MATH. SCAND. 67 (1990), 299–319.

# STABLE RANK OF FOURIER ALGEBRAS AND AN APPLICATION TO KOROVKIN THEORY

MICHAEL PANNENBERG

**Abstract.**

We generalize the formula $\mathrm{Bsr}(A) = \left[\frac{d}{2}\right] + 1$, which connects the Bass stable rank of a commutative unital Banach algebra $A$, which is regular or possesses a symmetric involution, to the covering dimension $d$ of its spectrum, to the non-unital case: The formula remains true if the covering dimension is replaced by Dowker's local covering dimension, which coincides with $d$ in case $A$ has paracompact spectrum, e.g. for separable $A$ or for Eymard's Fourier algebra of a locally compact group. We also calculate Rieffel's topological stable rank of a Fourier algebra (which coincides with Bass stable rank). Following an idea of Brown and Pedersen, we define the real rank of an involutive Banach algebra, prove some of its elementary properties and calculate its value for the disc algebra and any Fourier algebra. We finally use our results to solve a problem in Korovkin approximation theory: if $G$ is an LCA-group, the associated group algebra $L^1(G)$ possesses a finite universal Korovkin system, iff $L^1(G)$ is separable and has finite (stable or real) rank.

## 1. Introduction.

Let $A$ be a complex commutative Banach algebra, which is not necessarily unital. Let $\Delta_A$ denote is spectrum, i.e. the space of all non-trivial multiplicative linear functionals on $A$ endowed with the Gelfand topology. Then $\Delta_A$ is a locally compact Hausdorff space, which is compact if $A$ is unital. Generalizing a pioneering result of Vasershtein ([30]), Corach and Larotonda proved that the inequality

$$\mathrm{Bsr}(A) \leqq \left[\frac{d}{2}\right] + 1 \tag{1}$$

holds for every commutative *unital* Banach algebra $A$, and if additionally $A$ is regular, we even have equality:

$$\mathrm{Bsr}(A) = \left[\frac{d}{2}\right] + 1; \tag{2}$$

Received February 1, 1990.

compare [8, 9]. Here Bsr$(A)$ denotes the Bass stable rank of the commutative unital ring $A$ ([6], [30]); $d$ is the Cech-Lebesgue covering dimension of the compact Hausdorff space $\Delta_A$ ([14], [26], [30]) and [ ] denotes "integer part of". The relation (2) also holds if $A$ is endowed with a symmetric involution instead of being regular (this is implicitely proved in [30]).

It is the purpose of this note to generalize relations (1) and (2) to the non-unital case, using the definition of Bass stable rank for not necessarily unital rings given in [30]: We show that (1) and (2) remain true for non-unital $A$, if e.g. $\Delta_A$ is normal and weakly paracompact. The latter assertion is always true for separable $A$, or if $A$ is the Fourier algebra $\mathscr{A}(X)$ of a locally compact group $X$, as introduced by Eymard ([15]).

If no assumptions on the topology of $\Delta_A$ are made, (1) and (2) remain true if the (modified) Cech-Lebesgue covering dimension of the (locally compact Hausdorff, hence) completely regular space $\Delta_A$ is replaced by Dowker's local dimension of $\Delta_A$.

We also calculate the topological stable rank (tsr) defined by Rieffel ([27]) of $\mathscr{A}(X)$: It turns out that it coincides with the Bass stable rank, i.e. the formula

$$\text{Bsr}(\mathscr{A}(X)) = \left[\frac{d}{2}\right] + 1 = \text{tsr}(\mathscr{A}(X)) \tag{3}$$

holds for every locally compact group $X$ of covering dimension $d$.

Following Brown and Pedersen [4] we introduce the concept of real rank of an involutive Banach algebra by focusing attention only on self- adjoint elements in the definition of topological stable rank. After some preliminary observations concerning this rank function, we calculate the real rank of a Fourier algebra $\mathscr{A}(X)$: It coincides with the dimension of $X$.

We are particularly interested in Fourier algebras, since the results displayed above allow to establish an unexpected connection between the theory of each of these rank functions and Korovkin approximation theory: Using a result of [25], we may show that the group algebra $L^1(G)$ of a locally compact abelian group $G$ possesses a finite universal Korovkin system, iff $L^1(G)$ is separable and the value of one (hence of each) of these rank functions for $L^1(G)$ is finite.

## 2. Notations and Definitions.

Let $A$ be a complex Banach algebra, and denote by $A_+$ its unitization obtained by adjoining a unit element 1.

By GL$(n, A_+)$ we denote the group of invertible $n \times n$-matrices with entries in $A_+$; $1_n$ denotes its unit element. If $M_n(A)$ denotes the algebra of all $n \times n$-matrices over $A$, we write GL$(n, A)$ for the normal subgroup of GL$(n, A_+)$ consisting of all matrices in GL$(n, A_+)$ which are congruent to $1_n$ mod $M_n(A)$; if $A$ is unital, this group is isomorphic to the group of invertible matrices in $M_n(A)$.

For any $n \in \mathbb{N}$, let us set $\mathbf{e}_1 = (1, 0, 0, \ldots, 0) \in A_+^n$. Given elements $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ of $A_+^n$ we set

$$[\mathbf{b}, \mathbf{a}] := \sum_{i=1}^{n} b_i a_i \in A_+;$$

this corresponds to multiplying the $1 \times n$ matrix $\mathbf{b}$ by the $n \times 1$ matrix $\mathbf{a}$. We denote by $U_n(A_+, A)$ the set of all $A$-unimodular vectors in $A_+^n$, i.e. the set of all $\mathbf{a} \in A_+^n$ such that $\mathbf{a} \equiv \mathbf{e}_1 \bmod A^n$ and such that there exists $\mathbf{b} \in A_+^n$ which satisfies $\mathbf{b} \equiv \mathbf{e}_1 \bmod A^n$ and $[\mathbf{b}, \mathbf{a}] = 1$. Typical examples are given by the first column vector of any element of $\mathrm{GL}(n, A)$; however, in general not every unimodular vector arises in this way (compare [6, page 7] for a counterexample in case $A = \mathscr{C}(S^2)$).

If $A$ has a unit $e$, one usually calls $\mathbf{a} \in A^n$ unimodular ($\mathbf{a} \in U_n(A)$) iff there exists $\mathbf{b} \in A^n$ such that $[\mathbf{b}, \mathbf{a}] = e$ ([6], [9]); in this case

$$U_n(A) = \{\mathbf{a} \in A^n : (1 + a_1 - e, a_2, \ldots, a_n) \in U_n(A_+, A)\}$$

For non-unital $A$, both sets are related by the equation

$$U_n(A_+, A) = \{\mathbf{a} \in U_n(A_+) : \mathbf{a} \equiv \mathbf{e}_1 \bmod A^n\},$$

c.f. lemma 1 of [30].

In case $A$ is endowed with a continuous involution, we extend the involution canonically to $A_+$ and denote by $U_n(A_{sa})$ resp. $U_n(A_{+sa}, A_{sa})$ the intersection of the corresponding set with $A_{sa}^n$ resp. $A_{+sa}^n$, where $A_{sa}$ denotes the set of all self-adjoint elements of $A$.

The *Bass stable rank* $\mathrm{Bsr}(A)$ is the least $n \in \mathbb{N}$ for which the following condition holds ([30]):

(SR)$_n$ For any $\mathbf{a} = (a_1, \ldots, a_{n+1}) \in U_{n+1}(A_+, A)$ there exists $\mathbf{x} = (x_1, \ldots, x_n) \in A^n$ such that $(a_1 + x_1 a_{n+1}, \ldots, a_n + x_n a_{n+1}) \in U_n(A_+, A)$.

If no such $n$ exists, we set $\mathrm{Bsr}(A) = \infty$. In case $e \in A$, this coincides with the definition given in [8, 9] by the remark above.

