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Stable regions in the Earth's liquid core

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Summary. Slow cooling of the whole Earth can be responsible for the convection in the core that is required to generate the magnetic field. Previous studies have assumed the cooling rate to be high enough for the whole core to convect. Here we study the effects of a low rate of cooling by assuming the temperature at the base of the mantle to remain constant with an initially entirely molten, adiabatic core. We argue that, in such a situation, convection would stop at the top of the core, and calculate the consequent thermal evolution. A stable, density stratified layer grows downwards from the core mantle boundary reaching a thickness of 100–1000 km in a few thousands of millions of years. There is some geomagnetic evidence to support belief in the existence of such a stable layer.

1 Introduction

Verhoogen (1961) first argued that the Earth's magnetic field could be generated from energy released by cooling and gradual freezing of the core. Braginsky (1963) proposed that differentiation of the liquid core mixture would accompany the cooling and cause convection, and that this was the main cause of dynamo motion. These ideas have attracted more recent interest (Gubbins 1977a; Gubbins, Masters & Jacobs 1979; Loper 1978; Häge & Müller 1979; Verhoogen 1980). Gubbins (1977a) showed that the differentiation mechanism required much less energy than thermal convection to generate the same magnetic field. The calculation was extended by Gubbins *et al.* (1979, Paper I), using seismological results for core density from Masters (1979), in a thermal history calculation for the core which was controlled by a constant rate of cooling of the core–mantle boundary assuming that this cooling rate was fast enough to drive convection throughout the whole outer core. In this paper we study the consequences that follow from a very slow cooling rate of the core–mantle boundary. Convection cannot be maintained everywhere in the core and subadiabatic, density stratified regions develop.

Cooling of the core will depend on the thermal evolution of the mantle. It is most likely that the whole mantle convects although there is evidence of a separation of the convection into two or more layers from the focal mechanisms of deep earthquakes (Isacks & Molnar 1971) and depletion of the mantle source of ocean ridge basalts (Jacobsen & Wasserburg

1979; O'Nions, Evensen & Hamilton 1979). Models of mantle convection are now quite sophisticated and include the strong temperature dependence of viscosity (e.g. Tozer 1972; Sharpe & Peltier 1978, 1979; Schubert, Cassen & Young 1979; Davies 1980; Turcotte 1980; McKenzie & Richter 1981; McKenzie & Weiss 1980) but there is still no definite result for the cooling rate of the lower mantle. In fact, as McKenzie & Richter (1981) state, there is not even general agreement on whether the Earth's temperature is increasing or decreasing. The modest cooling rates of 30–200 K Ga^{-1} adopted in Paper I are consistent with modern views on mantle convection. Here we study the effect of a zero cooling rate; a simple case that enables us to study the development of stably stratified regions in the core. A more complete thermal history calculation, in which both mantle and core are included, will be reported on in the future (Mollett 1982).

To specify the model completely we need precise initial and boundary conditions. We follow Jacobs (1953) and suppose that the core was initially entirely liquid and convecting. Its temperature is everywhere adiabatic. The mantle temperature is constant at T_c. The core continues to cool and all the heat is carried away by mantle convection. The calculation is for a one-component liquid only; the important influence of a second component is considered in the discussion.

This model is obviously very specialized but it has some interesting consequences that may lead to observable phenomena. Clearly if there are no radioactive heat sources then convection in the core must eventually stop. The adiabatic gradient varies with depth, being steepest near the core–mantle boundary, mainly because of the higher gravitational acceleration there. Therefore convection is most likely to stop first at the top of the core. We imagine that a stable region forms near the mantle boundary, through which heat passes by conduction alone, and this zone grows as the core cools (see also Gubbins 1976). Estimates of the temperature gradient are about 0.1 K km^{-1} below that of the adiabat which is enough to inhibit all penetration of convection into the stable zone. Ruff & Anderson (1980) have given a qualitative discussion of fluid flow in a stable core driven by laterally heterogeneous heat sources in the mantle. We have ignored the vertical heat transported by these baroclinic instabilities.

