_t(B_r(z))$ defined as in (2.4). It follows from Lemma 2.11 that there exists a uniform constant $c > 0$ and $z_0 \in B_r(z)$ with

$$C_{\mathrm{samp}}(p, D_+, z_0, G) \leq c \quad \text{and} \quad C^a_{\mathrm{poinc}}(p, B_r(z), z_0) \leq c \cdot C_{\mathrm{poinc}}(p, B_r(z)),$$

where we have put $G := |F_2| - |F_1|$.

Now, it remains to employ Theorem 2.2 to get a suitable estimate on the quantity (3.3) for $p = 2$ which, together with Theorem 3.1, yields the desired result. □

More general results can be obtained for domains D which are diffeomorphic to $B_r(z)$ in an obvious way. The resulting bounds will depend on upper and lower bounds of the Jacobian of the mapping which maps D to $B_r(z)$.

A similar result can also be established for general convex domains D where r in the theorem above may be replaced by $\mathrm{diam}(D)$ and the constant c may depend on the geometry of D.

We omit the details.

3.2 Example II: An Annulus

To make the general result of Theorem 3.1 more accessible and to give an idea of the quantitative nature of the stability constant $C(z_0, p, D_+, D_0)$, we treat here the case of an annulus $D_+ = B_{r,s}(z)$ and $D_0 = B_r(z)$ for $s > r > 0$ and some $z \in \mathbb{C}$. It is interesting to observe the dependence of the stability constant on the size of the "lagoon" D_0 on which the phaseless measurements are allowed to be arbitrarily small. We have the following result.

Theorem 3.5 *Suppose that $F_1 \in \mathcal{H}(B_s(z), B_r(z), \delta, \Delta)$ for $s > r > 0$. Furthermore, let $F_2 \in C^1(B_s(z))$ be such that there exists a continuous function $\eta : B_s(z) \to \mathbb{C}$ for which both functions $\eta \cdot F_1,\ \eta \cdot F_2 \in \mathcal{O}(B_s(z))$.*

Then, there exist a continuous function $\rho : [0, 1) \to \mathbb{R}_+$ *with* $\lim_{\rho\to 1_-} = \infty$ *and a uniform constant* $c > 0$ *such that the following estimate holds.*

$$\begin{aligned}\inf_{\alpha\in\mathbb{R}} \left\| F_1 - e^{\mathrm{i}\alpha} F_2 \right\|_{L^2(B_s(z))} &\\ \leq c \cdot (1 + \rho(r/s)\cdot s) \cdot \Big(1 + r^{1/2}\cdot \rho(r/s)\cdot (s_j^{1/2} + s_j^{-1/2})\cdot \mathrm{var}(\eta, B_r(z))\Big) &\\ \cdot \frac{\Delta^2}{\delta^2} \| |F_1| - |F_2| \|_{W^{1,2}(B_{r,s}(z))} \cdot & \qquad (3.5)\end{aligned}$$

Proof We first observe the elementary fact that $D_+ = B_{r,s}(z)$ and that, by Lemma 2.10, there exists a uniform constant $c > 0$ with

$$s_{1/2}(B_{r,s}(z)) \geq c(s - r).$$

Using Lemma 2.11 and setting $G := |F_1| - |F_2|$, this implies the existence of $z_0 \in B_{r,s}(z)$ and a uniform constant c with

$$\begin{aligned} C_{\mathrm{samp}}(p, D_+, z_0, G) \leq c \quad \text{and} \quad & C^a_{\mathrm{poinc}}(p, B_{r,s}(z), z_0) \\ & \leq c \cdot \frac{1}{(1 - r/s)^{1/2}} C_{\mathrm{poinc}}(p, B_{r,s}(z)).\end{aligned}$$

All further constants may be estimated from Theorems 2.3, 2.7 and 2.5 which, together with Theorem 3.1, yield the desired result. □

Theorem 3.5 shows that stability can still be retained, even if the function F_1 is allowed to be small on a large set. Again, more general results can be derived for domains which are diffeomorphic to an annulus.