The *topological stable rank* $\mathrm{tsr}(A)$ of a Banach algebra $A$ has been introduced by Rieffel: If $A$ is unital, $\mathrm{tsr}(A)$ is defined to be the least integer $n \in \mathbb{N}$ for which $U_n(A)$ is dense in $A^n$ ([27], [10]). In the non-unital case, $\mathrm{tsr}(A)$ may be defined to be the least integer $n \in \mathbb{N}$ for which $U_n(A_+, A)$ is dense in $\{\mathbf{a} \in A_+^n : \mathbf{a} \equiv \mathbf{e}_1 \bmod A^n\}$; compare [27] proposition 4.2.

If $A$ is an involutive Banach algebra, we adopt an idea of Brown and Pedersen [4] and take an appropriate form of their definition for C*-algebras as definition for the general case:

The real rank RR$(A)$ is the least integer $n \in \mathbf{N}_0$ for which $U_{n+1}(A_{sa})$ is dense in $A_{sa}^{n+1}$ (in the unital case) resp. $U_{n+1}(A_{+sa}, A_{sa})$ is dense in $\{\mathbf{a} \in A_{+sa}^{n+1} : \mathbf{a} \equiv \mathbf{e}_1 \bmod A^{n+1}\}$ (in the non-unital case).

Bass stable rank and topological stable rank are related by

$$\mathrm{Bsr}(A) \leqq \mathrm{tsr}(A);$$

equality does not hold in general but holds for C*-algebras ([10, 18]).

The *Cech-Lebesgue covering dimension* dim $X$ of a non-empty normal space $X$ is the least integer $n \in \mathbf{N}_0$ such that every finite open cover of $X$ has a finite open refinement of order $\leqq n$. We refer to [13, 14] and [26] for the relevant properties of this dimension function and mention only the fact that for normal spaces this definition is consistent with the one given by Vasershtein in [30] (cf. Chap. 3, §3 of [26]).

When dealing with a locally compact, not necessarily normal space the *modified covering dimension* mod dim, which is relatively wellbehaved on the class of completely regular spaces, may be used: Its definition is obtained by just replacing "open" by "functionally open" (complement of a zero set) in the above definition, cf. [14. p. 222] and chap. 10, § 1 of [26]. We will use this modified covering dimension, which coincides with dim on normal spaces, only in proposition 2.

Dowker's *local covering dimension* loc dim ([12]) of a non-empty topological space $X$ is defined to be the least integer $n \in \mathbf{N}_0$ such that for every point $x \in X$ there is some open set $U \subset X$ containing $x$ such that dim $\bar{U} \leqq n$. We refer to [12] resp. [26, chapter 5], for its properties.

Finally, we recall that a topological space $X$ is *weakly paracompact* ([13, 14]) if $X$ is a Hausdorff space and every open cover of $X$ has a point-finite open refinement (the terms metacompact and point-paracompact are also commonly used).

## 3. The Bass stable rank of a commutative Banach algebra.

Let $A$ be a complex commutative Banach algebra. We denote the one-point compactification of its spectrum $\Delta = \Delta_A$ by $\Delta_\infty$. Our strategy consists in reducing the assertion we want to prove to the unital case by considering $A_+$. It is well known that tsr$(A) =$ tsr$(A_+)$ ([27] definition 1.4, and proposition 4.2.); since we couldn't find the analogous assertion for the Bass stable rank in the literature we include a proof:

LEMMA 1. Bsr$(A) =$ Bsr$(A_+)$

PROOF. Since $A$ is an ideal in $A_+$ and $A_+/A = \mathbf{C}$, this follows immediately from a result of Vasershtein. Indeed, each unimodular vector $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbf{C}^n$

has at least one entry non-zero, so there exists $T \in \mathrm{GL}(n, \mathbf{C}) \subset \mathrm{GL}(n, A_+)$ such that $T\mathbf{z} = (1, 0, \ldots, 0) \in \mathbf{C}^n$; consequently by theorem 4 of [30] we get

$$\mathrm{Bsr}(A_+) = \max(\mathrm{Bsr}(A), \mathrm{Bsr}(A_+/A)) = \mathrm{Bsr}(A),$$

since $\mathrm{Bsr}(A) \geqq 1 = \mathrm{Bsr}(\mathbf{C})$.

Setting $d_\infty = \dim \Delta_\infty$, this immediately implies

PROPOSITION 1. *We have* $\mathrm{Bsr}(A) \leqq \left[\frac{d_\infty}{2}\right] + 1$; *equality holds if $A$ is regular.*

The proof is accomplished by noting that $\Delta_{A_+} \cong \Delta_\infty$ and using (1) resp. (2), if $A$ and consequently also $A_+$ is regular.

REMARK 1. $d_\infty$ may be computed without actually knowing the space $\Delta_\infty$: It is known that $\dim \Delta_\infty$ is the supremum of the covering dimensions of the compact subspaces of $\Delta$ (prop. 15 on p. 103 of [16]).

This easily implies

LEMMA 2. *Let $X$ be a non-empty locally compact Hausdorff space with one-point compactification $X_\infty$. Then*

$$\operatorname{loc\,dim} X = \dim X_\infty.$$

PROOF. Let $C \subset X$ be a compact subspace. Since $X$ is an open subspace of $X_\infty$, the monotonicity of loc dim on closed resp. on open subsets ([26] propositions 5.2.1. and 5.2.2.) yields

$$\operatorname{loc\,dim} C \leqq \operatorname{loc\,dim} X \leqq \operatorname{loc\,dim} X_\infty.$$

Since $C$ and $X_\infty$ are compact Hausdorff spaces, their dimension coincides with their local dimension. Now an application of remark 1 yields the desired equality.

Just to abbreviate the statements to follow, we call a locally compact Hausdorff space *good*, iff $X$ is a normal space which additionally is weakly paracompact, or the union of a countable family of closed weakly paracompact subsets, or the union of a locally finite family of weakly paracompact subsets all but at most one of which are closed.

LEMMA 3. *Let $X$ be a non-empty good locally compact Hausdorff space with one-point compactification $X_\infty$. Then*

$$\dim X = \dim X_\infty.$$

PROOF. Since $X$ is good, proposition 5.3.4. and corollary 5.3.5. of [26] imply $\operatorname{loc\,dim} X = \dim X$, so that the result is clear by lemma 2.

Avoiding the notion of local dimension, a direct proof of the above equality may be given for weakly paracompact, normal $X$. Since $X_\infty = X \cup \{\infty\}$, a sum theorem of Dowker ([14, prop. 3.1.7.]) gives $\dim X_\infty \leqq \dim X$. But $X$ is a weakly paracompact, locally strongly paracompact subspace of $X_\infty$, so that $\dim X \leqq \dim X_\infty$ by a result of Lifanov and Pasynkov ([14, prop. 3.1.24]). The remaining two cases follow from the countable sum theorem resp. locally finite sum theorem for dim ([14, prop. 3.1.8. and 3.1.11.]).