Several interesting questions arise from this mode of cooling which can only be answered by calculation. First, there is the question of how fast the stable zone grows in relation to the age of the core, and how this 'blanket' of fluid, through which heat can only pass by conduction, slows cooling and freezing of the core. Secondly, we cannot be sure if the thermal time constant is short enough to allow the temperature to depart significantly from the initial adiabatic value. A typical value for the thermal diffusivity κ, taken from Paper I, gives a time constant l^2/κ of 4.5 Ga for a length scale of $l = 1000$ km. If the temperature has remained 'frozen' near the adiabat there would be little chance of detecting the stable zone from seismology.

2 Mathematical formulation

The relevant temperature gradients are shown in Fig. 1. $T_i(r)$ is the temperature gradient at time zero for the initially liquid, adiabatic core. If the temperature at the core–mantle boundary (r_c) falls from T_c at A to a lower value at B, and the core continues to convect adiabatically, the new temperature will be the adiabat $T_a(r)$. The inner core begins to form wherever the temperature falls below the local melting point, and we make the simplifying but artificial assumption that freezing starts at the centre at time zero. This requires that the melting curve $T_m(r)$ passes through $T_i(0)$. At any time the radius of the inner core is given by the intersection of T_m with the true temperature curve. This is the cooling regime used in Paper I.

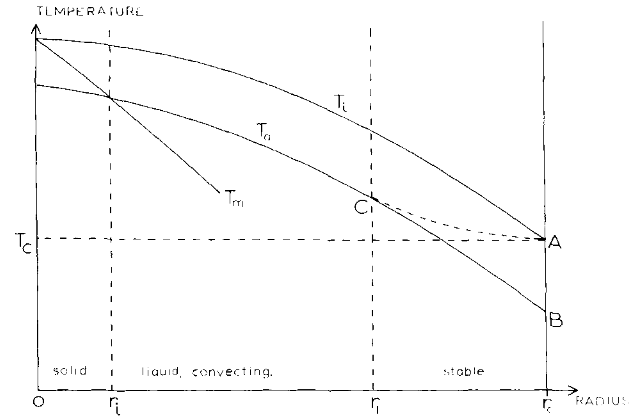

Figure 1. Relevant temperature gradients in the core; T_m is the variation of the melting point with radius (pressure), T_i the initial adiabatic temperature distribution, T_a an adiabat drawn through $T(r_i)$ at some later time.

Next consider what happens when the mantle temperature remains fixed at point A. A subadiabatic region develops between r_1 and r_c in which the temperature satisfies the thermal conduction equation and follows the dashed curve from C to A. Convection continues below the level r_1 where the temperature is adiabatic and follows $T_a(r)$. At point C, the top of the convecting region, both temperature and temperature gradient must be continuous. The heat flux is equal to that conducted down the temperature gradient, and if the convection is steady it will also equal the rate of generation of heat beneath r_1, which comes from latent heat of freezing at the inner core boundary and specific heat due to cooling of the whole core.

In the conducting region, then, the temperature obeys the equation

$$\frac{\partial T}{\partial t} = \frac{\kappa}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) \qquad r_1 \leqslant r \leqslant r_c \tag{1}$$

which is integrated forwards in time from prescribed initial conditions $T(r, 0) = T_a(r, 0)$ and boundary conditions:

$$T(r_c) = T_c$$

$$\left(\frac{\partial T}{\partial r}\right) = -\left(\frac{g\gamma}{\phi}\right)_{r_1} T \qquad \text{at } r = r_1 \tag{2}$$

where γ and ϕ are the Gruneisen and seismic parameters and g the acceleration due to gravity. The problem is more complex than the usual heat flow calculation because the bottom boundary condition must be applied at the variable depth r_1, which is in turn determined by the heat sources deeper down. The growth rate of the inner core radius is related to the cooling rate through the equation:

$$\frac{dr_i}{dt} = -\left(\frac{T}{X\rho g}\right)_{r_i}\left(\frac{1}{T}\frac{\partial T}{\partial t}\right)_r \tag{3}$$

where $X = (\partial T/\partial P)_m - (\partial T/\partial P)_a$; m applies to melting and a to adiabatic.