3.3 Phase Retrieval from Gabor Measurements

For a window $g \in L^2(\mathbb{R})$, define the windowed Fourier transform of $f \in L^2(\mathbb{R})$ as

$$V_g f(x, y) := \int_{\mathbb{R}} f(t)\overline{g(t - x)} e^{-2\pi \mathrm{i} t y} dt. \qquad (3.6)$$

The Gabor transform is defined as the windowed Fourier transform with window $\varphi(t) := e^{-\pi t^2}$. The following result is well known [22].

Theorem 3.6 *For* $z_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C}$ *and with* $\eta_{z_0}(z) := e^{\pi(|z-z_0|^2/2 - \mathrm{i}\cdot(x+x_0)\cdot(y-y_0))}$, *the function*

$$F(z) := V_\varphi f(x, -y) \quad \textit{where } z = x + \mathrm{i}y$$

satisfies that $\eta_{z_0} \cdot F \in \mathcal{O}(\mathbb{C})$.

Now, consider the problem of stably reconstructing a function from the absolute values of its Gabor transform. By Theorem 3.6, we are in a position to apply Theorem 3.1 directly.

Theorem 3.7 *Suppose that* $f \in L^2(\mathbb{R})$. *Suppose that* $V_\varphi f$ *is an atoll function associated with* $D_j := B_{s_j}(z_j)$ *and* $D_{0,j} := B_{r_j}(z_j)$ *for* $0 \leq r_j < s_j$ *and* $z_j \in \mathbb{C}$ *for* $j = 1, \dots, k$, *meaning that*

$$(V_\varphi f)\Big|_{D_j} \in \mathcal{H}(D_j, D_{0,j}, \delta_j, \Delta_j) \quad \forall j \in \{1, \dots, k\}.$$

Then, there exists a continuous function $\rho : [0, 1) \to \mathbb{R}_+$ *and a constant* $c > 0$ *so that for all* $g \in L^2(\mathbb{R})$ *the following estimate holds:*

$$\inf_{\alpha_1,\dots,\alpha_k \in \mathbb{R}} \sum_{j=1}^{k} \left\| V_\varphi f - e^{\mathrm{i}\alpha_j} V_\varphi g \right\|_{L^2(D_j)} \leq c \cdot \left(\sum_{j=1}^{k} \frac{\Delta_j^2}{\delta_j^2} (1 + \rho(r_j/s_j) \cdot s_j) \cdot \left(1 + r_j^{1/2} \cdot \rho(r_j/s_j) \cdot (s_j^{1/2} + s_j^{-1/2}) \cdot e^{r_j^2 \pi/2}\right) \right) \cdot \left\| |V_\varphi f| - |V_\varphi g| \right\|_{W^{1,2}\left(\bigcup_{j=1}^{k} (D_j)_+\right)}.$$

Proof The proof follows directly from Theorem 3.5 together with observing that $\mathrm{var}(\eta_{z_j}, B_{r_j}(z_j)) \leq c \cdot e^{r_j^2 \pi/2}$ for a uniform constant $c > 0$. □

We are now ready to conclude the proof of Theorem 1.5, as announced in Sect. 1.5.

Proof of Theorem 1.5 It is well known that the Gabor transform $V_\varphi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ is an isometry on its range, see [22]. By assumption, the functions f_j, g_j are ε_j-concentrated in D_j (see Definition 1.8). Therefore,

$$\left\| f_j - e^{\mathrm{i}\alpha_j} g_j \right\|_{L^2(\mathbb{R})} \leq \left\| V_\varphi f - e^{\mathrm{i}\alpha_j} V_\varphi g \right\|_{L^2(D_j)} + \varepsilon_j.$$

Now, the statement of Theorem 1.5 is a direct consequence of Theorem 3.7. □

3.4 Phase Retrieval from Cauchy Wavelet Measurements

For $g \in L^2(\mathbb{R})$, define the wavelet transform of $f \in L^2(\mathbb{R})$ as

$$W_g f(x, y) := \frac{1}{|y|^{1/2}} \int_{\mathbb{R}} f(t) \overline{g((t-x)/y)} dt. \tag{3.7}$$

Define the Cauchy wavelet of order $s \in \mathbb{N}$ via its Fourier transform $\widehat{\psi}(\omega) = \omega^s e^{-2\pi\omega} \chi_{\omega>0}(\omega)$. The following result is well known [5].