PROPOSITION 2. *Let $A$ be a complex commutative Banach algebra with non-empty spectrum $\Delta$. Set $l := \operatorname{loc\,dim} \Delta$, $m := \operatorname{mod\,dim} \Delta$. Then*

$$\operatorname{Bsr}(A) \leqq \left[\frac{l}{2}\right] + 1 \leqq \left[\frac{m}{2}\right] + 1.$$

*If $A$ is regular, we have*

$$\operatorname{Bsr}(A) = \left[\frac{l}{2}\right] + 1$$

*If $A$ is regular and $\Delta$ is a good space with $d = \dim \Delta$, we have*

$$\operatorname{Bsr}(A) = \left[\frac{l}{2}\right] + 1 = \left[\frac{m}{2}\right] + 1 = \left[\frac{d}{2}\right] + 1.$$

*Each of these equalities also holds if $A$ is assumed to have a symmetric involution instead of being regular.*

PROOF. By proposition 1 and lemma 2, we get $\operatorname{Bsr}(A) \leqq \left[\frac{l}{2}\right] + 1$, and equality holds for regular $A$. Now $l = \dim \Delta_\infty = \operatorname{mod\,dim} \Delta_\infty \leqq m$ (reason: The modified covering dimension cannot be increased by adding just one point, [13, ex. 7.1.C]), so the first chain of inequalities is proved. If $\Delta$ is a good space, $m = d$ since $\Delta$ is normal, and $l = d$ by lemma 2 and lemma 3; now the last chain of equalities is a direct consequence of proposition 1. To prove that the same equalities also hold in the symmetric case, we first observe that by lemma 1 and the above arguments we may assume that $A$ is unital. But $\operatorname{Bsr}(A) = \operatorname{Bsr}(\hat{A})$ by elementary Gelfand theory, where $\hat{A}$ is the algebra of Gelfand transforms, and $\operatorname{Bsr}(\hat{A}) = \left[\frac{d}{2}\right] + 1$ by theorem 7 of [30], since $\hat{A}$ is closed under complex conjugation. This finishes the proof.

Let $C_0(X)$ resp. $C_b(X)$ be the Banach algebra of all continuous complex-valued functions on a locally compact Hausdorff space $X$, which vanish at infinity resp. are bounded. Proposition 2 immediately yields.

COROLLARY 1. *Let $X$ be a locally compact Hausdorff space. Then*

$$\operatorname{Bsr}(C_0(X)) = \left[\frac{l}{2}\right] + 1 \leqq \left[\frac{m}{2}\right] + 1 = \operatorname{Bsr}(C_b(X))$$

*where $l = \operatorname{loc}\dim X$ and $m = \operatorname{mod}\dim X$. If $X$ is a good space and $d = \dim X$, we have*

$$\operatorname{Bsr}(C_0(X)) = \left[\frac{d}{2}\right] + 1 = \operatorname{Bsr}(C_b(X))$$

*Bass stable rank may be replaced by topological stable rank in each of these equations.*

PROOF. One just has to observe that the spectrum of $C_b(X)$ is the Stone-Cech compactification $\beta X$ of $X$, which satisfies $\dim \beta X = \operatorname{mod}\dim X$ ([14]), [26, theorem 10.1.4]); consequently $\operatorname{Bsr}(C_b(X)) = \left[\frac{m}{2}\right] + 1$ for every locally compact Hausdorff space. The same equalities hold for tsr, since it coincides with Bsr for C*-algebras (cf. [18] for the unital case and use $A_+$ to deduce the non-unital case).

REMARK 2. Some extra topological hypothesis except local compactness is needed for the above results: There are examples of locally compact normal spaces $X$ with

$$\operatorname{loc}\dim X = \dim X_\infty < \dim X = \operatorname{mod}\dim X,$$

cf. [16]. p. 103, example 14 and proposition 15 as well as [26], chap 5, §4.

REMARK 3. Examples of weakly paracompact, normal spaces are given by paracompact (Hausdorff) spaces (which, in contrast to weakly paracompact ones, are always normal [13]); in particular, locally compact (Hausdorff) groups and metrizable locally compact spaces are always paracompact ([17, theorem 8.13], [21, lemma 1.1], [13]) and hence good in our sense.

COROLLARY 2 *Let $A$ be a regular complex commutative Banach algebra, which is separable or possesses a countable approximate identity. If $d = \dim \Delta_A$, we have*

$$\operatorname{Bsr}(A) = \left[\frac{d}{2}\right] + 1$$

PROOF. If $A$ is separable, $\Delta_A$ is metrizable, and the asserted equality follows from proposition 2. If A possesses a countable approximate identity, its spectrum $\Delta_A$ is $\sigma$-compact ([20, theorem 2.2]). Consequently, $\Delta_A$ is paracompact, and again the asserted equality follows from proposition 2.

We remark that the existence of an element $x \in A$, whose Gelfand transform never vanishes on $\Delta_A$, also implies the $\sigma$-compactness of $\Delta_A$. In the presence of a symmetric involution, both assertions are equivalent; cf. [20] for a proof and a discussion of related matters.

## 4. The real rank of an involutive Banach algebra.

Let $A$ be a complex Banach algebra with a continuous involution. Checking the arguments leading to the equation $\operatorname{tsr}(A) = \operatorname{tsr}(A_+)$ (propostion 4.2 of [27]), one easily sees that $\mathrm{RR}(A) = \mathrm{RR}(A_+)$. If $x_1,\ldots,x_n,y_1,\ldots,y_n$ are self-adjoint elements of a C*-algebra $A$, one easily checks that $\max\{\|x_k - y_k\| : 1 \leqq k \leqq n\}$ is small iff $\|\sum_k (x_k - y_k)^2\|$ is small; this shows that our definition of real rank is consistent with the one given by Brown and Pedersen for C*-algebras.

For any compact Hausdorff space $X$, on has

$$\mathrm{RR}(C(X)) = \dim X$$

according to [4] proposition 1.1. A consideration of the one-point compactification of a locally compact Hausdorff space yields

$$\mathrm{RR}(C_0(X)) = \operatorname{loc}\dim X$$

and the argument used to prove corollary 1 shows

$$\mathrm{RR}(C_b(X)) = \operatorname{mod}\dim X$$

so that for good $X$ both real ranks equal $\dim X$.

We first observe that proposition 1.2. of [4] carries over to the general case:

PROPOSITION 3. *Let $A$ be a Banach algebra with a continuous involution. Then with $r := \mathrm{RR}(A)$ we get*

$$\mathrm{RR}(A) \leqq 2\operatorname{tsr}A) - 1 \quad \textit{and} \quad \left[\frac{r}{2}\right] + 1 \leqq \operatorname{tsr}(A).$$

PROOF. Without loss of generality we assume that $A$ is unital, the involution is isometric and $\operatorname{tsr} A) = n < \infty$.

Let $x_1,\ldots,x_{2n}$ be self-adjoint in $A$, and put $\xi_k := x_k + i\,x_{n+k}$ $(1 \leqq k \leqq n)$. By assumption, there exists a unimodular vector $\eta_k = y_k + i\,y_{n+k}$ $(y_k, y_{n+k}$ self-adjoint, $1 \leqq k \leqq n)$ in $A^n$ arbitrarily close to $(\xi_1,\ldots,\xi_n)$. Then $(y_1,\ldots,y_{2n})$ is arbitrarily close to $(x_1,\ldots,x_{2n})$, and the left ideal generated by $y_1,\ldots,y_{2n}$ contains $\eta_1,\ldots,\eta_n$; hence coincides with $A$, so $(y_1,\ldots,y_{2n}) \in U_{2n}(A_{\mathrm{sa}})$. Therefore $U_{2n}(A_{\mathrm{sa}})$ is dense in $A_{\mathrm{sa}}^{2n}$, which by our indexing convention yields the first inequality. The second inequality obviously follows.

In the case of a commutative Banach algebra with a symmetric involution, we get the following stronger result:

PROPOSITION 4. *Let A be a commutative Banach algebra with a symmetric involution. Then*

$$\operatorname{tsr}(A) = \left[\frac{r}{2}\right] + 1$$

*where* $r = \mathrm{RR}(A)$.

PROOF. We again assume that A is unital, the involution is isometric and $r < \infty$. Put $n = \left[\frac{r}{2}\right] + 1$ and consider $x_1, \ldots, x_n \in A$. Decomposing in real and imaginary parts, we observe that $2n \geqq r + 1$ so that we may approximate $(x_1, \ldots, x_n)$ by a vector $(y_1, \ldots, y_n)$ whose real and imaginary parts form a unimodular vector of $2n$ self-adjoint elements. If $y_1, \ldots, y_n$ would be contained in a maximal ideal of $A$, the same would be true for real and imaginary parts (since by the symmetry of the involution maximal ideals are automatically self-adjoint) which generate $A$ as an ideal – this contradiction shows that $(y_1, \ldots, y_n)$ is unimodular and finishes the proof.