Equating these sources to the heat flowing across the surface $r = r_1$ gives

$$4\pi r_1^2 k \left(\frac{\partial T}{\partial r}\right)_{r_1} = 4\pi r_i^2 \rho(r_i) L \frac{dr_i}{dt} + C_p \int \rho T \, dV \left(\frac{1}{T}\frac{\partial T}{\partial t}\right)_{r_1}$$
$$= \left\{\frac{4\pi r_i^2 \rho(r_i)}{(X\rho g/T)_{r_1}} + C_p \int \rho T dV\right\} \left(\frac{1}{T}\frac{\partial T}{\partial t}\right)_{r_1} \quad (4)$$

where k is the thermal conductivity, L the latent heat and C_p the specific heat.

Further details of the derivation are in Paper I. The specific heat integral should be over the volume contained within radius r_1 but we have been able to take it over the whole core with little error. (4) provides the extra equation to determine r_1 as a function of time for use in boundary condition (2). The equations are integrated forwards in time by the numerical method described in the Appendix.

3 Results

Numerical estimates for the parameters used in the calculations are given in Table 1. Seismologically determined parameters (ϕ, ρ, g) are based on earth model PEMA (Dziewonski, Hales & Lapwood 1975). Gruneisen's ratio (γ) was taken to be independent of radius to simplify the analysis, and the quantities in brackets on the right side of (4), $\rho(r_i)$, $(X\rho g/T)_{r_i}$ and $C_p \int \rho T dV$, were approximated as being independent of time.

The parameter values for the first calculations, given in column 1 of Table 1, were taken mainly from Paper I but with the thermal conductivity adjusted to give a reasonable final inner core radius of around the present value of 1210 km. The evolution of the inner core radius (r_i) and the radius of the bottom of the conducting region (r_1), are plotted in Fig. 2.

Table 1. Parameters for the calculations (SI units).

	1	2	3 (Mercury)
k	15	240	60
C_p	700	175	700
κ	1.8×10^{-6}	1.1×10^{-4}	1.1×10^{-5}
L	10^6	10^6	10^6
X	1.4×10^{-9}	8.4×10^{-9}	5×10^{-9}
γ	1.27	0.32	2.0
T_c	3300	3300	1700

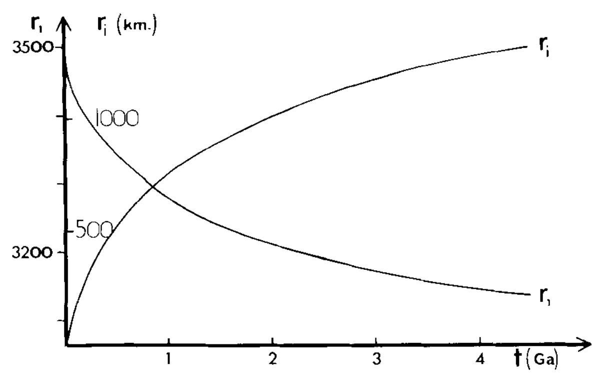

Figure 2. Evolution of inner core and conducting region for calculation with values of parameters taken from column 1 of Table 1.

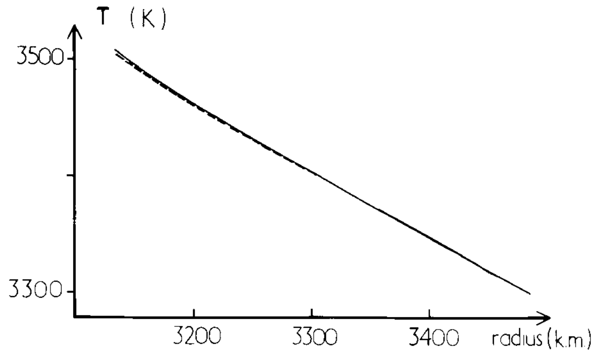

Figure 3. Temperature profile in the conduction region at the end of the calculation shown in Fig. 2. The dashed curve is the quasi-steady approximation for this curve.