Theorem 3.8 *For $\eta(z) := |1/y|^{s+1/2}$ and any $f \in L^2(\mathbb{R})$ with supp $\widehat{f} \subset \mathbb{R}_+$, the function*

$$F(z) := W_\psi f(x, y) \quad \text{where } z = x + \mathrm{i}y$$

satisfies that $\eta \cdot F \in \mathcal{O}(\mathbb{C}_+)$, where $\mathbb{C}_+ := \{x + \mathrm{i}y;\ y \geq 0\}$.

Proof For the convenience of the reader, we provide a proof. It is easy to check that, for f with supp $\widehat{f} \subset \mathbb{R}_+$, the function

$$G(z) := \int_{\mathbb{R}_+} \omega^s \widehat{f}(\omega) e^{-2\pi y\omega} e^{2\pi \mathrm{i} x\omega} d\omega, \quad \text{for } z = x + \mathrm{i}y \in \mathbb{C}_+$$

is holomorphic on $\mathbb{C}_+$. In fact, it is the holomorphic extension of the sth derivative of f, if the former exists.

Now, note that

$$W_\psi f(x, y) = f * \psi_y(x),$$

where

$$\psi_y(t) = \frac{1}{|y|^{1/2}} \psi(-t/y).$$

The Fourier transform of ψ_y is given as

$$\widehat{\psi_y}(\omega) = |y|^{1/2} \widehat{\psi}(y \cdot \omega) = |y|^{s+1/2} \omega^s e^{-2\pi y\omega} \chi_{\mathbb{R}_+}(\omega).$$

It follows that

$$G(z) = |y|^{-s-1/2} \cdot W_\psi f(x, y)$$

which proves the statement. □

Using Theorem 3.1, the statement of Theorem 3.8 immediately implies the following result related to the stability of phase retrieval from Cauchy wavelet measurements.

Theorem 3.9 *Suppose that $f \in L^2(\mathbb{R})$ with supp $\widehat{f} \subset \mathbb{R}_+$. Suppose that $W_\psi f$ is an atoll function associated with $D_j := B_{s_j}(z_j)$ and $D_{0,j} := B_{r_j}(z_j)$ for $0 \leq r_j < s_j$ and $z_j = x_j + \mathrm{i}y_j \in \mathbb{C}_+$ for $j = 1, \ldots, k$, meaning that*

$$(W_\psi f)\Big|_{D_j} \in \mathcal{H}(D_j, D_{0,j}, \delta_j, \Delta_j) \quad \forall j \in \{1, \ldots, k\}.$$

Then, for $g \in L^2(\mathbb{R})$ arbitrary with supp $\widehat{g} \subset \mathbb{R}_+$, the following estimate holds for a continuous function $\rho : [0,1) \to \mathbb{R}_+$ and a constant $c > 0$ that are both uniform.

$$\inf_{\alpha_1,\ldots,\alpha_k \in \mathbb{R}} \sum_{j=1}^{k} \left\| W_\psi f - e^{i\alpha_j} W_\psi g \right\|_{L^2(D_j)}$$
$$\leq c \cdot \left(\sum_{j=1}^{k} \frac{\Delta_j^2}{\delta_j^2} (1 + \rho(r_j/s_j) \cdot s_j) \cdot \left(1 + r_j^{1/2} \cdot \rho(r_j/s_j) \cdot (s_j^{1/2} + s_j^{-1/2}) \cdot \left| \frac{1}{1 - r_j/y_j} \right|^{s+1/2} \right) \right) \cdot$$
$$\left\| |W_\psi f| - |W_\psi g| \right\|_{W^{1,2}\left(\bigcup_{j=1}^{k} (D_j)_+\right)} .$$