Of course the argument heavily uses the commutativity of $A$: For non-commutative algebras, unimodularity is defined using one-sided ideals, which in general are not self-adjoint even for symmetric involutions.

If $A$ and $B$ are Banach algebras with continuous involutions and $L: A \to B$ is a continuous involutive algebra homomorphism with dense range, we may (by extending canonically to the unitizations) assume without loss of generality that $A, B$ and $L$ are unital. Then $L(U_{n+1}(A_{sa})) \subset U_{n+1}(B_{sa})$ and $L(A_{sa})$ is a dense subset of $B_{sa}$, so obviously $\mathrm{RR}(B) \leqq \mathrm{RR}(A)$.

Applying this observation to the map $L: A \to C_0(\Delta_A^*)$ obtained by composing the Gelfand transformation of a commutative Banach algebra $A$ with continuous involution with the restriction map from $C_0(\Delta_A)$ to $C_0(\Delta_A^*)$, where $\Delta_A$ resp. $\Delta_A^*$ is the locally compact space of all resp. all positive characters of $A$, we obtain the following estimate:

PROPOSITION 5. *Let A be a commutative Banach algebra with continuous involution. Then*

$$\operatorname{loc\,dim}(\Delta_A^*) \leqq \mathrm{RR}(A).$$

*If the involution is symmetric, we get*

$$\operatorname{loc\,dim}(\Delta_A) \leqq \mathrm{RR}(A).$$

It is well known that the involution $*$ is given by a homeomorphism $\varphi: \Delta_A \to \Delta_A$ of period 2 via $\varphi(m)(x) = m(x^*)^*$ resp. $(x^*)^\wedge = (\hat{x} \circ \varphi)^*$ for all characters $m \in \Delta_A$ resp. all $x \in A$, where $^\wedge$ denotes the Gelfand transformation. Using this description, $\Delta_A^*$ corresponds to the fixpoints of $\varphi$, so that the local dimension of the fixpoint-set of $\varphi$ is always dominted by the real rank of $A$. Simple examples for $C(X)$ endowed with an involution induced by a map without any or with only one fixpoint show that this estimate may be very rough. It may, however, give the exact value of the real rank as the following example shows:

EXAMPLE 1. Let $A$ be the disc algebra. Then it is well known that $\mathrm{Bsr}(A) = 1$ and $\mathrm{tsr}(A) = 2$. We now show that $\mathrm{RR}(A) = 1$ for any choice of the involution on $A$.

Indeed, standard Banach algebra techniques show that each involution on $A$ has the form

$$f^*(z) = (f(\varphi(z)))^* \quad (z \in D := \{w \in \mathbf{C} : |w| \leqq 1\};\ f \in A)$$

$$\varphi(z) = (a^* z^* - i\beta)(i\beta z^* + a)^{-1} \quad (z \in D)$$

for some constants $a \in \mathbf{C}$, $\beta \in \mathbf{R}$ related by $|a|^2 - \beta^2 = 1$.

The positive part of the spectrum is given by evaluations at

$$\Delta_A^* \cong \{z \in D : \varphi(z) = z\}$$

which geometrically is a circle intersecting the unit circle orthogonally. Therefore, $\mathrm{RR}(A) \geqq \dim \Delta_A^* = 1$ by proposition 5.

On the other hand, if $f_1, f_2$ are self-adjoint elements of $A$, we approximate these functions by polynomials $p_1, p_2$, where by adding a small constant if necessary we assume that $p_1 \neq 0$. If $Z$ is the set of zeroes of $p_1$, we choose a constant $\lambda \in \mathbf{R} \setminus p_2(Z)$ of small modulus and observe that $(p_1, p_2 - \lambda) \in U_2(A_{sa})$ approximates $(f_1, f_2)$- this shows $\mathrm{RR}(A) \leqq 1$ and finishes the proof.

This example also shows that the inequality of proposition 3 (which is sharp e.g. for $C([0,1])$) may be strict.

It further demonstrates that the symmetry assumption may not be dropped in proposition 4 – that the commutativity assumption is vital is an immediate consequence of the existence of C*-algebras with real rank zero but infinite topological stable rank ([4]).

Finally, we have $\mathrm{RR}(A) < \dim \Delta_A$ for the disc algebra, so that the symmetry assumption is needed in proposition 5.

## 5. The Bass stable rank of Fourier algebras.

We now determine the Bass stable rank of the Fourier algebra $\mathscr{A}(X)$ of any locally compact group $X$ (see [15] for a definition and the relevant properties):

THEOREM 1. *Let $X$ be a locally compact group, and set $d := \dim X$. Then*

$d = \text{loc dim } X$ *and*

$$\text{Bsr}(\mathscr{A}(X)) = \left[\frac{d}{2}\right] + 1$$

PROOF. It is known that $\mathscr{A}(X)$ is a regular complex commutative Banach algebra with $\Delta_{\mathscr{A}(X)} \cong X$ ([15]): now the assertion follows from proposition 2, since $X$ is paracompact ([17, theorem 8.13], [21, lemma 1.1.]). The involution given by complex conjugation is symmetric on $\mathscr{A}(X)$; this gives a second way to deduce the assertion from proposition 2.

Before stating our next corollary, we recall that the *torsion-free rank* of an abelian group $G$ is defined to be the cardinal number of any maximal linearly independent subset of the Z-module $G$ (cf. the appendix of [17]).

COROLLARY 4. *Let $G$ be locally compact abelian group with character group $X = \hat{G}$. Then the following assertions are valid:*

1) *If $d := \dim X$, we have*

$$\text{Bsr}(L^1(G)) = \left[\frac{d}{2}\right] + 1$$

2) *If $r$ is the torsion-free rank of $G$, we have*

$$\text{Bsr}(L^1(G)) \leqq \left[\frac{r}{2}\right] + 1$$

3) *If $G$ is discrete and $r$ is the torsion-free rank of $G$, we have*

$$\text{Bsr}(L^1(G)) = \left[\frac{r}{2}\right] + 1$$

PROOF. Since $\mathscr{A}(X)$ is the set of all Gelfand transforms of functions in $L^1(G)$ ([15]) and $L^1(G)$ is semisimple, the algebras $\mathscr{A}(X)$ and $L^1(G)$ are algebraically isomorphic. Hence $\text{Bsr}(\mathscr{A}(X)) = \text{Bsr}(L^1(G))$ and the first assertion follows from theorem 1.

By a result of Pontryagin ([17, theorem 24.28], the torsion-free rank $r$ of a discrete abelian group $G$ equals the dimension of the compact character group $X$; hence the third assertion is a consequence of the first one. To deduce the second assertion, we observe that $r$ is the torsion-free rank of $G_d$, the group $G$ endowed with the discrete topology; hence $r$ equals the covering dimension of $(G_d)^\wedge$, which is the Bohr compactification $bX$ of $X$ ([17, 19]). Using 1), we just need to show that $\dim X \leqq \dim bX$ to prove 2). This is not too obvious, since the natural map $\iota\colon X \to bX$ is not a homeomorphism onto its image $\iota(X)$ in general ([19]), but easy to prove: Choose an open neighbourhood $U$ of the identity in

$X$ with compact closure $K$; now $\dim X = \dim K$ since the covering dimension of $X$ equals its local version (cf. also the following remark 4). But since $\iota$ is continuous and injective, its restriction to $K$ is a homeomorphism onto $\iota(K)$. Therefore,

$$\dim X = \dim K = \dim \iota(K) \leqq \dim \mathrm{b}X$$

by the monotonicity of dim on compact subsets of $\mathrm{b}X$, and we are done.