The conduction region has grown to a final thickness of about 350 km. The temperature inside the conducting region is plotted by the full line in Fig. 3.

The numerical values of the relevant core parameters are very poorly known and we have performed many calculations to explore the effects of different values on the solutions. The thermal conductivity scales the time variable through κ in the conduction equation and in the 'boundary condition' (4). In fact we have made use of this scaling in adjusting k to bring about a plausible rate of growth to the inner core. γ controls the adiabatic gradient and is the most uncertain of the parameters determining the temperature. Reducing γ lowers the heat loss because the temperature gradients are lower, but more interestingly it allows the subadiabatic region to grow more rapidly in relation to the inner core radius. Changing the specific heat C_p alters the time-scale in the conduction equation but the heat sources in (4) are usually dominated by the latent heat term and so the overall effect is not a complete scaling of time but an adjustment of the thermal response time. Increasing the latent heat slows the growth of the inner core for the same cooling rate. X is a very poorly known quantity because it is the difference between two temperature gradients. Even the sign is uncertain (Higgins & Kennedy 1971). The figure of $1.4 \times 10^{-9}\,\mathrm{K\,Pa^{-1}}$ in Table 1 comes from the two estimates

$$\left(\frac{\partial T}{\partial P}\right)_m = 4 \times 10^{-9}\,\mathrm{K\,Pa^{-1}}, \quad \left(\frac{\partial T}{\partial P}\right)_a = 2.6 \times 10^{-9}\,\mathrm{K\,Pa^{-1}}$$

made in Paper I. Reducing γ by a factor of 3, say, will reduce $(\partial T/\partial P)_a$ and hence increase X. The melting gradient is also uncertain by at least a factor of 3 so that X can take any value from zero to about $10^{-8}\,\mathrm{K\,Pa^{-1}}$. X controls the growth rate of the inner core. Finally, changing T_c will change the adiabatic gradient and have a similar effect to changing γ.

A plausible if rather extreme set of parameters is given in column 2 of Table 1, and the solution is displayed in Figs 4 and 5. γ and C_p have been reduced and X increased to slow the growth of the inner core relative to the conduction region, and k has been increased to scale the time sensibly. The stable zone occupies almost the entire liquid core.

The temperature gradients in Figs 3 and 5 have departed significantly from their initial adiabatic form, suggesting that the thermal time constant is surprisingly short for this problem. If the adiabatic gradient in the convecting part of the liquid is continued to the core–mantle boundary, the temperature there is found to be 3260 and 3140 K for the two calculations. Therefore the mantle temperature would have to drop by at least 40 or 160 K respectively in 4.5 Ga to maintain convection throughout the whole core. These are the sort

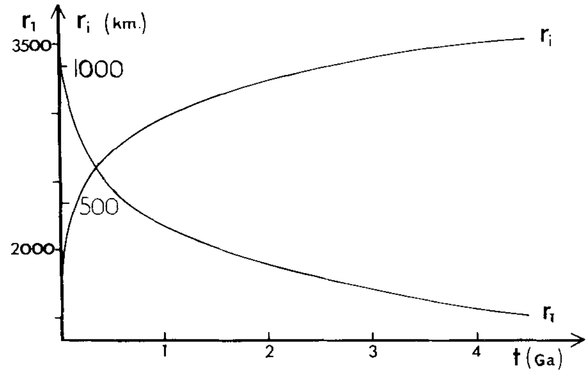

Figure 4. Evolution of inner core and conducting region for calculation with values of parameters taken from column 2 of Table 1.