Proof We have that $\operatorname{var}(\eta, B_{r_j}(z_j)) \leq c \cdot |1 - r_j/y_j|^{-s-1/2}$ for a uniform constant $c > 0$, so that the statement is a direct consequence of Theorem 3.5. □

Remark 3.10 It is interesting to observe how the stability bounds in Theorem 3.7 and 3.9 deteriorate as the size of the lagoons grows, that is, as the parameter r_j grows. In the case of Gabor measurements, this growth is of order $e^{r_j^2 \pi/2}$, while in the case of Cauchy wavelets with s vanishing moments, the growth is of order $(\frac{1}{1-r_j/y_j})^{s+1/2}$, becoming worse as the number of vanishing moments increases.

Interpreting these quantities in geometric terms, we note that the area of a lagoon in the parameter space of the Gabor transform is of order $r_j^2 \pi$, that is, the stability decays exponentially in the area of the lagoon.

For the wavelet transform, the natural notion of area in the upper half-plane is given by the Poincaré metric, i.e. by

$$\operatorname{area}_{\mathbb{C}_+}(B) := \int_B \frac{dxdy}{y^2}$$

and a simple calculation gives

$$\operatorname{area}_{\mathbb{C}_+}(B_{r_j}(z_j)) = \int_0^{2\pi} \int_0^{r_j} \frac{1}{(y_j + \rho \sin \phi)^2} \rho \, d\rho \, d\phi = 2\pi \left(\frac{1}{\sqrt{1 - r_j^2/y_j^2}} - 1 \right),$$

so that

$$\frac{\pi}{\sqrt{2}} \left(\frac{1}{\sqrt{1 - r_j/y_j}} - 1 \right) \leq \operatorname{area}_{\mathbb{C}_+}(B_{r_j}(z_j)) \leq 2\pi \left(\frac{1}{\sqrt{1 - r_j/y_j}} - 1 \right).$$

This shows that the stability of the phase retrieval from Cauchy wavelet measurements decays only polynomially in the area of the lagoon.

This behaviour is most likely related to the fact that Gabor systems are much more well localized in the time–frequency plane than Cauchy wavelets and that the localization properties of Cauchy wavelets increase as the number s of vanishing moments increases.

It is known that strong localization properties of the measurement system are an obstruction to stable phase retrieval [8] and in the light of this the stability behaviour of Theorems 3.7 and 3.9 is not really surprising.

4 Proof of Theorem 3.1

This section is devoted to prove Theorem 3.1 which is the main result of this paper. The proof follows several steps and relies on the following key lemma, see also [13] for related results.

Lemma 4.1 *Suppose that* $F \in \mathcal{O}(D)$*, then*

$$|F'(z)| = \big|\nabla|F|(x,y)\big| \qquad \forall\, z = x + iy \in D.$$

Proof Let u and v denote the real and imaginary part of F, respectively, i.e. $F(x,y) = u(x,y) + \mathrm{i}v(x,y)$. Then,

$$\begin{aligned}\partial_x\,|F| &= \partial_x(\sqrt{u^2+v^2})\\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{u^2+v^2}}\cdot(2u\cdot u_x + 2v\cdot v_x)\\ &= \frac{u\cdot u_x + v\cdot v_x}{|F|}.\end{aligned}$$

Similarly,

$$\partial_y|F| = \frac{uu_y + vv_y}{|F|} = \frac{-uv_x + vu_x}{|F|},$$

where the last equality follows from Cauchy–Riemann equations. Therefore,

$$\begin{aligned}\big|\nabla|F|\big|^2 = (\partial_x|F|)^2 + (\partial_y|F|)^2 &= \frac{(uu_x + vv_x)^2 + (-uv_x + vu_x)^2}{|F|^2}\\ &= \frac{(u^2+v^2)(u_x^2+v_x^2)}{|F|^2} = u_x^2 + v_x^2 = |F'(z)|^2.\end{aligned}$$

□

Having Lemma 4.1 at hand, we may now proceed to the proof of Theorem 3.1, which we restate here for convenience of the reader.