Inequality 2) may be a very rough estimate, as already the additive group of real numbers shows: We have $\mathrm{Bsr}(L^1(\mathsf{R})) = 1$, whereas $\mathsf{R}$ has infinite torsion-free rank.

If $X_1, X_2$ are two locally compact groups, a result of Nagami on the dimension of factor spaces ([21]) assures in particular the validity of the logarithmic formula

$$\dim(X_1 \times X_2) = \dim X_1 + \dim X_2$$

which in general does not hold even for separable metric spaces. This observation allows to get rid of the "integer part of" – function involved in the formula proved above and to calculate exactly the dimension of the locally compact group $X$ once certain Bass stable ranks are known:

Corollary 5. *Let $X$ be a locally compact group. Then*

$$\begin{aligned}\dim X &= \mathrm{Bsr}(\mathscr{A}(X \times X)) - 1 \\ &= \mathrm{Bsr}(\mathscr{A}(X)) + \mathrm{Bsr}(\mathscr{A}(\mathsf{R} \times X)) - 2\end{aligned}$$

*If $G$ is a locally compact abelian group with character group $X = \hat{G}$, this gives*

$$\begin{aligned}\dim X &= \mathrm{Bsr}(L^1(G \times G)) - 1 \\ &= \mathrm{Bsr}(L^1(G)) + \mathrm{Bsr}(L^1(\mathsf{R} \times G)) - 2\end{aligned}$$

*For a discrete abelian group with torsion-free rank $r_0(G)$ we get*

$$\begin{aligned}r_0(G) &= \mathrm{Bsr}(L^1(G \times G)) - 1 \\ &= \mathrm{Bsr}(L^1(G)) + \mathrm{Bsr}(L^1(\mathsf{Z} \times G)) - 2\end{aligned}$$

Proof. The assertions are direct consequences of therem 1, the product theorem for the dimension of locally compact groups cited above and the trivial formulas $\left[\frac{d}{2}\right] + \left[\frac{d+1}{2}\right] = d = \left[\frac{2d}{2}\right]$. Of course, any other locally compact group $H$ such that $\dim H = 1$ (for the first assertion) resp. $\dim \hat{H} = 1$ (for the abelian case) serves as well as $\mathsf{R}$ and $\mathsf{Z}$ do.

Remark 4. If $X$ is a locally compact group, its covering dimension coincides with its local as well as with its modified form; by a conjecture of Alexandroff proved by Pasynkov and generalized by Nagami ([21]) $\dim X$ also coincides with the small as well as with the large inductive dimension of $X$. Finally, it follows from the definitions and results of [23] and [7] that the covering dimension of $X$ coincides with the homological dimension of $X$ as well as with its cohomological counterpart, so "any reasonable" definition of dimension may be used to calculate the Bass stable rank of the Fourier algebra of $X$.

## 6. The topological stable rank and the real rank of Fourier algebras.

Let $X$ be a locally compact group with Fourier algebra $\mathscr{A}(X)$. Since $\operatorname{Bsr}(\mathscr{A}(X)) \leqq \operatorname{tsr}(\mathscr{A}(X))$, we get

$$\left[\frac{d}{2}\right] + 1 \leqq \operatorname{tsr}(\mathscr{A}(X)),$$

where $d = \dim X$. From proposition 5 we know that

$$d \leqq \operatorname{RR}(\mathscr{A}(X)).$$

Setting $n := \left[\frac{d}{2}\right] + 1$, we shall prove that $\operatorname{RR}(\mathscr{A}(X)) \leqq d$ resp. $\operatorname{tsr}(\mathscr{A}(X)) \leqq n$ by exhibiting a dense subset of $\mathscr{A}(X)_{sa}^{d+1}$ resp. $\mathscr{A}(X)^n$, each element of which has a nowhere dense spectrum in $\mathsf{R}^{d+1}$ resp. $\mathsf{C}^n$, and then applying an anlogue of a result of Corach and Suarez ([10]) for the real rank resp. non-unital case. Of course, by proposition 4 it is sufficient to prove the equality for the real rank.

Let $A$ be a commutative Banach algebra, and let $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$ ($n \in \mathsf{N}$) be given. We set $\sigma_A(\mathbf{a}) := \{(m(a_1), \ldots, m(a_n)) : m \in \Delta_A\}$; if $A$ is unital, this coincides with the joint spectrum of $A$ used in [10] and is never empty. For a non-unital $A$, $\sigma_{A_+}(\mathbf{a})$ is given by $\sigma_A(\mathbf{a}) \cup \{0\}$ ($\mathbf{a} \in A^n$); of course, $\sigma_A(\mathbf{a})$ is empty if $A$ is a radical algebra.

LEMMA 4. *Let $A$ be a commutative Banach algebra and $n$ a positive integer. Then* $\operatorname{tsr}(A) \leqq n$, *if and only if there exists a dense subset $D$ of $A^n$ such that $\sigma_A(\mathbf{a})$ has no interior for every $\mathbf{a} \in D$.*

PROOF. If $A$ is unital, this is a particular case of corollary 1.10 of [10]. In the non-unital case, we have $\operatorname{tsr}(A) = \operatorname{tsr}(A_+)$; so we just need to show that the second assertion is true or false simultaneously for $A$ and $A_+$.

Denote by $p_1 : A_+^n \to A^n$, $p_2 : A_+^n \to \mathsf{C}^n$ the canonical projections given by $p_1(\mathbf{a} + \lambda) = \mathbf{a}$ resp. $p_2(\mathbf{a} + \lambda) = \lambda (\mathbf{a} \in A^n,\ \lambda \in \mathsf{C}^n)$. If $D^+$ is a dense subset of $A_+^n$ such that each element has nowhere dense joint spectrum in $\mathsf{C}^n$, we set $D := p_1(D_+)$ and observe that $D$ is a dense subset of $A^n$ such that for each $\mathbf{a} \in D$ $\sigma_A(\mathbf{a})$ has no interior, since $\sigma_{A_+}(\mathbf{b}) = (\sigma_A(p_1(\mathbf{b})) + p_2(\mathbf{b})) \cup \{p_2(\mathbf{b})\}$ holds for each $\mathbf{b} \in A_+^n$. Conversely, the same equation shows that, if $D$ is a dense subset of $A^n$ such that $\sigma_A(\mathbf{a})$ has no interior for each $\mathbf{a} \in D$, the set $D_+ := p_1^{-1}(D)$ is a dense subset of $A^n_+$, each element of which has nowhere dense joint spectrum in $\mathbf{C}^n$.

LEMMA 5. *Let $A$ be a commutative Banach algebra with a symmetric involution and $d \in \mathbf{N}_0$. Then $\mathrm{RR}(A) \leqq d$, if and only if the set $\{\mathbf{a} \in A_{sa}^{d+1} : \sigma(\mathbf{a})$ nowhere dense in $\mathbf{R}^{d+1}\}$ is dense in $A_{sa}^{d+1}$.*

PROOF. As seen above, we may assume without loss of generality that $A$ is unital. But then the arguments of [10] carry over: We use proposition 1.3 of [10] for one implication (noting that the scalars may be chosen to be real by the symmetry of the involution) and observe for the other implication that the inequality $\mathrm{RR}(A) \leqq d$ and Baire's theorem yield the density of

$$M := \bigcap_{\xi \in \mathbf{Q}^{d+1}} U_{d+1}(A_{sa}) + \xi$$

in $A_{sa}^{d+1}$, where an easy argument shows that the dense open set $M$ is just

$$M = \{\mathbf{a} \in A_{sa}^{d+1} : \sigma(\mathbf{a}) \subset \mathbf{R}^{d+1} \setminus \mathbf{Q}^{d+1}\}.$$