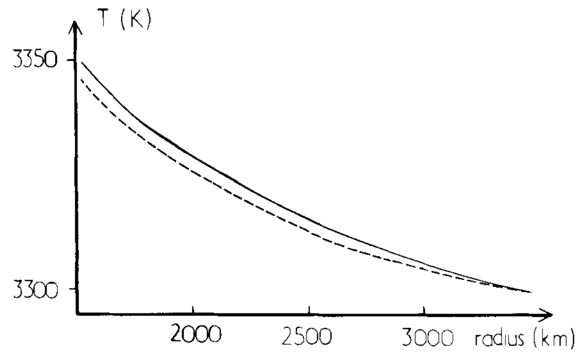

Figure 5. Temperature profile in the conducting region at the end of the calculation shown in Fig 4. The dashed curve is the quasi-steady approximation for this curve.

of cooling rates that were used in Paper I. The required rates are larger for other combinations of parameters, particularly if the adiabatic gradient is large. In all cases the overall loss of heat is smaller for a fixed temperature boundary condition than if the mantle cooled to maintain an adiabat. The reduction in heat loss depends on how fast the mantle cools, but for slow rates just enough to maintain convection through the entire core it is 30 per cent for the first calculation and 36 per cent for the second.

If the thermal time constant was very short in relation to the age of the Earth the temperatures in the conducting regions would be close to the solutions of the steady state conduction equation. The dashed lines in Figs 3 and 5 gives the solutions for $\kappa \gg 1$ in which the term $\partial T/\partial t$ has been dropped from equation (1). The approximation is better for the first calculation.

A calculation has also been performed for the planet Mercury. Assuming a central core of density $8.5\,\mathrm{g\,cm^{-3}}$, radius 1600 km and a mantle of density $3.7\,\mathrm{g\,cm^{-3}}$ yields a model for g and the hydrostatic pressure. Other parameters are listed in column 3 of Table 1 (see also Gubbins 1977b). The higher value of the adiabatic gradient leads to a very small conduction zone. The whole core freezes in 2.5 Ga and the conduction zone never exceeds 70 km in depth. This depth depends initially on the pressure variation of the melting temperature, which is better known than for the Earth's core because of the lower pressure in Mercury. This suggests that the development of stably stratified layers has been unimportant in the thermal histories of the smaller planets.

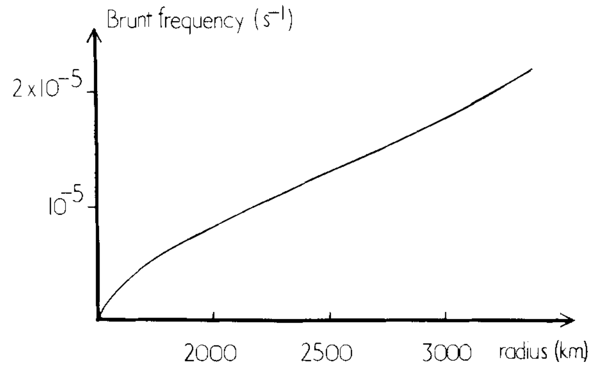

Figure 6. Brunt frequency, in s^{-1}, corresponding to the temperature gradient in Fig. 5 and the adiabat chosen for the calculation.

4 Stratified regions

The fluid in the conducting zone $r_1 \leqslant r \leqslant r_c$ will be stably stratified. The stratification is measured by the Brunt-Väisala frequency $N^2 = \alpha g \tau$, where τ is the difference between the true and adiabatic temperature gradients, which is plotted for calculation 2 in Fig. 6. The Brunt period varies from 10 hr upwards and asymptotes to infinity at $r = r_1$.

The existence of these stable zones is a matter for speculation. Masters (1979) has inverted free oscillation data for a stability parameter that is very similar to the Brunt-Väisala frequency. The data cannot resolve a stable layer such as those derived from our thermal calculations although improvements in the recordings of large earthquakes may change the situation. Core motions will affect the secular change of the magnetic field, and geomagnetic observations may give evidence for stable regions. It was once believed that the high electrical conductivity of the core would screen all magnetic field changes except those in the uppermost few hundred kilometres. This led to the familiar dipole representation (Lowes & Runcorn 1951). The theory of MAC waves (Hide 1966; Braginsky 1967) showed the screening argument to be invalid in a fluid and the secular variation is now thought to be a manifestation of wave motions in the core.