Theorem 3.1 *Suppose that* F_1 *belongs to a class of atoll functions as in Definition 1.3, i.e.* $F_1 \in \mathcal{H}(D, (D_0^i)_{i=1}^l, \delta, \Delta)$*. Assume further that* $F_2 \in C^1(D)$ *such that there exists a continuous function* $\eta : D \to \mathbb{C}$ *for which both functions* $\eta\cdot F_1,\ \eta\cdot F_2 \in \mathcal{O}(D)$*. Suppose that* $1 \le p \le \infty$.

Pick $z_0 \in D_+$. We denote $C_{\text{samp}} := C_{\text{samp}}(p, D_+, z_0, |F_1| - |F_2|)$, meaning that

$$\| |F_1(z_0)| - |F_2(z_0)| \|_{L^p(D_+)} \leq C_{\text{samp}} \cdot \| |F_1| - |F_2| \|_{L^p(D_+)}. \tag{4.1}$$

Then, the following estimate holds:

$$\inf_{\alpha \in \mathbb{R}} \left\| F_1 - e^{\mathrm{i}\alpha} F_2 \right\|_{L^p(D)} \leq C(z_0, p, D_+, (D_0^i)_{i=1}^l) \frac{\Delta^2}{\delta^2} \| |F_1| - |F_2| \|_{W^{1,p}(D_+)}. \tag{4.2}$$

We recall that for the constant $C(z_0, p, D_+, (D_0^i)_{i=1}^l)$ one may choose (with a suitably large but uniform constant $c > 0$):

$$\begin{aligned} C(z_0, p, D_+, (D_0^i)_{i=1}^l) = c \cdot (C_{\text{poinc}}^a(D_+) + C_{\text{samp}} \\ + \sum_{i=1}^{l} C_{\text{bound}}(D_0^i) \cdot \operatorname{var}(\eta, D_0^i) \cdot \\ C_{\text{trace}}(D_+)(C_{\text{poinc}}^a(D_+) + C_{\text{samp}})), \end{aligned} \tag{4.3}$$

where we have omitted the dependence of the various constants on p, z_0 and denote

$$\operatorname{var}(\eta, D_0^i) := \frac{\max_{z \in \partial D_0^i} |\eta(z)|}{\min_{z \in D_0^i} |\eta(z)|}, \quad i = 1, \ldots, l.$$

Proof of Theorem 3.1 Without loss of generality, we let $l = 1$ and put $D_0 := D_0^1$ (the general case being not more difficult). We need to bound the quantity

$$\left\| F_2(z) - e^{\mathrm{i}\alpha} F_1(z) \right\|_{L^p(D)} \leq \left\| F_2(z) - e^{\mathrm{i}\alpha} F_1(z) \right\|_{L^p(D_+)} + \left\| F_2(z) - e^{\mathrm{i}\alpha} F_1(z) \right\|_{L^p(D_0)} \tag{4.4}$$

for suitable $\alpha \in \mathbb{R}$, and we will develop separate arguments for the two terms on the RHS of the above.