To exhibit an explicite example of a subset $D$ of $\mathscr{A}(X)$ having the property required above we need the notion of a regular function with compact support on $X$, as introduced by Bruhat ([5]): Roughly spoken, these are the finite sums of translates of those continuous functions with compact support on $X_1$ that are canonically induced by $C^\infty$-functions with compact support on a Lie-group of the form $X_1/K$, where $X_1$ is an open Lie-projective subgroup of $X$ and $K$ is a compact normal subgroup of $X_1$. We refer the reader to [5] for the details. It has been proved by Eymard that the set $D(X)$ of all regular functions with compact support on $X$ is dense in the Banach algebra $\mathscr{A}(X)$ for every locally compact group $X$ ([15]). The next lemma shows that $D(X)$ has indeed the properties required above:

LEMMA 6. *Let $X$ be a locally compact group with Fourier algebra $\mathscr{A}(X)$. Suppose $X$ has finite covering dimension $d$, and set $n := \left[\frac{d}{2}\right] + 1$.*

*Then $\sigma_{\mathscr{A}(X)}(f_1, \ldots, f_n)$ is a compact subset of $\mathbf{C}^n$ with empty interior for every $n$-tuple $(f_1, \ldots, f_n) \in D(X)^n$.*

*If $(h_1, \ldots, h_{d+1}) \in D(X)^{d+1}$ are real-valued, $\sigma_{\mathscr{A}(X)}(h_1, \ldots, h_{d+1})$ is a compact subset of $\mathbf{R}^{d+1}$ with empty interior.*

PROOF. We first prove the assertion under the additional hypothesis that $X$ be separable metric and then deduce the assertion from this special case.

So suppose $X$ is separable metric. Let us choose an open, Lie-projective subgroup $X_1$ of $X$; then the family $\mathscr{F}$ of compact normal subgroups $K$ of $X_1$, whose corresponding factor group $X_1/K$ is a Lie group, has only the unit of $X$ in its intersection. Suppose we are given $f_1,\ldots,f_n \in D(X)$; for each fixed index $1 \leqq j \leqq n$, there exists a compact normal subgroup $K_j \in \mathscr{F}$ such that $f_j$ is a finite sum of translates of functions of the form $\varphi \circ \pi_j$, where $\pi_j\colon X_1 \to X_1/K_j$ is the quotient map and $\varphi$ is a complex-valued $C^\infty$-function with compact support on the Lie group $X_1/K_j$. Since $\mathscr{F}$ is directed downward, we may assume that one $K$ serves for every $j$; let $\pi$ denote the corresponding quotient map. We may choose a family $x_1X_1,\ldots,x_sX_1$ of disjoint left cosets and $C^\infty$-functions $\varphi_{jk}\colon X_1/K \to \mathrm{C}$ having compact support such that each $f_j$ has the form

$$f_j(x) = \begin{cases} \varphi_{jk}(\pi(x_k^{-1}x)), & x \in x_kX_1, 1 \leqq k \leqq s \\ 0 & , \; else \end{cases}$$

Now $\sigma_{\mathscr{A}(X)}(f_1,\ldots,f_n) = \{(f_1(x),\ldots,f_n(x)) : x \in X\}$, since $X$ is the spectrum of $\mathscr{A}(X)$. The explicit form established above of the functions involved yields

$$\sigma_{\mathscr{A}(X)}(f_1,\ldots,f_n) = \bigcup_{k=1}^{s} \{(\varphi_{1k}(w),\ldots,\varphi_{nk}(w)) : w \in X_1/K\}$$

But each $\varphi_k\colon X_1/K \to \mathrm{C}^n$, $w \mapsto (\varphi_{1k}(w),\ldots,\varphi_{nk}(w))$ is a $C^\infty$-function with compact support from the separable metric manifold $X_1/K$ of dimension $\dim X_1/K \leqq \dim X_1 \leqq d$ into the $2n$ – dimensional manifold $\mathrm{C}^n$. Now our choice of $n$ implies $2n > d$; consequently by Sard's theorem ([11, cor. 16.23.2]) we know that each set $\mathrm{C}^n \setminus \varphi_k(X_1/K)$ is a dense subset of $\mathrm{C}^n$, which is also open since each $\varphi_k$ has compact support. Therefore the union of the sets $\varphi_k(X_1/K)$ is nowhere dense in $\mathrm{C}^n$, and $\sigma_{\mathscr{A}(X)}(f_1,\ldots,f_n)$ is a compact subset of $\mathrm{C}^n$ with no interior.

Observe that in case we started with real-valued functions $h_1,\ldots,h_{d+1}$ the functions $\varphi_k$ constructed as above have their image contained in $\mathrm{R}^{d+1}$, so again Sard's theorem may be applied and yields the desired conclusion.

If $X$ is an arbitrary locally compact group, we adopt an argument of Dixmier and Eymard ([15, p. 218, 219]) to reduce the general case to the separable metric case: Let $f_1,\ldots,f_n \in D(X)$ be given. We may find an open, $\sigma$-compact subgroup $H$ of $X$ such that each $f_k$ vanishes outside $H$. Now the restrictions $\varphi_1,\ldots,\varphi_n$ of our functions to $H$ belong to $D(H)$. The uniform continuity of $\varphi_1,\ldots,\varphi_n$ and the $\sigma$-compactness of $H$ allow, by a theorem of Kodaira and Kakutani ([17, theorem 8.7]), to find a compact normal subgroup $N$ of $H$ such that $H/N$ is separable metric and each $\varphi_k$ is constant on left cosets of $N$. Denoting by $\psi_1,\ldots,\psi_n$ the functions induced on $H/N$ by $\varphi_1,\ldots,\varphi_n$, we have $\psi_1,\ldots,\psi_n \in D(H/N)$, and by construction we get

$$\sigma_{\mathscr{A}(X)}(f_1,\ldots,f_n) = \sigma_{\mathscr{A}(H/N)}(\psi_1,\ldots,\psi_n)$$

Since $H/N$ is separable metric of dimension $\leqq d$, an application of the first part is possible; this also settles the real-valued case and finishes the proof. Now the stage is set:

THEOREM 2. *Let $X$ be a locally compact group, and set $d := \dim X$. Then* $\mathrm{RR}(\mathscr{A}(X)) = d$ *and* $\mathrm{tsr}(\mathscr{A}(X)) = \left[\dfrac{d}{2}\right] + 1$.

PROOF. We may assume that $n$ and $d$ are finite. Set $D := D(X)_{\mathrm{sa}}^{d+1}$; then $D$ is dense in $\mathscr{A}(X)_{\mathrm{sa}}^{d+1}$, so an application of lemmas 5 and 6 yields the first equality. The second is deduced in the same way, or may be obtained as a consequence of proposition 4 and the first equality.

REMARK 5. The assertions of corollary 4 and 5 hold with the Bass stable rank replaced by the topological stable rank. For a locally compact abelian group $G$, we have $\dim \hat{G} = \mathrm{RR}(L^1(G))$, which yields $r_0(G) = \mathrm{RR}(L^1(G))$ in the discrete case.

COROLLARY 6. *Let $X$ be a locally compact group of covering dimension $d$. Then the quasi-invertible elements of $\mathscr{A}(X)$ are dense in $\mathscr{A}(X)$, if and only if $d \leqq 1$. The real-valued quasi-invertible functions in $\mathscr{A}(X)$ are dense in the set of all real-valued functions of $\mathscr{A}(X)$, if and only if $d = 0$.*

PROOF. By the above theorem, $d \leqq 1$ if and only if $\mathrm{tsr}(\mathscr{A}(X)) = 1$, i.e. iff $U_1(\mathscr{A}(X)_+, \mathscr{A}(X))$ is dense in the set $\{1 - f : f \in \mathscr{A}(X)\}$. Now the definition of $U_1(\mathscr{A}(X)_+, \mathscr{A}(X))$ and the elementary fact that an element $f \in \mathscr{A}(X)$ is quasi-invertible in $\mathscr{A}(X)$ iff $1 - f$ is invertible in $\mathscr{A}(A)_+$ yield the desired result. The second assertion is deduced in the same way.