Density stratification will affect the waves and may have some influence on the observed secular change. Hide (1969) has considered waves influenced by stratification (**N**), rotation (**Ω**) and magnetic fields ($\mathbf{B}_0$). The appropriate numerical values in our case are $N \sim \Omega \gg \mathbf{B}_0 \cdot \mathbf{k}$ where **k** is the wave vector of the wave, for which Hide's general dispersion relation (Hide 1969, equation 2.16) reduces to:

$$\omega^2 \doteqdot \frac{(\mathbf{N} \times \mathbf{k})^2 \, (\mathbf{B}_0 \cdot \mathbf{k})^2}{(2\boldsymbol{\Omega} \cdot \mathbf{k})^2 + (\mathbf{N} \times \mathbf{k})^2}. \tag{5}$$

This frequency is comparable with the Alfven frequency of about $5 \times 10^{-7} s^{-1}$ or a period of a few months. Such oscillations would be screened from our view by the mantle. The only long-period waves will be those that are horizontally polarized because these are not influenced by the stratification. The ordinary MAC or 'slow' waves are transverse and the fluid motion can be horizontal even though the wave propagates vertically through the stable region. Waves incident on the bottom of the conduction region with vertical polarization will be reflected, absorbed or converted to different wave types such as (5).

One consequence of stable density stratification is that there are no radial motions on time-scales long enough to affect magnetic fields. At the core surface the gradient of vertical velocity will vanish, and if the core fluid can be treated as incompressible this implies zero horizontal divergence of the flow. Prompted by these thermal calculations, Whaler (1980) has shown that the horizontal divergence can be derived from observations of the secular change at extrema of the magnetic field, and concludes that the data are entirely consistent with a stable zone at the top of the core. This striking result lends credence to the idea that the top of the core is stratified, although errors in the data may still allow a weakly convecting fluid. The question is still being investigated.

5 Discussion

We have argued that a stratified layer might form at the top of the liquid core if the mantle is cooling at a rather slow rate. The layer can be up to 2000 km thick, depending on the numerical values of the relevant parameters in the core. The calculations are based on a one-component liquid core model. The solutions will be different for a two-component model in which compositional convection plays a dominant role (e.g. Gubbins 1977a). Light material, released by freezing of the heavy fraction at the inner core boundary, rises up driving convection and this will reduce the size of the stable region. Calculations in Paper I suggested that the thermal effect on buoyancy was stabilizing although convection was still maintained by the compositional differences. If there were still a stable zone at the top of the core then there would be the possibility of 'salt fingering' (see Turner 1973). Alternatively the light material might be released in blobs by the crystallization process at the inner core boundary, and if these blobs were large and concentrated enough they could rise to the top without diffusing completely, forming a stable region rich in light material. This possibility will be studied further. Fearn & Loper (1981) have analysed the stability of a two-component fluid in a gravitational field in conductive equilibrium and find that the top 70 km are stably stratified. Such a conductive state would take many times the age of the Earth to set up, and it is not clear how this stable zone relates to one that evolves from an initially convecting core.

All discussions of modes of core convection will remain speculative until some direct or indirect observational evidence is found. The requirements of dynamo theory are satisfied provided some part of the core continues to convect. In fact Bullard & Gubbins (1977) found some kinematic dynamo models that worked more efficiently when the outer fluid was stationary. We have shown the argument that stable fluid will screen magnetic variations to be false because horizontally polarized MAC waves can still propogate. The main evidence for the stable zone comes from Whaler's (1980) study of the secular change, but even this result is open to other interpretations. Seismology is not yet sensitive enough to detect stable regions (Masters 1979) but there is hope for future improvements.

The principal assumption made here is that the temperature at the core–mantle boundary has remained constant. This depends on how the mantle convects and how the whole Earth cools, as well as the distribution of radioactive heat sources. Our model will be improved by combining a simple mantle convection model (e.g. McKenzie & Richter 1981) with a general core model.

Acknowledgments

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Appendix: numerical solution

We have integrated equations (1)–(4) forwards in time from prescribed initial conditions. The initial temperature is the adiabat passing through $T_i(r_c) = T_c$ and the initial value of r_i is zero. The melting temperature of iron is assumed arbitrarily to be $T_m(0) = T_i(0)$ so that freezing of the liquid core begins immediately.