Step 1. As a first step, we start by developing a basic estimate. Consider

$$F := F_2 / F_1.$$

By assumption, we have that $\eta \cdot F_1, \eta \cdot F_2 \in \mathcal{O}(D)$ and $|F_1(z)| \geq \delta$ for $z \in D_+$. Consequently, $F \in \mathcal{O}(D_+)$. Pick $\alpha \in \mathbb{R}$ such that

$$|F_2(z_0) - e^{\mathrm{i}\alpha} F_1(z_0)| = ||F_2(z_0)| - |F_1(z_0)||. \tag{4.5}$$

Now, consider for $z \in D$ arbitrary

$$\begin{aligned} |F_2(z) - e^{\mathrm{i}\alpha} F_1(z)| &= |F_1(z)||F(z) - e^{\mathrm{i}\alpha}| \\ &\leq |F_1(z)| \left(|F(z) - F(z_0)| + |F(z_0) - e^{\mathrm{i}\alpha}| \right) \end{aligned}$$

$$= |F_1(z)|\left(|F(z)-F(z_0)| + \frac{1}{|F_1(z_0)|}|F_2(z_0) - e^{\mathrm{i}\alpha}F_1(z_0)|\right)$$
$$= |F_1(z)|\left(|F(z)-F(z_0)| + \frac{1}{|F_1(z_0)|}||F_2(z_0)| - |F_1(z_0)||\right)$$
$$\leq \Delta\left(|F(z)-F(z_0)| + \frac{1}{\delta}||F_2(z_0)| - |F_1(z_0)||\right). \tag{4.6}$$

Step 2. In this step, we focus on the second term of (4.4) and show that it can actually be absorbed by an estimate on D_+. By the analyticity of $\eta \cdot F_1$ and $\eta \cdot F_2$ on D, we can apply Theorem 2.6 to obtain

$$\|\eta \cdot (F_2(z) - e^{\mathrm{i}\alpha}F_1(z))\|_{L^p(D_0)} \leq C_{\mathrm{bound}}(p, D_0)\|\eta \cdot (F_2(z) - e^{\mathrm{i}\alpha}F_1(z))\|_{L^p(\partial D_0)}$$

and, therefore, we get

$$\left\|F_2(z) - e^{\mathrm{i}\alpha}F_1(z)\right\|_{L^p(D_0)} \leq C_{\mathrm{bound}}(p, D_0) \cdot \mathrm{var}(\eta, D_0) \cdot \left\|F_2(z) - e^{\mathrm{i}\alpha}F_1(z)\right\|_{L^p(\partial D_0)}.$$

We may now estimate further, using (4.6), that

$$\left\|F_2(z) - e^{\mathrm{i}\alpha}F_1(z)\right\|_{L^p(D_0)} \leq C_{\mathrm{bound}}(p, D_0) \cdot \mathrm{var}(\eta, D_0) \cdot\left(\Delta\,\|F(z)-F(z_0)\|_{L^p(\partial D_+)} + \frac{\Delta}{\delta}\,\||F_1(z_0)| - |F_2(z_0)|\|_{L^p(\partial D_+)}\right).$$

Applying the trace theorem (Theorem 2.4), we further get that

$$\left\|F_2(z) - e^{\mathrm{i}\alpha}F_1(z)\right\|_{L^p(D_0)} \leq C_{\mathrm{bound}}(p, D_0) \cdot \mathrm{var}(\eta, D_0) \cdot C_{\mathrm{trace}}(p, D_+) \cdot\left(\Delta\,\|F(z)-F(z_0)\|_{W^{1,p}(D_+)} + \frac{\Delta}{\delta}\,\||F_1(z_0)| - |F_2(z_0)|\|_{L^p(D_+)}\right),$$

where we have used that $\||F_1(z_0)| - |F_2(z_0)|\|_{W^{1,p}(D_+)} = \||F_1(z_0)| - |F_2(z_0)|\|_{L^p(D_+)}$ because the function is constant. With the assumption in (4.1), we further get

$$\left\|F_2(z) - e^{\mathrm{i}\alpha}F_1(z)\right\|_{L^p(D_0)} \leq C_{\mathrm{bound}}(p, D_0) \cdot \mathrm{var}(\eta, D_0) \cdot C_{\mathrm{trace}}(p, D_+) \cdot\left(\Delta\,\|F(z)-F(z_0)\|_{W^{1,p}(D_+)} + \frac{\Delta}{\delta}C_{\mathrm{samp}}\,\||F_1| - |F_2|\|_{L^p(D_+)}\right).$$