COROLLARY 7. *Let $G$ be a discrete abelian group. Then the invertible elements of $L^1(G)$ are dense in $L^1(G)$, if and only if $G$ has torsion-free rank at most 1. The invertible elements in $L^1(G)_{\mathrm{sa}}$ are dense in $L^1(G)_{\mathrm{sa}}$, if and only if $G$ is a torsion group.*

## 7. An application to Korovkin approximation theory.

Let $G$ be a locally compact abelian group with group algebra $L^1(G)$ and character group $X = \hat{G}$. A subset $T$ of $L^1(G)$ is called a *universal Korovkin system* iff it has the following property:

(uKs) For every commutative Banach algebra $B$ with symmetric involution $*$, every $*$-algebra-homomorphism $L$: $L^1(G) \to B$ and every net $L_\alpha$: $L^1(G) \to B$ of positive contractions, the convergence $\rho(L_\alpha g - Lg) \to 0$ $(g \in T)$ already implies $\rho(L_\alpha f - Lf) \to 0$ for all $f \in L^1(G)$

Here $\rho$ denotes the spectral radius of $B$. Of course, the above definition makes sense with $L^1(G)$ replaced by an arbitrary commutative Banach algebra with symmetric involution. We refer to [2] for background on the classical Korovkin theorem and to [1] resp [23, 24] for generalizations to Banach algebras. It is well known that $B$ may be replaced by the Banach algebra $C_0(X)$, $X$ locally compact Hausdorff, in condition (uKs); in this case, $\rho$ is just the uniform norm.

By a result due to Altomare, $L^1(G)$ contains a *finite* universal Korovkin system iff the following property is fulfilled:

(P) There exist finitely many functions $f_1, \ldots, f_n \in L^1(G)$ whose Fourier transforms $\hat{f}_1, \ldots, \hat{f}_n$ strongly separate the points of $X$; cf. [25] and [1]. Explicitely, in this case the set

$$T := \left\{ f_1, \ldots, f_n, \sum_{k=1}^{n} f_k * f_k^* \right\}$$

is a finite universal Korovkin system for $L^1(G)$.

It has been proved in [25] that (P) holds iff $X$ is a finite-dimensional separable metric group. Using our computations of various notions of rank for $L^1(G)$ we may now characterize the fact that $L^1(G)$ possesses a finite universal Korovkin system by the separability of $L^1(G)$ and the finiteness of the stable rank $\operatorname{sr}(L^1(G)) := \operatorname{Bsr}(L^1(G)) = \operatorname{tsr}(L^1(G)$ resp. the real rank $\operatorname{RR}(L^1(G))$:

THEOREM 3. *Let $G$ be a locally compact abelian group. Then the following assertions are equivalent:*

i) *$L^1(G)$ possesses a finite universal Korovkin system*
ii) *$L^1(G)$ is separable and has finite stable rank*
iii) *$L^1(G)$ is separable and has finite real rank*

PROOF. By theorem 1 of [25], i) is equivalent to $X$ being a finite-dimensional separable metric space. By remark 5, $X$ is finite-dimensional iff $\operatorname{sr}(L^1(G)) < \infty$ iff $\operatorname{RR}(L^1(G)) < \infty$; since it is well known that $L^1(G))$ is separable iff $X$ is separable metric this finishes the proof.

REMARK 6. If $G$ is a discrete abelian group, it follows that $L^1(G)$ is separable and has finite stable rank, iff $G$ is countable and has finite torsion-free rank: Some elementary characterizations of these groups may be found in [25].

For a countable discrete group $G$, we have

*$L^1(G)$ has finite stable rank, if and only if $G$ contains a finite Kronecker-set which is maximal among Kronecker-sets.*

We recall that a subset $E$ of a discrete group $G$ is a Kronecker-set iff for every family $\{\alpha_g : g \in E\}$ of complex numbers of absolute value one and every $\varepsilon > 0$ there exists a character $\chi \in \hat{G}$ such that

$$\sup \{ |\alpha_g - \chi(g)| : g \in E \} < \varepsilon;$$

cf. [28] section 5.1. Now the equivalence stated above follows from the fact that the finite, linearly independent subsets of the Z-module $G$ are exactly the finite Kronecker-sets of $G$ ([28] corollary 5.1.3 and theorem 5.1.4).

REMARK 7. Let $\mathcal{M}(G)$ denote the commutative unital Banach algebra (under convolution) of complex, finite, regular Borel measures on a locally compact abelian group $G$. Though the involution of $\mathcal{M}(G)$ is not symmetric if $G$ is not discrete ([29]), one may consider Korovkin approximation of unital algebra homomorphisms by spectral contractions on $\mathcal{M}(G)$ in the sense of [23]. It turns out that $\mathcal{M}(G)$ does not possess a finite universal Korovkin system, if $G$ is not discrete:

*Let $G$ be a locally compact abelian group. Then the following assertions are equivalent:*

i) *$\mathcal{M}(G)$ possesses a finite universal Korovkin system*

ii) *$G$ is a countable discrete group of finite torsion-free rank.*

For a proof of the implication ii) $\Rightarrow$ i), just note that $\mathcal{M}(G) = L^1(G)$ since $G$ is discrete. To see that i) $\Rightarrow$ ii), we only need to show that $G$ must be discrete: By hypothesis, the spectrum of $\mathcal{M}(G)$ must coincide with its Choquet as well as with its Silov boundary ([23, proposition 8.1.]), but this is never the case for a nondiscrete group ([29, § 10.5]).

A Korovkin -type theorem for continuous functions on a compact abelian group, which shows that the universal Korovkin system $\{1, \cos, \sin\}$ for the space of all real-valued, $2\pi$-periodic continuous functions on R may be replaced by a set of generators of the character group in the general case, may be found in [3, theorem 2]). We conclude this paper by giving a generalization (with completely different proof) of this theorem and characterizing those locally compact abelian groups $G$ for which the commutative C*-algebra $\mathcal{AP}(G)$ of all continuous almost periodic complex-valued functions on $G$([17, 19]) possesses a finite universal Korovkin system in the sense of (uKs) with $L^1(G)$ replaced by $\mathcal{AP}(G)$, or equivalently for the approximation of unital algebra homomorphisms by contractions as in [23].

PROPOSITION 3. *Let $G$ be a locally compact abelian group with character group $X$. Then any set $T \subset X$ generating the group $X$ algebraically is a universal Korovkin system for $\mathcal{AP}(G)$.*

PROOF. The elements of $T$ are, via Gelfand-transformation, continuous functions on the spectrum of $\mathcal{AP}(G)$, which coincides with the Bohr compactification b$G$ of $G$ ([19, theorem 10.7]). Viewing the elements of $T$ as elements of the group $X_d$, which is $X$ endowed with the discrete topology, one sees that the functions in $T$ are characters on b$G$, since the latter group is the character group of $X_d$

([17, 19]). Now $T$ generates $X$; therefore each character of $X_d$ is uniquely determined by its values on $T$, which shows that the functions in $T$ separate the points of b$G$, the spectrum of $\mathscr{AP}(G)$. Since $T$ contains the unit and consists only of functions of absolute value one, the assertion follows from corollary 4.3 of [1].