Consider the time step from t to $t + \delta t$, and define

$$T_1 = T\,[r_1(t + \delta t), t + \delta t]\,.$$

Suppose we already know $r_1(t + \delta t)$ and T_1. Then the conduction equation (1) is stepped forward by an explicit time step. Radial derivatives are found by second-order differences on a uniformly spaced grid between $r = r_1(t + \delta t)$ and $r = r_c$ with spacing δr. The temperatures at the two end points (T_c and T_1) are not stepped but their values are used to find derivatives at the adjacent interior points. The new value of $r_i(t + \delta t)$ is found by integrating (3) forwards by one explicit time step.

The method so far is perfectly straightforward but we need values for r_1 and T_1. These come from equations (2) and (4). Consider (2) first and define

$$T_2 = T\,[r_1(t) + \delta r, t + \delta t]$$

$$T_3 = T\,[r_1(t) + 2\delta r, t + \delta t]\,.$$

Then by a Taylor expansion in r about $r_1(t + \delta t)$ at time $t + \delta t$ we have

$$T_2 = T_1 + \frac{\partial T}{\partial r}(\delta r - \dot{r}_1 \delta t) + \frac{\partial^2 T}{\partial r^2}\frac{(\delta r - \dot{r}_1 \delta t)^2}{2} + 0\,(\delta r^2) \tag{A1}$$

where the derivatives of T are evaluated at $r_1(t + \delta t)$ and time $(t + \delta t)$ and dot denotes differentiation with respect to time.

Now we expect to take δt to be of order $\kappa^{-1}\delta r^2$ for the integration of (1) to be stable, so anticipating that the conduction region will grow on the thermal diffusion time-scale let

$$p\delta r^2 = -\dot{r}_1 \delta t \tag{A2}$$

with p, $\dot{r}_1$ of order one. Now all the difference equations are reduced consistently to second-order accuracy in δr^2. (A1) together with a similar equation for T_3 are combined to give

$$\begin{aligned}\frac{\partial T}{\partial t} = \delta r^{-1}\,[&T_2(2 - p\delta r + p^2\delta r^2) + \tfrac{1}{2}T_3(1 + \tfrac{1}{2}p\delta r - \tfrac{1}{4}p^2\delta r^2)\\ &- \tfrac{1}{2}T_1(3 - \tfrac{5}{2}p\delta r + \tfrac{9}{4}p^2\delta r^2)] + 0\,(\delta r^2).\end{aligned} \tag{A3}$$

This is the left side of (2). The right side may be written as:

$$-\gamma\left[\left(\frac{g}{\phi}\right)_{r_1(t)} - \left\{\frac{\partial}{\partial r}\left(\frac{g}{\phi}\right)\right\}_{r_1(t)} p\delta r^2\right] T_1 + 0\,(\delta r^4) = \frac{\partial T}{\partial r}. \tag{A4}$$

(A3) and (A4) give an equation, relating the unknowns p (or $\dot{r}_1$) and T_1, that is linear in T_1.

Equation (4) is solved by explicit forward time differencing using:

$$\left(\frac{\partial T}{\partial t}\right)_{r_1(t+\delta t),\, t+\delta t} = \delta t^{-1}\left\{T_1 - T\,[r_1(t), t] - \left(\frac{\partial T}{\partial r}\right)_{r_1(t),\, t} p\delta r^2\right\}.$$

This gives a second equation in T_1 and p, this time linear in both unknowns. T_1 can be easily eliminated from the two equations leaving a quadratic in p. The appropriate choice of root is obvious. T_1 is found by substituting for p in (A3).

We now have r_1 and T_1 at time $t+\delta t$ for solving the conduction equation. The temperatures at the previous time step are known on an evenly spaced grid from $r_1(t)$ to r_c and these are linearly interpolated onto the new grid between $r_1(t+\delta t)$ and r_c. Solution then proceeds as usual by explicit time differencing.

The method was found to be stable with

$$\delta t = \delta r^2/4\kappa.$$

A typical calculation required several thousand time steps.