Lastly, we apply the analytic Poincaré inequality (2.2) and obtain the estimate

$$\left\|F_2(z) - e^{\mathrm{i}\alpha}F_1(z)\right\|_{L^p(D_0)} \leq C_{\mathrm{bound}}(p, D_0) \cdot \mathrm{var}(\eta, D_0) \cdot C_{\mathrm{trace}}(p, D_+) \cdot\left(\Delta(C^a_{\mathrm{poinc}}(p, D_+, z_0))\,\left\|F'\right\|_{L^p(D_+)} + \frac{\Delta}{\delta}C_{\mathrm{samp}}\,\||F_1| - |F_2|\|_{L^p(D_+)}\right). \tag{4.7}$$

Step 3. In this step, we focus on an estimate for the first term in (4.4). Using (4.6), we see that

$$\left\| F_2(z) - e^{\mathrm{i}\alpha} F_1(z) \right\|_{L^p(D_+)} \le \Delta \left\| F(z) - F(z_0) \right\|_{L^p(D_+)} + \frac{\Delta}{\delta} C_{\mathrm{samp}} \left\| |F_1| - |F_2| \right\|_{L^p(D_+)} .$$

Yet another application of the analytic Poincaré inequality yields

$$\left\| F_2(z) - e^{\mathrm{i}\alpha} F_1(z) \right\|_{L^p(D_+)} \le \Delta C^a_{\mathrm{poinc}}(p, D_+, z_0) \left\| F' \right\|_{L^p(D_+)} + \frac{\Delta}{\delta} C_{\mathrm{samp}} \left\| |F_1| - |F_2| \right\|_{L^p(D_+)} . \quad (4.8)$$

Step 4. In equations (4.7) and (4.8), we now have achieved estimates of both terms in (4.4). A close look at these estimates reveals that we only need to get a bound on $\left\| F' \right\|_{L^p(D_+)}$ in terms of $\left\| |F_1| - |F_2| \right\|_{W^{1,p}(D)}$ to finish the proof. This is where our key lemma, Lemma 4.1, comes into play, stating that

$$\left\| F' \right\|_{L^p(D_+)} = \left\| \nabla |F| \right\|_{L^p(D_+)} .$$

It thus remains to achieve a bound for $\left\| \nabla |F| \right\|_{L^p(D_+)}$. To this end, we consider

$$\frac{\partial}{\partial x}|F| = \frac{|F_1| \frac{\partial}{\partial x}|F_2| - |F_2| \frac{\partial}{\partial x}|F_1|}{|F_1|^2} = \frac{\frac{\partial}{\partial x}|F_1|(|F_1| - |F_2|) + |F_1|(\frac{\partial}{\partial x}|F_2| - \frac{\partial}{\partial x}|F_1|)}{|F_1|^2},$$

and hence,

$$\left| \frac{\partial}{\partial x}|F| \right| \le \frac{\Delta}{\delta^2} \left(\left| |F_1| - |F_2| \right| + \left| \frac{\partial}{\partial x}|F_2| - \frac{\partial}{\partial x}|F_1| \right| \right),$$

valid uniformly on D_+. A similar estimate holds for $\left| \frac{\partial}{\partial y}|F| \right|$, and thus there exists a universal constant $c > 0$ with

$$\left\| F' \right\|_{L^p(D_+)} \le c \cdot \frac{\Delta}{\delta^2} \left\| |F_1| - |F_2| \right\|_{W^{1,p}(D_+)} . \quad (4.9)$$

Step 5. We finish the proof by substituting the estimate (4.9) into Eqs. (4.7) and (4.8) (and noting that $\frac{\Delta}{\delta} \ge 1$), and then use Lemma 2.11 to remove the dependency on z_0, which gives the desired result. □

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