By the assertion just proved, $\mathscr{AP}(G)$ possesses a finite universal Korovkin system if $X$ is finitely generated; in this case the group $X$ is algebraically isomorphic to $\mathbf{Z}^n \times D$ for some $n \in \mathbf{N}$ and some finite abelian group $D$. Since any countable locally compact (Hausdorff) group is discrete (a consequence of Baire's theorem), this isomorphism is also topological, and therefore, $X$ is finitely generated if and only if $G \cong \mathbf{T}^n \times H$, where $\mathbf{T}$ is the torus group and $H = \hat{D}$ is a finite abelian group. However, there are lots of other groups for which $\mathscr{AP}(G)$ possesses a finite universal Korovkin system. This point is settled by our last result, which also contains a (rather obvious) formula for the stable rank of the commutative C*-algebra $\mathscr{AP}(G)$:

THEOREM 4. *Let G be locally compact abelian group with character group X and Bohr compactification* b$G$. *Then*

$$\mathrm{sr}\,(\mathscr{AP}(G)) = \left[\frac{\rho}{2}\right] + 1 = \mathrm{sr}\,(L^1(X_d))$$

*where $\rho$ is the torsion-free rank of $X$, which coincides with* dim b$G$.

*We have* $\mathrm{RR}(\mathscr{AP}(G)) = \rho = \mathrm{RR}\,(L^1(X_d))$
*Further on, the following assertions are equivalent:*
i) *$\mathscr{AP}(G)$ possesses a finite universal Korovkin system*
ii) *$C_b(G)$ possesses a finite universal Korovkin system*
iii) *$G$ is a compact metric group of finite covering dimension*
iv) *$X$ is a countable discrete group of finite torsion-free rank*
v) *$L^1(X_d)$ possesses a finite universal Korovkin system*

PROOF. Since $\mathscr{AP}(G) \cong C(\mathrm{b}G)$, the formula for the stable ranks is an immediate consequence of Vasershtein's theorem and corollary 4; here the torsion-free rank of $X$ coincides with the covering dimension of $\mathrm{b}G \cong (X_d)^\wedge$ by Pontryagin's theorem ([17, theorem 24.28]).

The equivalence of iii) and iv) is clear again by Pontryagin's theorem. If iii) holds, then obviously $\mathscr{AP}(G) = C(G) = C_b(G)$, so that i) and ii) are immediate consequences of section 8 of [23] and the embedding theorem for separable metric spaces ([14, 1.11.4]). Conversely, if i) holds we get (by section 8 of [23]) that b$G$ is metrizable and has finite covering dimension, hence $X_d \cong (\mathrm{b}G)^\wedge$ is a countable group of finite torsion-free rank. But $X$ must be discrete, since $X$ is countable; therefore i) implies iv). If ii) holds, $\beta G$ must be metrizable. Since no point of $\beta G \setminus G$ is a $G_\delta$-set by a well-known result of Cech, this forces $G$ to be compact metrizable; therefore ii) implies iii). Finally, the equivalence of iv) and v) follows from proposition 1 of [25] and the fact already used above that any countable locally compact (Hausdorff) group must be discrete.

The commutative C*-algebras $\mathscr{AP}(G)$ and $C_b(G)$ yield another example of the fact that there is no relationship between the stable ranks resp. real ranks of a Banach algebra and its (involutive) subalgebras (cf. [27] page 314): Just observe that

$$\mathrm{sr}\,(\mathscr{AP}(\mathsf{R})) = \infty = \mathrm{RR}\,(\mathscr{AP}(\mathsf{R})),$$

since $\mathsf{R} \cong \hat{\mathsf{R}}$ has infinite torsion-free rank, whereas $C_b(\mathsf{R})$ has stable and real rank 1 by Vasershtein's theorem.

## REFERENCES


1. F. Altomare, *On the universal convergence sets*, Annali di Matematica IV 88, (1984), 223–243.
2. H. Bauer, *Approximation and abstract boundaries*, Amer. Math. Monthly 85 (1978), 632–647.
3. W. R. Bloom, J. F. Sussich, *Positive linear operators and the approximation of continuous functions on locally compact abelian groups*, J. Austral. Math. Soc. Ser. A 30 (1980), 180–186.
4. L. G. Brown, G. K. Pedersen, *C*-algebras with real rank zero*, Preprint Series 1989 No. 21, Københavns Universitet, Matematisk Institut, September 1989.
5. F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques*, Bull. Soc. Math. France 89, (1961), 43–75.
6. H. Bass, *Introduction to some methods of algebraic K-theory*, CBMS Regional Conference series No. 20. Providence, Rhode Island; American Mathematical Society 1970.
7. H. Cohen, *A cohomological definition of dimension for locally compact spaces*, Duke Math. J. 21 (1954), 209–222.
8. G. Corach, A. R. Larotonda, *Stable Range in Banach algebras*, J. Pure Appl. Algebra 32 (1984), 289–300.
9. G. Corach, F. D. Suarez, *Extension problems and stable rank in commutative Banach algebras*, Topology Appl. 21 (1985), 1–8.
10. G. Corach, F. D. Suarez, *Thin spectra and Stable Range Conditions*, J. Funct. Anal. 81 (1988), 432–442.
11. J. Dieudonné, *Treatise on Analysis Vol.* III, Pure and Applied Mathematics 10. New York, Academic Press 1972.
12. C. H. Dowker, *Local dimension of normal spaces*. Quart. J. Math. Oxford (2) 6 (1955), 101–120.
13. R. Engelking, *Outline of General Topology*, Amsterdam, North-Holland Publ, 1968.
14. R. Engelking, *Dimension Theory*, North-Holland Math. Library Vol. 19. Amsterdam, North-Holland, 1978.
15. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181–236.
16. J. R. Isbell, *Uniform Spaces*, Mathematical Surveys 12. Providence, Rhode Island, American Mathemetical Society, 1964.
17. E. Hewitt, K. Ross, *Abstract Harmonic Analysis* I, Grund-lehren der mathematischen Wissenschaften 115. Berlin-Heidelberg-New York, Springer 2nd ed., 1979.

18. R. Herman, L. N. Vasershtein, *The stable range of C*-algebras*, Invent. Math. 77 (1984), 553–555.
19. R. Larsen, *Banach Algebras*, Pure and Applied Mathematics 24, New York, Marcel Dekker, 1973.
20. G. Maltese, R. Wille-Fier, *A characterization of homomorphisms in certain Banach involution algebras*, Studia Math. 89 (1988), 133–143.
21. K. Nagami, *Dimension-theoretical structure of locally compact groups*, J. Math. Soc. Japan 14 (1962), 379–396.
22. J. D. Newburgh, *Metrization of finite dimensional groups*, Duke Math. J. 20 (1953), 287–293; c.f. MR 14 949 (1953).
23. M. Pannenberg, *Korovkin Approximation in Waelbroeck Algebras*, Math. Ann. 274 (1986), 423–437.
24. M. Pannenberg, *Topics in qualitative Korovkin approximation*, Conferenze del Seminario di Matematica dell' Universita di Bari 226 (1988), 1–31.
25. M. Pannenberg, *A characterization of a class of locally compact abelian groups via Korovkin theory*, Math. Z. 204 (1990), 451–464.
26. A. R. Pears, *Dimension Theory of General Spaces*, Cambridge, Cambridge University Press, 1975.
27. M. A. Rieffel, *Dimension and stable rank in the K-theory of C*-algebras*, Proc. London Math. Soc. (3) 46 (1983), 301–333.
28. W. Rudin, *Fourier Analysis on Groups*, New York, Interscience Publ., 1962.
29. J. L. Taylor, *Measure Algebras*. CBMS Reg. Conf. Ser. Math. 16. Providence, Rhode Island, AMS 1973.
30. L. N. Vasershtein, *Stable rank of rings and dimensionality of topological spaces*, Functional Anal. Appl. 5 (1971), 102–110.

MATHEMATISCHES INSTITUT
WESTFÄLISCHEN WILHELMS UNIVERSITÄT
EINSTEINSTRASSE 62
D-4400 MÜNSTER
GERMANY